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▶ **To cite this version:**

Kim Klinger-Logan, Stephen D. Miller, Danylo Radchenko. The D^6R^4 interaction as a Poincaré series, and a related shifted convolution sum. 2022. hal-03954324

HAL Id: hal-03954324

https://cnrs.hal.science/hal-03954324

Preprint submitted on 24 Jan 2023

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THE D^6R^4 INTERACTION AS A POINCARÉ SERIES, AND A RELATED SHIFTED CONVOLUTION SUM

KIM KLINGER-LOGAN, STEPHEN D. MILLER, AND DANYLO RADCHENKO

ABSTRACT. We complete the program, initiated in a 2015 paper of Green, Miller, and Vanhove, of directly constructing the automorphic solution to the string theory D^6R^4 differential equation $(\Delta - 12)f = -E_{3/2}^2$ for $SL(2,\mathbb{Z})$. The construction is via a type of Poincaré series, and requires explicitly evaluating a particular double integral. We also show how to derive the predicted vanishing of one type of term appearing in f's Fourier expansion, confirming a conjecture made by Chester, Green, Pufu, Wang, and Wen motivated by Yang-Mills theory.

1. INTRODUCTION

Many quantities of interest in type IIB string theory are naturally automorphic functions on an arithmetic quotient of a symmetric space, such as $SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})/SO(2)$. For example, the coefficients of the low energy expansion of the 4-graviton scattering amplitude are functions on the coset space G/K, where G denotes the real points of the (algebraic) duality group and K is a maximal compact subgroup of G. Furthermore, U-duality implies that the scattering amplitude should obey transformation rules under a discrete group Γ. For example, in 10 space-time dimensions $G = SL(2,\mathbb{R})$ and the coefficients in the low energy expansion are modular functions. Furthermore, BPS symmetry implies some of the lower order coefficients are in fact eigenfunctions of the laplacian, making them automorphic forms. By now many examples are known and have been well-studied, for example [2, 3, 6, 7, 8, 11, 12, 13].

A good example is the R^4 term in the low energy expansion of maximally supersymmetric string theory. In 10 dimensions, the coefficient function for this term is given by

$$\mathcal{E}_{(1,0)}(z) = 2\zeta(3)E_{3/2}(z)\,, \tag{1.1}$$

where $E_s(z)$ is the non-holomorphic Eisenstein series

$$E_s(z) = \sum_{\gamma\in(B\cap\Gamma)\backslash\Gamma} \mathrm{Im}(\gamma z)^s, \quad \mathrm{Re}(s) > 1\,, \tag{1.2}$$

for $z = x + iy$ in the complex upper half plane $\mathbb{H} \cong SL(2,\mathbb{R})/SO(2)$, $\Gamma = SL(2,\mathbb{Z})$, and B is the subgroup of upper triangular matrices in $SL(2,\mathbb{R})$. In this case, $E_s(z)$ is an eigenfunction for the laplacian $\Delta = y^2\big(\partial_x^2 + \partial_y^2\big)$ and hence $\mathcal{E}_{(1,0)}$ is an automorphic form.

However, BPS protection only carries so far and some of the most intriguing and intricate coefficients are not automorphic forms themselves, but instead related to them

in a nonlinear way. In this paper, we will revisit the $\mathcal{E}_{(1,0)}$ term studied in [5, 10, 12, 14, 15, 16], which mathematically amounts to the solution to the inhomogeneous automorphic differential equation

$$(\Delta - 12)f(z) = -\big(2\zeta(3)E_{3/2}(z)\big)^2. \tag{1.3}$$

In [12], Green, Miller, and Vanhove give the full Fourier expansion for the solution $f(z)$. However, this solution, though explicit, was not computed directly – rather, it was correctly guessed and subsequently verified. Instead, the appendices of [12] describe three potential approaches to solving (1.3) directly, each with their own difficulties to extracting the full Fourier expansion for $f(z)$.

Given the abundance of automorphic differential equations generalizing (1.3) it is desirable to be able construct a solution via a method, rather than to rely on guesswork [1, 3, 6, 7, 8, 9]. In what follows, we resolve the issue that arose in the first of these three approaches, namely the Poincaré series construction in [12, Appendix A]. The obstacle there to obtaining the explicit form of the Fourier expansion for f was the difficulty of computing the integral presented in (A.40) there, i.e., the $a = 3/2$ case of (2.6) below. In Section 2, we generalize this Poincaré series method to equations of the form

$$(\Delta - \lambda)f(z) = -E_a(z)^2\,. \tag{1.4}$$

Specifically, the methods outlined are applicable to the cases when a is a half-integer or even integer and with specific choices of eigenvalue λ. In Section 4, we explicitly compute the complicated integral [12, (A.40)] (equivalently, (2.6)) which arose in the case of $a = 3/2$ and $\lambda = 12$. The method outlined in Section 4 extends (at least experimentally) to the Poisson summation integrals that arise for other half-integer values for a.

The method outlined in Section 4, yields the same Fourier expansion found in [12]:

$$f(z) = \sum_{n\in\mathbb{Z}} f_n(y)e^{2\pi i n x}\,, \tag{1.5}$$

where

$$\begin{aligned} f_n(y) = \delta_{n,0}\tilde{f}(y) &+ \alpha_n K_{7/2}(2\pi|n|y) \\ &+ \sum_{\substack{n_1+n_2=n\\(n_1,n_2)\neq(0,0)}} \sum_{i,j=0,1} M^{i,j}_{n_1,n_2}(\pi|n|y)K_i(2\pi|n_1|y)K_j(2\pi|n_2|y)\,, \end{aligned} \tag{1.6}$$

in which $\tilde{f}, M^{i,j}_{n_1,n_2}$, and α_n are polynomials in y and $\frac{1}{y}$ as defined in [12, Section 2.2]. Explicitly, each α_n can be expressed as

$$\alpha_n = \sum_{n_1+n_2=n} \alpha_{n_1,n_2}\,, \tag{1.7}$$

where $\alpha_{0,0} = 0$,

$$\alpha_{n_1,-n_1} = \frac{8\sigma_2(|n_1|)^2}{21n_1^6\pi^2y^3} \quad \text{for } n_1 \neq 0, \tag{1.8}$$

$$\alpha_{n,0} = \alpha_{0,n} = \frac{64\sigma_2(|n|)(n^2\pi^4 - 90\zeta(3))}{135|n|^{5/2}\pi} \quad \text{for } n \neq 0, \tag{1.9}$$

and

$$\begin{aligned}\alpha_{n_1,n_2} = \operatorname{sgn}(n_1+n_2)\frac{128\pi\sigma_2(|n_1|)\sigma_2(|n_2|)}{45n_1^2n_2^2|n_1+n_2|^{7/2}}\big(&n_1^5+n_2^5+15n_1^4n_2+15n_1n_2^4\\ &-80n_1^3n_2^2-80n_1^2n_2^3+60n_1^2n_2^2(n_1-n_2)\log\left(\left|\frac{n_1}{n_2}\right|\right)\big)\end{aligned} \tag{1.10}$$

for $n \neq 0$ and $n_1n_2 \neq 0$.

Each α_{n_1,n_2} arises from the homogeneous solution to a differential equation, or from the integral [12, (A.40)]; however, the sum (1.7) is of greater interest. Based on ideas from the AdS-CFT correspondence and Yang-Mills theory, Chester, Green, Pufu, Wang, and Wen made the surprising conjecture in [4, Section C.1(a)] that

$$\alpha_n = \sum_{n_1+n_2=n} \alpha_{n_1,n_2} = 0 \tag{1.11}$$

for each $n \neq 0$. In other words, they conjectured that the total sum of the Fourier coefficients in (1.7) corresponding to the homogeneous solution to (1.3) *vanishes*. This is not at all apparent from the formulas above, from which even the convergence of the sum is not manifest. In Section 5 we provide a formal proof of this vanishing conjecture of Chester, Green, Pufu, Wang, and Wen, based on formal (but nonrigorous) manipulations of Dirichlet series.

2. Solution to $(\Delta - \lambda)f = -E_a^2$

In this section, we outline the Poincaré series method for finding solutions of the automorphic differential equation

$$(\Delta - \lambda)f = -E_a^2 \tag{2.1}$$

for special values of a, recapping [12, Appendix A]. Using the definition of the Eisenstein series, one expands

$$E_a(z)^2 \;=\; \sum_{\substack{(m_1,n_1)\in\mathcal{S}\\(m_2,n_2)\in\mathcal{S}}} |m_1n_2 - n_1m_2|^{-2a}\,\mathcal{T}\left(\frac{m_1z+n_1}{m_2z+n_2}\right), \tag{2.2}$$

where

$$\mathcal{S} := \{(0,1)\} \sqcup \{(c,d) \in \mathbb{Z}\times\mathbb{Z} \mid c > 0 \,\&\, \gcd(c,d) = 1\},$$

and

$$\mathcal{T}(x+iy) = \sigma(x/y) := ((x/y)^2+1)^{-a}$$

(terms in (2.2) with $m_1n_2 = n_1m_2$ are replaced with an appropriate limit). This is a consequence of the identities

$$\frac{\operatorname{Re}(\gamma z)}{\operatorname{Im}(\gamma z)} = \frac{n_1n_2 + m_2n_1x + m_1n_2x + m_1m_2(x^2+y^2)}{y(\det\gamma)}$$

and

$$(n_1n_2 + m_2n_1x + m_1n_2x + m_1m_2(x^2+y^2))^2 + (y(\det\gamma))^2 = |m_1z+n_1|^2|m_2z+n_2|^2\,,$$

where $\gamma = \begin{bmatrix} m_1 & n_1 \\ m_2 & n_2 \end{bmatrix}$.

In terms of $u = x/y$ the differential equation (2.1) transforms to

$$\left(\frac{d}{du}\left((1+u^2)\frac{d}{du}\right) - b\right)h_{a,b}(u) = -(u^2+1)^{-a}\,. \tag{2.3}$$

There is an explicit solution to this differential equation satisfying the boundary conditions that $h_{a,b}(u) = O(|u|^{-2a})$ for large values of $u = x/y$. In terms of that solution we define $F(x+iy) := h_{a,b}(\frac{x}{y})$ so that $(\Delta - b)F = -\mathcal{T}$. Summing over $\mathcal{S}$, the solution to (2.1) is given by the sum

$$f(z) = \sum_{\substack{(m_1,n_1)\in\mathcal{S}\\(m_2,n_2)\in\mathcal{S}}} |m_1n_2 - n_1m_2|^{-2a}\, F\bigg(\frac{m_1z+n_1}{m_2z+n_2}\bigg), \tag{2.4}$$

which is absolutely convergent since $h_{a,b}$ is bounded by a constant multiple of σ, hence F is bounded by a constant multiple of $\mathcal{T}$, and the sum in (2.4) converges for $\mathrm{Re}(a) > 1$ since (2.2) does.

The Fourier modes

$$\hat{f}_n(y) = \sum_{n_1+n_2=n} \hat{f}_{n_1,n_2}(y) \tag{2.5}$$

of the solution $f(z) = \sum_{n\in\mathbb{Z}} \hat{f}_n(y)e^{2\pi inx}$ can be derived as in [12, Appendix A] by writing

$$f(z) = \Sigma^{0,0}(z) + \Sigma^{0,1}(z) + \Sigma^{1,0}(z) + \Sigma^{1,1}(z)$$

where

$$\begin{aligned}
\Sigma^{0,0}(z) &= \lim_{(m_1n_2-n_1m_2)\to 0} |m_1n_2 - n_1m_2|^{-2a}\, F\bigg(\frac{m_1z+n_1}{m_2z+n_2}\bigg),\\
\Sigma^{0,1}(z) &= \sum_{m_2=1}^{\infty} \sum_{(n_2,m_2)=1} |m_2|^{-2a}\, F\bigg(\frac{1}{m_2z+n_2}\bigg),\\
\Sigma^{1,0}(z) &= \sum_{m_1=1}^{\infty} \sum_{(n_1,m_1)=1} |m_1|^{-2a}\, F(m_1z+n_1),\\
\Sigma^{1,1}(z) &= \sum_{m_1=1}^{\infty} \sum_{(n_1,m_1)=1} \sum_{m_2=1}^{\infty} \sum_{(n_2,m_2)=1} |m_1n_2 - n_1m_2|^{-2a}\, F\bigg(\frac{m_1z+n_1}{m_2z+n_2}\bigg),
\end{aligned}$$

and applying Poisson summation. Note that the limit in $\Sigma^{0,0}$ is well-defined. In terms of the customary notation $e(x) = e^{2\pi ix}$ it is straightforward to see

$$\Sigma^{0,1}(z) = \Sigma^{1,0}(z) = \sum_{n\in\mathbb{Z}} \widehat{\Sigma}^{0,1}_n(y)e(nx) \quad \text{and} \quad \Sigma^{1,1}(z) = \sum_{n_1\in\mathbb{Z}}\sum_{n_2\in\mathbb{Z}} \widehat{\Sigma}^{1,1}_{n_1,n_2}(y)e((n_1+n_2)x),$$

while inputting Ramanujan sums as in [12, (A.28)] gives the formula

$$\begin{aligned}\Sigma^{0,1}(z) &= \sum_{m_2=1}^{\infty} \frac{1}{m_2^{2a}} \sum_{n_2 \in (\mathbb{Z}/m_2\mathbb{Z})^*} \sum_{n\in\mathbb{Z}} e\bigg(n(x+\frac{n_2}{m_2})\bigg) y\, \widehat{h}(ny) \\ &= \frac{1}{\zeta(2a)} \sum_{n\in\mathbb{Z}} e(nx) n^{1-2a} \sigma_{2a-1}(|n|) y\, \widehat{h}(ny)\,.\end{aligned}$$

Note that in the case of $a = 3/2$ and $\lambda = 12$ studied in [12], both h and $\widehat{h}$ are explicitly solved.

However, the term $\Sigma^{1,1}$ is harder because double-Poisson summation gives the more complicated formula

$$\Sigma^{1,1}(z) = \sum_{n_1,n_2\in\mathbb{Z}} \frac{\sigma_{2a-1}(|n_1|)\sigma_{2a-1}(|n_2|)}{|n_1n_2|^{2a-1}\zeta(2a)^2 y} e((n_1+n_2)x)\, \mathcal{I}(n_1,n_2;y)\,,$$

where

$$\mathcal{I}(n_1,n_2;y) := \int_{\mathbb{R}^2} \frac{h\Big(\frac{r_1r_2+1}{r_2-r_1}\Big)}{|r_2-r_1|^{2a}} e(-(n_1yr_1+n_2yr_2))\, dr_1\, dr_2 \tag{2.6}$$

is the key integral in this approach, as it determines the general Fourier modes via the formulas

$$\begin{aligned}\hat{f}_{0,0}(y) &= \Sigma^{0,0}(y) + 2\widehat{\Sigma}_0^{0,1}(y) + \widehat{\Sigma}_{0,0}^{1,1}(y) \\ \hat{f}_{n,0}(y) &= \hat{f}_{0,n}(y) = \widehat{\Sigma}_n^{0,1}(y) + \widehat{\Sigma}_{n,0}^{1,1}(y) \qquad\qquad \text{for } n\neq 0 \\ \sum_{n_1\neq 0,n} \widehat{f}_{n_1,n-n_1}(y) &= \frac{1}{\zeta(2a)^2 y} \sum_{n_1\neq 0,n} \frac{\sigma_{2a-1}(|n_1|)\sigma_{2a-1}(|n-n_1|)}{|n-n_1|^{2a-1}} \mathcal{I}(n_1,n-n_1;y)\end{aligned} \tag{2.7}$$

as in [12, (A.44)].

However, in [12] Green, Miller and Vanhove were unable to compute the integral $\mathcal{I}(\cdot,\cdot;y)$, instead resorting to guessing the solutions for the Fourier coefficients. In the next two sections we explicitly compute (2.6) for $a = 3/2$, thereby completing the Poincaré series approach and providing a method for computing $\mathcal{I}$ for $a \in \frac{1}{2}\mathbb{Z}_{>0}$.

3. Solutions h to (2.3)

Euler's method of variation of parameters gives solutions to (2.3) of the form

$$\begin{aligned}h_{a,b}(u) =& P_{b_-}(iu) \int_1^u \frac{i(1-\sqrt{4b+1})(k_1^2+1)^{-a} Q_{b_-}(ik_1)}{2b\big(P_{b_+}(ik_1)Q_{b_-}(ik_1) - P_{b_-}(ik_1)Q_{b_+}(ik_1)\big)}\, dk_1 \\ &+ Q_{b_-}(iu) \int_1^u \frac{i(1-\sqrt{4b+1})(k_2^2+1)^{-a} P_{b_-}(ik_2)}{2b\big(P_{b_-}(ik_2)Q_{b_+}(ik_2) - P_{b_+}(ik_2)Q_{b_-}(ik_2)\big)}\, dk_2 \\ &+ c_1 P_{b_-}(iu) + c_2 Q_{b_-}(iu),\end{aligned} \tag{3.1}$$

where $b_+ := \frac{1}{2}\big(\sqrt{4b+1}+1\big)$, $b_- := \frac{1}{2}\big(\sqrt{4b+1}-1\big)$, and P_n and Q_n are Legendre functions of the first and second kind. The latter are defined for integral values of n (the main case of interest in this paper) as

$$P_n(x) := \frac{1}{2^n}\sum_{\ell=0}^{\lfloor \frac{n}{2}\rfloor}(-1)^\ell \binom{n}{\ell}\binom{2n-2\ell}{n}x^{n-2\ell} \tag{3.2}$$

and

$$Q_n(x) := \frac{1}{2}P_n(x)\ln\left(\frac{1+x}{1-x}\right) - \sum_{k=1}^{n}\frac{1}{k}P_{k-1}(x)\cdot P_{n-k}(x). \tag{3.3}$$

The last two terms of (3.1) are solutions to the homogenous part of (2.3). In [12, (A.7)], $h(u)$ is given explicitly for $a = 3/2$ and $b = 12$ as

$$h(u) = \frac{7+44u^2+40u^4}{3\sqrt{1+u^2}} - \frac{16}{3\pi}\left(\frac{4}{3}+5u^2+u(3+5u^2)\arctan(u)\right). \tag{3.4}$$

Note that the first term $\frac{7+44u^2+40u^4}{3\sqrt{1+u^2}}$ already solves (2.3), but the second term is necessary to ensure the boundary condition $h(u) = O(|u|^{-3})$.

3.1. **Half-integer a.** We next examine the more general case where $a \in \frac{1}{2}\mathbb{Z}$ and evaluate $h_{a,b}(u)$. When b_+ and b_- are integers, the denominators of the integrands of $h_{a,b}$ in (3.1) are constant: for $n \in \mathbb{Z}_{>0}$, the Wronskian becomes

$$P_n(x)Q_{n-1}(x) - P_{n-1}(x)Q_n(x) = \frac{1}{n}\,,$$

and one can compute the integrals in (3.1) using trigonometric substitution. In that case $h_{a,b}$ will have a form similar to that of (3.4):

Proposition 3.1. *For $n \in \mathbb{Z}_{\geq 0}$, let $a = n+\frac{1}{2}$ and $b = (2n+1)(2n+2)$. Then all solutions to equation (2.3) have the form*

$$\begin{aligned} h_{a,b}(u) &= c_1 P_{2n+1}(iu) + c_2 Q_{2n+1}(iu) + Q_{2n+1}(iu)\sum_{k=0}^{n} A_k\Big((1+u^2)^{-k+\frac{1}{2}} - 2^{-k+\frac{1}{2}}\Big) \\ &\quad + P_{2n+1}(iu)\left(\sum_{k=0}^{n} B_k\Big[\frac{\arctan u}{(u^2+1)^{k-\frac{1}{2}}} - \frac{\pi}{2^{k+\frac{3}{2}}}\Big] + \sum_{k=0}^{2n+1} C_k\left[\left(\frac{u^2}{1+u^2}\right)^{k+\frac{1}{2}} - \frac{1}{2^{k+1/2}}\right]\right), \end{aligned} \tag{3.5}$$

where A_k, B_k and C_k are absolute constants, and $c_1, c_2 \in \mathbb{C}$ are arbitrary.

In practice, for $a = \frac{2n+1}{2}$ and $b = (2n+1)(2n+2)$ with $n \in \mathbb{Z}_{\geq 0}$, c_1 and c_2 in (3.5) may be chosen so $h_{a,b}(u) = O(|u|^{-2n-1})$. In fact, c_1 and c_2 may be chosen so that this solution matches (3.4) when $a = 3/2$. Notice that even for $a \neq 3/2$ the solution (3.5) already has a form similar to that of (3.4). Specifically, $h_{a,b}(u)$ for $a \in \frac{1}{2}\mathbb{Z}_{>0}$ is a linear combination of $\arctan u$, polynomials in u, and polynomials in $\frac{u}{\sqrt{u^2+1}}$. This form allows us to use the method outlined in Section 4 to compute the associated integral $\mathcal{I}$ in (2.6) when $a = \frac{2n+1}{2}$ and $b = (2n+1)(2n+2)$.

3.2. **Integer a.** The case when $a \in \mathbb{Z}_{>0}$ results in two different forms of $h_{a,b}$ depending on whether a is even or odd. Choosing $b = a(a-1)$ again forces the Wronskian to be a constant. However, computing the integrals in (3.1) yields different types of solutions (arising from different trigonometric identities) depending on whether a is odd or even.

For specific choices of even values of a, computing the integrals in (3.1) yields $h_{a,b}(u)$ of a similar form to that when $a = n + \frac{1}{2}$ (i.e., a linear combination of polynomials in $u, \frac{1}{\sqrt{1+u^2}}$, and $\arctan u$). In fact when, a is even it appears one can find a formula for $h_{a,b}(u)$ as a rational function satisfying the bound $h_{a,b}(u) = O(|u|^{-a})$, which is still adequate for absolute convergence for positive even integers a. The lack of the $\arctan u$ factor in the case when a is even allows for the integral $\mathcal{I}$ to be computed directly, without resorting to the method outlined in Section 4.

When a is odd, however, computing the integrals in (3.1) results in the appearance of factors of $\arctan^2 u$, making the computation of $\mathcal{I}$ more difficult. Our method below appears to require some modifications in order to work; for example, the integration by parts method applied in the next section runs into trouble with these $\arctan^2 u$ terms.

4. The integral $\mathcal{I}$

In this section we show how to explicitly compute the integral (2.6) for $a = n + \frac{1}{2}$ and $b = (2n+1)(2n+2)$, illustrated with the example of $(a, b) = (\frac{3}{2}, 12)$ studied in [11]. Recall that for this choice of a and b the integral is

$$\mathcal{I}(n_1, n_2; y) := \int_{\mathbb{R}^2} \frac{h\left(\frac{r_1 r_2 + 1}{r_2 - r_1}\right)}{|r_2 - r_1|^3} e(n_1 y r_1 + n_2 y r_2)\, dr_1\, dr_2,$$

with h given by (3.4). The main tool will be a seven-fold integration by parts (in general, a $(4a+1)$-fold integration by parts) which removes the $\arctan(u)$ term completely. To wit, define

$$H(r_1, r_2) := \frac{h\left(\frac{r_1 r_2 + 1}{r_2 - r_1}\right)}{|r_2 - r_1|^3}$$

and the differential operator

$$Df := \frac{\partial}{\partial r_1} f + \frac{\partial}{\partial r_2} f. \tag{4.1}$$

Then

$$D^7(e(-y(n_1 r_1 + n_2 r_2))) = 128i \cdot e(-y(n_1 r_1 + n_2 r_2))(n_1 + n_2)^7 \pi^7 y^7,$$

and seven-fold integration by parts yields the formula

$$\mathcal{I}(n_1, n_2; y) = -\frac{1}{128i(n_1 + n_2)^7 \pi^7 y^7} \int_{\mathbb{R}^2} D^7 H(r_1, r_2) e(-y(n_1 r_1 + n_2 r_2))\, dr_1\, dr_2 \tag{4.2}$$

(there are no boundary terms because of the decay conditions). To compute the integral we separate out terms in the integrand depending on whether or not they are smooth

at $r_1 = r_2$. Namely, we write

$$\frac{D^7 H(r_1, r_2)}{128i(n_1+n_2)^7\pi^7} = T_1(r_1, r_2) + T_2(r_1, r_2)\,,$$

where direct calculation shows

$$T_1(r_1, r_2) = \frac{|r_1 - r_2|}{(1+r_1^2)^7(1+r_2^2)^7} p_1(r_1, r_2) \tag{4.3}$$

$$T_2(r_1, r_2) = -\frac{3}{\pi(1+r_1^2)^{15/2}(1+r_2^2)^{15/2}} p_2(r_1, r_2), \tag{4.4}$$

with $p_1(r_1, r_2)$ and $p_2(r_1, r_2)$ explicitly-computable polynomials, and examine their respective contributions to the total integral. In particular certain complicated terms such as the arctangent from (3.4) disappear through this seven-fold differentiation.

4.1. **The first term T_1.** Let

$$\mathcal{I}_1(n_1, n_2, y) := -\frac{1}{y^7} \int_{\mathbb{R}^2} \frac{|r_1 - r_2| \cdot p_1(r_1, r_2)}{(1+r_1^2)^7(1+r_2^2)^7} \cdot e(-y(n_1 r_1 + n_2 r_2)) dr_1\, dr_2\,.$$

After changing variables $r_1 \to r_1 + r_2$ and then breaking up the integration into two ranges depending on the sign of r_1, we write

$$\begin{aligned}
\frac{4}{y\, n_1^2\, n_2^2} \mathcal{I}_1(n_1, n_2, y) &= \int_{\mathbb{R}^2} \frac{3r_1^4(r_1+2r_2) \cdot \widetilde{p}_1(r_1, r_2) \cdot e(-y(n_1(r_1+r_2) + n_2 r_2))}{\pi n_1^2 n_2^2 (r_2^2+1)^7 y^8 |r_1|^3 ((r_1+r_2)^2+1)^7}\, dr_1\, dr_2 \\
&= \int_{\mathbb{R}} \int_0^\infty \frac{3r_1(r_1+2r_2) \cdot \widetilde{p}_1(r_1, r_2) \cdot e(-y(n_1(r_1+r_2) + n_2 r_2))}{\pi n_1^2 n_2^2 (r_2^2+1)^7 y^8 ((r_1+r_2)^2+1)^7}\, dr_1\, dr_2 \\
&\quad + \int_{\mathbb{R}} \int_0^\infty \frac{3r_1(2r_2-r_1) \cdot \widetilde{p}_1(-r_1, r_2) \cdot e(-y(n_1(r_2-r_1) + n_2 r_2))}{\pi n_1^2 n_2^2 (r_2^2+1)^7 y^8 ((r_1-r_2)^2+1)^7}\, dr_1\, dr_2\,,
\end{aligned}$$

where $(r_1 + 2r_2) \cdot \widetilde{p}_1(r_1, r_2) := p_1(r_1 + r_2, r_2)$.

Assume $n_1 + n_2 \neq 0$ and change the order of integration, so that the r_2-integrals can be computed by shifting the contours up or down depending on whether $n_1 + n_2$ is negative or positive. After picking up poles this results in the expression

$$\frac{4}{y\, n_1^2\, n_2^2} \mathcal{I}_1(n_1, n_2, y) = \int_0^\infty \sum_{k=3}^{6} \frac{2\pi c_k e^{-2\pi(n_1+n_2)y}}{\pi^{k+2} y^{k+2} n_1^2 n_2^2 (n_1+n_2)^{k+1}} N(n_1, n_2) \cdot A_k(r_1) r_1^{-k}\, dr_1 \tag{4.5}$$

where $c_3 = -8/3$, $c_4 = 16$, $c_5 = 40$, $c_6 = -40$,

$$N(n_1, n_2) = 15 + 2(n_1+n_2)\pi y[15 + 4(n_1+n_2)\pi y(3 + (n_1+n_2)\pi y)] \tag{4.6}$$

and

$$A_k(r_1) = \begin{cases} \frac{1}{2}e^{-2i\pi n_1 r_1 y} + \frac{1}{2}e^{2i\pi n_1 r_1 y} + \frac{1}{2}e^{-2i\pi n_2 r_1 y} + \frac{1}{2}e^{2i\pi n_2 r_1 y} & \text{for } k \text{ odd} \\ \frac{1}{2}ie^{-2i\pi n_1 r_1 y} - \frac{1}{2}ie^{2i\pi n_1 r_1 y} + \frac{1}{2}ie^{-2i\pi n_2 r_1 y} - \frac{1}{2}ie^{2i\pi n_2 r_1 y} & \text{for } k \text{ even.} \end{cases}$$

This integral converges, but the integrals $\int_0^\infty A_k r_1^{-k}\,dr_1$ do not. We instead evaluate

$$\int_0^\infty A_k r_1^{-k} e^{-\epsilon r_1} r_1^s\,dr_1$$

which is holomorphic for $\mathrm{Re}(s) > 6$ and $\mathrm{Re}(\epsilon) > 0$, and calculate the full sum by analytic continuation. Namely, we multiply the integrand in (4.5) by $e^{-\varepsilon r_1} r_1^a$ to get

$$\begin{aligned}\sum_{k=3}^{6} \frac{2\pi c_k e^{-2\pi(n_1+n_2)y}}{\pi^{k+2} y^{k+2} n_1^2 n_2^2 (n_1+n_2)^{k+1}} N(n_1,n_2) \int_0^\infty A_k r_1^{-k} e^{-\epsilon r_1} r_1^a\,dr_1 \\ = \frac{e^{-2\pi y(n_1+n_2)} N(n_1,n_2)}{\pi^7 n_1^2 n_2^2 y^8 (n_1+n_2)^7} \sum_{k=3}^{6} \pi^{6-k} c_k\,\Gamma(a-k+1)(n_1+n_2)^{6-k} A_k(a,\epsilon)\end{aligned}$$

where

$$A_k(a,\epsilon) = \begin{cases} i^{k-1} \sum_{\ell=1}^2 \Big[(2\pi n_\ell y + i\epsilon)^{k-1}(\epsilon - 2i\pi n_\ell y)^{-a} \\ \qquad\qquad +(2\pi n_\ell y - i\epsilon)^{k-1}(\epsilon + 2i\pi n_\ell y)^{-a}\Big], & k \text{ odd}, \\ i^{k-1} \sum_{\ell=1}^2 \big[(\epsilon - 2i\pi n_\ell y)^{k-1-a} + i(2\pi n_\ell y - i\epsilon)^{k-1}(\epsilon + 2i\pi n_\ell y)^{-a}\big], & k \text{ even}. \end{cases}$$

Not all of the terms in this sum are holomorphic at $a = \epsilon = 0$, but the full sum is and evaluates there to give

$$\begin{aligned}\mathcal{I}_1(n_1,n_2,y) = \frac{2e^{-2\pi y(n_1+n_2)}}{45\pi^2 y^2 (n_1+n_2)^7} N(n_1,n_2)\ \times \\ \Big[(n_1+n_2)\big(-94n_1^2n_2^2 + 14n_1^3n_2 + n_1^4 + 14n_1n_2^3 + n_2^4\big)\ + \\ 30n_1^2n_2^2(n_1-n_2)\big(\log(-in_1y) + \log(in_1y) - \log(-in_2y) - \log(in_2y)\big)\Big].\end{aligned}$$

This indeed matches the homogeneous part of the solution found by Green, Miller and Vanhove in [12] and given by the α_{n_1,n_2} term in [12, (2.40)]. Explicitly,

$$\frac{4}{y|n_1|^2|n_2|^2}\sigma_2(|n_1|)\sigma_2(|n_2|)\mathcal{I}_1(n_1,n_2,y) = \alpha_{n_1,n_2}\sqrt{y}K_{7/2}(2\pi|n_1+n_2|y) \tag{4.7}$$

for $y > 0$, $n_1 + n_2 \neq 0$ and α_{n_1,n_2} as in [12, (2.40)].

4.2. **The second term T_2.** We write $p_2(r_1,r_2)$ as $\sum_{a,b=0}^{12} d(a,b) r_1^a r_2^b$ for some coefficients $d(a,b)$. The integral

$$\mathcal{I}_2(n_1,n_2,y) = \frac{3}{\pi y^7}\int_{\mathbb{R}^2} \frac{p_2(r_1,r_2)\cdot e(-y(r_1n_1+r_2n_2))}{(1+r_1^2)^{15/2}(1+r_2^2)^{15/2}} dr_1\,dr_2$$

can be evaluated using the function

$$\mathcal{K}(\xi) := \int_{\mathbb{R}} \frac{1}{(1+r^2)^{15/2}} e^{-2\pi i r\xi}\,dr = \frac{256\pi^7|\xi|^7 K_7(2\pi|\xi|)}{135135} \tag{4.8}$$

and its derivatives using the formula

$$\frac{\partial^k}{\partial \xi^k}\left(\frac{e(-r\xi)}{(1+r^2)^{15/2}}\right) = \frac{(-2\pi i)^k r^k e(-r\xi)}{(1+r^2)^{15/2}}\,. \tag{4.9}$$

Assume $n_1 \neq 0$, $n_2 \neq 0$ and $n_1 + n_2 \neq 0$. The change of variables $n_1 \mapsto \frac{\xi_1}{y}, n_2 \mapsto \frac{\xi_2}{y}$ along with (4.8)–(4.9) together give

$$\mathcal{I}_2(n_1,n_2,y) = \frac{3}{\pi y^7}\int_{\mathbb{R}^2} \frac{p_2(r_1,r_2)\, e(-r_1\xi_1 - r_2\xi_2)}{(1+r_1^2)^{15/2}(1+r_2^2)^{15/2}} dr_1\, dr_2 \tag{4.10}$$

$$= \frac{3}{\pi y^7}\sum_{a,b=0}^{12} d(a,b)\int_{\mathbb{R}} \frac{r_1^a\, e(-r_1\xi_1)}{(1+r_1^2)^{15/2}} dr_1 \cdot \int_{\mathbb{R}} \frac{r_2^b\, e(-r_2\xi_2)}{(1+r_2^2)^{15/2}}\, dr_2 \tag{4.11}$$

$$= \frac{3}{\pi y^7}\sum_{a,b=0}^{12} \frac{d(a,b)}{(-2\pi i)^{a+b}} \cdot \frac{\partial^a \mathcal{K}}{\partial \xi_1^a}(\xi_1) \cdot \frac{\partial^b \mathcal{K}}{\partial \xi_2^b}(\xi_2). \tag{4.12}$$

Using the derivative formula $K_\nu'(z) = \frac{1}{2}(-K_{\nu-1}(z) - K_{\nu+1}(z))$ as well as the recurrence $K_\nu(z) = K_{\nu-2}(z) + \frac{2(\nu-1)}{z}K_{\nu-1}(z)$, all K-Bessel functions arising can be expressed in terms of combinations of K_0 and K_1. Applying the change of variables $\xi_i \mapsto y n_i$ we find that (4.10) equals

$$\begin{aligned}
&\frac{8n_1n_2}{3y^5(n_1+n_2)^7}\times \\
&\Big[\Big(-n_1^4\big(1-16\pi^2n_2^2y^2\big) - n_1^3\big(14n_2 - 40\pi^2n_2^3y^2\big) + 2n_1^2n_2^2\big(8\pi^2n_2^2y^2+47\big) \\
&\qquad - 4\pi^2n_1^5n_2y^2 - 2n_1n_2^3\big(2\pi^2n_2^2y^2+7\big) - n_2^4\Big)y^5(n_1+n_2) \\
&\qquad \times \operatorname{sgn}(n_1y)\operatorname{sgn}(n_2y)K_1(2\pi|n_1y|)K_1(2\pi|n_2y|) \\
&\quad + \Big(-n_1^3\big(30-98\pi^2n_2^2y^2\big) + 2n_1^2n_2\big(19\pi^2n_2^2y^2+15\big) + 27\pi^2n_1^4n_2y^2 \\
&\qquad - 19\pi^2n_1^5y^2 - 15\pi^2n_1n_2^4y^2 - \pi^2n_2^5y^2\Big)\frac{n_2y^4}{\pi} \\
&\qquad \times \operatorname{sgn}(n_1y)K_1(2\pi|n_1y|)K_0(2\pi|n_2y|) \\
&\quad + \Big(-\pi^4y^5(n_1+n_2)^2\big(13n_1^2n_2 + n_1^3 - 65n_1n_2^2 + 19n_2^3\big) - 30\pi^2n_2^2y^3(n_2-n_1)\Big) \\
&\qquad \times \frac{n_1y}{\pi^3}\operatorname{sgn}(n_2y)K_0(2\pi|n_1y|)K_1(2\pi|n_2y|) \\
&\quad + \Big(5n_1^2\big(4\pi^2n_2^2y^2-3\big) + 8\pi^2n_1^3n_2y^2 - 2\pi^2n_1^4y^2 + 2n_1n_2\big(4\pi^2n_2^2y^2+15\big) \\
&\qquad - n_2^2\big(2\pi^2n_2^2y^2+15\big)\Big)2n_1n_2y^5(n_1+n_2) \\
&\qquad \times K_0(2\pi|n_1y|)K_0(2\pi|n_2y|)\Big]
\end{aligned} \tag{4.13}$$

Our computation shows that $\mathcal{I}_2$ matches the principal part of the solution found in [12]. Specifically, when $n_1 \neq 0$, $n_2 \neq 0$ and $n_1 + n_2 \neq 0$,

$$\frac{4}{y|n_1|^2|n_2|^2}\sigma_2(|n_1|)\sigma_2(|n_2|)\mathcal{I}_2(n_1, n_2, y) = \hat{f}^P_{n_1,n_2}(y)\,,$$

where $f^P_{n_1,n_2}$ is given in [12, (2.35)] and $\mathcal{I}_2$ computed as above. In other words, our computation matches the principal part of the original solution in regions (iii) and (iv).

Note that when $n_1 = 0$ and $n_2 = 0$, T_2 can be computed directly and is 0. Similarly, when just one of $n_1 = 0$ or $n_2 = 0$, the integral without the exponential may be computed directly while the integral with the exponential may be computed as above.

When $n_1 + n_2 = 0$ (but $n_1 \neq 0$, $n_2 \neq 0$), one can use a limiting argument to make this match the solution provided in the 2015 paper in region (v). The other cases of $n_1 = 0$ or $n_2 = 0$ are given as limits as well.

5. A conjecture of Chester, Green, Pufu, Wang and Wen

The solution to the differential equation (1.3) was computed in [12] and above here in terms of instanton/anti-instanton pairs. In particular, the solution involves a $K_{7/2}$-Bessel function term for the homogeneous solutions, which is manifestly nonzero for particular pairs α_{n_1,n_2} in (1.8)–(1.10). However, when summed together to give the Fourier modes of the solution itself, they give a rather complicated expression. Chester, Green, Pufu, Wang and Wen have conjectured, based on ideas from the AdS-CFT correspondence and Yang-Mills theory, that this total sum actually *vanishes* [4].

We present a mathematical calculation which explains the vanishing in (1.11). However, our explanation falls short of a proof because it involves manipulations of divergent series. We will use these to derive the equivalent statement

$$\sum_{\substack{m+n=r\\ mn\neq 0}} \alpha_{m,n} = -\sum_{\substack{m+n=r\\ mn=0}} \alpha_{m,n} = -\alpha_{0,r} - \alpha_{r,0}. \tag{5.1}$$

First, the desired identity (1.11) can be more succinctly stated in terms of

$$\widetilde{\alpha}_{m,n} = \frac{45|m+n|^{5/2}}{128\pi}\alpha_{m,n} \;= \sigma_2(|m|)\sigma_2(|n|)\Big(g\Big(\frac{2m}{m+n}\Big) + g\Big(\frac{2n}{m+n}\Big)\Big) \quad \text{for } m, n \neq 0,$$

where

$$g(x) = 60x\log|x| - 60\log|x| - 60 + \frac{24}{x} + \frac{4}{x^2}\,.$$

In terms of $\widetilde{\alpha}_{m,n}$, note that (1.9) gives

$$\widetilde{\alpha}_{r,0} = \widetilde{\alpha}_{0,r} = \sigma_2(r)\big(r^2\zeta(2) + 60\zeta'(-2)\big)$$

since $\zeta(2) = \frac{\pi^2}{6}$ and $\zeta'(-2) = \frac{\zeta(3)}{4\pi^2}$. Thus (1.11) is equivalent to

$$\sum_{\substack{m+n=r\\ mn\neq 0}} \widetilde{\alpha}_{m,n} = -2\sigma_2(r)\big(r^2\zeta(2) + 60\zeta'(-2)\big). \tag{5.2}$$

We will explain how

$$A_r := \sum_{c,d>0} (cd)^2 \sum_{\substack{ad-bc=r \\ ab\neq 0}} \Big(g\Big(\frac{2ad}{r}\Big) + g\Big(\frac{2bc}{-r}\Big)\Big) \ = -2\sigma_2(r)\big(r^2\zeta(2) + 60\zeta'(-2)\big). \tag{5.3}$$

which is equivalent to (5.2) by reindexing the sum.

The rest of this section is devoted to a formal manipulation of divergent series to demonstrate (5.3). We stress at the outset that although parts of this argument can be made rigorous, appropriately and correctly normalizing all the divergences involved is beyond the scope of this paper.

Let

$$T_r(c,d) := \sum_{ad-bc=r, ab\neq 0} g\Big(-\frac{2bc}{r}\Big), \tag{5.4}$$

so that

$$A_r \ = \ 2\sum_{c,d>0} c^2\, d^2\, T_r(c,d)\,. \tag{5.5}$$

To analyze $T_r(c,d)$ for fixed $c,d>0$ we pick a solution $a=a^*, b=b^*$ to $ad-bc=r$ with $|b^*|$ minimal. The general solution to $ad-bc=r$ can be parameterized as

$$a = a^* + \frac{m}{(c,d)}c,\ \ b = b^* + \frac{m}{(c,d)}d, \qquad m\in\mathbb{Z}, \tag{5.6}$$

since $(a-a^*)\frac{d}{(c,d)} - (b-b^*)\frac{c}{(c,d)} = 0$ forces these coefficients $a-a^*$ and $b-b^*$ of the coprime integers $\frac{d}{(c,d)}$ and $\frac{c}{(c,d)}$ to have that form.

Our next goal is to perform Poisson summation on $T_r(c,d)$ over $m\in\mathbb{Z}$ (in terms of the parametrization in (5.6)), keeping in mind that the conditions that neither a nor b vanish may constrain m. Vanishing of a (resp. b) in the equation $ad-bc=r$ forces c (resp. d) to divide r. We break into four cases:

(1) Both $c|r$ and $d|r$: In this case there is a solution to $ad-bc=r$ with $a=0$, as well as a solution with $b=0$. Thus $b^*=0$, hence $a^*=\frac{r}{d}$, implying that the term with $m=0$ must be excluded as well as the term with $m=-\frac{r(c,d)}{cd}$ and $g(-\frac{2bc}{r}) = g(2)$ (note this m is in fact an integer).
(2) $d|r$ but $c\nmid r$: In this case again $b^*=0$, but no solution with $a=0$; thus only $m=0$ is excluded.
(3) $c|r$ but $d\nmid r$: Only the term $g(2)$, corresponding to the value of $-\frac{2bc}{r}$ at the solution to $ad-bc=r$ with $a=0$, is omitted.
(4) Both $c\nmid r$ and $d\nmid r$: Here neither a nor b ever vanishes and no $m\in\mathbb{Z}$ is excluded.

Let $x_{c,d} := -\frac{b(c,d)}{d}$, and $y_{c,d} := \frac{2cd}{r(c,d)}$. Formal Poisson summation over $m \in \mathbb{Z}$ shows

$$\begin{aligned}\sum_{n\in\mathbb{Z}} g(y_{c,d}(n+x_{c,d})) = &-\frac{30y_{c,d}}{\pi}\operatorname{Im}\operatorname{Li}_2(e(x_{c,d})) - 60\log|1-e(x_{c,d})| \\ &- \frac{24\pi i}{y_{c,d}}\frac{1+e(x_{c,d})}{1-e(x_{c,d})} - \frac{16\pi^2}{y_{c,d}^2}\frac{e(x_{c,d})}{(e(x_{c,d})-1)^2},\end{aligned} \tag{5.7}$$

for $x_{c,d} \notin \mathbb{Z}$. This can be seen from the formal integral computation that the function

$$x \mapsto |x|^{s-1}\operatorname{sgn}(x)^\eta, \quad \eta \in \{0,1\},$$

has Fourier transform

$$|r|^{-s}\operatorname{sign}(-r)G_\eta(s),$$

where

$$G_0(s) = 2(2\pi)^{-s}\Gamma(s)\cos(\pi s/2) \quad \text{and} \quad G_1(s) = 2i(2\pi)^{-s}\Gamma(s)\sin(\pi s/2),$$

then applying Poisson summation (to this function and its s-derivatives), and finally invoking the customary calculus of treating the $r=0$ expression for the Fourier transform as zero (which is valid for $\operatorname{Re}(s) \ll 0$). When $x_{c,d} \in \mathbb{Z}$ we instead use the limiting case

$$\sum_{n\neq 0} g(ny_{c,d}) \ = 60\log\Big|\frac{y_{c,d}}{2\pi}\Big| + 60 + \frac{4\pi^2}{3y_{c,d}^2}\,, \tag{5.8}$$

which can be seen more directly by noting that $g(x)+g(-x) = 8\big(-15\log(|x|) + \frac{1}{x^2} - 15\big)$.

It follows that

$$T_r(c,d) = 60\log\Big|\frac{cd}{r(c,d)\pi}\Big| + 60 + \frac{r^2\pi^2(c,d)^2}{3(cd)^2} - g(2)\delta_{c|r} \tag{5.9}$$

for $d|r$, and

$$\begin{aligned}T_r(c,d) = -g(2)\delta_{c|r} &- \frac{60cd}{r(c,d)\pi}\operatorname{Im}\operatorname{Li}_2(e(x_{c,d})) - 60\log|1-e(x_{c,d})| \\ &- \frac{12r(c,d)\pi i}{cd}\frac{1+e(x_{c,d})}{1-e(x_{c,d})} - \frac{4r^2(c,d)^2\pi^2}{c^2d^2}\frac{e(x_{c,d})}{(e(x_{c,d})-1)^2}\end{aligned} \tag{5.10}$$

for $d \nmid r$.

Next define the double Dirichlet series

$$L_r(s,w) := \sum_{c,d>0} c^{-s}d^{-w}T_r(c,d) \ = \sum_{d\geq 1}\frac{L_{d,r}(s)}{d^w} \quad \text{where} \quad L_{d,r}(s) := \sum_{c>0} c^{-s}T_r(c,d)$$

so that $A_r = 2L_r(-2,-2)$. We first formally compute $L_{d,r}(s)$ using the following Lemma.

Lemma 5.1. *For $\operatorname{Re}(s)$ sufficiently large one has*

$$\sum_{c\geq 1}\frac{\log|(c,d)|}{c^s} = \zeta(s)\Big(\sum_{\ell|d}\Lambda(\ell)\ell^{-s}\Big)$$

and

$$\sum_{c\geq 1}\frac{(c,d)^k}{c^{s+k}}=\zeta(s+k)\prod_{p|d}\Big(1+(1-p^{-k})(p^{-s}+\cdots+p^{-v_p(d)s})\Big),$$

where Λ is the von Mangoldt function and $v_p(n)$ denotes the valuation of n at prime p.

Proof. The first formula directly follows from the identity $\log|(c,d)|=\sum_{l|(c,d)}\Lambda(l)$ and changing the order of summation in the resulting double sum. To prove the second formula, we note that the Dirichlet series on the left has multiplicative coefficients, and so it suffices to prove that

$$\sum_{n\geq 0}\frac{p^{k\min(n,v_p(d))}}{p^{n(s+k)}}=(1-p^{-s-k})^{-1}\Big(1+(1-p^{-k})(p^{-s}+\cdots+p^{-v_p(d)s})\Big),$$

which is easily verified after multiplying both sides by $(1-p^{-s-k})$. □

From (5.9) and the above Lemma, it follows that

$$L_{d,r}(-2)=-60\zeta'(-2)-r^2\zeta(2)-g(2)\sigma_2(r)\qquad\text{for } d|r$$

since $\zeta'(s)=-\sum_{n\geq 2}\frac{\log(n)}{n^s}$. The case when $d\nmid r$ is more complicated. We claim

$$L_{d,r}(-2)=-g(2)\sigma_2(r)\qquad\text{for } d\nmid r \tag{5.11}$$

from which it formally follows that

$$\sum_{d\geq 1}\frac{L_{d,r}(-2)}{d^w}=-g(2)\sigma_2(r)\zeta(w)-\sigma_{-w}(r)(r^2\zeta(2)+60\zeta'(-2)),$$

and hence (5.3) by specializing $w=-2$.

To obtain (5.11) we appeal to (5.10) and set $c'=\frac{c}{(c,d)}$, $d'=\frac{d}{(c,d)}$, and $r'=\frac{r}{(c,d)}$, and recall that $x_{c,d}\equiv-\frac{b(c,d)}{d}\equiv\frac{\alpha}{d'}\pmod 1$ is uniquely determined by $\alpha c'=r'\pmod{d'}$. Thus for $d\nmid r$

$$\begin{aligned}L_{d,r}(s)=\sum_{\substack{0<c<d\\(c,d)|r}}\Big[&-\frac{60\operatorname{Im}\operatorname{Li}_2(e(x_{c,d}))}{r(c,d)d^{s-2}\pi}\zeta(s-1,\frac{c}{d})-\frac{60\log|1-e(x_{c,d})|}{d^s}\zeta(s,\frac{c}{d})\\&-\frac{12r(c,d)\pi i}{d^{s+2}}\frac{1+e(x_{c,d})}{1-e(x_{c,d})}\zeta(s+1,\frac{c}{d})-\frac{4r^2(c,d)^2\pi^2}{d^{s+4}}\frac{e(x_{c,d})}{(e(x_{c,d})-1)^2}\zeta(s+2,\frac{c}{d})\Big]\\&-g(2)\sigma_{-s}(r).\end{aligned}$$

When this is evaluated at $s=-2$, the Hurwitz zeta-functions for c and $d-c$ combine together to give vanishing of the sum, and (5.11) is established as claimed.

Acknowledgements. We wish to thank Guillaume Bossard, Eric D'Hoker, Dorian Goldfeld, Michael B. Green, Henryk Iwaniec, Axel Kleinschmidt, Daniel Persson, Boris Pioline, Peter Sarnak, Pierre Vanhove, and Don Zagier for their helpful conversations. K. K-L. acknowledges support from NSF Grant number DMS-2001909, while S.D.M. acknowledges support from NSF Grant number DMS-1801417.

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