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DIOPHANTINE SETS AND DIRICHLET IMPROVABILITY

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ABSTRACT. This note pushes further the discussion about relations between Dirichlet improvable, badly approximable and singular points held in [1] by considering Diophantine sets extending the notion of badly approximability.

1. INTRODUCTION

The aim of this paper is to extend slightly the main result of [1], questioning relations between Dirichlet improvability, badly approximability and singularity. We first provide a short description of this setting and definitions.

For $x \in \mathbb{R}$, we denote $\langle x \rangle := \min\{|x - z| \mid z \in \mathbb{Z}\}$ the distance from x to a nearest integer. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we denote $\mathbf{x} \cdot \mathbf{y} := x_1 y_1 + \cdots + x_n y_n$ the usual scalar product.

The celebrated *Dirichlet's Theorem*, root of Diophantine approximation, reads as follows.

Theorem 1.1 (Dirichlet, 1842). *Let $\mathbf{x} \in \mathbb{R}^n$. For every $Q > 1$ there exists $\mathbf{q} \in \mathbb{Z}^n$ such that*

$$0 < |\mathbf{q}| \leq Q \quad \text{and} \quad \langle \mathbf{x} \cdot \mathbf{q} \rangle \leq Q^{-n}.$$

A main interest in Diophantine approximation is to question when and how Dirichlet's theorem can be improved. This lead Davenport and Schmidt [6] to call a given $\mathbf{x} \in \mathbb{R}^n$ *Dirichlet improvable* if there exists $\varepsilon \in (0, 1)$ such that for all sufficiently large Q , there exists \mathbf{q} with

$$0 < |\mathbf{q}| \leq Q \quad \text{and} \quad \langle \mathbf{x} \cdot \mathbf{q} \rangle \leq \varepsilon Q^{-n}. \quad (\text{DI})$$

We denote by $\mathbf{DI}_n(\varepsilon)$ the set of \mathbf{x} satisfying (DI), so that the set of Dirichlet improvable numbers is

$$\mathbf{DI}_n = \bigcup_{\varepsilon \in (0, 1)} \mathbf{DI}_n(\varepsilon).$$

Furthermore, $\mathbf{x} \in \mathbb{R}^n$ is called *singular* if it is in $\mathbf{DI}_n(\varepsilon)$ for ε arbitrarily small. Denote

$$\mathbf{Sing}_n := \bigcap_{\varepsilon \in (0, 1)} \mathbf{DI}_n(\varepsilon).$$

On the opposite, we say that $\mathbf{x} \in \mathbb{R}^n$ is *badly approximable* if there exists $\varepsilon = \varepsilon(\mathbf{x}) \in (0, 1)$ such that

$$\langle \mathbf{q} \cdot \mathbf{x} \rangle \geq \varepsilon |\mathbf{q}|^{-n}$$

for all $\mathbf{q} \in \mathbb{R}^n$. We denote the set of badly approximable numbers \mathbf{Bad}_n .

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A very natural question is the size and relation between these sets. Regarding sizes, it is well known that \mathbf{DI}_n , \mathbf{Bad}_n and \mathbf{Sing}_n have 0 Lebesgue measure. The set \mathbf{Bad}_n has full Hausdorff dimension, hence \mathbf{DI}_n as well:

$$\dim_H(\mathbf{Bad}_n) = \dim_H(\mathbf{DI}_n) = n.$$

For \mathbf{Sing}_n , the study of the Hausdorff dimension is much more involved. The result requires the powerful variational principle in parametric geometry of numbers by Das, Fishman, Simmons and Urbański [7], extending to a wider setting (including ours: approximation to a linear form) a brilliant result by Cheung and Chevallier [4] for simultaneous approximation.

$$\dim_H(\mathbf{Sing}_n) = \frac{n^2}{n+1}, \quad \text{for } n \geq 2.$$

Regarding relations, we have the inclusion of the disjoint union $\mathbf{Bad}_n \sqcup \mathbf{Sing}_n \subset \mathbf{DI}_n$. When $n = 1$, singular reals are the rationals and the last inclusion is an equality. In higher dimension, one can ask about the set

$$\mathcal{FS}_n := \mathbf{DI}_n \setminus (\mathbf{Bad}_n \sqcup \mathbf{Sing}_n).$$

This set is studied in [1], where it is proved to be uncountable. However, it is probably far from the truth and one conjectures

Conjecture 1.2. $\dim_H(\mathcal{FS}_n) = n$.

Actually, in [1] the result is finer than just uncountability. This requires the definition of exponents of Diophantine approximation. Namely, for $\mathbf{x} \in \mathbb{R}^m$, we define $\omega(\mathbf{x})$ as the supremum of positive reals w such that

$$0 < |\mathbf{q}| \leq Q, \quad \langle \mathbf{q} \cdot \mathbf{x} \rangle \leq Q^{-w}$$

has integer solution \mathbf{q} for arbitrarily large Q .

This exponent is usually referred to as exponent of Diophantine approximation to a linear form. It can take any value in the interval $[n, \infty]$. The main result in [1] asserts that there are uncountably many $\mathbf{x} \in \mathbb{R}^n$ in

$$(\mathbf{DI}_n(\varepsilon) \setminus \mathbf{DI}_n(c_n \varepsilon)) \setminus (\mathbf{Bad}_n \sqcup \mathbf{Sing}_n)$$

with prescribed exponent $\omega(\mathbf{x}) \in [n, \infty]$. Here c_n is an explicit constant. The exclusion of \mathbf{Sing}_n is already included in the exclusion of $\mathbf{DI}_n(c_n \varepsilon)$, but we express this set to enlighten its link to \mathcal{FS}_n .

The motivation of this paper is the obvious observation that $\mathbf{x} \notin \mathbf{Bad}_n$ if $\omega(\mathbf{x}) > n$. This leads to consider the following notion : fix dimension n and let $\varepsilon \in (0, 1)$ and finite $w \geq n$. We define the (ε, w) -Diophantine set

$$\mathcal{D}_w(\varepsilon) := \{\mathbf{x} \in \mathbb{R}^n \mid \langle \mathbf{q} \cdot \mathbf{x} \rangle \geq \varepsilon |\mathbf{q}|^{-w} > 0, \quad \text{for all } \mathbf{q} \in \mathbb{N}^n\}$$

and denote $\mathcal{D}_w := \bigcup_{\varepsilon \in (0, 1)} \mathcal{D}_w(\varepsilon)$. One can see that $\mathcal{D}_n = \mathbf{Bad}_n$.

These Diophantine sets play a role in dynamical systems. For example they describe the Diophantine condition in small divisors problems with applications to KAM theory, Aubry-Mather theory, conjugation of circle diffeomorphisms, and so on (see [2, 5, 10, 13, 20]).

Our main theorem reads as follows.

Theorem 1.3. *Fix the dimension $n \geq 2$, a finite exponent $w > n$, parameters $\varepsilon, \nu \in (0, 1)$ and the constants $c_n = e^{-20(n+1)^3(n+10)}$ and $c'_n = e^{-20(n+1)^2(w+1)(n+10)}$. Then*

- *there exists uncountably many*

$$\mathbf{x} \in (\mathbf{DI}_n(\varepsilon) \setminus \mathbf{DI}_n(c_n \varepsilon)) \setminus \mathcal{D}_w$$

with $\omega(\mathbf{x}) = w$,

- *there exists uncountably many*

$$\mathbf{x} \in (\mathbf{DI}_n(\varepsilon) \setminus \mathbf{DI}_n(c_n\varepsilon)) \cap (\mathcal{D}_w(\nu) \setminus \mathcal{D}_w(c'_n\nu)).$$

As discussed in [1], the constants c_n and c'_n are not optimized.

So far, we considered only approximation to a linear form. An analogous argumentation applies for simultaneous approximation, where we replace (DI) by

$$0 < q < Q \text{ and } \|q\mathbf{x}\| \leq \varepsilon Q^{-1/m}$$

and $\mathcal{D}_w(\varepsilon)$ by

$$\mathcal{D}_\lambda(\varepsilon) := \{\mathbf{x} \in \mathbb{R}^m \mid \|q\mathbf{x}\| \geq \varepsilon |q|^{-\lambda} > 0\}$$

where $\|\cdot\|$ is the distance to a nearest integer point. We get uncountably many points $\mathbf{x} \in \mathbb{R}^m$ in

$$\mathbf{x} \in (\mathbf{DI}_n(\varepsilon) \setminus \mathbf{DI}_n(c_n\varepsilon)) \setminus \mathcal{D}_\lambda$$

with exponent of simultaneous approximation λ and uncountably many $\mathbf{x} \in \mathbb{R}^m$ in the intersection

$$\mathbf{x} \in (\mathbf{DI}_n(\varepsilon) \setminus \mathbf{DI}_n(c_n\varepsilon)) \cap (\mathcal{D}_\lambda(\nu) \setminus \mathcal{D}_\lambda(c'_n\nu)).$$

One could also consider approximation by rational subspaces of dimension exactly d , for $0 \leq d < n$, as introduced by Laurent [9] following Schmidt [14]. However, their usual study via compounds convex bodies involves constants depending on \mathbf{x} (See [12, Proposition 3.1] or [3, §4]) that seem to break our proof. See [1] for more discussion about the sets \mathbf{DI}_n , \mathbf{Bad}_n and \mathbf{Sing}_n related to these approximation settings.

The proof of Theorem 1.3 relies on the parametric geometry of numbers and Roy's fundamental theorem [11]. We provide a short introduction to parametric geometry of numbers in Section 2, and prove our main theorem in Section 3.

2. PARAMETRIC GEOMETRY OF NUMBERS

Parametric geometry of numbers was developped by Schmidt and Summerer [18, 19], answering a question of Schmidt [15]. It was pushed by a fundamental theorem of Roy [11], that was quantified and extended to a matrix setting by Das, Fishman, Simmons and Urbański [7].

Fix dimension $n > 1$ and $\mathbf{x} \in \mathbb{R}^n$. Consider the convex body with parameter $q \geq 0$

$$\mathcal{C}_\mathbf{x}(e^q) := \{\mathbf{y} \in \mathbb{R}^{n+1} \mid |y_i| \leq 1, |\mathbf{y} \cdot (1, \mathbf{x})| \leq e^{-q}\}$$

and its d th successive minima

$$\lambda_{\mathbf{x},d}(q) := \lambda_d(\mathbb{Z}^{n+1}, \mathcal{C}_\mathbf{x}(e^q))$$

for any d between 1 and $n+1$. Following Schmidt and Summerer, we consider the successive minima map consisting of their logs

$$\mathbf{L}_\mathbf{x} : [0, \infty) \rightarrow \mathbb{R}^{n+1} : q \mapsto \mathbf{L}_\mathbf{x}(q) := (L_{\mathbf{x},1}(q), \dots, L_{\mathbf{x},n+1}(q)),$$

where $L_{\mathbf{x},d}(q) := \log \lambda_{\mathbf{x},d}(q)$. It appears that the map $\mathbf{L}_\mathbf{x}$ encodes Diophantine properties of \mathbf{x} .

Proposition 2.1. *Fix dimension n and $\mathbf{x} \in \mathbb{R}^n$.*

- $\mathbf{x} \in \mathbf{DI}_n(\varepsilon)$ if and only if for all sufficiently large q

$$\frac{q}{n+1} - L_{\mathbf{x},1}(q) \geq -\frac{\log(\varepsilon)}{n+1} \tag{2.1}$$

- $\mathbf{x} \in \mathcal{D}_w(\varepsilon)$ if and only if for all sufficiently large q

$$\frac{q}{w+1} - L_{\mathbf{x},1}(q) \leq -\frac{\log(\varepsilon)}{w+1} \quad (2.2)$$

- $\liminf_{q \rightarrow \infty} \frac{L_{\mathbf{x},1}(q)}{q} = \frac{1}{1+\omega(\mathbf{x})}$

See [12, Proposition 3.1] and [1, Lemma 2.1] for proofs.

The following notion of *system* was introduced by Roy [12, Definition 4.5]. The latter approach exactly the family of successive minima maps.

Definition 2.2. Be I a subinterval of $[0, \infty]$ with nonempty interior. A system on I is a continuous linear map $\mathbf{P} = (P_1, \dots, P_{n+1}) : I \rightarrow \mathbb{R}^{n+1}$ with the following properties.

- (i) For each $q \in I$, $0 \leq P_1(q) \leq P_2(q) \leq \dots \leq P_{n+1}(q)$ and $P_1(q) + P_2(q) + \dots + P_{n+1}(q) = q$.
- (ii) If $I' \subset I$ is a nonempty open subinterval on which \mathbf{P} is differentiable, then there exists integers r_1, r_2 with $1 \leq r_1 \leq r_2 \leq n+1$ such that P_{r_1}, \dots, P_{r_2} coincide on I' and have slope $\frac{1}{r_2 - r_1 + 1}$ while all other component P_i are constant on I' .
- (iii) If q is an interior point of I at which \mathbf{P} is not differentiable, and if r_1, r_2, s_1, s_2 are integers such that

$$P'_i(q^-) = \frac{1}{r_2 - r_1 + 1} \quad (r_1 \leq i \leq r_2) \text{ and } P'_j(q^+) = \frac{1}{s_2 - s_1 + 1} \quad (s_1 \leq j \leq s_2)$$

and if $r_1 \leq s_2$ then $P_{r_1}(q) = \dots = P_{s_2}(q)$

Roy's fundamental theorem reads as follows [12, Corollary 4.7], [11, Theorems 1.3 & 1.8].

Theorem 2.3 (Roy, 2015). *Fix dimension $n \geq 1$ and $q_0 \geq 0$. For each $\mathbf{x} \in \mathbb{R}^n$, there exists a system $\mathbf{P} : [q_0, \infty) \rightarrow \mathbb{R}^{n+1}$ such that $L_{\mathbf{x}} - \mathbf{P}$ is bounded on $[q_0, \infty)$. Conversely, for each system $\mathbf{P} : [q_0, \infty) \rightarrow \mathbb{R}^{n+1}$, there exists a $\mathbf{x} \in \mathbb{R}^n$ such that $L_{\mathbf{x}} - \mathbf{P}$ is bounded. In particular, for each $q \geq q_0$ and a constant R_n*

$$|L_{\mathbf{x}} - \mathbf{P}| \leq R_n.$$

The constant R_n induces the constants $c_n = e^{4(n+1)R_n}$ and $c'_n = e^{4(w+1)R_n}$ in Theorem 1.3.

We call two systems \mathbf{P}_1 and \mathbf{P}_2 non-equivalent if there exists q such that $|\mathbf{P}_1(q) - \mathbf{P}_2(q)| > 2C_n$. No point in \mathbb{R}^n has successive minima map close to two non-equivalent systems in the sense of Theorem 2.3.

3. PROOF OF THEOREM 1.3

Fix dimension $n > 1$, exponent $w > n$ and $\varepsilon, \nu \in (0, 1)$. In view of Theorem 2.3 and Proposition 2.1, the proof of Theorem 1.3 reduces to the construction of uncountably many non-equivalent systems satisfying

- $\liminf_{q \rightarrow \infty} \frac{P_1(q)}{q} = \frac{1}{1+w},$
- $\liminf_{q \rightarrow \infty} \left(\frac{q}{n+1} - P_1(q) \right) = -\frac{\log(\varepsilon)}{n+1} + 2R_n,$
- Either $\limsup_{q \rightarrow \infty} \left(\frac{q}{w+1} - P_1(q) \right) = \left(-\frac{\log(\nu)}{w+1} + 2R_n \right)$
or $\limsup_{q \rightarrow \infty} \left(\frac{q}{w+1} - P_1(q) \right) = \infty.$

Note that the second and third inequalities provide the definition of the constants $c_n := e^{4(n+1)R_n}$ and $c'_n = e^{4(w+1)R_n}$.

Choose a parameter $\delta \in (0, 1)$ that will provide uncountability. We first construct elementary systems \mathbf{P}_k^δ on intervals $[q_k, q_{k+1}]$. Denote $\alpha := -\frac{\log(\varepsilon)}{n+1} + 2R_n$ and $\beta_k := (-\frac{\log(\nu)}{w+1} + 2R_n)$ or $\log q_k$. Figure 1 illustrates the construction.

At q_k , we fix $P_1(q_k) = \dots = P_n(q_k) = \frac{q_k}{n+1} - \alpha$ and $P_{n+1}(q_k) = \frac{q_k}{n+1} + n\alpha$.

Define

$$\begin{aligned} p_k &= q_k \frac{w+1}{n+1} - (w+1)(\alpha - \beta_k), & q_{k+1} &= \frac{w}{n} q_k + \frac{(w-1)(n+1)}{n} (\alpha - \beta_k), \\ r_k &= q_k + (n^2 - 1)\alpha, & u_k &= p_k - (n+1)\alpha, \\ s_k^M &= r_k + n \log q_k, & s_k^m &= r_k + \log q_k, \\ s_k &= \delta s_k^m + (1 - \delta) s_k^M \in [s_k^m, s_k^M] & t_k &= s_k + (n-1)(s_k - r_k), \end{aligned}$$

and note that q_{k+1} does not depend on δ .

On the interval $[q_k, r_k]$, the $n-1$ components $P_2 = \dots = P_n$ coincide and have slope $\frac{1}{n-1}$ while P_1 and P_{n+1} are constant. By definition of r_k , $P_2(r_k) = P_{n+1}(r_k)$.

On the interval $[r_k, s_k]$, P_{n+1} has slope 1 while all other component P_i are constant and on $[s_k, t_k]$ the $n-1$ components $P_2 = \dots = P_n$ coincide and have slope $\frac{1}{n-1}$ while P_1 and P_{n+1} are constant. By definition of t_k , $P_{n+1}(t_k) = P_2(t_k)$.

On the interval $[t_k, u_k]$, the n components $P_2 = \dots = P_{n+1}$ coincide and have slope $1/n$, while P_1 is constant.

On the interval $[u_k, p_k]$, P_{n+1} has slope 1 while all other component P_i are constant. By definition of u_k and p_k , at p_k we have $P_{n+1}(p_k) - P_n(p_k) = (n+1)\alpha$ and $P_1(p_k) = \frac{p_k}{w+1} - \beta_k$.

On the interval $[p_k, q_{k+1}]$, P_1 has slope 1 while all other components are constant. By definition of q_{k+1} we have $P_1(q_{k+1}) = \dots = P_n(q_{k+1}) = \frac{q_{k+1}}{n+1} - \alpha$ and $P_{n+1}(q_{k+1}) = \frac{q_{k+1}}{n+1} + n\alpha$.

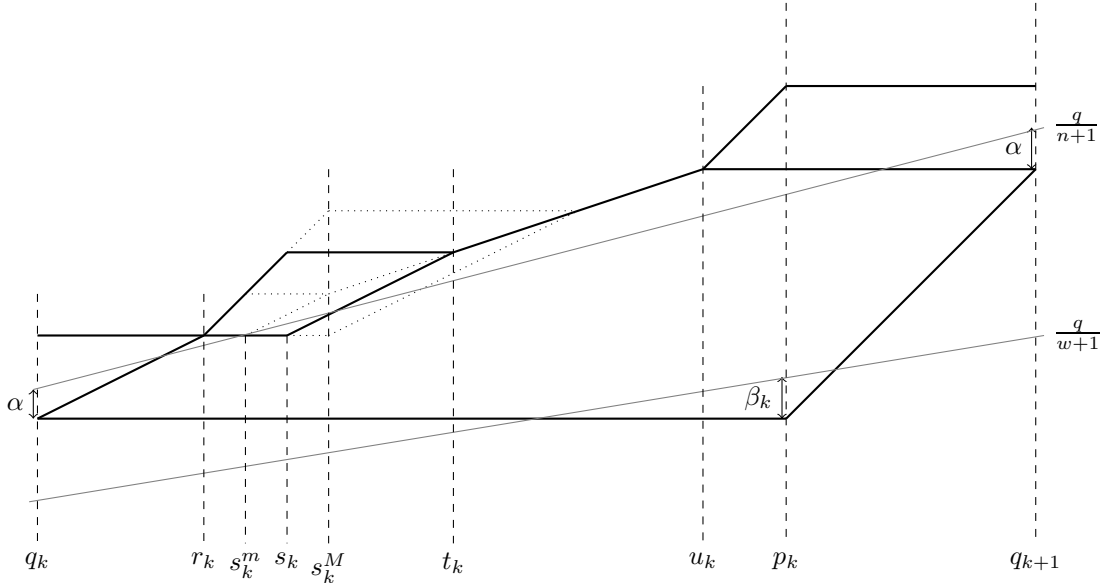


FIGURE 1. Generic system \mathbf{P}_k^δ , dotted are the extremal cases $\delta = 0$ and 1 .

Choose q_1 large enough so that $t_1 < u_1$. Since $w > n$, the inductive sequence $(q_k)_{k \geq 1}$ tends to infinity and we define $\mathbf{P}^\delta = \bigcup_{k \geq 1} \mathbf{P}_k^\delta$ the concatenation of the systems \mathbf{P}_k^δ . It is a system, as properties (i) - (iii) holds, in particular at q_k .

By construction,

$$\begin{aligned} \liminf_{q \rightarrow \infty} \frac{P_1^\delta(q)}{q} &= \liminf_{k \rightarrow \infty} \frac{P_1^\delta(p_k)}{p_k} = \frac{1}{1+w} \\ \min_{[q_k, q_{k+1}]} \frac{q}{n+1} - P_1^\delta(q) &= \alpha = -\frac{\log(\varepsilon)}{n+1} + 2R_n \\ \limsup_{q \rightarrow \infty} \left(\frac{q}{w+1} - P_1^\delta(q) \right) &= \limsup_{k \rightarrow \infty} \left(\frac{p_k}{w+1} - P_1^\delta(p_k) \right) = \limsup_{k \rightarrow \infty} \beta_k \end{aligned}$$

and the requested properties hold. Furthermore, $P_1^\delta(s_k^M) = P_{n+1}(q_k) + \delta \log q_k$, so that

$$\|\mathbf{P}^\delta - \mathbf{P}^{\delta'}\| \geq |\mathbf{P}^\delta(s_k^M) - \mathbf{P}^{\delta'}(s_k^M)| = |\delta - \delta'| \log q_k.$$

Hence, for $\delta \neq \delta'$ the systems \mathbf{P}^δ and $\mathbf{P}^{\delta'}$ are non-equivalent. Uncountability follows. \square

For simultaneous approximation, we rather use the setting and notation of Schmidt and Summerer for parametric geometry of numbers, studying the convex body

$$\mathcal{C}'_{\mathbf{x}}(q) := \left\{ \mathbf{y} \in \mathbb{R}^{m+1} \mid |y_1| \leq e^{mq}, \max_{1 \leq i \leq n} |y_1 x_i - y_{i+1}| \leq e^{-q} \right\}$$

See for example [18, 19, 16, 17]. The analogous proof relies on the construction of dual systems depicted by Figure 2.

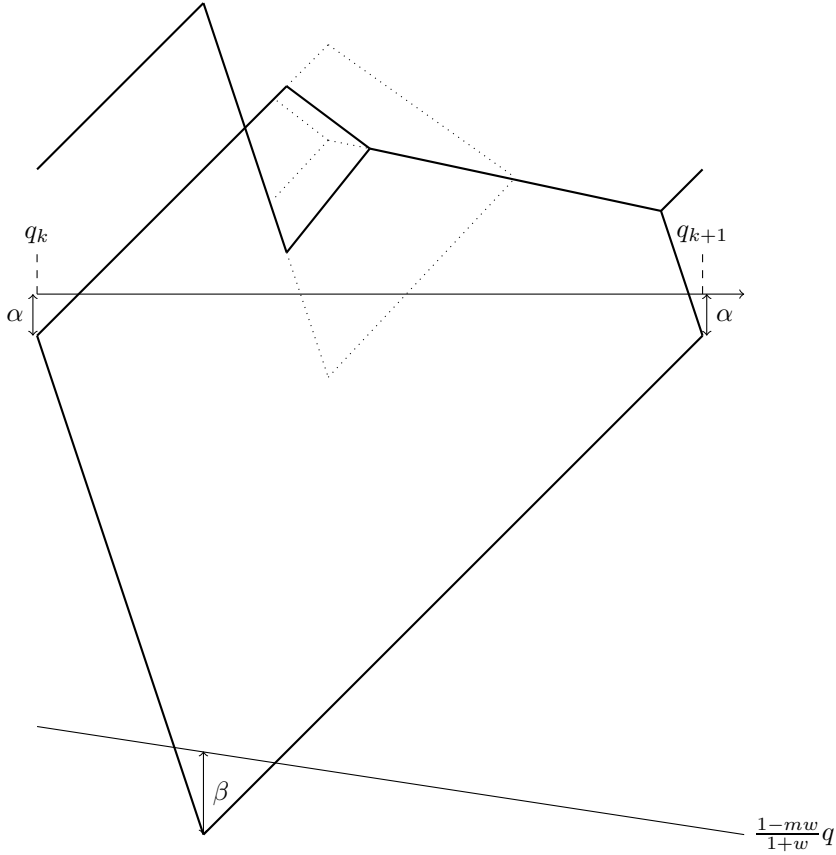


FIGURE 2. Generic system for simultaneous approximation.

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