Unifying Graded Linear Logic and Differential Operators

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Abstract

Linear Logic refines Intuitionnistic Logic by taking into account the resources used during the proof and program computation. In the past decades, it has been extended to various frameworks. The most famous are indexed linear logics which can describe the resource management or the complexity analysis of a program. From an other perspective, Differential Linear Logic is an extension which allows the linearization of proofs. In this article, we merge these two directions by first defining a differential version of Graded linear logic: this is made by indexing exponential connectives with a monoid of differential operators. We prove that it is equivalent to a graded version of previously defined extension of finitary differential linear logic. We give a denotational model of our logic, based on distribution theory and linear partial differential operators with constant coefficients.

1 Introduction

Linear logic (LL) [18] and its differential counterpart [12] give a framework to study resource usages of proofs and programs. These logics were invented by enriching the syntax of proofs with new constructions observed in denotational models of λ-calculus [19, 9]. The exponential connective ! introduces non-linearity in the context of linear proofs and encapsulate the notion of resource usage. This notion was refined into *parametrised exponentials* [20, 11, 15, 17], where exponential connectives are indexed by annotations specifying different behaviors. Our aim here is to follow Kerjean's former works [23] by indexing formulas of Linear Logic with Differential Operators. Thanks to the setting of Bounded Linear Logic, we formalize and deepen the connection between Differential Linear Logic and Differential Operators.

The fundamental linear decomposition of LL is the decomposition of the usual non-linear implication $\Rightarrow$ into a linear one $\multimap$ from a set of resources represented by new connective !: $(A \Rightarrow B) \equiv (!A \multimap B)$. Bounded Linear Logic (BLL) [20] was introduced as the first attempt to use typing systems for complexity analysis. But our interest for this logic stems from the fact that it extends LL with several exponential connectives which are indexed by *polynomially bounded intervals*. Since then, some other indexations of LL have been developed for many purposes, for example IndLL [11] where the exponential modalities are indexed by some functions, or the graded logic $\mathsf{B}_{\mathcal{S}}\mathsf{LL}$ [6, 17, 25] where they are indexed by the elements of a semiring $\mathcal{S}$. This theoretical development finds applications in programming languages [1, 14].

Differential linear logic [12] (DiLL) consists in an a priori distinct approach to linearity, and base itself on the denotational semantics of linear proofs in terms of linear functions. In the syntax of LL, the dereliction rule states that if a proof is linear, one can then forget its linearity

and consider it as non-linear. To encapture differentiation, DiLL is based on a codereliction rule which is the syntactical opposite of the dereliction. It states that from a non-linear proof (or a non-linear function) one can extract a linear approximation of it, which, in terms of functions, is exactly the differential (one can notice that here, the analogy with resources does not work). Then, models of DiLL interpret the codereliction by different kinds of differentiation [8, 3].

A first step towards merging the graded and the differential extension of LL was made by Kerjean in 2018 [23]. In this paper, she defines an extension of DiLL, named D-DiLL, in which the exponential connectives ? and ! are indexed with a *fixed* linear partial differential operator with constant coefficients (LPDOcc) D. There, formulas $!_D A$ and $?_D A$ are respectively interpreted in a denotational model as spaces of functions or distributions which are solutions of the differential equation induced by D. The dereliction and codereliction rules then represent respectively the resolution of a differential equation and the application of a differential operator. This is a significant step forward in our aim to make the theory of programming language and functional analysis closer, with a Curry-Howard perspective. In this work, we will generalize D-DiLL to a logic indexed by a monoid of LPDOcc.

Contributions. This work considerably generalizes, corrects and consolidates the extention of DiLL to differential operators sketched in [23]. It extends D-DiLL in the sense that the logic is now able to deal with all LPDOcc and combine their action. It corrects D-DiLL as the denotational interpretation of indexed exponential $?_D$ and $!_D$ are changed, leaving the interpretation of *inference rules* unchanged but reversing their type in a way that is now compatible with graded logics. Finally, this work consolidate D-DiLL as it proved a cut-elimination procedure in the general graded case, making use of a splitting additive property on the monoid of LPDOcc.

Outline. We begin this paper in Section 2 by reviewing Differential Linear Logic and its semantics in terms of functions an distributions. We also recall the definition of $\mathsf{B}_{\mathcal{S}}\mathsf{LL}$. Section 3 focuses on the definition of an extension of $\mathsf{B}_{\mathcal{S}}\mathsf{LL}$, where we construct a finitary differential version for it and prove a cut-elimination theorem. The cut-elimination procedure mimicks partly the one of DiLL or $\mathsf{B}_{\mathcal{S}}\mathsf{LL}$, but also deals with completely new interactions with inference rules. Then, Section 4 generalizes D-DiLL into a framework whith several indexes and shows that it corresponds to our finitary differential $\mathsf{B}_{\mathcal{S}}\mathsf{LL}$ indexed by a monoid of LPDOcc. It formally constructs a denotational model for it. This gives in particular a new semantics for $\mathsf{B}_{\mathcal{S}}\mathsf{LL}$. Finally, Section 5 discusses the addition of an indexed promotion to differential $\mathsf{B}_{\mathcal{S}}\mathsf{LL}$ and possible definitions for a semiring of differential operators.

2 Linear logic and its extensions

Linear Logic refines Intuitionistic Logic by introducing a notion of linear proofs. Formulas are defined according to the following grammar (omitting neutral elements which do not play a role here): $A, B := A \otimes B \mid A \parr B \mid A \,\&\, B \mid A \oplus B \mid \ ?A \mid !A$.

The linear negation $(_)^\perp$ of a formula is *defined* on the syntax and is involutive, with in particular $(!A)^\perp := ?(A)^\perp$. The new connector ! enjoys structural rules, respectively called weakening w, contraction c, dereliction d and promotion p:

$$\frac{\Gamma, \vdash \Delta}{\Gamma, !A \vdash \Delta}\, w \qquad \frac{\Gamma, !A, !A \vdash \Delta}{\Gamma, !A \vdash \Delta}\, c \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma, !A \vdash \Delta}\, d \qquad \frac{!\Gamma \vdash A}{!\Gamma \vdash !A}\, p$$

These structural rules can be understood in terms of resources: a proof of $A \vdash B$ uses exactly once the hypothesis A while a proof of $!A \vdash B$ might use A an arbitrary number of times. Notice that the dereliction allows to forget the linearity of a proof by making it non-linear. Differentiation is then introduced through a "codereliction" rule $\bar{d}$, which is symmetrical to d and allows to linearize a non-linear proof [12]. To express the cut-elimination with the promotion rule,

other costructural rules are needed, which find a natural interpretation in terms of differential calculus. Note that the first version of DiLL, called DiLL_0 does not feature the promotion rule, which was introduced in later versions [26]. The exponential rules of DiLL_0 are then w, c, d with the following coweakening $\bar{w}$, cocontraction $\bar{c}$ and codereliction $\bar{d}$ rules, given here in a one-sided flavour.

$$\frac{\vdash}{\vdash !A}\,\bar{w} \qquad \frac{\vdash \Gamma, !A \qquad \vdash \Delta, !A}{\vdash \Gamma, \Delta, !A}\,\bar{c} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, !A}\,\bar{d}$$

2.1 Distribution theory as a semantical interpretation of DiLL

DiLL originates from vectorial refinements of models of LL [9], which mainly keep their discrete structure. However, the exponential connectives and rules of DiLL can also be understood as operations on smooth functions or distributions [27]. In the whole paper, $(_)' := \mathcal{L}(_, \mathbb{R})$ is the dual of a (topological) vector space, and *distributions with compact support* are by definition linear continuous maps on the space of smooth scalar maps, that is elements of $(\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}))'$. Distributions are sometimes described as "generalized functions"[1] Let us recall the notation for dirac operator, which is a distribution with compact support and used a lot in the rest of the paper: $\delta : v \in \mathbb{R}^n \mapsto (f \mapsto f(v)) \in (\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}))'$.

Recently, Kerjean [23] gave an interpretation of the connective ? by a space of smooth scalar functions, while ! is interpreted as the space of linear maps acting on those functions, that is a space of *distributions*:

$$[\![?A]\!] := \mathcal{C}^\infty([\![A]\!]', \mathbb{R}) \qquad [\![!A]\!] := \mathcal{C}^\infty([\![A]\!], \mathbb{R})'.$$

Note that the language of distributions applies to all models of DiLL as noticed by Ehrhard on Köthe spaces [8]. Nicely, every exponential rule of DiLL has an interpretation in terms of functions and distributions, through the following natural transformations. In the whole paper, E and F denote topological vector spaces, which will represent the interpretation $[\![A]\!]$ and $[\![B]\!]$ of formulas A, B of DiLL. For the sake of readability, we will denote the natural transformations (*e.g.* $d, \bar{d}$) by the same label as the deriving rule they interpret, and likewise for connectors (*e.g.* $?, \otimes, !$) and functors.

- The weakening $w : \mathbb{R} \to ?E$ maps $1 \in \mathbb{R}$ to the constant function at 1, while the coweakening $\bar{w} : \mathbb{R} \to !E$ maps $1 \in \mathbb{R}$ to dirac distribution at 0: $\delta_0 : f \mapsto f(0)$.
- The dereliction $d : E' \to ?(E')$ maps a linear function to itself while the codereliction $\bar{d} : E \to !E$ maps a vector v to the distribution mapping a function to its differential at 0 according to the vector v :

$$d : \ell \mapsto \ell \qquad \bar{d} : v \mapsto (D_0(_)(v) : f \mapsto D_0(f)(v))\,.$$

- The contraction $c : ?E \otimes ?E \to ?E$ maps two scalar functions f, g to their scalar multiplication $f.g$ while the cocontraction $\bar{c} : !E \otimes !E \to !E$ maps two distributions ψ and ϕ to their *convolution product* $\psi * \phi : f \mapsto \psi\,(x \mapsto \phi(y \mapsto f(x+y)))$.

The natural transformations $w, \bar{w}, d, \bar{d}$ can also be directly constructed from the biproduct on topological vector spaces and Schwartz' Kernel Theorem expressing Seely isomorphisms.

It is particularly important to us that these operations take place in the dual setting of Fréchet or Df Nuclear topological vector spaces [22]. Positive formulas (left stable by $\otimes$!) are

[1] Indeed, any function with compact support $g \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ acts as a distribution $T_g \in (\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}))'$ with compact support, through integration: $T_g : f \mapsto \int gf$. It is indeed a distribution, as it acts linearly (and continuously) on smooth functions.

interpreted as Nuclear DF spaces while Negative formulas (left stable by ⅋ ?) are interpreted by Nuclear Fréchet spaces. We will not dive into the details of these definitions, but the reader should keep in mind that the formulas are always interpreted as *reflexive topological vector spaces*, that is spaces E which are isomorphic to their double dual E''. The model of functions and distribution is thus a model of *classical* DiLL, in which $[\![(_)^\perp]\!] := (_)'$.

2.2 Differential operators as an extension of DiLL

A first advance in merging the graded and the differential extensions of LL was made by Kerjean in 2018 [23]. In this paper, she defines an extension of DiLL named D-DiLL. This logic is based on a *fixed single* linear partial differential operator D, which appears as a single index in exponential connectives $!_D$ and $?_D$.

The abstract interpretation of ? and ! as spaces of functions and distributions respectively allows to generalize them to spaces of solutions and parameters of differential equations. To do so, we generalize the action of $D_0(_)$ in the interpretation of $\bar{d}$ to another differential operator D. The interpretation $\bar{d}$ then corresponds to the application of a differential operator while the interpretation of d corresponds to the resolution of a differential equation (which is ℓ itself when the equation is $D_0(_) = \ell$, but this is specifically due to the involutivity of D_0).

In D-DiLL, the exponential connectives can be indexed by a fixed differential operator. It admits a denotational semantics for a specific class of those, whose resolution is particularly easy thanks to the existence of a fundamental solution. A *Linear Partial Differential Operator with constant coefficients (LPDOcc)* acts linearly on functions $f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$, and by duality acts also on distributions. In what follows, each a_α will be an element of $\mathbb{R}$. By definition, only a finite number of such a_α are non-zero[2].

$$D : f \mapsto \left(z \mapsto \sum_{\alpha \in \mathbb{N}^n} a_\alpha \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(z) \right) \qquad \hat{D} : \phi \mapsto \left(f \mapsto \phi \left(\sum_{\alpha \in \mathbb{N}^n} (-1)^{|\alpha|} a_\alpha \frac{\partial^{|\alpha|} f}{\partial x^\alpha} \right) \right) \tag{1}$$

We make D act on distributions through the following equation:

$$D(\phi) := \left(\phi \circ \widehat{D} : f \mapsto \phi(D(f)) \right) \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})'.$$

Definition 2.1. *Let D be a LPDOcc. A* fundamental solution *of D is a distribution $\Phi_D \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})'$ such that $\hat{D}(\Phi_D) = \delta_0$.*

Proposition 2.2 (Hormander, 1963)**.** *LPDOcc* distribute over convolution*, meaning that $\hat{D}(\phi * \psi) = \hat{D}(\phi) * \psi = \phi * \hat{D}(\psi)$ for any $\phi, \psi \in !E$.*

The previous proposition is easy to check and means that knowing the fundamental solution of D gives access to the solution $\psi * \Phi_D$ of the equation $\hat{D}(_) = \psi$. It is also the reason why indexation with several differential operators is possible. Luckily for us, LPDOcc are particularly well-behaved and always have a fundamental solution. The proof of the following well-known theorem can for example be found in [21, 3.1.1].

Theorem 2.3 (Malgrange-Ehrenpreis)**.** *Every linear partial differential operator with constant coefficients admits exactly one fundamental solution.*

Using this result, D-DiLL gives new definitions for d and $\bar{d}$, depending of a LPDOcc D:

$$d_D : f \mapsto \Phi_D * f \qquad \bar{d}_D : \phi \mapsto \phi \circ D$$

[2]The coefficients $(-1)^{|\alpha|}$ in equation 1 originates from the intuition of distributions as generalized functions. With this intuition, it is natural to want that for each smooth function f, $D(T_f) = T_{D(f)}$ Hence, the calculus of $T_{D(f)}$ with partial integration reveals the $(-1)^{|\alpha|}$.

2.3 Indexed linear logics: resources, effects and coeffects

Since Girard's original BLL [20], several systems have implemented indexed exponentials to keep track of resource usage [7, 13]. More recently, several authors [17, 15, 6] have defined a modular (but a bit less expressive) version $\mathrm{B}_{\mathcal{S}}\mathrm{LL}$ where the exponentials are indexed (more specifically "graded", as in graded algebras) by elements of a given semiring $\mathcal{S}$.

Definition 2.4. *A* semiring $(\mathcal{S}, +, 0, \times, 1)$ *is given by a set* $\mathcal{S}$ *with two associative binary operations on* $\mathcal{S}$*: a sum* $+$ *which is commutative and has a neutral element* $0 \in \mathcal{S}$ *and the product* $\times$ *which is distributive over the sum and has a neutral element* $1 \in \mathcal{S}$*.*
Such a semiring is said to be commutative *when the multiplication is commutative.*
An ordered semiring *is a semiring endowed with a partial order* $\leq$ *such that the sum and the product are monotonic.*

This type of indexation, named *grading*, has been used in particular to study effects and coeffects, as well as resources [6, 5, 15]. The main feature is to use this grading in a type system where some types are indexed by elements of the semiring. This is exactly what is done in the logic $\mathsf{B}_{\mathcal{S}}\mathsf{LL}$, where $\mathcal{S}$ is an ordered semiring. The exponential rules of $\mathsf{B}_{\mathcal{S}}\mathsf{LL}$ are adapted from those of LL, and agree with the intuitions that the index x in $!_x A$ is a witness for the usage of resources of type A during the proof/program.

$$\frac{\Gamma \vdash B}{\Gamma, !_0 A \vdash B}\ w \qquad \frac{\Gamma, !_x A, !_y A \vdash B}{\Gamma, !_{x+y} A \vdash B}\ c \qquad \frac{\Gamma, A \vdash B}{\Gamma, !_1 A \vdash B}\ d \qquad \frac{!_{x_1} A_1, \ldots, !_{x_n} A_n \vdash B}{!_{x_1 \times y} A_1, \ldots, !_{x_n \times y} A_n \vdash !_y B}\ p$$

Finally, a subtyping rule is also added, which uses the order of $\mathcal{S}$. In Section 3, we will use an order induced by the additive rule of $\mathcal{S}$, and this subtyping rule will stand for a generalized dereliction.

$$\frac{\Gamma, !_x A \vdash B \quad x \leq y}{\Gamma, !_y A \vdash B}\ d_I$$

3 A differential $\mathsf{B}_{\mathcal{S}}\mathsf{LL}$

In this section, we extend a graded linear logic with indexed coexponential rules. We define and prove correct a cut-elimination procedure.

Formulas and proofs We define a differential version of $\mathsf{B}_{\mathcal{S}}\mathsf{LL}$ by extending its set of exponential rules. Here, we will restrict ourselves to a version without promotion, as it has been done for DiLL originally. Following the ideas behind DiLL, we add *costructural* exponential rules: a coweakening $\bar{w}$, a cocontraction $\bar{c}$, an indexed codereliction $\bar{d}_I$ and a codereliction $\bar{d}$. The set of exponential rules of our new logic $\mathsf{DB}_{\mathcal{S}}\mathsf{LL}$ is given in Figure 1. Note that by doing so we study a *classical* version of $\mathsf{B}_{\mathcal{S}}\mathsf{LL}$, with an involutive linear duality.

Remark 3.1. *In* $\mathsf{B}_{\mathcal{S}}\mathsf{LL}$*, we consider a semiring* $\mathcal{S}$ *as set of indices. With* $\mathsf{DB}_{\mathcal{S}}\mathsf{LL}$*, we do not need a semiring: since this is a promotion-free version, only one operation (the sum) is important. Hence, in* $\mathsf{DB}_{\mathcal{S}}\mathsf{LL}$*,* $\mathcal{S}$ *will only be a monoid. This modification requires two precisions:*

- *The indexed dereliction uses the fact that* $\mathcal{S}$ *is an ordered semiring. Here, the order will* always *be defined through the sum:* $\forall x, y \in \mathcal{S},\ x \leq y \iff \exists x' \in \mathcal{S},\ x + x' = y$*. This is due to the fact that for compatiblity with coexponential rules, we always need* $\forall x, 0 \leq x$*.*
- *In* $\mathsf{B}_{\mathcal{S}}\mathsf{LL}$*, the dereliction is indexed by* 1*, the neutral element of the product. In* $\mathsf{DB}_{\mathcal{S}}\mathsf{LL}$*, we will remove this index since we do not have a product operation and simply use* $!$ *and* $?$ *instead of* $!_1$ *and* $?_1$*.*

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?_0 A}\, w \qquad \frac{\vdash \Gamma, ?_x A, ?_y A}{\vdash \Gamma, ?_{x+y} A}\, c \qquad \frac{\vdash \Gamma, ?_x A \qquad x \leq y}{\vdash \Gamma, ?_y A}\, d_I \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\, d$$

$$\frac{\vdash}{\vdash !_0 A}\, \bar{w} \qquad \frac{\vdash \Gamma, !_x A \qquad \vdash \Delta, ?_y A}{\vdash \Gamma, \Delta, !_{x+y} A}\, \bar{c} \qquad \frac{\vdash \Gamma, !_x A \qquad x \leq y}{\vdash \Gamma, !_y A}\, \bar{d}_I \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, !A}\, \bar{d}$$

Figure 1: Exponential rules of $\mathsf{DB}_{\mathcal{S}}\mathsf{LL}$

Since every element of $\mathcal{S}$ is greater than 0, we have two admissible rules which will appear in the cut elimination procedure: an indexed weakening w_I and an indexed coweakening $\bar{w}_I$:

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?_x A}\, w_I \quad := \quad \frac{\dfrac{\vdash \Gamma}{\vdash \Gamma, ?_0 A}\, w}{\vdash \Gamma, ?_x A}\, d_I \qquad \frac{\vdash}{\vdash !_x A}\, \bar{w}_I \quad := \quad \frac{\dfrac{\vdash}{\vdash !_0 A}\, \bar{w}}{\vdash !_x A}\, d_I \ .$$

Definition of the cut elimination procedure In order to define the cut elimination procedure of $\mathsf{DB}_{\mathcal{S}}\mathsf{LL}$, we have to consider the cases of cuts after each costructural rule that we have introduced, since the cases of cuts after MALL rules or after w, c, d_I and d are already known. An important point is that we will use the formerly introduced indexed (co)weakening rather than the usual one.

Before giving the formal rewriting of each case, we will divide them into three groups. Since $\mathsf{DB}_{\mathcal{S}}\mathsf{LL}$ is highly inspired from DiLL, one can try to adapt the cut-elimination procedure from DiLL. This adaptation would mean that the structure of the rewriting is exactly the same, but the exponential connextives have to be indexed. For most cases, this method works and there is exactly one possible way to index these connectives, since w_I, $\bar{w}_I$, c, $\bar{c}$, d and $\bar{d}$ do not require a choice of the index (at this point, one can think that there is a choice in the indexing of w_I and $\bar{w}_I$, but this is a forced choice thanks to the other rules).

However, the case of the cut between a contraction and a cocontraction will require some work on the indexes because these two rules use the addition of the monoid. The index of the principal formula x (resp. x') of a contraction (resp. cocontraction) rule is the sum of two indexes x_1 and x_2 (resp. x_3 and x_4). But $x = x'$ does not imply that $x_1 = x_3$ and $x_2 = x_4$. We will then have to use a technical algebraic notion to decorate the indexes of the cut elimination between c and $\bar{c}$ in DiLL: the additive splitting.

Definition 3.2. *A monoid $(\mathcal{M}, +, 0)$ is* addtive splitting *if for each $x_1, x_2, x_3, x_4 \in \mathcal{M}$ such that $x_1 + x_2 = x_3 + x_4$, there are elements $x_{1,3},\ x_{1,4},\ x_{2,3},\ x_{2,4} \in \mathcal{M}$ such that*

$$x_1 = x_{1,3} + x_{1,4} \qquad x_2 = x_{2,3} + x_{2,4} \qquad x_3 = x_{1,3} + x_{2,3} \qquad x_4 = x_{1,4} + x_{2,4}.$$

This notion appears in [5], for describing particular models of $\mathsf{B}_{\mathcal{S}}\mathsf{LL}$, based on the relational model. Here the purpose is different: it appears from a syntactical point of view. In the rest of this section, we will not only require $\mathcal{S}$ to be a monoid, but to be additive splitting as well.

Now that we have raised some fundamental difference in a possible cut-elimination procedure, one can note that we do not have mentioned how to rewrite the cuts following an indexed (co)dereliction. This is because the procedure from DiLL cannot be adapted at all in order to eliminate those cuts, as d_I and $\bar{d}_I$ have nothing in common with the exponential rules of DiLL. The situation is even worse: these cuts cannot be eliminated since these rules are not deterministic because of the use of the order relation. These considerations lead to the following division between the cut elimination cases.

Group 1: The cases where DiLL can naively be decorated. These will be cuts involving two exponential rules, with at least one being an indexed (co)weakening or a non-indexed (co)dereliction.

Group 2: The case where DiLL can be adapted using algebraic technicality, which is the cut between a contraction and a cocontraction.

Group 3: The cases highly different from DiLL. Those are the ones involving an indexed dereliction or an indexed codereliction.

The formal rewritings for the cases of group 1 and 2 are given in Figure 2. The cut-elimination for contraction and a cocontraction uses the additive splitting property with the notations of Definition 3.2. Finally, the last possible case of an occurrence of a cut in a proof is the one where d_I or $\bar{d}_I$ is applied before the cut: the group 3. The following definition introduces rewritings where these rules goes up in the derivation tree, and which will be applied before the cut elimination procedure. This technique is inspired from subtyping ideas, which make sense since d_I is originally defined as a subtyping rule.

Definition 3.3. *The rewriting procedures $\rightsquigarrow_{d_I}$ and $\rightsquigarrow_{\bar{d}_I}$ are defined on proof trees of $\mathsf{DB}_{\mathcal{S}}\mathsf{LL}$.*

1. *When d_I (resp. $\bar{d}_I$) is applied after a rule r and r is either from MALL (except the axiom) or r is $\bar{w}_I$, $\bar{c}$, $\bar{d}_I$ (resp. d_I), $\bar{d}$ or d, the rewriting $\rightsquigarrow_{d_I,1}$ (resp. $\rightsquigarrow_{\bar{d}_I,1}$) exchanges r and d_I (resp. $\bar{d}_I$) which is possible since r and d_I do not have the same principal formula.*

2. *When d_I or $\bar{d}_I$ is applied after a contraction, the rewriting is*

$$\dfrac{\dfrac{\Pi}{\vdash \Gamma, ?_{x_1}A, ?_{x_2}A}\, c}{\vdash \Gamma, ?_{x_1+x_2}A}\, d_I \quad \rightsquigarrow_{d_I,2} \quad \dfrac{\dfrac{\Pi}{\vdash \Gamma, ?_{x_1}A, ?_{x_2}A}\, d_I}{\vdash \Gamma, ?_{x_1+x_2+x_3}A}\, c$$

$$\dfrac{\dfrac{\Pi_1 \qquad \Pi_2}{\vdash \Gamma, !_{x_1}A \qquad \vdash \Delta, !_{x_2}A}\, \bar{c}}{\vdash \Gamma, \Delta, !_{x_1+x_2}A}\, \bar{d}_I \quad \rightsquigarrow_{\bar{d}_I,2} \quad \dfrac{\dfrac{\Pi_1}{\vdash \Gamma, !_{x_1}A}\, \bar{d}_I \qquad \Pi_2}{\vdash \Gamma, \Delta, !_{x_1+x_2+x_3}A}\, \bar{c}$$

3. *If it is applied after an indexed weakening, the rewriting is*

$$\dfrac{\dfrac{\Pi}{\vdash \Gamma}\, w_I}{\vdash \Gamma, ?_{x}A}\, d_I \quad \rightsquigarrow_{d_I,3} \quad \dfrac{\Pi}{\vdash \Gamma, ?_{x+y}A}\, w_I \qquad \dfrac{\dfrac{\Pi}{\vdash}\, \bar{w}_I}{\vdash !_{x}A}\, \bar{d}_I \quad \rightsquigarrow_{\bar{d}_I,3} \quad \dfrac{\Pi}{\vdash !_{x+y}A}\, \bar{w}_I$$

4. *And if it is after an axiom, we define*

$$\dfrac{\dfrac{}{\vdash !_{x}A, ?_{x}A^{\perp}}\, ax}{\vdash !_{x}A, ?_{x+y}A^{\perp}}\, d_I \quad \rightsquigarrow_{d_I,4} \quad \dfrac{\dfrac{\dfrac{}{\vdash !_{x}A, ?_{x}A^{\perp}}\, ax}{\vdash !_{x}A, ?_{x}A^{\perp}, ?_{y}A^{\perp}}\, w_I}{\vdash !_{x}A, ?_{x+y}A^{\perp}}\, c$$

$$\dfrac{\dfrac{}{\vdash !_{x}A, ?_{x}A^{\perp}}\, ax}{\vdash !_{x+y}A, ?_{x}A^{\perp}}\, \bar{d}_I \quad \rightsquigarrow_{\bar{d}_I,4} \quad \dfrac{\dfrac{}{\vdash !_{x}A, ?_{x}A^{\perp}}\, ax \qquad \dfrac{\vdash}{\vdash !_{y}A}\, \bar{w}_I}{\vdash !_{x+y}A, ?_{x}A^{\perp}}\, \bar{c}$$

$$\dfrac{\dfrac{\Pi_1}{\vdash \Gamma}\, w_I}{\vdash \Gamma, ?_x A}\quad \dfrac{\dfrac{\Pi_2}{\vdash}\, \bar{w}_I}{\vdash !_x A^\perp}\, cut \quad \rightsquigarrow_{cut} \quad \dfrac{\Pi_1}{\vdash \Gamma}$$

$$\dfrac{\dfrac{\Pi_1}{\vdash \Gamma, A}\, d}{\vdash \Gamma, ?A}\quad \dfrac{\dfrac{\Pi_2}{\vdash \Delta, A^\perp}\, \bar{d}}{\vdash \Delta, !A^\perp}\, cut \quad \rightsquigarrow_{cut} \quad \dfrac{\Pi_1 \qquad \Pi_2}{\vdash \Gamma, A \qquad \vdash \Delta, A^\perp}\, cut$$

$$\dfrac{\dfrac{\Pi_1}{\vdash \Gamma, ?_x A, ?_y A}\, c}{\vdash \Gamma, ?_{x+y} A}\quad \dfrac{\dfrac{\Pi_2}{\vdash}\, \bar{w}_I}{\vdash !_{x+y} A^\perp}\, cut \quad \rightsquigarrow_{cut} \quad \dfrac{\dfrac{\Pi_1}{\vdash \Gamma, ?_x A, ?_y A}\quad \dfrac{\dfrac{\Pi_2}{\vdash}\, \bar{w}_I}{\vdash !_y A^\perp}\, cut}{\vdash \Gamma, ?_x A}\quad \dfrac{\dfrac{\Pi_2}{\vdash}\, \bar{w}_I}{\vdash !_x A^\perp}\, cut}{\vdash \Gamma}$$

$$\dfrac{\dfrac{\Pi_1 \qquad \Pi_2}{\vdash \Gamma, !_x A \qquad \vdash \Delta, !_y A}\, \bar{c}}{\Gamma, \Delta, !_{x+y} A}\quad \dfrac{\dfrac{\Pi_3}{\vdash \Xi}\, w_I}{\vdash \Xi, ?_{x+y} A^\perp}\, cut \quad \rightsquigarrow_{cut}$$

$$\dfrac{\dfrac{\Pi_1}{\vdash \Gamma, !_x A}\quad \dfrac{\dfrac{\Pi_3}{\vdash \Xi}\, w_I}{\vdash \Xi, ?_x A^\perp}\, cut}{\vdash \Gamma, \Xi}\, w_I}{\vdash \Gamma, \Xi, ?_y A^\perp}\quad \dfrac{\Pi_2}{\vdash \Delta, !_y A}\, cut$$

$$\dfrac{\dfrac{\Pi_1}{\vdash \Gamma, ?_{x_1} A^\perp, ?_{x_2} A^\perp}\, c}{\vdash \Gamma, ?_{x_1+x_2} A^\perp}\quad \dfrac{\dfrac{\Pi_2 \qquad \Pi_3}{\vdash \Delta, !_{x_3} A \qquad \vdash \Xi, !_{x_4} A}\, \bar{c}}{\vdash \Delta, \Xi, !_{x_3+x_4=x_1+x_2} A}\, cut \quad \rightsquigarrow_{cut}$$

$$\dfrac{\dfrac{\Pi_2^{c\bar{c}}}{\vdash \Gamma, ?_{x_{1,4}} A^\perp, ?_{x_{2,4}} A^\perp, ?_{x_3} A^\perp} \qquad \dfrac{\Pi_2}{\vdash \Delta, !_{x_3} A}\, cut}{\vdash \Gamma, \Delta, ?_{x_{1,4}} A^\perp, ?_{x_{2,4}} A^\perp}\, c \qquad \dfrac{\Pi_3}{\Xi, !_{x_4} A}\, cut}{\vdash \Gamma, \Delta, ?_{x_4} A^\perp}$$

$$\vdash \Gamma, \Delta, \Xi$$

in which $\Pi_1^{c\bar{c}}$ and $\Pi_2^{c\bar{c}}$ are as follows:

$$\Pi_1^{c\bar{c}} = \dfrac{\dfrac{\dfrac{}{\vdash ?_{x_{2,3}} A^\perp, !_{x_{2,3}} A}\, ax \qquad \dfrac{}{\vdash ?_{x_{2,4}} A^\perp, !_{x_{2,4}} A}\, ax}{\vdash ?_{x_{2,3}} A^\perp, ?_{x_{2,4}} A^\perp, !_{x_2} A}\, \bar{c} \qquad \dfrac{\Pi_1}{\vdash \Gamma, ?_{x_1} A^\perp, ?_{x_2} A^\perp}\, cut}{\vdash \Gamma, ?_{x_{2,3}} A^\perp, ?_{x_{2,4}} A^\perp, ?_{x_1} A^\perp}$$

$$\Pi_2^{c\bar{c}} = \dfrac{\dfrac{\dfrac{\Pi_1^{c\bar{c}}}{\vdash \Gamma, ?_{x_{2,3}} A^\perp, ?_{x_{2,4}} A^\perp, ?_{x_1} A^\perp} \qquad \dfrac{\dfrac{}{\vdash ?_{x_{1,3}} A^\perp, !_{x_{1,3}} A}\, ax \qquad \dfrac{}{\vdash ?_{x_{1,4}} A^\perp, !_{x_{1,4}} A}\, ax}{?_{x_{1,3}} A^\perp, ?_{x_{1,4}} A^\perp, !_{x_1} A}\, \bar{c}}{\vdash \Gamma, ?_{x_{2,3}} A^\perp, ?_{x_{2,4}} A^\perp, ?_{x_{1,3}} A^\perp, ?_{x_{1,4}} A^\perp}\, cut}{\vdash \Gamma, ?_{x_{1,4}} A^\perp, ?_{x_{2,4}} A^\perp, ?_{x_3} A^\perp}\, c$$

Figure 2: Cut elimination for $\mathsf{DB}_{\mathcal{S}}\mathsf{LL}$: group 1 and group 2

One defines $\rightsquigarrow_{d_I}$ (resp. $\rightsquigarrow_{\bar{d}_I}$) as the transitive closure of the union of the $\rightsquigarrow_{d_I,i}$ (resp. $\rightsquigarrow_{\bar{d}_I,i}$).

Even if this definition is non-deterministic, this is not a problem. Every indexed (co)dereliction goes up in the tree, without meeting an other one. This implies that this rewriting is confluent: the result of the rewriting does not depend on the choices made.

Remark 3.4. *It is easy to define a forgetful functor U, which transforms a formula (resp. a proof) of $\mathsf{DB}_{\mathcal{S}}\mathsf{LL}$ into a formula (resp. a proof) of DiLL. For a formula A of $\mathsf{DB}_{\mathcal{S}}\mathsf{LL}$, $U(A)$ is A where each $!_x$ (resp. $?_x$) is transformed into $!$ (resp. $?$), which is a formula of DiLL. For a proof-tree without any d_I and $\bar{d}_I$, the idea is the same: when an exponential rule of $\mathsf{DB}_{\mathcal{S}}\mathsf{LL}$ is applied in a proof-tree Π, the same rule but not indexed is applied in $U(\Pi)$, which is a proof-tree in DiLL. Moreover, we notice that if $\Pi_1 \rightsquigarrow_{cut} \Pi_2$, $U(\Pi_1) \rightsquigarrow_{\mathsf{DiLL}} U(\Pi_2)$ with the cut-elimination in [10].*

We can now define a cut-elimination procedure:

Definition 3.5. *The rewriting $\rightsquigarrow$ is defined on derivation trees. For a tree Π, we apply $\rightsquigarrow_{d_I}$, $\rightsquigarrow_{\bar{d}_I}$ and $\rightsquigarrow_{cut}$ as long as it is possible. When there and no more cuts, the rewriting ends.*

Theorem 3.6. *The rewriting procedure $\rightsquigarrow$ terminates on each derivation tree, and reaches an equivalent tree with no cut.*

In order to prove this theorem, we first need to prove a lemma, which shows that the (co)derelictions elimination is well defined.

Lemma 3.7. *For each derivation tree Π, if we apply $\rightsquigarrow_{d_I}$ and $\rightsquigarrow_{\bar{d}_I}$ to Π, this procedure terminates such that $\Pi \rightsquigarrow_{d_I} \Pi_1 \rightsquigarrow_{\bar{d}_I} \Pi_2$ without any d_I and $\bar{d}_I$ in Π_2.*

Proof. Let Π be a proof-tree. Each rule has a height (using the usual definition for nodes in a tree). We define the *depth* of a node as the height of the tree minus the height of this node. The procedure $\rightsquigarrow_{d_I}$ terminates on Π: let $c_1, \ldots, c_n$ be the contraction rules in Π, ordered by their height (c_1 has the smallest height). For each c_i, let $d(\Pi, c_i)$ be the number of indexed dereliction rules applied after c_i, $d(\Pi)$ be the number of indexed derelictions in Π and $a(\Pi)$ is the sum of the depths of each indexed dereliction rule in Π. Now, let us define

$$H(\Pi) = (d(\Pi), d(\Pi, c_1), \ldots, d(\Pi, c_n), a(\Pi)) \in \mathbb{N}^{n+2}.$$

For each $\Pi_a \rightsquigarrow_{d_I,i} \Pi_b$ (for $1 \leq i \leq 4$), we have $H(\Pi_a) <_{lex} H(\Pi_b)$ where $<_{lex}$ is the lexicographic order on $\mathbb{N}^{n+2}$:

1. If $\Pi_a \rightsquigarrow_{d_I,1} \Pi_b$, the depth of the indexed dereliction rule implied on the rewriting decreases. For the other d_I, the depth does not change. Moreover, the number of d_I does not change and the number of d_I after a specific contraction either decreases or does not change (if the rule r is a contraction it decreases, else it does not change). Hence, $H(\Pi_a) <_{lex} H(\Pi_b)$.

2. If $\Pi_a \rightsquigarrow_{d_I,2} \Pi_b$, let c_i be the contraction implied in the rewriting. The total number of d_I does not change, and for each $j \neq i$, this rewriting does not change the number of d_I after c_j. But the number of d_I after c_i decreases with the rewriting which implies that $H(\Pi_a) <_{lex} H(\Pi_b)$.

3. If $\Pi_a \rightsquigarrow_{d_I,3} \Pi_b$, the total number of d_I decreases, so $H(\Pi_a) <_{lex} H(\Pi_b)$.

4. If $\Pi_a \rightsquigarrow_{d_I,4} \Pi_b$, $H(\Pi_a) <_{lex} H(\Pi_b)$ because the total number of d_I decreases as well as in the previous case.

Using this property and the fact that $<_{lex}$ is a well-founded order on $\mathbb{N}^{n+2}$, this rewriting procedure has to terminates on a tree Π_1. Moreover, if there is a dereliction in Π_1, this dereliction is below an other rule, so $\rightsquigarrow_{d_I,i}$ for $1 \leq i \leq 4$ can be applied which leads to a contradiction with the definition of Π_1. Then, there is no dereliction in Π_1.

Using similar arguments, the rewriting procedure $\rightsquigarrow_{\bar{d}_I}$ on Π_1 ends on a tree Π_2 where there is no codereliction (and no dereliction because the procedure $\rightsquigarrow_{\bar{d}_I}$ does not introduce any derelictions). □

Proof of Theorem 3.6. If we apply our procedure $\rightsquigarrow$ on a tree Π we will, using Lemma 3.7, have a tree $\Pi_{d_I,\bar{d}_I}$ such that $\Pi \rightsquigarrow_{d_I} \Pi_{d_I} \rightsquigarrow_{\bar{d}_I} \Pi_{d_I,\bar{d}_I}$ and there is no dereliction and no codereliction in $\Pi_{d_I,\bar{d}_I}$. Hence, the procedure $\rightsquigarrow$ applied on Π gives a rewriting

$$\Pi \rightsquigarrow_{d_I} \Pi_d \rightsquigarrow_{\bar{d}_I} \left(\Pi_{d_I,\bar{d}_I} = \Pi_0\right) \rightsquigarrow_{cut} \Pi_1 \rightsquigarrow_{cut} \dots$$

Applying the forgetful functor U from Remark 3.4 on each tree Π_i (for $i \in \mathbb{N}$), the cut-elimination theorem of DiLL [26] implies that this rewriting terminates at a rank n, because the cut-elimination rules of $\mathsf{DB}_{\mathcal{S}}\mathsf{LL}$ which are used in Π_0 are those of DiLL when the indexes are removed. Then, $\Pi \rightsquigarrow^* \Pi_n$ where Π_n is cut-free. □

Remark 3.8. *Notice that while DiLL is famous for introducing* formal sums of proofs *with its cut-elimination, we have none of that here. The syntactical reason is that, as exponential are labelled with indices, there is no non-deterministic choices to make here. The semantical reason is that sum is introduced while operating a cut between codereliction and contraction (differentiating a scalar multiplication of functions) or a cut between a dereliction and cocontraction (applying a convolution product of distributions to a linear map). As detailed in section 4, LPDOcc do not behave like this and fundamental solutions or differential operators are painlessly propagated into the first argument of a distribution or function.*

4 An indexed differential linear logic

In the previous section, we have defined a logic $\mathsf{DB}_{\mathcal{S}}\mathsf{LL}$ as the syntactical differential of an indexed linear logic $\mathsf{B}_{\mathcal{S}}\mathsf{LL}$, with its cut elimination procedure. Here we will take an other point of view: starting from differential linear logic, we will index it into a logic named IDiLL, and then study the relation between $\mathsf{DB}_{\mathcal{S}}\mathsf{LL}$ and IDiLL.

4.1 IDiLL: a generalization of D-DiLL

As we saw in Section 2, Kerjean generalized $\bar{d}$ and d in previous work [23], with the idea that in DiLL, the codereliction corresponds to the application of the differential operator D_0 whereas the dereliction corresponds to the resolution of the differential equation associated to D_0. This led to a logic D-DiLL, where $\bar{d}$ and d have the same effect but with a LPDOcc D instead of D_0, and where the exponential connectives are indexed by this operator D. One would expect that this work could be connected to $\mathsf{DB}_{\mathcal{S}}\mathsf{LL}$, but these definitions clash with the traditional intuitions of graded logics. The first reason is syntactical: in graded logics, the exponential connectives are indexed by elements of an algebraic structure, whereas in D-DiLL only one operator is used as an index. We then change the logic D-DiLL into a logic IDiLL, which is much closer to what is done in the graded setting. In this new framework, we will consider the composition of two LPDOcc as our monoidal operation. Indeed, thanks to Proposition 2.2, we have that $\hat{D_1}(\phi) * \hat{D_2}(\psi) = \widehat{D_1 \circ D_2}(\phi * \psi)$. The convolution $*$ being the interpretation of the cocontraction rule $\bar{d}$, the composition is the monoidal operation on the set of LPDOcc that we are looking for.

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?_D A}\, w_I \qquad \frac{\vdash \Gamma, ?_{D_1} A, ?_{D_2} A}{\vdash \Gamma, ?_{D_1 \circ D_2} A}\, c \qquad \frac{\vdash \Gamma, ?_{D_1} A}{\vdash \Gamma, ?_{D_1 \circ D_2} A}\, d_I$$

$$\frac{\vdash}{\vdash !_D A}\, \bar{w}_I \qquad \frac{\vdash \Gamma, !_{D_1} A \qquad \vdash \Delta, !_{D_2} A}{\vdash \Gamma, \Delta, !_{D_1 \circ D_2} A}\, \bar{c} \qquad \frac{\vdash \Gamma, !_{D_1} A}{\vdash \Gamma, !_{D_1 \circ D_2} A}\, \bar{d}_I$$

Figure 3: Exponential rules of IDiLL

We describe the exponential rules of IDiLL in Figure 3. The indexed rules d_D and $\bar{d}_D$ are generalized to rules d_I and $\bar{d}_I$ involving a variety of LPDOcc. The interpretations of $?_D A$ and $!_D A$, and hence the typing of d_I and $\bar{d}_I$ are changed from what D-DiLL would have directly enforced (see remark 4.1). Our new interpretations for $?_D A$ and $!_D A$ are now compatible with the intuition that in graded logics, rules are supposed to add information.

$$[\![?_D A]\!] := \{g \mid \exists f \in [\![?A]\!],\ D(g) = f\} \qquad [\![!_D A]\!] := ([\![?_D A]\!]')' = \hat{D}([\![!A]\!])$$
$$d_I : [\![?_{D_1} A]\!] \to [\![?_{D_1 \circ D_2} A]\!] \qquad \bar{d}_I : [\![!_{D_1} A]\!] \to [\![!_{D_1 \circ D_2} A]\!]$$

The reader might note that these new definitions have an other benefit: they ensure that the dereliction (resp. the codereliction) is well typed when it consists in solving (resp. applying) a differential equation. This will be detailed in Section 4.3.

Notice that a direct consequence of Proposition 2.2 is that for two LPDOcc D_1 and D_2, $\Phi_{D_1 \circ D_2} = \Phi_{D_1} * \Phi_{D_2}$. It expresses that our monoidal law is also well-defined w.r.t. the interpretation of the indexed dereliction.

Remark 4.1. *Our definition for indexed connectives and thus for the type of d_D and $\bar{d}_D$ differs from the original one in D-DiLL [23]. Through the involutory duality, each $v \in E$ corresponds to a unique $\delta_v \in E'' \simeq E$, and $\bar{d}$ is then interpreted as $\phi \in E'' \mapsto \phi \circ D_0$. Then Kerjean considered that $E'' = (D_0(?(E'), \mathbb{R}))'$ and generalized it by replacing D_0 with D, defining $?_{D,old} E := D(\mathcal{C}^\infty(E', \mathbb{R}))$. This gave types $d_D : ?_{D,old} E' \to ?E'$ and $\bar{d}_D : !_{D,old} E \to !E$. However, it does not agree with general grading intuitions, and made it impossible to see D-DiLL as a particular case of $\mathsf{DB}_{\mathcal{S}}\mathsf{LL}$.*

4.2 Grading linear logic with differential operators

In this section, we will show that IDiLL consists of admissible rules of $\mathsf{DB}_{\mathcal{S}}\mathsf{LL}$ for the monoid of LPDOcc. In order to connect IDiLL with our results from Section 3, we have to study the algebraic struture of the set of linear partial differential operators with constant coefficients $\mathcal{D}$. More precisely, our goal is to prove the following theorem.

Theorem 4.2. *The set $\mathcal{D}$ of LPDOcc is an additive splitting monoid under composition, with the identity operator id as the identity element.*

To prove this result, we will use multivariates polynomials: $\mathbb{R}[X^{(\omega)}] := \bigcup_{n \in \mathbb{N}} \mathbb{R}[X_1, \ldots, X_n]$. It is well known that $(\mathbb{R}[X^{(\omega)}], +, \times, 0, 1)$ is a commutative ring. It is isomorphic to $(\mathcal{D}, \circ, id)$, the LPDOcc endowed with composition, through the following monoidal isomorphism

$$\chi : \begin{cases} (\mathcal{D}, \circ) \to (\mathbb{R}[X^{(\omega)}], \times) \\ \displaystyle\sum_{\alpha \in \mathbb{N}^n} a_\alpha \frac{\partial^\alpha (_)}{\partial x^\alpha} \mapsto \sum_{\alpha \in \mathbb{N}^n} a_\alpha X^{\alpha_1} \ldots X_n^{\alpha_n} \end{cases}$$

The following proposition is crucial in the indexation of $\mathsf{DB}_{\mathcal{S}}\mathsf{LL}$ by differential operators, since the monoid in $\mathsf{DB}_{\mathcal{S}}\mathsf{LL}$ has to be additive splitting.

Proposition 4.3. *The monoid $(\mathbb{R}[X^{(\omega)}], \times, 1)$ is additive splitting.*

The proof requires some algebraic definitions to make it more readable.

Definition 4.4. *Let $\mathcal{R}$ be a non-zero commutative ring.*

1. *$\mathcal{R}$ is an integral domain if for each $x, y \in \mathcal{R}\backslash\{0\}$, $xy \neq 0$.*
2. *An element $u \in \mathcal{R}$ is a unit if there is $v \in \mathcal{R}$ such that $uv = 1$.*
3. *Two elements $x, y \in \mathcal{R}$ are associates if x divides y and y divides x.*
4. *$\mathcal{R}$ is a factorial ring if it is an integral domain such that for each $x \in \mathcal{R}\backslash\{0\}$ there is a unit $u \in \mathcal{R}$ and $p_1, \ldots, p_n \in \mathcal{R}$ irreducible elements such that $x = up_1 \ldots p_n$ and for every other decomposition $vq_1 \ldots q_m = up_1 \ldots p_n$ (with v unit and q_i irreducible for each i) we have $n = m$ and a bijection $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that p_i and $q_{\sigma(i)}$ are associated for each i.*

Proof of Proposition 4.3. For each integer n, the ring $\mathbb{R}[X_1, \ldots, X_n]$ is factorial. This classical proposition is for example proved in [4, 2.7 Satz 7].

Let us take four polynomials P_1, P_2, P_3 and P_4 in $\mathbb{R}[X^{(\omega)}]$ such that $P_1 \times P_2 = P_3 \times P_4$. There is $n \in \mathbb{N}$ such that $P_1, P_2, P_3, P_4 \in \mathbb{R}[X_1, \ldots, X_n]$.

If $P_1 = 0$ or $P_2 = 0$, then $P_3 = 0$ or $P_4 = 0$, since $\mathbb{R}[X_1, \ldots, X_n]$ has integral domain. If for example $P_1 = 0$ and $P_3 = 0$, one can define

$$P_{1,3} = 0 \qquad P_{1,4} = P_4 \qquad P_{2,3} = P_2 \qquad P_{2,4} = 1$$

which gives a correct decomposition. And we can reason symmetrically for the other cases.

Now, we suppose that each polynomials P_1, P_2, P_3 and P_4 are non-zero. By factoriality of $\mathbb{R}[X_1, \ldots, X_n]$, we have a decomposition

$$P_i = u_i Q_{n_{i-1}+1} \times \ldots Q_{n_i} \qquad \text{(for each } 1 \leq i \leq 4\text{)}$$

where $n_0 = 0 \leq n_1 \cdots \leq n_4$, u_i are units and Q_i are irreducible. Then, the equality $P_1P_2 = P_3P_4$ gives

$$u_1u_2Q_1 \ldots Q_{n_2} = u_3u_4Q_{n_2+1} \ldots Q_{n_4}.$$

Since u_1u_2 and u_3u_4 are units, the factoriality implies that $n_2 = n_4 - n_2$ and that there is a bijection $\sigma : \{1, \ldots, n_2\} \to \{n_2 + 1, \ldots, n_4\}$ such that Q_i and $Q_{\sigma(i)}$ are associates for each $1 \leq i \leq n_2$. It means that for each $1 \leq i \leq n_2$, there is a unit v_i such that $Q_{\sigma(i)} = v_iQ_i$. Hence, defining two sets $A_3 = \sigma^{-1}(\{n_2 + 1, \ldots, n_3\})$ and $A_4 = \sigma^{-1}(\{n_3 + 1, \ldots, n_4\})$ we can rewrite our polynomials P_1 and P_2 using:

$$\begin{array}{llll}
A_{1,3} = A_3 \cap \{1, \ldots, n_1\} = p_1, \ldots, p_{m_1} & & R_{1,3} = Q_{p_1} \ldots Q_{p_{m_1}} & v_{1,3} = v_{p_1} \ldots v_{p_{m_1}} \\
A_{1,4} = A_4 \cap \{1, \ldots, n_1\} = q_1, \ldots, q_{m_2} & & R_{1,4} = Q_{q_1} \ldots Q_{q_{m_2}} & v_{1,4} = v_{q_1} \ldots v_{q_{m_2}} \\
A_{2,3} = A_3 \cap \{n_1 + 1, \ldots, n_2\} = r_1, \ldots, r_{m_3} & & R_{2,3} = Q_{r_1} \ldots Q_{r_{m_3}} & v_{2,3} = v_{r_1} \ldots v_{r_{m_3}} \\
A_{2,4} = A_4 \cap \{n_1 + 1, \ldots, n_2\} = s_1, \ldots, s_{m_4} & & R_{2,4} = Q_{s_1} \ldots Q_{s_{m_4}} & v_{2,4} = v_{s_1} \ldots v_{s_{m_4}}
\end{array}$$

which leads to

$$P_1 = u_1R_{1,3}R_{1,4} \qquad P_2 = u_2R_{2,3}R_{2,4} \qquad P_3 = u_3v_{1,3}R_{1,3}v_{2,3}R_{2,3} \qquad P_4 = u_4v_{1,4}R_{1,4}v_{2,4}R_{2,4}$$

Finally, we define our new polynomials

$$P_{1,3} = u_1 R_{1,3} \qquad P_{1,4} = R_{1,4} \qquad P_{2,3} = \frac{u_3 v_{1,3} v_{2,3}}{u_1} R_{2,3} \qquad P_{2,4} = \frac{u_1 u_2}{u_3 v_{1,3} v_{2,3}} R_{2,4}$$

gives the wanted decomposition: this is straightforward for P_1, P_2 and P_3 (the coefficients are chosen for that), and for P_4, it comes from the fact that $u_1 u_2 = u_3 u_4$ (which is in the definition of a factorial ring), and that $v_{1,a} v_{1,b} v_{2,a} v_{2,b} = 1$ which is easy to see using our new polynomials $R_{1,3}, R_{1,4}, R_{2,3}, R_{2,4}$ and the equality $P_1 P_2 = P_3 P_4$. □

This result ensures that $(\mathcal{D}, \circ, id)$ is an additive splitting monoid. Then, $\mathcal{D}$ induces a logic $\mathsf{DB}_{\mathcal{D}}\mathsf{LL}$. In this logic, since the order of the monoid is defined through the composition rule, for D_1 and D_2 in $\mathcal{D}$ we have

$$D_1 \leq D_2 \iff \exists D_3 \in \mathcal{D},\ D_2 = D_1 \circ D_3$$

which expresses that the rules d_I and $\bar{d}_I$ from IDiLL and those from $\mathsf{DB}_{\mathcal{D}}\mathsf{LL}$ are exactly the same. In addition, the weakening and the coweakening from $\mathsf{DB}_{\mathcal{D}}\mathsf{LL}$ are rules which exists in IDiLL (the (co)weakening with $D = id$), and a weakening (resp. a coweakening) in IDiLL can be expressed in $\mathsf{DB}_{\mathcal{D}}\mathsf{LL}$ as an indexed weakening (resp. an indexed coweakening). In fact, this indexed weakening is the one that appears in the cut elimination procedure of $\mathsf{DB}_{\mathcal{S}}\mathsf{LL}$. Hence, this gives the following proposition.

Proposition 4.5. *Each rule of IDiLL is admissible in $\mathsf{DB}_{\mathcal{D}}\mathsf{LL}$, and each rule of $\mathsf{DB}_{\mathcal{D}}\mathsf{LL}$ except d and $\bar{d}$ is admissible in IDiLL.*

With this proposition, Theorem 3.6 ensures that IDiLL enjoys a cut elimination procedure, which is the same as the one defined for $\mathsf{DB}_{\mathcal{S}}\mathsf{LL}$. This procedure will even be easier in the case of IDiLL. One issue in the definition of the cut elimination of $\mathsf{DB}_{\mathcal{S}}\mathsf{LL}$ is to define w_I and $\bar{w}_I$. This is no longer a problem in IDiLL because these rules already exist in this framework.

4.3 A concrete semantics for IDiLL

Now that we have defined the rules and the cut elimination procedure for a logic able to deal with the interaction between differential operators in its syntax, we should express how it semantically acts on smooth maps and distributions. For MALL formulas and rules, the interpretation is the same as the one for DiLL (or D-DiLL), given in section 2. First, we give the interpretation of our indexed exponential connectives. Beware that we are still here in a *finitary setting*, in wich exponential connectives only apply to finite dimensional vector spaces, meaning that $[\![A]\!] = \mathbb{R}^n$ for some n in equation (2) below. This makes sense syntactically as long as we do not introduce a promotion rule, and corresponds to the denotational model exposed originally by Kerjean. As mentioned in the conclusion, we think that work in higher dimensional analysis should provide an higher-order interpretation for indexed exponential connectives [16].

Consider $D \in \mathcal{D}$. Then D applies independently to any $f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ for any n, by injecting smoothly $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}) \subseteq \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{R})$ for any $m \geq n$. We give the following interpretation of graded exponential connectives:

$$\begin{aligned} [\![!_D A]\!] &:= (\{f \in \mathcal{C}^\infty([\![A]\!], \mathbb{R}) \mid \exists g \in \mathcal{C}^\infty([\![A]\!], \mathbb{R}),\ D(f) = g\})' = \hat{D}([\![!A]\!]) \\ [\![?_D A]\!] &:= \{f \in \mathcal{C}^\infty([\![A']\!], \mathbb{R}) \mid \exists g \in \mathcal{C}^\infty([\![A']\!], \mathbb{R}),\ D(f) = g\} = D^{-1}([\![?A]\!]) \end{aligned} \tag{2}$$

From this definition, one can note that when $D = id$, we get

$$[\![!_{id} A]\!] = (\mathcal{C}^\infty([\![A]\!], \mathbb{R}))' = [\![!A]\!] \qquad [\![?_{id} A]\!] = \mathcal{C}^\infty([\![A]\!]', \mathbb{R}) = [\![?A]\!].$$

Remark 4.6. *One can notice that, as differential equations always have solutions in our case, the space of solutions* $[\![?_D A]\!]$ *is* isomorphic *to the function space* $[\![?A]\!]$*. The isomorphism in question is plainly the dereliction* $d_D : f \mapsto \Phi_D * f$*. While our setting might be seen as too simple from the point of view of analysis, it is a first and necessary step before extending* IDiLL *to more intricate differential equations. If we were to explore the abstract categorical setting for our model, these isomorphisms would be relevant in a* bicategorical *setting.*

The next step is to give a semantical interpretation of the exponential rules. Most of these interpretations will be quite natural, in the sense that they will be based on the intuitions given in Section 4.1 and on the model of DiLL described in previous work [23]. However, the contraction rule will require some refinements. The contraction takes two formulas $?_{D_1}E$ and $?_{D_2}E$, and contracts them into a formula $?_{D_1\circ D_2}E$. In our model, it corresponds to the contraction of two functions $f \in \mathcal{C}^\infty(E', \mathbb{R})$ such that $D_1(f) \in \mathcal{C}^\infty(E', \mathbb{R})$ and $g \in \mathcal{C}^\infty(E', \mathbb{R})$ such that $D_2(g) \in \mathcal{C}^\infty(E', \mathbb{R})$ into a function $h \in \mathcal{C}^\infty(E', \mathbb{R})$ such that $D_1 \circ D_2(h) \in \mathcal{C}^\infty(E', \mathbb{R})$. In differential linear logic, the contraction is interpreted as the pointwise product of functions. This is not possible here, since we do not know how to compute $D_1 \circ D_2(f.g)$. We will then use the fundamental solution, which has the property that $D(\Phi_D * f) = f$. This leads to the following definition.

Definition 4.7. *We define the interpretation of each exponential rule of* IDiLL *by:*

$$w\colon \begin{cases} \mathbb{R} \to ?_{id}E \\ 1 \mapsto cst_1 \end{cases} \qquad \bar{w}\colon \begin{cases} \mathbb{R} \to !_{id}E \\ 1 \mapsto \delta_0 \end{cases}$$

$$c\colon \begin{cases} ?_{D_1}E \hat{\otimes} ?_{D_2}E \to ?_{D_1\circ D_2}E \\ \qquad f \otimes g \mapsto \Phi_{D_1\circ D_2} * (D_1(f).D_2(g)) \end{cases} \qquad \bar{c}\colon \begin{cases} !_{D_1}E \hat{\otimes} !_{D_2}E \to !_{D_1\circ D_2}E \\ \qquad \psi \otimes \phi \mapsto \psi * \phi \end{cases}$$

$$d_I\colon \begin{cases} ?_{D_1}E \to ?_{D_1\circ D_2}E \\ \qquad f \mapsto \Phi_{D_2} * f \end{cases} \qquad \bar{d}_I\colon \begin{cases} !_{D_1}E \to !_{D_1\circ D_2}E \\ \qquad \psi \mapsto \psi \circ D_2 \end{cases}$$

Remark 4.8. *One can note that we only have define the interpretation of the (co)weakening when it is indexed by the identity. This is because, as well as for* $\mathsf{DB}_\mathcal{S}\mathsf{LL}$*, the one of* w_I *and* $\bar{w}_I$ *can be deduced from this one, using the definition of* d_I *and* $\bar{d}_I$*. This leads to*

$$w_I : 1 \mapsto \Phi_D * cst_1 = cst_{\Phi_D(cst_1)} \qquad \bar{w}_I : 1 \mapsto \delta_0 \circ D = (f \mapsto D(f)(0)).$$

The interpretation for $\bar{c}$ and c is justified by the fact that in Nuclear Fréchet or Nuclear DF spaces [23], both the $\parr$ and $\otimes$ connectors of LL are interpreted by the same completed topological tensor product $\hat{\otimes}$. They however do not apply to the same kind of spaces, as $?E$ is Fréchet while $!E$ isn't. Thus, basic operations on the interpretation of $A \parr B$ or $A \otimes B$ are first defined on elements $a \otimes b$ on the tensor product, and then extended by linearity and completion.

In order to ensure that Definition 4.7 gives a correct model of IDiLL, we should verify the well-typedness of each morphism. First, this is obvious for the weakening and the coweakening. The function cst_1 defined on the interpretation of E is smooth, and δ_0 is the canonical example of a distribution. Moreover, we interpret w and $\bar{w}$ in the same way as in the model of DiLL on which our intuitions are based. The indexed dereliction is well-typed, because for $f \in ?_{D_1}E$, there is $g \in \mathcal{C}^\infty(E', \mathbb{R})$ such that $D_1(f) = g$ by definition. Hence, $D_1 \circ D_2(\Phi_{D_2} * f) = D_1(f) = g \in \mathcal{C}^\infty(E', \mathbb{R})$ so $d_I(f) \in ?_{D_1\circ D_2}E$. For the contraction, if $f \in ?_{D_1}E$ and $g \in ?_{D_2}E$, $D_1(f)$ and $D_2(g)$ are in $\mathcal{C}^\infty(E', \mathbb{R})$, and so is there scalar product. Hence, $D_1 \circ D_2(c(f, g)) = D_1(f).D_2(g) \in \mathcal{C}^\infty(E', \mathbb{R})$. The indexed codereliction is also well-typed: for $\psi \in !_{D_1}E$, equation (2) ensures that $\psi = \hat{D}_1(\psi_1)$ with $\psi_1 \in !E$, so $\psi \circ D_2 = (\psi_1 \circ D_1) \circ D_2 \in !_{D_1\circ D_2}E$. Finally, using similar

arguments for the cocontraction, if $\psi \in !_{D_1}E$ and $\phi \in !_{D_2}E$, then $\psi = \hat{D_1}(\psi_1)$ and $\phi = \hat{D_2}(\phi_1)$, with $\psi_1, \phi_1 \in !E$. Hence,

$$\psi * \phi = (\psi_1 \circ D_1) * (\phi_1 \circ D_2) = (\psi_1 * \phi_1) \circ (D_1 \circ D_2) = \widehat{D_1 \circ D_2}(\psi_1 * \phi_1) \in !_{D_1 \circ D_2}E.$$

We have then proved the following proposition.

Proposition 4.9. *Each morphism $w, \bar{w}, c, \bar{c}, d_I$ and $\bar{d}_I$ is well-typed.*

Another crucial point to study is the compatibility between this model and the cut elimination procedure $\rightsquigarrow$. In denotational semantics, one would expect that a model is invariant w.r.t. the computation. In our case, that would mean that for each step of rewriting of $\rightsquigarrow$, the interpretation of the proof-tree has the same value.

It is easy to see that this is true for the cut $w_I/\bar{w}_I$, since $D(\Phi_D * cst_1)(0) = cst_1(0) = 1$. For the cut between a contraction and an indexed coweakening, the interpretation before the reduction is $\delta_0(D_1 \circ D_2)(\Phi_{D_1 \circ D_2}(D_1(f).D_2(g))) = D_1(f)(0).D_2(g)(0)$, which is exactly the interpretation after the reduction[3].

Finally, proving the invariance of our semantics over the cut between a contraction or a weakening, and a cocontraction takes slightly more work. The weakening case is enforced by linearity of the distributions, while the contraction case relies on the density of $\{\delta_x \mid x \in E\}$ in $!E$.

Lemma 4.10. *The interpretation of $\mathsf{DB}_{\mathcal{S}}\mathsf{LL}$ with $\mathcal{D}$ as indexes is invariant over the $c/\bar{c}$ and the $\bar{c}/w_I$ cut-elimination rules, as given in Figure 2.*

Proof. Before cut-elimination, the interpretation of the $\bar{c}/w$ as given in Figure 2 is:

$$\begin{aligned}
&(\psi * \phi)(\Phi_{D_1 \circ D_2} * cst_1) \\
&= \psi(x \mapsto \phi(y \mapsto \Phi_{D_1} * (\Phi_{D_2} * cst_1)(x + y))) \\
&= \psi(x \mapsto \phi(y \mapsto \Phi_{D_1}(z \mapsto \Phi_{D_2} * cst_1(x + y - z)))) \\
&= \psi(x \mapsto \phi(y \mapsto \Phi_{D_1}(cst_{\Phi_{D_2}(cst_1)}))) \\
&= \psi(x \mapsto \phi(y \mapsto \Phi_{D_1}(\Phi_{D_2}(cst_1).cst_1))) \\
&= \psi(x \mapsto \phi(y \mapsto \Phi_{D_2}(cst_1).\Phi_{D_1}(cst_1))) && \text{(by homogeneity of } \phi) \\
&= \psi(x \mapsto \phi(cst_{\Phi_{D_2}(cst_1).\Phi_{D_1}(cst_1)})) \\
&= \psi(x \mapsto \phi(\Phi_{D_1}(cst_1).cst_{\Phi_{D_2}(cst_1)})) \\
&= \psi(x \mapsto \Phi_{D_1}(cst_1).\phi(cst_{\Phi_{D_2}(cst_1)})) && \text{(by homogeneity of } \phi) \\
&= \psi(cst_{\Phi_{D_1}(cst_1).\phi(cst_{\Phi_{D_2}(cst_1)})}) \\
&= \psi(\phi(cst_{\Phi_{D_2}(cst_1)}).cst_{\Phi_{D_1}(cst_1)}) \\
&= \phi(cst_{\Phi_{D_2}(cst_1)}).\psi(cst_{\Phi_{D_1}(cst_1)}) && \text{(by homogeneity of } \psi)
\end{aligned}$$

which corresponds to the interpretation of the proof after cut-elimination.

Let us tackle now the $\bar{c}/c$ cut-elimination case. Suppose that we have $D_1, D_2, D_3, D_4 \in \mathcal{D}$ such that $D_1 \circ D_2 = D_3 \circ D_4$. By the additive splitting property we have $D_{1,3}, D_{1,4}, D_{2,3}, D_{2,4} \in \mathcal{D}$ such that

$$D_1 = D_{1,3} \circ D_{1,4} \qquad D_2 = D_{2,3} \circ D_{2,4} \qquad D_3 = D_{1,3} \circ D_{2,3} \qquad D_4 = D_{1,4} \circ D_{2,4}.$$

The diagrammatic translation of the cut-elimination rule in Figure 2 is the following.

[3]The scalar product $(_._)$ appears as the tensor product in $\mathbb{R} \otimes \mathbb{R}$, and is transparent in sequent interpretation as $\mathbb{R} = [\![\bot]\!]$.

$$\begin{array}{ccc}
!_{D_1}E \otimes !_{D_2}E & \xrightarrow{c'_{D_{1,3},D_{1,4}} \otimes c'_{D_{2,3},D_{2,4}}} & !_{D_{1,3}}E \otimes !_{D_{1,4}}E \otimes !_{D_{2,3}}E \otimes !_{D_{2,4}}E \\
\downarrow \bar{c}_{D_1,D_2} & & \\
!_{D_1 \circ D_2}E = !_{D_3 \circ D_4}E & & \\
\downarrow c'_{D_3,D_4} & & \downarrow \\
!_{D_3}E \otimes !_{D_4}E & \xleftarrow[\bar{c}_{D_{1,3},D_{2,3}} \otimes \bar{c}_{D_{1,4},D_{2,4}}]{} & !_{D_{1,3}}E \otimes !_{D_{2,3}}E \otimes !_{D_{1,4}}E \otimes !_{D_{2,4}}E
\end{array}$$

As we interpret formulas by reflexive spaces, we can without loss of generality interpret contraction as a law $c'_{D_a,D_b} : !_{D_a \circ D_b}E \to !_{D_a}E \otimes !_{D_b}E$. Because we are working on finite dimensional spaces E, an application of Hahn-Banach theorem gives us that the span of $\{\delta_x \mid x \in E\}$ is dense in $!E$. As such, the interpretation of c' can be restricted to elements of the form $\delta_x \circ D_a \circ D_b \in !_{D_a \circ D_b}E$, and one checks easily that the dual of c (Definition 4.7) is : $c'_{D_a,D_b} : \delta_x \circ D_a \circ D_b \mapsto (\delta_x \circ D_a) \otimes (\delta_x \circ D_b)$. Remember that the convolution of dirac operators is the dirac of the sum of points, and as such we have : $\bar{c}_{D_a,D_b} : (\delta_x \circ D_a) \otimes (\delta_y \circ D_b) \mapsto (\delta_{x+y} \circ D_b \circ D_a)$. Now one can compute easily that the diagram above commutes on elements $(\delta_x \circ D_1) \otimes (\delta_y \circ D_2)$ of $!_{D_1}E \otimes !_{D_2}E$, and as such commutes on all elements by density and continuity of $\bar{c}$ and c'. $\square$

In order to ensure that this model is fully compatible with $\rightsquigarrow$, it also has to be invariant by $\rightsquigarrow_{d_I}$ and by $\rightsquigarrow_{\bar{d}_I}$. For $\rightsquigarrow_{d_I}$, the interpretation of the reduction step when the indexed dereliction meets a contraction is

$$\Phi_{D_3} * (\Phi_{D_1 \circ D_2} * (D_1(f).D_2(g))) = \Phi_{D_1 \circ D_2 \circ D_3} * (D_1 \circ D_3(\Phi_{D_3} * f).D_2(g))$$

which is the interpretation after the application of $\rightsquigarrow_{d_I,2}$. The case with a weakening translates the fact that $\Phi_{D_1 \circ D_2} = \Phi_{D_1} * \Phi_{D_2}$. Finally, the axiom rule introduces a distribution $\psi \in !_{D_1}E$ and a smooth map $f \in !_{D_1}E$, and $\rightsquigarrow_{d_I,4}$ corresponds to the equality $\Phi_{D_1 \circ D_2} * D_1(f) = \Phi_{D_2} * f$.

The remaining case is the procedure $\rightsquigarrow_{\bar{d}_I}$, which is quite similar to $\rightsquigarrow_{d_I}$. The invariance of the model with the cocontraction case follows from Proposition 2.2. For the weakening, this is just the associativity of the composition, and the axiom works because δ_0 is the neutral element of the convolution product. We can finally deduce that our model gives an interpretation which is invariant by the cut elimination procedure of Section 3.

Proposition 4.11. *Each morphism $w, \bar{w}, c, \bar{c}, d_I$ and $\bar{d}_I$ is compatible with the cut elimination procedure $\rightsquigarrow$.*

5 Promotion and higher-order differential operators

In the previous section, we have defined a differential extension of graded linear logic, which is interpreted thanks to exponentials indexed by a monoid of differential operators. This extension is done *up-to promotion*, meaning that we do not incorporate promotion in the set of rules. There are two reasons why it makes sense to leave promotion out of the picture:

- DiLL was historically introduced without it, with a then perfectly symmetric set of rules.
- Concerning semantics, LPDOcc are only defined when acting on functions with finite dimensional codomain: $D : \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}) \to \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$. Introducing a promotion rule would mean extending the theory of LPDOcc to higher-order functions.

In this section, we sketch a few of the difficulties one faces when trying to introduce promotion and dereliction rules indexed by differential operators, and explore possible solutions.

Graded dereliction Indexing the promotion goes hand-in-hand with indexing the *dereliction*. In Figure 1, we introduced a basic (not indexed) dereliction and codereliction rule d and $\bar{d}$. The original intuition of DiLL is that codereliction computes the differentiation at 0 of some proof. Following the intuition of D-DiLL, dereliction computes a solution to the equation $D_0(_) = \ell$ for some ℓ. Therefore, as indexes are here to *keep track* of the computations, and following equation (2), we should have (co)derelictions indexed by D_0 as below:

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, !A}\,\bar{d} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\,d \qquad\qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, !_{D_0}A}\,\bar{d}_{D_0} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?_{D_0}A}\,d_{D_0}$$

Mimicking what happens in graded logics, D_0 should be the identity element for the second law in the semiring interpreting the indices of exponentials in $\mathsf{DB}_{\mathcal{S}}\mathsf{LL}$. However, D_0 is *not* a linear partial differential operator (even less with constant coefficient). Let us briefly compare how a LPDOcc D and D_0 act on a function $f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$:

$$D : f \mapsto \left(y \in \mathbb{R}^n \mapsto \sum_{\alpha \in \mathbb{N}^n} a_\alpha \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(y) \right) \qquad D_0 : f \mapsto \left(y \in \mathbb{R}^n \mapsto \sum_{0 \leq i \leq n} y_i \frac{\partial f}{\partial x_i}(0) \right)$$

where $(x_i)_i$ is the canonical base of $\mathbb{R}^n$, y_i is the i-th coordinate of y in the base $(x_i)_i$, and $a_\alpha \in \mathbb{R}$. To include LPDOcc and D_0 in a single semiring structure, one would need to consider global differential operators generated by:

$$\mathsf{D} : f \mapsto \left((y, v) \mapsto \textstyle\sum_{\alpha \in \mathbb{N}^n} a_\alpha(v) \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(y) \right), \text{ with } a_\alpha \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}).$$

The algebraic structure of such a set would be more complicated, and the composition in particular would not be commutative, and as such not suitable for the first law of a semi-ring which is essential since it ensures the symmetry of the contraction and the cocontraction.

Graded promotion To introduce a promotion law in $\mathsf{DB}_{\mathcal{S}}\mathsf{LL}$, we need to define a multiplicative law $\odot$ on $\mathcal{D}$, with D_0 as a unit. We will write it under a digging form:

$$\frac{\vdash \Gamma, ?_{D_1}?_{D_2}A}{\vdash \Gamma, ?_{D_1 \odot D_2}A}\,digging$$

This relates with recent work by Kerjean and Lemay [24], inspired by preexisting mathematical work in infinite dimensional analysis [16]. They show that in particular quantitative models, one can define the exponential of elements of $!A$, such that $e^{D_0} : \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}) \to \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ is the identity. It hints at a possible definition of the multiplicative law as $D_1 \odot D_2 := D_1 \circ e^{D_2}$.

Even if one finds a semi-ring structure on the set of all LPDOcc, the introduction of promotion in the syntax means higher-order functions in denotational models. Indexed exponential connectives are defined so-far thanks to the action of LPDOcc on functions with a finite number of variable. To make LPDOcc act on higher order function (*e.g.* elements of $\mathcal{C}^\infty(\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}), \mathbb{R})$ and not only $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$) one would need to find a definition of partial differential operators independent from any canonical base, which seems difficult.

As a trick to bypass some of these issues, we could consider that the $!_D$ modalities are not composable. This is possible in a framework similar to the original BLL or that of IndLL [11], where indexes have a source and a target.

6 Conclusion

In this paper, we define a multi-operator version to D-DiLL, which turns out to be the finitary differential version of Graded Linear Logic. We describe the cut-elimination procedure and give a denotational model of this calculus in terms of differential operators. This provide a new and unexpected semantics for Graded Linear Logic, and tighten the links between Linear Logic and Functional Analysis.

There are several directions to explore now that the proof theory of $\mathsf{DB}_{\mathcal{S}}\mathsf{LL}$ has been established. The obvious missing piece in our work is the *categorical axiomatization* of our model. In a version with promotion, that would consist in a differential version of bounded linear exponentials [6]. While similarities with differential categories [2] will certainly exist in categorical models of $\mathsf{DB}_{\mathcal{S}}\mathsf{LL}$, differences between the dynamic of LPDOcc and of differentiation at 0 will certainly require adaptation. In particular, the treatment of the sum will require attention (proof do not need to be summed here while differential categories are additive). Finally, beware that our logic does not yet extend to higher-order and that without a concrete higher-model it might be difficult to design elegant categorical axioms.

Another line of research would consist in introducing more complex differential operators as indices of exponential connectives. Equations involving LPDOcc are extremely simple to manipulate as they are solved in a one step computation (by applying a convolution product with their fundamental solution). The vast majority of differential equations are difficult if not impossible to solve. One could introduce fixpoint operators within the theory of $\mathsf{DB}_{\mathcal{S}}\mathsf{LL}$, to try and modelize the resolution of differential equation by fixed point. This could also be combined with the study of particularly stable classes of differential operators, as *D-finite operators*.

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