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# Leading constants of rank deficient Gaussian elimination

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## 1 Introduction

Rank revealing factorizations play a key role in many applications, especially in producing compact generators for structured matrices [4, 1, 7, 8].

In the context of exact linear algebra, Gaussian elimination based factorizations prevail, since there is no numerical instability issue. We refer to [6, 2, 3, 9, 5] for further details on the various decompositions (LSP, LQUP, PLUQ, CUP, PLE, CRE, etc) and algorithms to compute them.

We focus here on algorithms reducing to fast matrix multiplication, with a rank sensitive cost. For instance, in [6, 2], cost analysis are given in the form  $O(mnr^{\omega-2})$  where  $\omega$  is a feasible exponent for matrix multiplication.

In order to compare algorithms more precisely, it is essential to estimate the leading constant in the cost of such algorithms. We will denote by  $T_{XX}(n)$  the leading term in the cost of running algorithm **XX** on dimension  $n$ , i.e. such that the cost is  $T_{XX}(n) + o(T_{XX}(n))$ . In the following we will assume that the product of two  $n \times n$  matrices over a field can be computed in  $T_{MM}(n) = C_\omega n^\omega$ , where  $2 < \omega \leq 3$ .

In [6, Table 1] and [2], such constants are shown but only under strong genericity assumptions: namely, that the matrices are square with full rank ( $m = n = r$ ) and must also sometimes have generic rank profile.

The purpose of this report is to state leading constants for the costs of the main algorithms when matrices are rank deficient and with no genericity assumption.

Unfortunately, the non-predictable rank distribution among the blocks being processed leads to overestimate some costs, which prevents us to produce tight constants (i.e. matching the known ones in the generic case).

Following [6, 2, 3], the rank revealing factorizations CUP, PLE, PLUQ, CRE can all be computed using either a uni- [6] or bi-dimensional [2] divide and conquer algorithm.

Recall from [6, Table 1] that the cost for solving  $n$  triangular systems of order  $m$  is

$$T_{\text{TRSM}}(m, n) = \frac{C_\omega}{2^{\omega-1} - 2} m^{\omega-1} n$$

## 2 Unidimensional recursive algorithms

Most divide and conquer Gaussian elimination follow split one of the two dimension of the work matrix in halves (either the row or column dimension). Refer to [6] for a survey on these algorithms and the related matrix factorizations being produced: CUP, PLE, LSP, LQUP, PLUQ, etc.

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**Lemma 1.**  $T_{\text{uni}}(m, n, r) = \alpha mn r^{\omega-2} + \beta mr^{\omega-1}$  with

$$\alpha = \frac{C_\omega}{2-2^{3-\omega}}, \quad \beta = \max\left(0, \frac{C_\omega(3-2^{\omega-1})}{2^{\omega-1}-2}\right).$$

Note that with  $(\omega, C_\omega) = (3, 2)$ , this becomes  $T_{\text{uni}}(m, n, r) = 2mnr^{\omega-2} - \frac{2}{3}mr^{\omega-1}$ . The constant overshoots by a factor of 2, the actual constant for  $m = n = r$  which is  $2/3$ .

*Proof.*

$$\begin{aligned} T_{\text{uni}}(m, n, r) &= T_{\text{uni}}\left(\frac{m}{2}, n, r_1\right) + T_{\text{uni}}\left(\frac{m}{2}, n - r_1, r_2\right) + \frac{C_\omega}{2^{\omega-1}-2} \frac{m}{2} r_1^{\omega-1} + C_\omega \frac{m}{2} (n - r_1) r_1^{\omega-2} \\ &\leq mn \left( \frac{\alpha}{2} (r_1^{\omega-2} + r_2^{\omega-2}) + \frac{C_\omega}{2} \right) + m \left( \frac{\beta}{2} (r_1^{\omega-1} + r_2^{\omega-1}) + \frac{C_\omega}{2} \left( \frac{3-2^{\omega-1}}{2^{\omega-1}-2} \right) r_1^{\omega-1} \right) \end{aligned}$$

since  $a^{\omega-2} + b^{\omega-2} \leq 2^{3-\omega}(a+b)^{\omega-2}$ . For any  $\omega \geq \log_2 3 + 1$ , we have  $3 - 2^{\omega-1} \leq 0$  and therefore setting  $\beta = 0$  we get

$$T_{\text{uni}}(m, n, r) \leq mn r^{\omega-2} \left( 2^{2-\omega} \alpha + \frac{C_\omega}{2} \right)$$

Lemma 1 is satisfied as soon as  $\alpha \geq 2^{2-\omega} \alpha + \frac{C_\omega}{2}$  which gives  $\alpha = \frac{C_\omega}{2-2^{3-\omega}}$ .

When  $\omega \leq 1 + \log_2 3$ , we have

$$T_{\text{uni}}(m, n, r) \leq mn r^{\omega-2} \left( 2^{2-\omega} \alpha + \frac{C_\omega}{2} \right) + mr^{\omega-1} \left( \frac{\beta}{2} + \frac{C_\omega(3-2^{\omega-1})}{2(2^{\omega-1}-2)} \right)$$

Similarly, Lemma 1 is satisfied for  $\alpha = \frac{C_\omega}{2-2^{3-\omega}}$  and  $\beta = \frac{C_\omega(3-2^{\omega-1})}{2^{\omega-1}-2}$ . □

### 3 Bi-dimensional recursive algorithms

Alternatively, a bi-dimensionnal split can be applied, which generates in the general case four recursive calls. Refer to [2, 3] for an algorithm and a survey on the related matrix decompositions: PLUQ, CRE, etc.

**Lemma 2.**  $T_{\text{bi}}(m, n, r) = \alpha mn r^{\omega-2} + \beta(m+n)r^{\omega-1}$  with

$$\alpha = \frac{3^{4-\omega} C_\omega}{(4-4^{3-\omega})}, \quad \beta = \max\left(0, \frac{C_\omega(3-2^{\omega-1})}{2^\omega - 4}\right).$$

*Proof.* Following [2]

$$\begin{aligned} T_{\text{bi}}(m, n, r) &\leq T_{\text{bi}}\left(\frac{m}{2}, \frac{n}{2}, r_1\right) + T_{\text{bi}}\left(\frac{m}{2} - r_1, \frac{n}{2}, r_2\right) + T_{\text{bi}}\left(\frac{m}{2}, \frac{n}{2} - r_1, r_3\right) + T_{\text{bi}}\left(\frac{m}{2}, \frac{n}{2} - r_2 - r_3, r_4\right) \\ &\quad + \frac{C_\omega}{2} \left( \frac{1}{2^{\omega-1}-2} - 1 \right) ((m+n)r_1^{\omega-1} + mr_2^{\omega-1} + nr_3^{\omega-1}) + \frac{C_\omega}{4} mn (3r_1^{\omega-2} + r_2^{\omega-2} + r_3^{\omega-2}) \\ &\leq mn r^{\omega-2} \left( \frac{4^{3-\omega} \alpha + 3^{4-\omega} C_\omega}{4} \right) \\ &\quad + (m+n)r^{\omega-1} \left( \frac{C_\omega(3-2^{\omega-1})}{2(2^\omega - 4)} \right) ((m+n)r_1^{\omega-1} + mr_2^{\omega-1} + nr_3^{\omega-1}) \\ &\quad + \frac{\beta}{2} (m+n) (r_1^{\omega-1} + r_2^{\omega-1} + r_3^{\omega-1} + r_4^{\omega-1}) \end{aligned}$$

For any  $\omega \geq \log_2 3 + 1$ , we have  $3 - 2^{\omega-1} \leq 0$  and therefore setting  $\beta = 0$  we get

$$T_{\text{bi}}(m, n, r) \leq mn r^{\omega-2} \left( \frac{4^{3-\omega} \alpha + 3^{4-\omega} C_\omega}{4} \right).$$

Lemma 2 is satisfied as soon as  $\alpha \geq \frac{4^{3-\omega} \alpha + 3^{4-\omega} C_\omega}{4}$ , which gives  $\alpha = \frac{3^{4-\omega} C_\omega}{(4-4^{3-\omega})}$

When  $\omega \leq \log_2 3 + 1$ , then the same value for  $\alpha$  and  $\beta = \frac{C_\omega(3-2^{\omega-1})}{2^\omega - 4}$  satisfy Lemma 2. □

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