



**HAL**  
open science

# General fourth order Chapman-Enskog expansion of lattice Boltzmann schemes

François Dubois, Bruce M Boghosian, Pierre Lallemand

► **To cite this version:**

François Dubois, Bruce M Boghosian, Pierre Lallemand. General fourth order Chapman-Enskog expansion of lattice Boltzmann schemes. 2023. hal-03981336

**HAL Id: hal-03981336**

**<https://cnrs.hal.science/hal-03981336>**

Preprint submitted on 13 Feb 2023

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution - NoDerivatives 4.0 International License

# General fourth-order Chapman-Enskog expansion of lattice Boltzmann schemes

François Dubois<sup>ab</sup>, Bruce M. Boghosian<sup>c</sup> and Pierre Lallemand<sup>d</sup>

<sup>a</sup> *Laboratoire de Mathématiques d'Orsay, Faculté des Sciences d'Orsay,  
Université Paris-Saclay, France.*

<sup>b</sup> *Conservatoire National des Arts et Métiers, LMSSC laboratory, Paris, France.*

<sup>c</sup> *Department of Mathematics, Tufts University, Medford, MA, 02155, USA.*

<sup>d</sup> *Beijing Computational Science Research Center, Haidian District, Beijing 100094, China.*

09 February 2023 \*

**Keywords:** partial differential equations, asymptotic analysis

**AMS classification:** 76N15, 82C20.

**PACS numbers:** 02.70.Ns, 47.10.+g

## Abstract

In order to derive the equivalent partial differential equations of a lattice Boltzmann scheme, the Chapman Enskog expansion is very popular in the lattice Boltzmann community. A main drawback of this approach is the fact that multiscale expansions are used without any clear mathematical signification of the various variables and operators. Independently of this framework, the Taylor expansion method allows to obtain formally the equivalent partial differential equations. The general equivalency of these two approaches remains an open question. In this contribution, we prove that both approaches give identical results with acoustic scaling for a very general family of lattice Boltzmann schemes and up to fourth-order accuracy. Examples with a single scalar conservation illustrate our purpose.

---

\* This contribution has been presented at the 31th International Conference on Discrete Simulation of Fluid Dynamics, Suzhou (China) the 22 August 2022.

## 1) Introduction

The Chapman-Enskog method is a fundamental approach developed for the asymptotic analysis of the Boltzmann equation. The book of Chapman and Cowling, first published in 1939 [4], contains the essential of this subject. When lattice gas automata were first developed in the 1970's [19] and the 1980's [17], the length of the lattice vectors was uniformly equal to unity. The asymptotic analysis for the emergence of the Navier Stokes equations was conducted by taking the size of the included bodies bigger and bigger. Employing this limit, a fundamental work was achieved by Hénon [21] for the determination of the viscosity of a lattice gas. With the lattice Boltzmann schemes in the 1990's [22, 23], the underlying paradigm of the Boltzmann equation in the approximation of Bhatnagar-Gross-Krook [2] collision operator became very popular. A method of analysis was developed by Chen-Doolen [5] and Qian-Zhou [32] based on a Chapman-Enskog expansion. This method was also used by d'Humières [24] when he introduced the multiple relaxation time variant of the lattice Boltzmann schemes. This approach involves a rather strange formal calculus of partial derivatives with respect to fast and slow time scales that can have noncommutative properties. Nevertheless, the approach has enjoyed significant success and is recommended in textbooks on the subject, e.g. [18, 26, 34].

When one of us began to work in the lattice Boltzmann community, lattice Boltzmann schemes were considered as a special finite-difference method on cartesian meshes. From this perspective, the classical approach to finding equivalent partial differential equations [30, 33, 35] provided a simple way to make explicit the continuum limit of a given algorithm. By adapting this method to lattice Boltzmann schemes, we created the Taylor expansion method [7, 8, 9, 10]. This method has predicted super-convergence of various lattice Boltzmann schemes [1, 11, 12, 14, 27, 28], and elucidated the specific behavior of the scheme for several sets of boundary conditions [14, 15, 16]. A natural question is that of precisely how these two approaches, the Chapman-Enskog and Taylor expansion methods, are related. This is the subject of this contribution.

In the second section, we review the Bhatnagar-Gross-Krook framework and the Chapman-Enskog analysis in the case of a single conserved quantity. Multi-resolution time lattice Boltzmann schemes are presented in Section 3. In the following section, a linear model with a single conserved quantity is presented for two spatial dimensions, and a preliminary result establishes the equivalence of Chapman Enskog and Taylor approaches in this specific case. The main result is presented at fourth-order accuracy for very general schemes in Section 5. The proof for orders three and four is detailed in the two last sections of the paper.

## 2) Bhatnagar-Gross-Krook framework

In this section, we follow the standard “BGK” framework [2] for lattice Boltzmann schemes. For completeness of our study and to make this work self-contained, we recall known results derived in [3, 5, 8, 32] relating to Chapman-Enskog and Taylor expansions for an advection-diffusion model.

At a vertex  $x$  of a discrete lattice  $\mathcal{L}$  and at discrete time  $t$ , a particle distribution with  $q$

velocities,  $f(x, t) = \{f_j(x, t) | 0 \leq j < q\}$ , is defined. Its evolution relative to time follows a classical algorithm. First an equilibrium particle distribution  $f_j^{\text{eq}}(x, t)$  is computed from the vector  $f(x, t)$ , according to a process which is not detailed at this step (see *e.g.* [31]). Then the nonlinear relaxation, parametrized by a relaxation time  $\tau$ , is achieved by locally modifying the particle distribution  $f$  into a new distribution  $f^*$ , defined by the relation

$$f_j^*(x, t) = f_j(x, t) + \frac{\tau_0}{\tau} (f_j^{\text{eq}}(x, t) - f_j(x, t)), \quad 0 \leq j < q,$$

with the introduction of a reference time scale  $\tau_0$ . The second step of the algorithm is pure linear advection of each component of the distribution at its corresponding velocity  $v_j$ : During a small time step  $\Delta t = \varepsilon \tau_0$ , the particles stream from the vertex  $x$  to the neighbouring vertex  $x + v_j \varepsilon \tau_0$  of the lattice. An iteration of the scheme is written

$$f_j(x + v_j \varepsilon \tau_0, t + \varepsilon \tau_0) = f_j^*(x, t).$$

Because  $f_j(x + v_j \varepsilon \tau_0, t + \varepsilon \tau_0) = f_j(x, t) + \frac{\tau_0}{\tau} (f_j^{\text{eq}} - f_j)$ , a discrete equation solved by the numerical scheme is easy to make explicit:

$$(1) \quad f_j(x, t) + \frac{\tau}{\tau_0} [f_j(x + v_j \varepsilon \tau_0, t + \varepsilon \tau_0) - f_j(x, t)] = f_j^{\text{eq}}(x, t).$$

With P. Coveney and one of us [3], the linear advection operator  $D_j \equiv \partial_t + v_j^\alpha \partial_\alpha$  in the direction number  $j$  of the lattice was introduced, with an implicit summation on the spatial index  $\alpha$ . Then we can express the linear advection in terms of the exponential of this operator  $\exp(\varepsilon \tau_0 D_j)$ ,

$$f_j(x + v_j \varepsilon \tau_0, t + \varepsilon \tau_0) = \exp(\varepsilon \tau_0 D_j) f_j(x, t).$$

With the notation  $\mathbb{I}$  for the identity operator and after a second-order expansion of the exponential operator relative to the small parameter  $\varepsilon$ , we obtain an approximate expression of the functional equation (1):

$$(2) \quad \left[ \mathbb{I} + \varepsilon \tau \left( D_j + \frac{\varepsilon}{2} (D_j)^2 + \mathcal{O}(\varepsilon^2) \right) \right] f_j = f_j^{\text{eq}}$$

• At this point, the Chapman-Enskog expansion proposed in [5, 32] introduces a formal multiple scale expansion for the time derivative,

$$\partial_t \equiv \partial_{t_1} + \varepsilon \partial_{t_2} + \mathcal{O}(\varepsilon^2).$$

Then the advection operator  $D_j$  can be expanded in terms of  $\varepsilon$ :

$$D_j = D_j^1 + \varepsilon \partial_{t_2} + \mathcal{O}(\varepsilon^2), \quad D_j^1 = \partial_{t_1} + v_j^\alpha \partial_\alpha.$$

We then suppose an *a priori* asymptotic expansion of the particle distribution,

$$f \equiv f^0 + \varepsilon f^1 + \varepsilon^2 f^2 + \mathcal{O}(\varepsilon^3),$$

in the approximate functional equation (2) satisfied by the scheme to obtain

$$(3) \quad \begin{cases} f_j^0 = f_j^{\text{eq}} \\ f_j^1 + \tau D_j^1 f_j^0 = 0 \\ f_j^2 + \tau D_j^1 f_j^1 + \tau [\partial_{t_2} + \frac{\tau_0}{2} (D_j^1)^2] f_j^0 = 0. \end{cases}$$

From these relations, we deduce various evolution equations for the distinct time scales  $\partial_{t_j}$ .

To fix these ideas, we detail the process for one conservation law. In this case, there is only one scalar conserved variable and we have typically

$$\sum_j f_j = \sum_j f_j^{\text{eq}} \equiv \rho$$

with the condition  $\sum_j v_j^\alpha f_j^{\text{eq}} \equiv \rho u^\alpha$ . Then  $\sum_j f_j^1 = \sum_j f_j^2 = 0$ . When we insert this condition in the second equation of (3), we obtain  $\tau \sum_j (\partial_{t_1} + v_j^\alpha \partial_\alpha) f_j^0 = 0$ . After division by  $\tau$ , the evolution equation at first order,

$$(4) \quad \partial_{t_1} \rho + u^\alpha \partial_\alpha \rho = 0,$$

is established.

We next insert the condition  $\sum_j f_j^2 = 0$  in the third relation of (3). After dividing by  $\tau$ , we obtain

$$(5) \quad \sum_j D_j^1 f_j^1 + \sum_j \partial_{t_2} f_j^{\text{eq}} + \frac{\tau_0}{2} \sum_j (D_j^1)^2 f_j^{\text{eq}} = 0.$$

We have from the second relation of (3):  $\sum_j D_j^1 f_j^1 = -\tau \sum_j D_j^1 (D_j^1 f_j^{\text{eq}}) = -\tau \sum_j (D_j^1)^2 f_j^{\text{eq}}$ , so the previous relation (5) can be written

$$(6) \quad \partial_{t_2} \rho + \left( \frac{\tau_0}{2} - \tau \right) \sum_j (D_j^1)^2 f_j^{\text{eq}} = 0,$$

where we also have

$$\begin{aligned} \sum_j (D_j^1)^2 f_j^{\text{eq}} &= \sum_j (\partial_{t_1} + v_j^\alpha \partial_\alpha) (\partial_{t_1} + v_j^\beta \partial_\beta) f_j^{\text{eq}} \\ &= \partial_{t_1}^2 \rho + 2 \partial_\alpha \partial_{t_1} \left( \sum_j v_j^\alpha f_j^{\text{eq}} \right) + \sum_j v_j^\alpha v_j^\beta \partial_\alpha \partial_\beta f_j^{\text{eq}} \\ &= \partial_{t_1} (-u^\alpha \partial_\alpha \rho) + 2 \partial_\alpha \partial_{t_1} (\rho u^\alpha) + \partial_\alpha \partial_\beta \left( \sum_{j,\alpha,\beta} v_j^\alpha v_j^\beta f_j^{\text{eq}} \right) \\ &= -u^\alpha \partial_\alpha (-u^\beta \partial_\beta \rho) + 2 u^\alpha \partial_\alpha (\partial_{t_1} \rho) + \partial_\alpha \partial_\beta \left( \sum_{j,\alpha,\beta} v_j^\alpha v_j^\beta f_j^{\text{eq}} \right) \\ &= \partial_\alpha \partial_\beta \left[ \sum_{j,\alpha,\beta} v_j^\alpha v_j^\beta f_j^{\text{eq}} - u^\alpha u^\beta \rho \right] \end{aligned}$$

Inserting this result in the relation (6), we obtain the evolution equation for the second time scale,

$$(7) \quad \partial_{t_2} \rho + \left( \frac{\tau_0}{2} - \tau \right) \partial_\alpha \partial_\beta \left[ \sum_j v_j^\alpha v_j^\beta f_j^{\text{eq}} - u^\alpha u^\beta \rho \right] = 0.$$

In this way, with the multiple-time representation  $\partial_t \equiv \partial_{t_1} + \varepsilon \partial_{t_2} + \text{O}(\varepsilon^2)$ , we have finally from the relations (4) and (7) a derivation of the equivalent partial differential equation up to second order,

$$(8) \quad \partial_t \rho + u^\alpha \partial_\alpha \rho - \varepsilon \left( \tau - \frac{1}{2} \right) \partial_\alpha \partial_\beta \left[ \sum_j v_j^\alpha v_j^\beta f_j^{\text{eq}} - u^\alpha u^\beta \rho \right] = \text{O}(\varepsilon^2).$$

- An alternative to the Chapman-Enskog expansion is the Taylor expansion framework proposed by one of us [8, 10]. With this paradigm, we do not consider multiple time scales

and we do not introduce any *a priori* asymptotic representation of the particle distribution. Using the BGK framework to fix the ideas, we solve the approximate functional equation (2) using a formal power series relative to the small parameter  $\varepsilon$ ,

$$\begin{aligned} f_j &= \left[ \mathbf{I} - \varepsilon \tau \left( D_j + \frac{\varepsilon \tau_0}{2} (D_j)^2 \right) + (\varepsilon \tau)^2 (D_j)^2 + \mathcal{O}(\varepsilon^3) \right] f_j^{\text{eq}} \\ &= \left[ \mathbf{I} - \varepsilon \tau D_j + \tau \varepsilon^2 \left( \tau - \frac{\tau_0}{2} \right) (D_j)^2 + \mathcal{O}(\varepsilon^3) \right] f_j^{\text{eq}} \\ &= f_j^{\text{eq}} - \varepsilon \tau D_j f_j^{\text{eq}} + \tau \varepsilon^2 \left( \tau - \frac{\tau_0}{2} \right) (D_j)^2 f_j^{\text{eq}} + \mathcal{O}(\varepsilon^3) \end{aligned}$$

For the case of one scalar conserved quantity, we have  $\sum_j f_j = \sum_j f_j^{\text{eq}} \equiv \rho$  and  $\sum_j v_j^\alpha f_j^{\text{eq}} \equiv \rho u^\alpha$ . Then after division by  $\tau \varepsilon$ , we have the asymptotic relation

$$(9) \quad \sum_j D_j f_j^{\text{eq}} - \varepsilon \left( \tau - \frac{\tau_0}{2} \right) \sum_j (D_j)^2 f_j^{\text{eq}} = \mathcal{O}(\varepsilon^2).$$

At first order, we have  $\sum_j D_j f_j^{\text{eq}} = \partial_t \rho + u^\alpha \partial_\alpha \rho = \mathcal{O}(\varepsilon)$ . To obtain a result at second order, we have the following calculation:

$$\begin{aligned} \sum_j (D_j)^2 f_j^{\text{eq}} &= \sum_j (\partial_t + v_j^\alpha \partial_\alpha) (\partial_t + v_j^\beta \partial_\beta) f_j^{\text{eq}} \\ &= \partial_t^2 \rho + 2 \sum_j v_j^\alpha \partial_\alpha \partial_t f_j^{\text{eq}} + \sum_j v_j^\alpha v_j^\beta \partial_\alpha \partial_\beta f_j^{\text{eq}} \\ &= \partial_t (-u^\alpha \partial_\alpha \rho) + 2 \partial_\alpha \partial_t \left( \sum_j v_j^\alpha f_j^{\text{eq}} \right) + \partial_\alpha \partial_\beta \left[ \sum_j v_j^\alpha v_j^\beta f_j^{\text{eq}} \right] + \mathcal{O}(\varepsilon) \\ &= -u^\alpha \partial_\alpha (-u^\beta \partial_\beta \rho) + 2 \partial_\alpha \partial_t (u^\alpha \rho) + \partial_\alpha \partial_\beta \left[ \sum_j v_j^\alpha v_j^\beta f_j^{\text{eq}} \right] + \mathcal{O}(\varepsilon) \\ &= \partial_\alpha \partial_\beta \left[ \sum_j v_j^\alpha v_j^\beta f_j^{\text{eq}} - u^\alpha u^\beta \rho \right] + \mathcal{O}(\varepsilon) \end{aligned}$$

We insert this result into the relation (9) and we recover exactly the relation (8).

In this section, in the specific case of a scalar equation in the BGK framework, we have established that the Chapman-Enskog methodology and the Taylor expansion method yield exactly the same equivalent partial differential equation at second-order accuracy. In the next sections, we generalize this result for an arbitrary number of conservation laws in the framework of multi-resolution times lattice Boltzmann schemes, and we establish agreement up to fourth-order accuracy.

### 3) Multi-resolution times lattice Boltzmann schemes

The multi-resolution times paradigm is an extension of the BGK collision operator presented in the previous section. It has been formalized by d'Humières [24]. As previously, a discrete particle distribution of  $q$  velocities  $f(x, t) = \{f_j(x, t) \mid 0 \leq j < q\}$  is defined with corresponding discrete velocities  $v_j \in \mathcal{V}$  at a vertex  $x$  of a discrete lattice  $\mathcal{L}$  and at discrete time  $t$ . One time iteration, leading to the evaluation of  $f_j(x, t + \Delta t)$ , is composed of two steps.

(i) Nonlinear relaxation. During this step, a local modification of the particle distribution  $f$ , denoted by  $f^*$ , is determined. First an invertible matrix  $M$  transforms the particle distribution  $f$  into moments  $m$ . We write  $m = M f$ , or in terms of components  $m_k \equiv \sum_{\ell} M_{k\ell} f_{\ell}$  for  $0 \leq k < q$ . We split this vector into two blocks. The first block  $W$  is composed by the conserved quantities or macroscopic moments, whereas the second block  $Y$  determines the nonconserved or microscopic moments,

$$(10) \quad m \equiv \begin{pmatrix} W \\ Y \end{pmatrix}.$$

After relaxation, the conserved moments do not change:  $W^* = W$ . Secondly, an equilibrium value  $Y^{\text{eq}}$  of the nonconserved moments is introduced; it is a function of the conserved moments,

$$(11) \quad Y^{\text{eq}} = \Phi(W).$$

This function  $W \mapsto \Phi(W)$  is required to be regular and can be seen as a discrete Gaussian in reference to the Boltzmann equation for gas dynamics. Nevertheless, it has *a priori* no direct algebraic relation with the Maxwell-Boltzmann distribution and is constrained only by symmetry considerations. Knowledge of the equilibrium function  $\Phi$  is essential for specifying the multi-resolution times lattice Boltzmann scheme. After relaxation the vector of microscopic moments is modified, and a new vector  $Y^*$  is created according to

$$(12) \quad Y^* = Y + S(Y^{\text{eq}} - Y).$$

The relation (12) introduces a relaxation matrix  $S$ . This is an invertible square matrix, usually chosen as diagonal,  $S = \text{diag}(s_k)$ . The relaxation coefficients  $s_k$  are dimensionless and strictly positive. They are also an essential specification of the multi-resolution times lattice Boltzmann scheme. The moments  $m^*$  after relaxation combine the two families of moments:  $m^* = (W, Y^*)^t$ . Then the particle distribution after relaxation  $f^*$  is simply determined from the moments after relaxation:  $f^* = M^{-1} m^*$ . Observe here that if all the relaxation coefficients  $s_k$  are identical, *id est* if  $s_k \equiv \frac{\tau_0}{\tau}$  for all indices  $k$ , then the multi-resolution times lattice Boltzmann scheme is identical to the BGK variant [29].

(ii) Linear advection. This step is identical to the BGK framework. Recall that the velocities  $v_j$  are chosen in such a way that after one time step  $\Delta t$ , a particle located in  $x \in \mathcal{L}$  arrives at a new vertex of the lattice:  $x + v_j \Delta t \in \mathcal{L}$ . With the previous notation introduced, we can formulate a compact description of the lattice Boltzmann advection scheme:

$$(13) \quad f_j(x, t + \Delta t) = f_j^*(x - v_j \Delta t, t), \quad v_j \in \mathcal{V}, \quad x \in \mathcal{L}.$$

As a final remark, the present paradigm of multiple relaxation schemes allows one to take into account multiple distributions of particles. Two or more particle distributions can be introduced in practice. We just observe that the mapping  $j \mapsto v_j$  is not necessarily injective.

- It should be pointed out that important hypotheses have been made for the asymptotic expansions proposed in this contribution. First, the discrete function  $f(x, t)$ , for  $x$  a vertex of the lattice and  $t$  the discrete time, is assumed to be the restriction to the lattice of a very regular function denoted in the same way  $f(x, t, \Delta t, s_k, \dots)$  at a point of the continuous

space  $x \in \mathbb{R}^d$  and continuous time  $t$ . The time step  $\Delta t$  is an infinitesimal quantity, and this is also the case for the spatial step  $\Delta x$ . Additionally, we adopt an acoustic scaling: The numerical velocity  $\lambda \equiv \frac{\Delta x}{\Delta t}$  is supposed fixed as  $\Delta x$  and  $\Delta t$  tend to zero. Last but not least, the relaxation parameters  $s_k$  are held fixed when the lattice  $\mathcal{L}$  is made finer and finer.

#### 4) A multi-resolution D2Q9 lattice Boltzmann scheme

In this section, we consider a D2Q9 scheme (see, inter alia, [5, 20, 29, 31]) for a single conservation law in the paradigm of multi-resolution times lattice Boltzmann schemes. This scheme is classic and has been studied in the detail in an article of Luo and one of us [29]. The nine velocities begin with  $v_0 = 0$  and are presented in the Figure 1. The moments  $m$  are named as follows in this contribution:

$$(14) \quad m^t = (\rho, J_x, J_y, \varepsilon, XX, XY, q_x, q_y, h).$$

The density  $\rho$  is a polynomial of degree zero relative to the velocities, the momentum  $(J_x, J_y)$  is composed by polynomials of degree 1, the energy  $\varepsilon$  and the moments  $XX$  and  $XY$  are polynomials of degree 2, the energy flux  $(q_x, q_y)$  is associated to polynomials of degree 3, and the second energy  $h$  is of degree 4. The explicit construction of the matrix  $M$  between particles and moments is detailed in the reference [29]. We have

$$M = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & \lambda & 0 & -\lambda & 0 & \lambda & -\lambda & -\lambda & \lambda \\ 0 & 0 & \lambda & 0 & -\lambda & \lambda & \lambda & -\lambda & -\lambda \\ -4\lambda^2 & -\lambda^2 & -\lambda^2 & -\lambda^2 & -\lambda^2 & 2\lambda^2 & 2\lambda^2 & 2\lambda^2 & 2\lambda^2 \\ 0 & \lambda^2 & -\lambda^2 & \lambda^2 & -\lambda^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda^2 & -\lambda^2 & \lambda^2 & -\lambda^2 \\ 0 & -2\lambda^3 & 0 & 2\lambda^3 & 0 & \lambda^3 & -\lambda^3 & -\lambda^3 & \lambda^3 \\ 0 & 0 & -2\lambda^3 & 0 & 2\lambda^3 & \lambda^3 & \lambda^3 & -\lambda^3 & -\lambda^3 \\ 4\lambda^4 & -2\lambda^4 & -2\lambda^4 & -2\lambda^4 & -2\lambda^4 & \lambda^4 & \lambda^4 & \lambda^4 & \lambda^4 \end{pmatrix}.$$

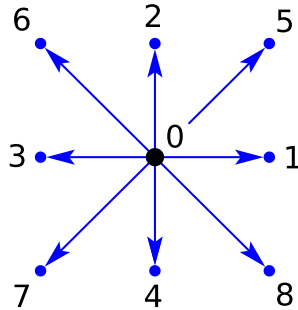


Figure 1: D2Q9 lattice Boltzmann scheme



- Advection operator in the basis of moments

From the velocities  $v_j$  and the matrix  $M$ , we introduce the momentum-velocity operator matrix (see [10]). It is defined by

$$(15) \quad \Lambda \equiv M \operatorname{diag} \left( \sum_{\alpha} v^{\alpha} \partial_{\alpha} \right) M^{-1}$$

with  $1 \leq \alpha \leq d = \text{space dimension}$ . It is simply the set of all advection operators defined for the lattice and resolved in the basis of the moments.

In the case of a single conservation law as studied in the previous section, there is only one conserved quantity  $W$  whereas the vector  $Y$  in the relation (10) is of dimension 8. We decompose the momentum-velocity operator matrix into four blocks  $A$ ,  $B$ ,  $C$ ,  $D$  in coherence with the decomposition (10),

$$(16) \quad \Lambda \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

and similarly for the various powers of  $\Lambda$ ,

$$\Lambda^2 \equiv \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}, \quad \Lambda^3 \equiv \begin{pmatrix} A_3 & B_3 \\ C_3 & D_3 \end{pmatrix}, \quad \Lambda^4 \equiv \begin{pmatrix} A_4 & B_4 \\ C_4 & D_4 \end{pmatrix}.$$

For the D2Q9 scheme and the advection-diffusion, we have after some lines of algebra detailed in [10] the following explicit form for the ‘‘ABCD’’ decomposition introduced in (16):

$$(17) \quad \Lambda_{D2Q9}^{\text{advection}} = \begin{pmatrix} 0 & \partial_x & \partial_y & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{2\lambda^2}{3} \partial_x & 0 & 0 & \frac{1}{6} \partial_x & \frac{1}{2} \partial_x & \partial_y & 0 & 0 & 0 \\ \frac{2\lambda^2}{3} \partial_y & 0 & 0 & \frac{1}{6} \partial_y & -\frac{1}{2} \partial_y & \partial_x & 0 & 0 & 0 \\ 0 & \lambda^2 \partial_x & \lambda^2 \partial_y & 0 & 0 & 0 & \partial_x & \partial_y & 0 \\ 0 & \frac{\lambda^2}{3} \partial_x & -\frac{\lambda^2}{3} \partial_y & 0 & 0 & 0 & -\frac{1}{3} \partial_x & \frac{1}{3} \partial_y & 0 \\ 0 & \frac{2}{3} \lambda^2 \partial_y & \frac{2}{3} \lambda^2 \partial_x & 0 & 0 & 0 & \frac{1}{3} \partial_y & \frac{1}{3} \partial_x & 0 \\ 0 & 0 & 0 & \frac{\lambda^2}{3} \partial_x & -\lambda^2 \partial_x & \lambda^2 \partial_y & 0 & 0 & \frac{1}{3} \partial_x \\ 0 & 0 & 0 & \frac{\lambda^2}{3} \partial_y & \lambda^2 \partial_y & \lambda^2 \partial_x & 0 & 0 & \frac{1}{3} \partial_y \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 \partial_x & \lambda^2 \partial_y & 0 \end{pmatrix}.$$

The structure of the operator matrix  $\Lambda$  follows ‘‘selection rules’’ somewhat similar to those involved in atoms coupled to the electromagnetic field (see, *e.g.*, the book of Cohen-Tannoudji, Diu and Laloë [6]). The moments at equilibrium are typically parametrized by the two components  $u$ ,  $v$  of the imposed velocity and by a coefficient  $\alpha$ :

$$(18) \quad \Phi(\rho) = (\rho u, \rho v, \alpha \rho \lambda^2, \rho(u^2 - v^2), \rho u v, 0, 0, 0)^t.$$

These moments at equilibrium (18) correspond to the following nonconserved distribution

of particles at equilibrium  $f^{\text{eq}} = M^{-1} \Phi$ , with

$$\left\{ \begin{array}{l} f_0^{\text{eq}} = \left(\frac{1}{9} - \frac{\alpha}{9}\right) \rho \\ f_1^{\text{eq}} = \left(\frac{1}{9} - \frac{\alpha}{36} + \frac{u}{6\lambda} + \frac{1}{4\lambda^2} (u^2 - v^2)\right) \rho \\ f_2^{\text{eq}} = \left(\frac{1}{9} - \frac{\alpha}{36} + \frac{v}{6\lambda} - \frac{1}{4\lambda^2} (u^2 - v^2)\right) \rho \\ f_3^{\text{eq}} = \left(\frac{1}{9} - \frac{\alpha}{36} - \frac{u}{6\lambda} + \frac{1}{4\lambda^2} (u^2 - v^2)\right) \rho \\ f_4^{\text{eq}} = \left(\frac{1}{9} - \frac{\alpha}{36} - \frac{v}{6\lambda} - \frac{1}{4\lambda^2} (u^2 - v^2)\right) \rho \\ f_5^{\text{eq}} = \left(\frac{1}{9} + \frac{\alpha}{18} + \frac{u+v}{6\lambda} + \frac{1}{4\lambda^2} uv\right) \rho \\ f_6^{\text{eq}} = \left(\frac{1}{9} + \frac{\alpha}{18} - \frac{u-v}{6\lambda} + \frac{1}{4\lambda^2} uv\right) \rho \\ f_7^{\text{eq}} = \left(\frac{1}{9} + \frac{\alpha}{18} - \frac{u+v}{6\lambda} - \frac{1}{4\lambda^2} uv\right) \rho \\ f_8^{\text{eq}} = \left(\frac{1}{9} + \frac{\alpha}{18} + \frac{u-v}{6\lambda} - \frac{1}{4\lambda^2} uv\right) \rho. \end{array} \right.$$

Then we have the relations

$$\left\{ \begin{array}{l} \sum_j f_j^{\text{eq}} = \rho, \quad \sum_j v_j^x f_j^{\text{eq}} = \rho u, \quad \sum_j v_j^y f_j^{\text{eq}} = \rho v \\ \partial_\alpha \partial_\beta [\sum_j v_j^\alpha v_j^\beta f_j^{\text{eq}} - u^\alpha u^\beta \rho] = \left(\frac{\alpha+4}{6} - \frac{1}{2}(u^2 + v^2)\right) \Delta \rho \end{array} \right.$$

where  $\Delta = \partial_x^2 + \partial_y^2$  is the Laplace operator. If all the relaxation times  $s_{jx}, s_{jy}, s_\varepsilon, s_{xx}, s_{xy}, s_{qx}, s_{qy}$  and  $s_h$  are equal to the ratio  $\frac{\tau_0}{\tau}$ , we have seen in Section 2 that the equivalent partial differential equation at second order derived in Chapman Enskog derived in (8) takes the form

$$(19) \quad \partial_t \rho + u \partial_x \rho + v \partial_y \rho - \varepsilon \left(\tau - \frac{\tau_0}{2}\right) \left[\frac{\alpha+4}{6} \lambda^2 - \frac{1}{2}(u^2 + v^2)\right] \Delta \rho = O(\varepsilon^2).$$

The question now is how to find the equivalent equation when the relaxation parameters differ. Before entering into the resolution of this question, we end this section with a general proposition for lattice Boltzmann schemes.

- Exponential expression of a multi-resolution times lattice Boltzmann scheme

We have an exact relation for a discrete time iteration, in the same spirit as for the relation (1). It explicitly uses the momentum-velocity operator defined in (15).

### Proposition 1: formal expression of one iteration of the scheme

A multi-resolution times lattice Boltzmann scheme (13) can be written in terms of the momentum-velocity operator  $\Lambda$  introduced in (15) through an exponential operator:

$$(20) \quad m(x, t + \varepsilon \tau_0) = \exp(-\varepsilon \tau_0 \Lambda) m^*(x, t).$$

The proof of Proposition 1 is given in [10]. We recall it here to make this contribution

self-contained. We have the following calculation:

$$\begin{aligned}
 m_k(x, t + \varepsilon \tau_0) &= \sum_j M_{kj} f_j^*(x - v_j \varepsilon \tau_0, t) \\
 &= \sum_{j\ell} M_{kj} (M^{-1})_{j\ell} m_\ell^*(x - v_j \varepsilon \tau_0, t) \\
 &= \sum_{j\ell} M_{kj} (M^{-1})_{j\ell} \sum_{n=0}^{\infty} \frac{1}{n!} (-\varepsilon \tau_0 \sum_{\alpha} v_j^\alpha \partial_\alpha)^n m_\ell^*(x, t) \\
 &= \sum_{\ell} \sum_{n=0}^{\infty} \frac{1}{n!} \sum_j M_{kj} (-\varepsilon \tau_0 \sum_{\alpha} v_j^\alpha \partial_\alpha)^n (M^{-1})_{j\ell} m_\ell^*(x, t) \\
 &= \sum_{\ell} \left[ \sum_{n=0}^{\infty} \frac{1}{n!} (-\varepsilon \tau_0 \Lambda)_{k\ell}^n \right] m_\ell^*(x, t) \\
 &= \sum_{\ell} \exp(-\varepsilon \tau_0 \Lambda)_{k\ell} m_\ell^*(x, t) \\
 &= (\exp(-\varepsilon \tau_0 \Lambda) m^*(x, t))_k
 \end{aligned}$$

and the relation (20) is established.  $\square$

## 5) Chapman-Enskog framework for multi-resolution times schemes

In this section, we introduce the Chapman-Enskog expansion in the context of multi-resolution times lattice Boltzmann schemes. We present at Proposition 2 the multiple times dynamics up to fourth order and we deduce general algebraic formulas that control the dynamics at various scales. We remark in Proposition 3, that both expansions give identical results. Then we prove Proposition 2 up to second-order accuracy.

In the kinetic theory of gases, the small parameter  $\varepsilon$  in the Chapman-Enskog expansion is the ratio of the mean free path, typically 65 nanometers under the usual conditions of temperature and pressure [25], and a characteristic dimension of the problem. Here, for multi-resolution times lattice Boltzmann schemes, we can set  $\varepsilon = \frac{\Delta t}{\tau_0}$  as previously, and the small parameter has a purely numerical interpretation: It is the ratio between the time step of the numerical scheme and the reference time scale. We then expand the particle distribution up to fourth order:

$$(21) \quad f = f^{\text{eq}} + \varepsilon f^1 + \varepsilon^2 f^2 + \varepsilon^3 f^3 + \text{O}(\varepsilon^4).$$

First, we consider an important hypothesis of such Chapman-Enskog expansion: The perturbation terms  $f^\ell$  are functions only of the equilibrium  $f = f^{\text{eq}}$  and its spatial derivatives. We apply the d'Humières matrix  $M$  to the expansion (21),

$$m = M f = M f^{\text{eq}} + \varepsilon M f^1 + \varepsilon^2 M f^2 + \varepsilon^3 M f^3 + \text{O}(\varepsilon^4).$$

We take the first conserved component of the previous relation. Then  $W = W + 0$  and the first components of  $M f^1$ ,  $M f^2$ , *etc.* are equal to zero.

Taking next the second nonconserved component, we obtain

$$Y = Y^{\text{eq}} + \varepsilon (M f^1)_Y + \varepsilon^2 (M f^2)_Y + \varepsilon^3 (M f^3)_Y + O(\varepsilon^4)$$

and the perturbation terms  $\varepsilon^\ell (M f^\ell)_Y$  depend only on the conserved moments  $W$  and their spatial derivatives. We introduce the specific notations  $\Psi_j$  for the previous expansion:

$$(22) \quad Y = \Phi(W) + S^{-1} (\varepsilon \tau_0 \Psi_1(W) + \varepsilon^2 \tau_0^2 \Psi_2(W) + \varepsilon^3 \tau_0^3 \Psi_3(W)) + O(\varepsilon^4),$$

with  $Y^{\text{eq}} = \Phi(W)$ , see (11). We suppose also as in [5, 32] a multi-scale approach for the time dynamics:

$$\partial_t = \partial_{t_1} + \varepsilon \partial_{t_2} + \varepsilon^2 \partial_{t_3} + \varepsilon^3 \partial_{t_4} + O(\varepsilon^4).$$

**Proposition 2: multiple time dynamics with the Chapman-Enskog expansion**

With the hypotheses presented previously, the conserved quantities  $W$  follow a multiple time dynamics :

$$(23) \quad \partial_{t_1} W + \Gamma_1(W) = 0, \quad \partial_{t_2} W + \tau_0 \Gamma_2(W) = 0, \quad \partial_{t_3} W + \tau_0^2 \Gamma_3(W) = 0, \quad \partial_{t_4} W + \tau_0^3 \Gamma_4(W) = 0.$$

The differential operators  $\Gamma_1(W)$ ,  $\Psi_1(W)$ ,  $\Gamma_2(W)$ ,  $\Psi_2(W)$ ,  $\Gamma_3(W)$ ,  $\Psi_3(W)$  and  $\Gamma_4(W)$  introduced in the relations (22) and (23) are determined recursively as functions of the data  $v_j$ ,  $M$ ,  $\Phi(W)$  and  $S$ . The operator  $\Gamma_1(W)$  establishes the first-order dynamics,

$$(24) \quad \Gamma_1 = A W + B \Phi(W).$$

After introducing the Hénon matrix

$$(25) \quad \Sigma \equiv S^{-1} - \frac{1}{2} \text{I}$$

that generalizes the expansion first presented by Hénon in [21], the differential operators  $\Psi_1(W)$  and  $\Gamma_2(W)$  have to be specified for the second-order evolution:

$$(26) \quad \begin{cases} \Psi_1 = d\Phi(W) \cdot \Gamma_1 - (C W + D \Phi(W)) \\ \Gamma_2 = B \Sigma \Psi_1. \end{cases}$$

At third order, we have

$$(27) \quad \begin{cases} \Psi_2(W) = \Sigma d\Psi_1 \cdot \Gamma_1 + d\Phi(W) \cdot \Gamma_2 - D \Sigma \Psi_1 \\ \Gamma_3(W) = B \Sigma \Psi_2 - \frac{1}{6} B d\Psi_1 \cdot \Gamma_1 + \frac{1}{12} B_2 \Psi_1, \end{cases}$$

and at fourth order

$$(28) \quad \begin{cases} \Psi_3(W) = \Sigma d\Psi_1 \cdot \Gamma_2 + d\Phi \cdot \Gamma_3 - D \Sigma \Psi_2 + \Sigma d\Psi_2 \cdot \Gamma_1 + \frac{1}{6} D d\Psi_1 \cdot \Gamma_1 \\ \quad - \frac{1}{12} D_2 \Psi_1(W) - \frac{1}{12} d(d\Psi_1 \cdot \Gamma_1) \cdot \Gamma_1 \\ \Gamma_4(W) = B \Sigma \Psi_3 + \frac{1}{4} B_2 \Psi_2 + \frac{1}{6} B D_2 \Sigma \Psi_1 - \frac{1}{6} A B \Psi_2 - \frac{1}{6} B d(d\Phi \cdot \Gamma_1) \cdot \Gamma_2 \\ \quad - \frac{1}{6} B d(d\Phi \cdot \Gamma_2) \cdot \Gamma_1 - \frac{1}{6} B \Sigma d(d\Psi_1 \cdot \Gamma_1) \cdot \Gamma_1. \end{cases}$$

- The proof of this proposition constitutes the remainder of this contribution. We first observe, however, that with the Taylor expansion method an asymptotic partial differential system is emerging [10]:

$$(29) \quad \partial_t W + \Gamma_1 + \Delta t \Gamma_2 + \Delta t^2 \Gamma_3 + \Delta t^3 \Gamma_4 = O(\Delta t^4).$$

The coefficients  $\Gamma_j$  in the expansion (29) are vectors obtained after  $j$  spatial derivations of the conserved moments  $W$  and the equilibrium vector  $\Phi(W)$ . For the non-conserved moments, we have

$$(30) \quad Y = \Phi(W) + S^{-1} (\Delta t \Psi_1 + \Delta t^2 \Psi_2 + \Delta t^3 \Psi_3) + O(\Delta t^4).$$

The differential operators  $\Psi_j$  are analogous to  $\Gamma_j$  but not with the same dimension. The explicit forms of the operators  $\Gamma_1(W)$ ,  $\Psi_1(W)$ ,  $\Gamma_2(W)$ ,  $\Psi_2(W)$ ,  $\Gamma_3(W)$ ,  $\Psi_3(W)$  and  $\Gamma_4(W)$  relative to the fourth-order Taylor expansion have been derived in our contribution [10]. We have the following result.

**Proposition 3: The Taylor and Chapman-Enskog expansions give identical results**  
 With the hypotheses presented at the end of Section 3, the precise algebraic expression of the operators  $\Gamma_1(W)$ ,  $\Psi_1(W)$ ,  $\Gamma_2(W)$ ,  $\Psi_2(W)$ ,  $\Gamma_3(W)$ ,  $\Psi_3(W)$  and  $\Gamma_4(W)$  are identical to the relations (24), (26), (27) and (28).

The proof of this proposition is obtained by comparing the results of Proposition 2 with the main result of our previous contribution. We just have to observe that the expressions (29) and (30) use the same notations as in the reference [10], and that the relations (24) to (28) are exactly the same as those proposed in this reference.  $\square$

- Example: advection-diffusion with the D2Q9 scheme

Before entering into the different steps of the proof of Proposition 2, we illustrate the previous expansion with the scalar conservation law studied in the previous section. The moments are still given by the relation (14) and the nonconserved moments at equilibrium by the formulas (18). The operator matrix  $\Lambda_{D2Q9}^{\text{advection}}$  for advection-diffusion has been made explicit in (17). The block decomposition (16) can be determined. We have  $A = 0$  and

$$(31) \quad B(j_x, j_y, \varepsilon, xx, yy, q_x, q_y, h)^t = \partial_x j_x + \partial_y j_y.$$

Then at first order, we have

$$\Gamma_1 = A W + B \Phi(W) = u \partial_x \rho + v \partial_y \rho.$$

Also, we have from (24) that  $\Psi_1 = d\Phi(W) \cdot \Gamma_1 - (C W + D \Phi(W))$  and

$$(\Psi_1)_{jx} = \frac{u^2 + v^2}{2} \partial_x \rho - \left( \frac{2}{3} + \frac{\alpha}{6} \right) \lambda^2 \partial_x \rho, \quad (\Psi_1)_{jy} = \frac{u^2 + v^2}{2} \partial_y \rho - \left( \frac{2}{3} + \frac{\alpha}{6} \right) \lambda^2 \partial_y \rho.$$

The Hénon matrix  $\Sigma \equiv S^{-1} - \frac{1}{2} I$  is a diagonal matrix and we impose isotropy conditions:  $\sigma_{jx} = \sigma_{jy} = \sigma_j$  and  $\sigma_{qx} = \sigma_{qy} = \sigma_q$ . We then have

$$\Sigma = \text{diag}(\sigma_j, \sigma_j, \sigma_e, \sigma_x, \sigma_x, \sigma_q, \sigma_q, \sigma_h)$$

with  $\sigma_j = \frac{1}{s_j} - \frac{1}{2}$ . Finally, at second order, we have  $\Gamma_2 = B \Sigma \Psi_1$ , and due to the structure (31) of the  $B$  differential operator, only the two first components  $(\Psi_1)_{jx}$  and  $(\Psi_1)_{jy}$  of the vector  $\Psi_1$  are used. Then we have  $B \Sigma \Psi_1 = \sigma_j (\partial_x (\Psi_1)_{jx} + \partial_y (\Psi_1)_{jy})$  and

$$\Gamma_2 = \sigma_j \left[ \frac{u^2 + v^2}{2} - \left( \frac{2}{3} + \frac{\alpha}{6} \right) \lambda^2 \right] \Delta \rho.$$

Finally, due to (23), the equivalent partial differential equation of the D2Q9 lattice Boltzmann scheme is written

$$\partial_t \rho + u \partial_x \rho + v \partial_y \rho - \varepsilon \tau_0 \left( \frac{1}{s_j} - \frac{1}{2} \right) \left[ \frac{\alpha + 4}{6} \lambda^2 - \frac{1}{2} (u^2 + v^2) \right] \Delta \rho = O(\varepsilon^2).$$

This equation is very similar to the equation (19) established previously for the BGK variant. Now, we know that with multi-resolution times lattice Boltzmann schemes, the coefficient of dissipation is exactly related to the relaxation coefficient  $s_j$  for the momentum  $J$ .

- Chapman-Enskog expansion: Study at order zero

We establish here that in the expansion (22), the first term is the equilibrium function  $Y^{\text{eq}} = \Phi(W)$ . We start from the formal expansion (20):  $m(t + \varepsilon \tau_0) = \exp(-\varepsilon \tau_0 \Lambda) m^*$ . At order zero, we can write

$$m + O(\varepsilon) = m^* + O(\varepsilon).$$

For the first component, we have:  $W + O(\varepsilon) = W^* + O(\varepsilon)$ . This relation provides no new information because  $W^* = W$ . For the second component we obtain  $Y + O(\varepsilon) = Y^* + O(\varepsilon)$ . Due to the relaxation (12), we have  $Y^* = Y + S(\Phi(W) - Y)$ . The matrix  $S$  is supposed fixed and invertible. Then

$$(32) \quad Y = \Phi(W) + O(\varepsilon), \quad Y^* = \Phi(W) + O(\varepsilon).$$

- Chapman-Enskog expansion : Study at order one

We consider the expansion (20) at order one with  $\partial_t = \partial_{t_1} + O(\varepsilon)$ . Then

$$(33) \quad m + \varepsilon \tau_0 \partial_{t_1} m + O(\varepsilon^2) = m^* - \varepsilon \tau_0 \Lambda m^* + O(\varepsilon^2)$$

with  $m^t = (W, Y)$  and  $\Lambda$  decomposed into four blocks according to (16). For the first component of the relation (33), we have

$$W + \varepsilon \tau_0 \partial_{t_1} W + O(\varepsilon^2) = W^* - \varepsilon \tau_0 (A W + B Y^*) + O(\varepsilon^2)$$

with  $W^* = W$  and  $Y^* = \Phi(W) + O(\varepsilon)$ . Then  $\partial_{t_1} W = -(A W + B \Phi(W))$  and the first relation of (23) is established, with  $\Gamma_1(W)$  given by the relation (24).  $\square$

- Chapman-Enskog expansion: End of the study at order one

We look now at the second component of the relation (33):

$$Y + \varepsilon \tau_0 \partial_{t_1} Y + O(\varepsilon^2) = Y^* - \varepsilon \tau_0 (C W + D Y^*) + O(\varepsilon^2).$$

Then  $Y - Y^* = -\varepsilon \tau_0 (\partial_{t_1} Y + (C W + D Y^*)) + O(\varepsilon^2)$ . If we take also into consideration the relation (12), we have the exact relation  $S(Y - \Phi(W)) = Y - Y^*$ . Then, after taking into consideration the expansions  $Y = \Phi(W) + O(\varepsilon)$  and  $Y^* = \Phi(W) + O(\varepsilon)$ , we have the following calculation:

$$\begin{aligned} S(Y - \Phi(W)) &= Y - Y^* \\ &= -\varepsilon \tau_0 \partial_{t_1} (\Phi(W) + O(\varepsilon)) - \varepsilon \tau_0 (C W + D (\Phi(W) + O(\varepsilon))) + O(\varepsilon^2) \\ &= \varepsilon \tau_0 [-d\Phi(W) \cdot \partial_{t_1} W - (C W + D \Phi(W))] + O(\varepsilon^2) \\ &= \varepsilon \tau_0 [d\Phi(W) \cdot \Gamma_1 - (C W + D \Phi(W))] + O(\varepsilon^2) \\ &= \varepsilon \tau_0 \Psi_1(W) + O(\varepsilon^2), \end{aligned}$$

with  $\Psi_1 = d\Phi(W) \cdot \Gamma_1 - (C W + D \Phi(W))$ , and the first relation of (26) is established.  $\square$

- Taking into account the Hénon matrix inside the expansion

From the relations (12) and (22), we have the two expansions at first order

$$Y = \Phi(W) + \varepsilon \tau_0 S^{-1} \Psi_1(W) + O(\varepsilon^2), \quad Y^* = \Phi(W) + \varepsilon \tau_0 (S^{-1} - I) \Psi_1(W) + O(\varepsilon^2).$$

With the matrix  $\Sigma$  introduced in (25), we have also the expansions at first order

$$(34) \quad \begin{cases} Y = \Phi(W) + \varepsilon \tau_0 (\Sigma + \frac{1}{2} \text{I}) \Psi_1(W) + \text{O}(\varepsilon^2) \\ Y^* = \Phi(W) + \varepsilon \tau_0 (\Sigma - \frac{1}{2} \text{I}) \Psi_1(W) + \text{O}(\varepsilon^2). \end{cases}$$

- Chapman-Enskog expansion : Study at order two

We consider again the expansion (20), but now at order two, and we obtain

$$m + \varepsilon \tau_0 \partial_t m + \frac{1}{2} \varepsilon^2 \tau_0^2 \partial_t^2 m + \text{O}(\varepsilon^3) = m^* - \varepsilon \tau_0 \Lambda m^* + \frac{1}{2} \varepsilon^2 \tau_0^2 \Lambda^2 m^* + \text{O}(\varepsilon^3).$$

We introduce the multiple scales for time evolution:  $\partial_t = \partial_{t_1} + \varepsilon \partial_{t_2} + \text{O}(\varepsilon^2)$ . Then

$$m + \varepsilon \tau_0 (\partial_{t_1} + \varepsilon \partial_{t_2}) m + \frac{1}{2} \varepsilon^2 \tau_0^2 (\partial_{t_1} + \text{O}(\varepsilon))^2 m + \text{O}(\varepsilon^3) = m^* - \varepsilon \tau_0 \Lambda m^* + \frac{1}{2} \varepsilon^2 \tau_0^2 \Lambda^2 m^* + \text{O}(\varepsilon^3)$$

and we have

$$(35) \quad m + \varepsilon \tau_0 \partial_{t_1} m + \varepsilon^2 \tau_0 \left( \partial_{t_2} m + \frac{\tau_0}{2} \partial_{t_1}^2 m \right) = m^* - \varepsilon \tau_0 \Lambda m^* + \frac{1}{2} \varepsilon^2 \tau_0^2 \Lambda^2 m^* + \text{O}(\varepsilon^3).$$

The square of the operator  $\Lambda$  satisfies  $\Lambda^2 = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \equiv \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}$  and

$$(36) \quad A_2 = A^2 + BC, \quad B_2 = AB + BD, \quad C_2 = CA + DC, \quad D_2 = CB + D^2.$$

and similar operators for higher powers of the matrix  $\Lambda$ . Then the first component of the relation (35) can be written

$$\begin{cases} W + \varepsilon \tau_0 \partial_{t_1} W + \varepsilon^2 \tau_0 \left( \partial_{t_2} W + \frac{\tau_0}{2} \partial_{t_1}^2 W \right) \\ = W - \varepsilon \tau_0 (AW + BY^*) + \frac{1}{2} \varepsilon^2 \tau_0^2 (A_2 W + B_2 Y^*) + \text{O}(\varepsilon^3). \end{cases}$$

The terms at order zero of the previous relation are eliminated. At order one, we have to take into account the relation  $Y^* = \Phi(W) + \varepsilon \tau_0 (\Sigma \Psi_1 - \frac{1}{2} \Psi_1) + \text{O}(\varepsilon^2)$ . Then we recover the relation  $\partial_{t_1} W + AW + B\Phi(W) = 0$  established previously. At second order a new relation is emerging:

$$(37) \quad \partial_{t_2} W + \frac{\tau_0}{2} \partial_{t_1}^2 W = -\tau_0 B \left( \Sigma \Psi_1 - \frac{1}{2} \Psi_1 \right) + \frac{\tau_0}{2} (A_2 W + B_2 \Phi).$$

From the relation (24), we have

$$\begin{aligned} \partial_{t_1}^2 W &= \partial_{t_1} (-\Gamma_1(W)) \\ &= -\partial_{t_1} (AW + B\Phi(W)) \\ &= A\Gamma_1 + B d\Phi(W). \Gamma_1 \\ &= A(AW + B\Phi) + B d\Phi(W). \Gamma_1 \\ &= A_2 W - BCW + AB\Phi + B d\Phi(W). \Gamma_1, \end{aligned}$$

due to (36). Then the relation (37) can be written

$$\begin{aligned} &\partial_{t_2} W + \tau_0 B \Sigma \Psi_1 \\ &= -\frac{\tau_0}{2} (A_2 W - BCW + AB\Phi + B d\Phi(W). \Gamma_1) + \frac{\tau_0}{2} B \Psi_1 + \frac{\tau_0}{2} (A_2 W + B_2 \Phi) \\ &= \frac{\tau_0}{2} (BCW - AB\Phi - B d\Phi(W). \Gamma_1 + B (d\Phi(W). \Gamma_1 - CW - D\Phi) + (AB + BD) \Phi) \\ &= 0. \end{aligned}$$

This last relation expresses exactly that  $\partial_{t_2} W + \tau_0 \Gamma_2(W) = 0$  with  $\Gamma_2(W) = B \Sigma \Psi_1(W)$ , and the second relation of (26) is established.  $\square$

## 6) Chapman-Enskog expansion at order three

The relations established in the previous section are very useful. For example, we have used them to study the ability to recover formally the compressible Navier-Stokes equations at second order with only one particle distribution [13]. In order to study finer properties of the lattice Boltzmann scheme, however, a higher precision is necessary. We have done this for specific problems in previous contributions [12, 13, 27, 28]. Here, we establish general formulas (27) for future works.

- Chapman-Enskog expansion: End of the study at order two

We first look to the second-order expansion (35). The second component can be written

$$\begin{cases} Y + \varepsilon \tau_0 \partial_{t_1} Y + \varepsilon^2 \tau_0 \partial_{t_2} Y + \frac{1}{2} \varepsilon^2 \tau_0^2 \partial_{t_1}^2 Y \\ = Y^* - \varepsilon \tau_0 (C W + D Y^*) + \frac{1}{2} \varepsilon^2 \tau_0^2 (C_2 W + D_2 Y^*) + O(\varepsilon^3). \end{cases}$$

Then we have

$$\begin{aligned} S(Y - \Phi(W)) &= Y - Y^* \\ &= -\varepsilon \tau_0 \partial_{t_1} Y - \varepsilon^2 \tau_0 (\partial_{t_2} Y + \frac{\tau_0}{2} \partial_{t_1}^2 Y) - \varepsilon \tau_0 (C W + D Y^*) \\ &\quad + \frac{1}{2} \varepsilon^2 \tau_0^2 (C_2 W + D_2 Y^*) + O(\varepsilon^3). \end{aligned}$$

We insert the representations (34) into the right-hand side of the previous expansion to obtain

$$\begin{aligned} S(Y - \Phi(W)) &= -\varepsilon \tau_0 \partial_{t_1} \Phi(W) - \varepsilon^2 \tau_0 [\tau_0 \partial_{t_1} (\Sigma \Psi_1 + \frac{\tau_0}{2} \Psi_1) + \partial_{t_2} \Phi(W) + \frac{\tau_0}{2} \partial_{t_1}^2 \Phi(W)] \\ &\quad - \varepsilon \tau_0 [C W + D (\Phi(W) + \varepsilon \tau_0 (\Sigma \Psi_1 - \frac{1}{2} \Psi_1))] + \frac{1}{2} \varepsilon^2 \tau_0^2 (C_2 W + D_2 \Phi) + O(\varepsilon^3). \end{aligned}$$

We have by definition  $S(Y - \Phi(W)) = \varepsilon \tau_0 \Psi_1 + \varepsilon^2 \tau_0^2 \Psi_2 + O(\varepsilon^3)$ . The first-order term relative to  $\varepsilon$  is proportional to  $d\Phi \cdot \Gamma_1 - (C W + D \Phi(W))$ , and we recover  $\Psi_1$  due to the first relation of (26). We can make explicit the second-order term from the previous calculation:

$$\Psi_2 = -\Sigma \partial_{t_1} \Psi_1 - \frac{1}{2} \partial_{t_1} \Psi_1 - \frac{1}{\tau_0} \partial_{t_2} \Phi(W) - \frac{1}{2} \partial_{t_1}^2 \Phi(W) - D (\Sigma \Psi_1 - \frac{1}{2} \Psi_1) + \frac{1}{2} C_2 W + \frac{1}{2} D_2 \Phi(W)$$

with

$$\begin{aligned} \partial_{t_1} \Psi_1 &= \partial_{t_1} (d\Phi \cdot \Gamma_1 - C W - D \Phi(W)) \\ &= \partial_{t_1} (d\Phi \cdot \Gamma_1) - C \partial_{t_1} W - D d\Phi \cdot \partial_{t_1} W \\ &= \partial_{t_1} (d\Phi \cdot \Gamma_1) + C \Gamma_1 + D d\Phi \cdot \Gamma_1, \end{aligned}$$

$$\frac{1}{\tau_0} \partial_{t_2} \Phi(W) = \frac{1}{\tau_0} d\Phi(W) \cdot \partial_{t_2} W = -d\Phi(W) \cdot \Gamma_2,$$

$$\partial_{t_1}^2 \Phi(W) = \partial_{t_1} (\partial_{t_1} \Phi(W)) = \partial_{t_1} (d\Phi \cdot \partial_{t_1} W) = -\partial_{t_1} (d\Phi \cdot \Gamma_1),$$

$C_2 = C A + B D$  and  $D_2 = C B + D^2$ . We deduce

$$\begin{aligned} \Psi_2 &= \Sigma d\Psi_1 \cdot \Gamma_1 - \frac{1}{2} (\partial_{t_1} (d\Phi \cdot \Gamma_1) + C \Gamma_1 + D d\Phi \cdot \Gamma_1) + d\Phi(W) \cdot \Gamma_2 + \frac{1}{2} \partial_{t_1} (d\Phi \cdot \Gamma_1) \\ &\quad - D \Sigma \Psi_1 + \frac{1}{2} D \Psi_1 + \frac{1}{2} C (A W + B \Phi) + \frac{1}{2} D (C W + B \Phi) \\ &= \Sigma d\Psi_1 \cdot \Gamma_1 - \frac{1}{2} D d\Phi \cdot \Gamma_1 + d\Phi(W) \cdot \Gamma_2 - D \Sigma \Psi_1 + \frac{1}{2} D \Psi_1 + \frac{1}{2} D (d\Phi \cdot \Gamma_1 - \Psi_1) \\ &= \Sigma d\Psi_1 \cdot \Gamma_1 + d\Phi(W) \cdot \Gamma_2 - D \Sigma \Psi_1 \end{aligned}$$



and the first relation of (27) is proven.  $\square$

• Chapman-Enskog expansion: Study at order three

We refer to the expansion (20) at order three, and we obtain

$$\begin{cases} m + \varepsilon \tau_0 \partial_t m + \frac{1}{2} \varepsilon^2 \tau_0^2 \partial_t^2 m + \frac{1}{6} \varepsilon^3 \tau_0^3 \partial_t^3 m \\ = m^* - \varepsilon \tau_0 \Lambda m^* + \frac{1}{2} \varepsilon^2 \tau_0^2 \Lambda^2 m^* - \frac{1}{6} \varepsilon^3 \tau_0^3 \Lambda^3 m^* + O(\varepsilon^4) \end{cases}$$

and

$$\begin{cases} m + \varepsilon \tau_0 (\partial_{t_1} + \varepsilon \partial_{t_2} + \varepsilon^2 \partial_{t_3}) m + \frac{1}{2} \tau_0^2 \varepsilon^2 (\partial_{t_1} + \varepsilon \partial_{t_2} + O(\varepsilon^2))^2 m + \frac{1}{6} \varepsilon^3 \tau_0^3 (\partial_{t_1} + O(\varepsilon))^3 m \\ = m^* - \varepsilon \tau_0 \Lambda m^* + \frac{1}{2} \varepsilon^2 \tau_0^2 \Lambda^2 m^* - \frac{1}{6} \varepsilon^3 \tau_0^3 \Lambda^3 m^* + O(\varepsilon^4). \end{cases}$$

We expand the various powers of  $\partial_t = \partial_{t_1} + \varepsilon \partial_{t_2} + \varepsilon^2 \partial_{t_3} + O(\varepsilon^3)$ , paying attention to the non commutation of these operators. For example,  $\partial_{t_1} \partial_{t_2} \neq \partial_{t_2} \partial_{t_1}$ :

$$(38) \quad \begin{cases} m + \varepsilon \tau_0 (\partial_{t_1} + \varepsilon \partial_{t_2} + \varepsilon^2 \partial_{t_3}) m + \frac{1}{2} \varepsilon^2 \tau_0^2 (\partial_{t_1}^2 + \varepsilon \partial_{t_1} \partial_{t_2} + \varepsilon \partial_{t_2} \partial_{t_1}) m \\ + \frac{1}{6} \varepsilon^3 \tau_0^3 \partial_{t_1}^3 m = m^* - \varepsilon \tau_0 \Lambda m^* + \frac{1}{2} \varepsilon^2 \tau_0^2 \Lambda^2 m^* - \frac{1}{6} \varepsilon^3 \tau_0^3 \Lambda^3 m^* + O(\varepsilon^4). \end{cases}$$

We consider the first component of the relation (38), relative to the conserved variables:

$$\begin{aligned} W + \varepsilon \tau_0 (\partial_{t_1} + \varepsilon \partial_{t_2} + \varepsilon^2 \partial_{t_3}) W + \frac{1}{2} \varepsilon^2 \tau_0^2 (\partial_{t_1}^2 + \varepsilon \partial_{t_1} \partial_{t_2} + \varepsilon \partial_{t_2} \partial_{t_1}) W + \frac{1}{6} \varepsilon^3 \tau_0^3 \partial_{t_1}^3 W \\ = W - \varepsilon \tau_0 (A W + B Y^*) + \frac{1}{2} \varepsilon^2 \tau_0^2 (A_2 W + B_2 Y^*) - \frac{1}{6} \varepsilon^3 \tau_0^3 (A_3 W + B_3 Y^*) + O(\varepsilon^4), \end{aligned}$$

with  $Y^* = \Phi(W) + \varepsilon \tau_0 (\Sigma \Psi_1 - \frac{1}{2} \Psi_1) + \varepsilon^2 \tau_0^2 (\Sigma \Psi_2 - \frac{1}{2} \Psi_2) + O(\varepsilon^3)$ . Then we obtain

$$\begin{cases} W + \varepsilon \tau_0 (\partial_{t_1} + \varepsilon \tau_0^2 \partial_{t_2} + \varepsilon^2 \partial_{t_3}) W + \frac{1}{2} \varepsilon^2 \tau_0^2 (\partial_{t_1}^2 + \varepsilon \partial_{t_1} \partial_{t_2} + \varepsilon \partial_{t_2} \partial_{t_1}) W + \frac{1}{6} \varepsilon^3 \tau_0^3 \partial_{t_1}^3 W \\ = W - \varepsilon \tau_0 A W - \varepsilon \tau_0 B [\Phi(W) + \varepsilon \tau_0 (\Sigma \Psi_1 - \frac{1}{2} \Psi_1) + \varepsilon^2 \tau_0^2 (\Sigma \Psi_2 - \frac{1}{2} \Psi_2)] \\ + \frac{1}{2} \varepsilon^2 \tau_0^2 A_2 W + \frac{1}{2} \varepsilon^2 \tau_0^2 B_2 [\Phi(W) + \varepsilon \tau_0 (\Sigma \Psi_1 - \frac{1}{2} \Psi_1)] \\ - \frac{1}{6} \varepsilon^3 \tau_0^3 (A_3 W + B_3 \Phi) + O(\varepsilon^4). \end{cases}$$

We identify the third-order terms of the previous relation:

$$\begin{cases} \frac{1}{\tau_0^2} \partial_{t_3} W + \frac{1}{2 \tau_0} (\partial_{t_1} \partial_{t_2} W + \partial_{t_2} \partial_{t_1} W) + \frac{1}{6} \partial_{t_1}^3 W \\ = -B (\Sigma \Psi_2 - \frac{1}{2} \Psi_2) + \frac{1}{2} B_2 (\Sigma \Psi_1 - \frac{1}{2} \Psi_1) - \frac{1}{6} (A_3 W + B_3 \Phi) \end{cases}$$

with

$$\begin{aligned} \frac{1}{\tau_0} \partial_{t_1} \partial_{t_2} W &= \partial_{t_1} (-B \Sigma \Psi_1) = -B \Sigma d\Psi_1 \cdot \partial_{t_1} W = B \Sigma d\Psi_1 \cdot \Gamma_1, \\ \frac{1}{\tau_0} \partial_{t_2} \partial_{t_1} W &= \frac{1}{\tau_0} \partial_{t_2} (-A W - B \Phi) = A \Gamma_2 - \frac{1}{\tau_0} B d\Phi \cdot \partial_{t_2} W = A B \Sigma \Gamma_1 + B d\Phi \cdot \Gamma_2. \end{aligned}$$

Then

$$\begin{aligned} \frac{1}{\tau_0^2} \partial_{t_3} W + \frac{1}{2} B \Sigma d\Psi_1 \cdot \Gamma_1 + \frac{1}{2} (A B \Sigma \Gamma_1 + B d\Phi \cdot \Gamma_2) + \frac{1}{6} \partial_{t_1}^3 W \\ = -B (\Sigma \Psi_2 - \frac{1}{2} \Psi_2) + \frac{1}{2} B_2 (\Sigma \Psi_1 - \frac{1}{2} \Psi_1) - \frac{1}{6} (A_3 W + B_3 \Phi). \end{aligned}$$

We observe that

$$\begin{aligned} \partial_{t_1}^3 W &= \partial_{t_1} (A \Gamma_1 + B d\Phi \cdot \Gamma_1) = \partial_{t_1} (A (A W + B \Phi) + B d\Phi \cdot \Gamma_1) \\ &= -A (A \Gamma_1 - B d\Phi \cdot \Gamma_1) - B d(d\Phi \cdot \Gamma_1) \cdot \Gamma_1 \\ &= -A^2 \Gamma_1 - A B d\Phi \cdot \Gamma_1 - B d(d\Phi \cdot \Gamma_1) \cdot \Gamma_1, \end{aligned}$$

$$A_3 = A_2 A + B_2 C, \quad B_3 = A_2 B + B_2 D,$$

$$\begin{aligned} A_3 W + B_3 \Phi &= A_2 (A W + B \Phi) + B_2 (C W + D \Phi) \\ &= (A^2 + B C) \Gamma_1 + (A B + B D) (d\Phi \cdot \Gamma_1 - \Psi_1) \end{aligned}$$

$$\begin{aligned}
 &= A(A\Gamma_1 + B d\Phi.\Gamma_1) + B(C\Gamma_1 + D d\Phi.\Gamma_1) - B_2 \Psi_1 \\
 &= A(A\Gamma_1 + B d\Phi.\Gamma_1) + B(d(d\Phi.\Gamma_1).\Gamma_1 - d\Psi_1.\Gamma_1) - B_2 \Psi_1 \\
 &= -\partial_{t_1}^3 W - B d\Psi_1.\Gamma_1 - B_2 \Psi_1,
 \end{aligned}$$

$$\Psi_2 = \Sigma d\Psi_1.\Gamma_1 + d\Phi.\Gamma_2 - D \Sigma \Psi_1.$$

In consequence, we have

$$\begin{aligned}
 \frac{1}{\tau_0} \partial_{t_3} W &= -\frac{1}{2} (B \Sigma d\Psi_1.\Gamma_1 + A B \Sigma \Gamma_1 + B d\Phi.\Gamma_2) - B \Sigma \Psi_2 + \frac{1}{2} B \Psi_2 \\
 &\quad + \frac{1}{2} B_2 (\Sigma \Psi_1 - \frac{1}{2} \Psi_1) + \frac{1}{6} (B d\Psi_1.\Gamma_1 + B_2 \Psi_1) \\
 &= -\frac{1}{2} B \Sigma d\Psi_1.\Gamma_1 - \frac{1}{2} A B \Sigma \Gamma_1 - \frac{1}{2} B d\Phi.\Gamma_2 - B \Sigma \Psi_2 \\
 &\quad + \frac{1}{2} B (\Sigma d\Psi_1.\Gamma_1 + d\Phi.\Gamma_2 - D \Sigma \Psi_1) + \frac{1}{2} B_2 \Sigma \Psi_1 - \left(\frac{1}{4} - \frac{1}{6}\right) B_2 \Psi_1 + \frac{1}{6} B d\Psi_1.\Gamma_1 \\
 &= -B \Sigma \Psi_2 - \frac{1}{12} B_2 \Psi_1 + \frac{1}{6} B d\Psi_1.\Gamma_1
 \end{aligned}$$

because  $B_2 = AB + BD$ . Then the third relation of (23) is established, and  $\Gamma_3$  is given by the second relation of (27).  $\square$

## 7) Chapman-Enskog expansion at order four

We establish the first relation of (28) and make explicit the expression for  $\Psi_3$ . Then we extract the value of  $\Gamma_4$  from (20) and establish the second relation of (28).

- Chapman-Enskog expansion: End of the study at order three

We consider the second component of the relation (38):

$$\begin{aligned}
 Y + \varepsilon \tau_0 (\partial_{t_1} + \varepsilon \partial_{t_2} + \varepsilon^2 \partial_{t_3}) Y + \frac{1}{2} \varepsilon^2 \tau_0^2 (\partial_{t_1}^2 + \varepsilon \partial_{t_1} \partial_{t_2} + \varepsilon \partial_{t_2} \partial_{t_1}) Y + \frac{1}{6} \varepsilon^3 \tau_0^3 \partial_{t_1}^3 Y \\
 = Y^* - \varepsilon \tau_0 (C W + D Y^*) + \frac{1}{2} \varepsilon^2 \tau_0^2 (C_2 W + D_2 Y^*) - \frac{1}{6} \varepsilon^3 \tau_0^3 (C_3 W + D_3 Y^*) + O(\varepsilon^4).
 \end{aligned}$$

We insert in this relation the representation at order two of the nonconserved variables  $Y$  and  $Y^*$

$$\begin{cases} Y = \Phi(W) + \varepsilon \tau_0 (\Sigma \Psi_1 + \frac{1}{2} \Psi_1) + \varepsilon^2 \tau_0^2 (\Sigma \Psi_2 + \frac{1}{2} \Psi_2) + O(\varepsilon^3) \\ Y^* = \Phi(W) + \varepsilon \tau_0 (\Sigma \Psi_1 - \frac{1}{2} \Psi_1) + \varepsilon^2 \tau_0^2 (\Sigma \Psi_2 - \frac{1}{2} \Psi_2) + O(\varepsilon^3). \end{cases}$$

Then

$$\begin{aligned}
 Y + \varepsilon \tau_0 \partial_{t_1} [\Phi(W) + \varepsilon \tau_0 (\Sigma \Psi_1 + \frac{1}{2} \Psi_1) + \varepsilon^2 \tau_0^2 (\Sigma \Psi_2 + \frac{1}{2} \Psi_2)] \\
 + \varepsilon^2 \tau_0 \partial_{t_2} [\Phi(W) + \varepsilon \tau_0 (\Sigma \Psi_1 + \frac{1}{2} \Psi_1)] + \varepsilon^3 \tau_0 \partial_{t_3} \Phi(W) \\
 + \frac{1}{2} \varepsilon^2 \tau_0^2 \partial_{t_1}^2 [\Phi(W) + \varepsilon \tau_0 (\Sigma \Psi_1 + \frac{1}{2} \Psi_1)] + \frac{1}{2} \varepsilon^3 \tau_0^2 \partial_{t_1} \partial_{t_2} \Phi(W) + \frac{1}{2} \varepsilon^3 \tau_0^2 \partial_{t_2} \partial_{t_1} \Phi(W) \\
 + \frac{1}{6} \varepsilon^3 \tau_0^3 \partial_{t_1}^3 \Phi(W) = Y^* - \varepsilon \tau_0 C W \\
 - \varepsilon \tau_0 D [\Phi(W) + \varepsilon \tau_0 (\Sigma \Psi_1 - \frac{1}{2} \Psi_1) + \varepsilon^2 \tau_0^2 (\Sigma \Psi_2 - \frac{1}{2} \Psi_2)] + \frac{1}{2} \varepsilon^2 \tau_0^2 C_2 W \\
 + \frac{1}{2} \varepsilon^2 \tau_0^2 D_2 [\Phi(W) + \varepsilon \tau_0 (\Sigma \Psi_1 - \frac{1}{2} \Psi_1)] - \frac{1}{6} \varepsilon^3 \tau_0^3 (C_3 W + D_3 \Phi) + O(\varepsilon^4)
 \end{aligned}$$

We identify the terms relative to  $\varepsilon^3$  in the relation

$$Y - Y^* = S(Y - \Phi) = \varepsilon \tau_0 \Psi_1 + \varepsilon^2 \tau_0^2 \Psi_2 + \varepsilon^3 \tau_0^3 \Psi_3 + O(\varepsilon^4),$$

and we deduce

$$\begin{cases} \Psi_3 = -\partial_{t_1} (\Sigma \Psi_2 + \frac{1}{2} \Psi_2) - \frac{1}{\tau_0} \partial_{t_2} (\Sigma \Psi_1 + \frac{1}{2} \Psi_1) - \frac{1}{\tau_0^2} \partial_{t_3} \Phi - \frac{1}{2} \partial_{t_1}^2 (\Sigma \Psi_1 + \frac{1}{2} \Psi_1) \\ \quad - \frac{1}{2\tau_0} \partial_{t_1} \partial_{t_2} \Phi(W) - \frac{1}{2\tau_0} \partial_{t_2} \partial_{t_1} \Phi(W) - \frac{1}{6} \partial_{t_1}^3 \Phi - D (\Sigma \Psi_2 - \frac{1}{2} \Psi_2) \\ \quad + \frac{1}{2} D_2 (\Sigma \Psi_1 - \frac{1}{2} \Psi_1) - \frac{1}{6} (C_3 W + D_3 \Phi). \end{cases}$$

We have the auxiliary relations

$$\begin{aligned}
 -\partial_{t_1}(\Sigma \Psi_2 + \frac{1}{2}\Psi_2) &= (\Sigma + \frac{1}{2}) d\Psi_2.\Gamma_1 \\
 -\frac{1}{\tau_0} \partial_{t_2}(\Sigma \Psi_1 + \frac{1}{2}\Psi_1) &= (\Sigma + \frac{1}{2}) d\Psi_1.\Gamma_2 \\
 -\frac{1}{\tau_0^2} \partial_{t_3} \Phi &= d\Phi.\Gamma_3 \\
 -\frac{1}{2} \partial_{t_1}^2(\Sigma \Psi_1 + \frac{1}{2}\Psi_1) &= \frac{1}{2}(\Sigma + \frac{1}{2}) \partial_{t_1}(d\Psi_1.\Gamma_1) = -\frac{1}{2}\Sigma d(d\Psi_1.\Gamma_1).\Gamma_1 - \frac{1}{4}d(d\Psi_1.\Gamma_1).\Gamma_1 \\
 -\frac{1}{2\tau_0} \partial_{t_1} \partial_{t_2} \Phi(W) &= \frac{1}{2} \partial_{t_1}(d\Phi.\Gamma_2) \\
 -\frac{1}{2\tau_0} \partial_{t_2} \partial_{t_1} \Phi(W) &= \frac{1}{2} \partial_{t_2}(d\Phi.\Gamma_1) \\
 -D(\Sigma \Psi_2 - \frac{1}{2}\Psi_2) &= -D\Sigma \Psi_2 + \frac{1}{2}D\Psi_2 \\
 \frac{1}{2}D_2(\Sigma \Psi_1 - \frac{1}{2}\Psi_1) &= \frac{1}{2}D_2\Sigma \Psi_1 - \frac{1}{4}D_2\Psi_1
 \end{aligned}$$

and we obtain

$$(39) \quad \begin{cases} \Psi_3 = (\Sigma + \frac{1}{2}) d\Psi_2.\Gamma_1 + (\Sigma + \frac{1}{2}) d\Psi_1.\Gamma_2 + d\Phi.\Gamma_3 - \frac{1}{2}\Sigma d(d\Psi_1.\Gamma_1).\Gamma_1 \\ \quad -\frac{1}{4}d(d\Psi_1.\Gamma_1).\Gamma_1 + \frac{1}{2}\partial_{t_1}(d\Phi.\Gamma_2) + \frac{1}{2}\partial_{t_2}(d\Phi.\Gamma_1) - \frac{1}{6}\partial_{t_1}^3\Phi - D\Sigma \Psi_2 + \frac{1}{2}D\Psi_2 \\ \quad + \frac{1}{2}D_2\Sigma \Psi_1 - \frac{1}{4}D_2\Psi_1 - \frac{1}{6}(C_3W + D_3\Phi). \end{cases}$$

We have also the final intermediate relation

$$-\frac{1}{6}[\partial_{t_1}^3\Phi + (C_3W + D_3\Phi)] = \frac{1}{6}(D_2\Psi_1 + Dd\Psi_1.\Gamma_1 + d(d\Psi_1.\Gamma_1).\Gamma_1).$$

This last relation comes from the following calculation:

$$\partial_{t_1}\Phi = -d\Phi.\Gamma_1 = -\Psi_1 - CW - D\Phi$$

$$\partial_{t_1}^2\Phi = d\Psi.\Gamma_1 + C\Gamma_1 + Dd\Phi.\Gamma_1$$

$$= d\Psi.\Gamma_1 + C(AW + B\Phi) + D(\Psi_1 + CW + D\Phi)$$

$$= C_2W + D_2\Phi + D\Psi_1 + d\Psi.\Gamma_1 \quad \text{because } C_2 = CA + DC, \quad D_2 = CB + D^2$$

$$\partial_{t_1}^3\Phi = -C_2\Gamma_1 - D_2(\Psi_1 + CW + D\Phi) - Dd\Psi.\Gamma_1 - d(d\Psi_1.\Gamma_1).\Gamma_1$$

$$= -C_2(AW + B\Phi) - D_2\Psi_1 - D_2(CW + D\Phi) - Dd\Psi.\Gamma_1 - d(d\Psi_1.\Gamma_1).\Gamma_1$$

$$= -C_3W - D_3\Phi - D_2\Psi_1 - Dd\Psi.\Gamma_1 - d(d\Psi_1.\Gamma_1).\Gamma_1$$

$$\text{because } C_3 = C_2A + D_2C, \quad D_3 = C_2B + D_2D$$

and  $-\partial_{t_1}^3\Phi - C_3W - D_3\Phi = D_2\Psi_1 + Dd\Psi.\Gamma_1 + d(d\Psi_1.\Gamma_1).\Gamma_1$ . We insert this relation into the expression (39) and we obtain

$$\begin{aligned}
 \Psi_3 &= d\Phi.\Gamma_3 + \Sigma d\Psi_1.\Gamma_2 + \frac{1}{2}d\Psi_1.\Gamma_2 + \Sigma d\Psi_2.\Gamma_1 + \frac{1}{2}d\Psi_2.\Gamma_1 - \frac{1}{2}\Sigma d(d\Psi_1.\Gamma_1).\Gamma_1 - \frac{1}{4}d(d\Psi_1.\Gamma_1).\Gamma_1 \\
 &\quad + \frac{1}{2}\partial_{t_1}(d\Phi.\Gamma_2) + \frac{1}{2}\partial_{t_2}(d\Phi.\Gamma_1) - D\Sigma \Psi_2 + \frac{1}{2}D\Psi_2 + \frac{1}{2}D_2\Sigma \Psi_1 - \frac{1}{4}D_2\Psi_1 \\
 &\quad + \frac{1}{6}[D_2\Psi_1 + Dd\Psi_1.\Gamma_1 + d(d\Psi_1.\Gamma_1).\Gamma_1] \\
 &= d\Phi.\Gamma_3 + \Sigma d\Psi_1.\Gamma_2 + \frac{1}{2}d(d\Phi.\Gamma_1 - CW - D\Phi).\Gamma_2 + \Sigma d\Psi_2.\Gamma_1 + \frac{1}{2}d(\Sigma d\Psi_1.\Gamma_1 \\
 &\quad + d\Phi.\Gamma_2 - D\Sigma \Psi_1).\Gamma_1 - \frac{1}{2}\Sigma d(d\Psi_1.\Gamma_1).\Gamma_1 - \frac{1}{12}d(d\Psi_1.\Gamma_1).\Gamma_1 + \frac{1}{2}\partial_{t_1}(d\Phi.\Gamma_2) + \frac{1}{2}\partial_{t_2}(d\Phi.\Gamma_1) \\
 &\quad - D\Sigma \Psi_2 + \frac{1}{2}D(\Sigma d\Psi_1.\Gamma_1 + d\Phi.\Gamma_2 - D\Sigma \Psi_1) + \frac{1}{2}D_2\Sigma \Psi_1 - \frac{1}{12}D_2\Psi_1 + \frac{1}{6}Dd\Psi_1.\Gamma_1 \\
 &= d\Phi.\Gamma_3 + \Sigma d\Psi_1.\Gamma_2 - \frac{1}{2}CB\Sigma \Psi_1 + \Sigma d\Psi_2.\Gamma_1 - \frac{1}{12}d(d\Psi_1.\Gamma_1).\Gamma_1 - D\Sigma \Psi_2 - \frac{1}{2}D^2\Sigma \Psi_1 \\
 &\quad + \frac{1}{2}D_2\Sigma \Psi_1 - \frac{1}{12}D_2\Psi_1 + \frac{1}{6}Dd\Psi_1.\Gamma_1
 \end{aligned}$$

after combining several terms. Then we have finally

$$\Psi_3(W) = d\Phi.\Gamma_3 + \Sigma d\Psi_1.\Gamma_2 + \Sigma d\Psi_2.\Gamma_1 - \frac{1}{12}d(d\Psi_1.\Gamma_1).\Gamma_1 - D\Sigma \Psi_2 - \frac{1}{12}D_2\Psi_1 + \frac{1}{6}Dd\Psi_1.\Gamma_1$$

because  $D_2 = CB + D^2$ . In this way, the first relation of (28) is established.  $\square$

• Chapman-Enskog expansion: Study at order four

We look now to the expansion (20) at order four:

$$\begin{aligned} m + \varepsilon \tau_0 \partial_t m + \frac{1}{2} \varepsilon^2 \tau_0^2 \partial_t^2 m + \frac{1}{6} \varepsilon^3 \tau_0^3 \partial_t^3 m + \frac{1}{24} \varepsilon^4 \tau_0^4 \partial_t^4 m \\ = m^* - \varepsilon \tau_0 \Lambda m^* + \frac{1}{2} \varepsilon^2 \tau_0^2 \Lambda^2 m^* - \frac{1}{6} \varepsilon^3 \tau_0^3 \Lambda^3 m^* + \frac{1}{24} \varepsilon^4 \tau_0^4 \Lambda^4 m^* + O(\varepsilon^5), \end{aligned}$$

and, as usual with the Chapman-Enskog expansion, we replace the time derivative operator  $\partial_t$  by  $\partial_{t_1} + \varepsilon \partial_{t_2} + \varepsilon^2 \partial_{t_3} + \varepsilon^3 \partial_{t_4} + O(\varepsilon^4)$  and deduce

$$\begin{aligned} m + \varepsilon \tau_0 (\partial_{t_1} + \varepsilon \partial_{t_2} + \varepsilon^2 \partial_{t_3} + \varepsilon^3 \partial_{t_4} + O(\varepsilon^4)) m + \frac{1}{2} \varepsilon^2 \tau_0^2 (\partial_{t_1} + \varepsilon \partial_{t_2} + \varepsilon^2 \partial_{t_3} + O(\varepsilon^3))^2 m \\ + \frac{1}{6} \varepsilon^3 \tau_0^3 (\partial_{t_1} + \varepsilon \partial_{t_2} + O(\varepsilon^2))^3 m + \frac{1}{24} \varepsilon^4 \tau_0^4 (\partial_{t_1} + O(\varepsilon))^4 m \\ = m^* - \varepsilon \tau_0 \Lambda m^* + \frac{1}{2} \varepsilon^2 \tau_0^2 \Lambda^2 m^* - \frac{1}{6} \varepsilon^3 \tau_0^3 \Lambda^3 m^* + \frac{1}{24} \varepsilon^4 \tau_0^4 \Lambda^4 m^* + O(\varepsilon^5). \end{aligned}$$

We expand the powers of the noncommutative operators,

$$\begin{aligned} m + \varepsilon \tau_0 (\partial_{t_1} + \varepsilon \partial_{t_2} + \varepsilon^2 \partial_{t_3} + \varepsilon^3 \partial_{t_4} + O(\varepsilon^4)) m \\ + \frac{1}{2} \varepsilon^2 \tau_0^2 [\partial_{t_1}^2 + \varepsilon (\partial_{t_1} \partial_{t_2} + \partial_{t_2} \partial_{t_1}) + \varepsilon^2 (\partial_{t_2}^2 + \partial_{t_1} \partial_{t_3} + \partial_{t_3} \partial_{t_1})] m \\ + \frac{1}{6} \varepsilon^3 \tau_0^3 [\partial_{t_1}^3 m + \varepsilon (\partial_{t_2} \partial_{t_1}^2 + \partial_{t_1} \partial_{t_2} \partial_{t_1} + \partial_{t_1}^2 \partial_{t_2})] m + \frac{1}{24} \varepsilon^4 \tau_0^4 \partial_{t_1}^4 m \\ = m^* - \varepsilon \tau_0 \Lambda m^* + \frac{1}{2} \varepsilon^2 \tau_0^2 \Lambda^2 m^* - \frac{1}{6} \varepsilon^3 \tau_0^3 \Lambda^3 m^* + \frac{1}{24} \varepsilon^4 \tau_0^4 \Lambda^4 m^* + O(\varepsilon^5). \end{aligned}$$

We take the first component of the above relation,

$$\begin{aligned} W + \varepsilon \tau_0 (\partial_{t_1} + \varepsilon \partial_{t_2} + \varepsilon^2 \partial_{t_3} + \varepsilon^3 \partial_{t_4} + O(\varepsilon^4)) W \\ + \frac{1}{2} \varepsilon^2 \tau_0^2 [\partial_{t_1}^2 + \varepsilon (\partial_{t_1} \partial_{t_2} + \partial_{t_2} \partial_{t_1}) + \varepsilon^2 (\partial_{t_2}^2 + \partial_{t_1} \partial_{t_3} + \partial_{t_3} \partial_{t_1})] W \\ + \frac{1}{6} \varepsilon^3 \tau_0^3 [\partial_{t_1}^3 W + \varepsilon (\partial_{t_2} \partial_{t_1}^2 + \partial_{t_1} \partial_{t_2} \partial_{t_1} + \partial_{t_1}^2 \partial_{t_2})] W + \frac{1}{24} \varepsilon^4 \tau_0^4 \partial_{t_1}^4 W = W - \varepsilon \tau_0 (AW + BY^*) \\ + \frac{1}{2} \varepsilon^2 \tau_0^2 (A_2 W + B_2 Y^*) - \frac{1}{6} \varepsilon^3 \tau_0^3 (A_3 W + B_3 Y^*) + \frac{1}{24} \varepsilon^4 \tau_0^4 (A_4 W + B_4 Y^*) + O(\varepsilon^5) \end{aligned}$$

and we introduce the relation

$$Y^* = \Phi(W) + \varepsilon \tau_0 \left( \Sigma - \frac{1}{2} \right) \Psi_1 + \varepsilon^2 \tau_0^2 \left( \Sigma - \frac{1}{2} \right) \Psi_2 + \varepsilon^3 \tau_0^3 \left( \Sigma - \frac{1}{2} \right) \Psi_3 + O(\varepsilon^4)$$

into the right-hand side of the previous expression. Then we obtain the identity

$$\begin{aligned} W + \varepsilon \tau_0 (\partial_{t_1} + \varepsilon \partial_{t_2} + \varepsilon^2 \partial_{t_3} + \varepsilon^3 \partial_{t_4} + O(\varepsilon^4)) W \\ + \frac{1}{2} \varepsilon^2 \tau_0^2 [\partial_{t_1}^2 + \varepsilon (\partial_{t_1} \partial_{t_2} + \partial_{t_2} \partial_{t_1}) + \varepsilon^2 (\partial_{t_2}^2 + \partial_{t_1} \partial_{t_3} + \partial_{t_3} \partial_{t_1})] W \\ + \frac{1}{6} \varepsilon^3 \tau_0^3 [\partial_{t_1}^3 W + \varepsilon (\partial_{t_2} \partial_{t_1}^2 + \partial_{t_1} \partial_{t_2} \partial_{t_1} + \partial_{t_1}^2 \partial_{t_2})] W + \frac{1}{24} \varepsilon^4 \tau_0^4 \partial_{t_1}^4 W \\ = W - \varepsilon \tau_0 A W - \varepsilon \tau_0 B [\Phi(W) + \varepsilon \tau_0 (\Sigma - \frac{1}{2}) \Psi_1 + \varepsilon^2 \tau_0^2 (\Sigma - \frac{1}{2}) \Psi_2 + \varepsilon^3 \tau_0^3 (\Sigma - \frac{1}{2}) \Psi_3] \\ + \frac{1}{2} \varepsilon^2 \tau_0^2 A_2 W + \frac{1}{2} \varepsilon^2 \tau_0^2 B_2 [\Phi(W) + \varepsilon \tau_0 (\Sigma - \frac{1}{2}) \Psi_1 + \varepsilon^2 \tau_0^2 (\Sigma - \frac{1}{2}) \Psi_2] \\ - \frac{1}{6} \varepsilon^3 \tau_0^3 A_3 W - \frac{1}{6} \varepsilon^3 \tau_0^3 B_3 [\Phi(W) + \varepsilon \tau_0 (\Sigma - \frac{1}{2}) \Psi_1] + \frac{1}{24} \varepsilon^4 \tau_0^4 (A_4 W + B_4 \Phi) + O(\varepsilon^5), \end{aligned}$$

and we identify the fourth-order terms relative to  $\varepsilon$ :

$$\begin{aligned} \partial_{t_4} W + \frac{1}{2 \tau_0^2} (\partial_{t_2}^2 + \partial_{t_1} \partial_{t_3} + \partial_{t_3} \partial_{t_1}) W + \frac{1}{6 \tau_0} (\partial_{t_2} \partial_{t_1}^2 + \partial_{t_1} \partial_{t_2} \partial_{t_1} + \partial_{t_1}^2 \partial_{t_2}) W + \frac{1}{24} \partial_{t_1}^4 W \\ + B (\Sigma - \frac{1}{2}) \Psi_3 - \frac{1}{2} B_2 (\Sigma - \frac{1}{2}) \Psi_2 + \frac{1}{6} B_3 (\Sigma - \frac{1}{2}) \Psi_1 - \frac{1}{24} (A_4 W + B_4 \Phi) = 0. \end{aligned}$$

Due to the fourth relation of (23), we can write

$$\begin{aligned} \Gamma_4 = B (\Sigma - \frac{1}{2}) \Psi_3 + \frac{1}{2 \tau_0^2} (\partial_{t_2}^2 + \partial_{t_1} \partial_{t_3} + \partial_{t_3} \partial_{t_1}) W - \frac{1}{2} B_2 (\Sigma - \frac{1}{2}) \Psi_2 \\ + \frac{1}{6 \tau_0} (\partial_{t_2} \partial_{t_1}^2 + \partial_{t_1} \partial_{t_2} \partial_{t_1} + \partial_{t_1}^2 \partial_{t_2}) W + \frac{1}{6} B_3 (\Sigma - \frac{1}{2}) \Psi_1 + \frac{1}{24} [\partial_{t_1}^4 W - (A_4 W + B_4 \Phi)]. \end{aligned}$$

We make explicit the following algebraic expressions for intermediate terms:

$$\begin{aligned}
 \frac{1}{\tau_0^2} \partial_{t_2}^2 W &= -\frac{1}{\tau_0} \partial_{t_2}(\Gamma_2) = -\frac{1}{\tau_0} d\Gamma_2 \cdot \partial_{t_2} W = d\Gamma_2 \cdot \Gamma_2, \\
 \frac{1}{\tau_0^2} \partial_{t_1} \partial_{t_3} W &= -\partial_{t_1} \Gamma_3 = -d\Gamma_3 \cdot \partial_{t_1} W = d\Gamma_3 \cdot \Gamma_1, \\
 \frac{1}{\tau_0^2} \partial_{t_3} \partial_{t_1} W &= d\Gamma_1 \cdot \Gamma_3 = A\Gamma_3 + B d\Phi \cdot \Gamma_3, \\
 \frac{1}{\tau_0} \partial_{t_2} \partial_{t_1}^2 W &= \frac{1}{\tau_0} \partial_{t_2}(d\Gamma_1 \cdot \Gamma_1) = \frac{1}{\tau_0} \partial_{t_2}(A\Gamma_1 + d\Phi \cdot \Gamma_1) = -A d\Gamma_1 \cdot \Gamma_2 - B d(d\Phi \cdot \Gamma_1) \cdot \Gamma_2, \\
 \frac{1}{\tau_0} \partial_{t_1} \partial_{t_1} \partial_{t_2} W &= \partial_{t_1}(d\Gamma_1 \cdot \Gamma_2) = \partial_{t_1}(A\Gamma_2 + B d\Phi \cdot \Gamma_2) = -A d\Gamma_2 \cdot \Gamma_1 - B d(d\Phi \cdot \Gamma_2) \cdot \Gamma_1, \\
 \frac{1}{\tau_0} \partial_{t_1}^2 \partial_{t_2} W &= \partial_{t_1}(d\Gamma_2 \cdot \Gamma_1) = \partial_{t_1}(B \Sigma d\Psi_1 \cdot \Gamma_1) = -B \Sigma d(d\Psi_1 \cdot \Gamma_1) \cdot \Gamma_1,
 \end{aligned}$$

and in this way we obtain

$$(40) \quad \begin{cases} \Gamma_4 = B(\Sigma - \frac{1}{2})\Psi_3 + \frac{1}{2}[d\Gamma_2 \cdot \Gamma_2 + d\Gamma_3 \cdot \Gamma_1 + A\Gamma_3 + B d\Phi \cdot \Gamma_3 - B_2(\Sigma - \frac{1}{2})\Psi_2] \\ -\frac{1}{6}[A d\Gamma_1 \cdot \Gamma_2 + B d(d\Phi \cdot \Gamma_1) \cdot \Gamma_2 + A d\Gamma_2 \cdot \Gamma_1 + B d(d\Phi \cdot \Gamma_2) \cdot \Gamma_1 \\ + B \Sigma d(d\Psi_1 \cdot \Gamma_1) \cdot \Gamma_1 - B_3(\Sigma - \frac{1}{2})\Psi_1] + \frac{1}{24}[\partial_{t_1}^4 W - (A_4 W + B_4 \Phi)]. \end{cases}$$

We now establish the identity  $\partial_{t_1}^4 W - (A_4 W + B_4 \Phi) = B_3 \Psi_1 + B_2 d\Psi_1 \cdot \Gamma_1 + B d(d\Psi_1 \cdot \Gamma_1) \cdot \Gamma_1$ .

We have the following relations:

$$\partial_{t_1} W = -\Gamma_1 = -(A W + B \Phi),$$

$$\begin{aligned}
 \partial_{t_1}^2 W &= A\Gamma_1 + B d\Phi \cdot \Gamma_1 \\
 &= A\Gamma_1 + B(\Psi_1 + C W + D \Phi) \\
 &= A(A W + B \Phi) + B \Psi_1 + B C W + B D \Phi \\
 &= A_2 W + B_2 \Phi + B \Psi_1
 \end{aligned}$$

$$\begin{aligned}
 \partial_{t_1}^3 W &= -A_2 \Gamma_1 - B_2 d\Phi \cdot \Gamma_1 - B d\Psi_1 \cdot \Gamma_1 \\
 &= -A_2(A W + B \Phi) - B_2(\Psi_1 + C W + D \Phi) - B d\Psi_1 \cdot \Gamma_1 \\
 &= -A_3 W - B_3 \Phi - B_2 \Psi_1 - B d\Psi_1 \cdot \Gamma_1
 \end{aligned}$$

$$\begin{aligned}
 \partial_{t_1}^4 W &= A_3 \Gamma_1 + B_3 d\Phi \cdot \Gamma_1 + B_2 d\Psi_1 \cdot \Gamma_1 + B d(d\Psi_1 \cdot \Gamma_1) \cdot \Gamma_1 \\
 &= A_3(A W + B \Phi) + B_3(\Psi_1 + C W + D \Phi) + B_2 d\Psi_1 \cdot \Gamma_1 + B d(d\Psi_1 \cdot \Gamma_1) \cdot \Gamma_1 \\
 &= A_4 W + B_4 \Phi + B_3 \Psi_1 + B_2 d\Psi_1 \cdot \Gamma_1 + B d(d\Psi_1 \cdot \Gamma_1) \cdot \Gamma_1
 \end{aligned}$$

and  $\partial_{t_1}^4 W - (A_4 W + B_4 \Phi) = B_3 \Psi_1 + B_2 d\Psi_1 \cdot \Gamma_1 + B d(d\Psi_1 \cdot \Gamma_1) \cdot \Gamma_1$ . We replace this relation in the expression (40) to find

$$\begin{aligned}
 \Gamma_4 &= B \Sigma \Psi_3 - \frac{1}{2} B \Psi_3 + \frac{1}{2}(d\Gamma_2 \cdot \Gamma_2 + d\Gamma_3 \cdot \Gamma_1 + A\Gamma_3 + B d\Phi \cdot \Gamma_3) - \frac{1}{2} B_2 \Sigma \Psi_2 + \frac{1}{4} B_2 \Psi_2 \\
 &\quad - \frac{1}{6}[A d\Gamma_1 \cdot \Gamma_2 + B d(d\Phi \cdot \Gamma_1) \cdot \Gamma_2 + A d\Gamma_2 \cdot \Gamma_1 + B d(d\Phi \cdot \Gamma_2) \cdot \Gamma_1 + B \Sigma d(d\Psi_1 \cdot \Gamma_1) \cdot \Gamma_1] \\
 &\quad + \frac{1}{6} B_3 \Sigma \Psi_1 - \frac{1}{12} B_3 \Psi_1 + \frac{1}{24}[B_3 \Psi_1 + B_2 d\Psi_1 \cdot \Gamma_1 + B d(d\Psi_1 \cdot \Gamma_1) \cdot \Gamma_1] \\
 &= B \Sigma \Psi_3 - \frac{1}{2} B[\Sigma d\Psi_1 \cdot \Gamma_2 + d\Phi \cdot \Gamma_3 - D \Sigma \Psi_2 + \Sigma d\Psi_2 \cdot \Gamma_1 + \frac{1}{6} D d\Psi_1 \cdot \Gamma_1 - \frac{1}{12} D_2 \Psi_1 \\
 &\quad - \frac{1}{12} d(d\Psi_1 \cdot \Gamma_1) \cdot \Gamma_1] + \frac{1}{2} d\Gamma_2 \cdot \Gamma_2 + \frac{1}{2} d\Gamma_3 \cdot \Gamma_1 + \frac{1}{2} A[B \Sigma \Psi_2 + \frac{1}{12} B_2 \Psi_1 - \frac{1}{6} B d\Psi_1 \cdot \Gamma_1] + \frac{1}{2} B d\Phi \cdot \Gamma_3 \\
 &\quad - \frac{1}{2} B_2 \Sigma \Psi_2 + \frac{1}{4} B_2 \Psi_2 - \frac{1}{6} A d\Gamma_1 \cdot \Gamma_2 - \frac{1}{6} B d(d\Phi \cdot \Gamma_1) \cdot \Gamma_2 - \frac{1}{6} A d\Gamma_2 \cdot \Gamma_1 - \frac{1}{6} B d(d\Phi \cdot \Gamma_2) \cdot \Gamma_1 \\
 &\quad - \frac{1}{6} B \Sigma d(d\Psi_1 \cdot \Gamma_1) \cdot \Gamma_1 + \frac{1}{6} B_3 \Sigma \Psi_1 - \frac{1}{24} B_3 \Psi_1 + \frac{1}{24} B_2 d\Psi_1 \cdot \Gamma_1 + \frac{1}{24} B d(d\Psi_1 \cdot \Gamma_1) \cdot \Gamma_1
 \end{aligned}$$

due to the expressions (27) for  $\Gamma_3$  and (28) for  $\Psi_3$ . We see that ten terms disappear from the previous expression because  $\Sigma d\Psi_1 \cdot \Gamma_2 = d\Gamma_2 \cdot \Gamma_2$ , and  $BD + AB = B_2$  and  $BD_2 + AB_2 = B_3$ .

Then, after elementary simplification of some fractions, we replace  $\Gamma_3$  in the expression  $d\Gamma_3.\Gamma_1$  by its expression given by (27) to obtain

$$\begin{aligned}\Gamma_4 &= B \Sigma \Psi_3 - \frac{1}{2} B \Sigma d\Psi_2.\Gamma_1 - \frac{1}{12} B D d\Psi_1.\Gamma_1 + \frac{1}{2} d[B \Sigma \Psi_2 + \frac{1}{12} B_2 \Psi_1 - \frac{1}{6} B d\Psi_1.\Gamma_1].\Gamma_1 \\ &\quad - \frac{1}{12} A B d\Psi_1.\Gamma_1 + \frac{1}{4} B_2 \Psi_2 - \frac{1}{6} A d\Gamma_1.\Gamma_2 - \frac{1}{6} B d(d\Phi.\Gamma_1).\Gamma_2 - \frac{1}{6} A d\Gamma_2.\Gamma_1 - \frac{1}{6} B d(d\Phi.\Gamma_2).\Gamma_1 \\ &\quad + \frac{1}{6} B_3 \Sigma \Psi_1 - \frac{1}{6} B \Sigma d(d\Psi_1.\Gamma_1).\Gamma_1 + \frac{1}{24} B_2 d\Psi_1.\Gamma_1 + \frac{1}{12} B d(d\Psi_1.\Gamma_1).\Gamma_1.\end{aligned}$$

Four terms clearly vanish and four others may be eliminated because  $BD + AB = B_2$ . Then

$$\begin{aligned}\Gamma_4 &= B \Sigma \Psi_3 + \frac{1}{4} B_2 \Psi_2 - \frac{1}{6} A d\Gamma_1.\Gamma_2 - \frac{1}{6} B d(d\Phi.\Gamma_1).\Gamma_2 - \frac{1}{6} B d(d\Phi.\Gamma_2).\Gamma_1 \\ &\quad + \frac{1}{6} (A B_2 + B D_2) \Sigma \Psi_1 - \frac{1}{6} B \Sigma d(d\Psi_1.\Gamma_1).\Gamma_1 \\ &= B \Sigma \Psi_3 + \frac{1}{4} B_2 \Psi_2 - \frac{1}{6} A (A B \Sigma \Psi_1 + B d\Phi.\Gamma_2) - \frac{1}{6} B d(d\Phi.\Gamma_1).\Gamma_2 - \frac{1}{6} A B \Sigma d\Psi_1.\Gamma_1 \\ &\quad - \frac{1}{6} B d(d\Phi.\Gamma_2).\Gamma_1 + \frac{1}{6} A B_2 \Sigma \Psi_1 + \frac{1}{6} B D_2 \Sigma \Psi_1 - \frac{1}{6} B \Sigma d(d\Psi_1.\Gamma_1).\Gamma_1 \\ &= B \Sigma \Psi_3 + \frac{1}{4} B_2 \Psi_2 + \frac{1}{6} A B D \Sigma \Psi_1 - \frac{1}{6} A B d\Phi.\Gamma_2 - \frac{1}{6} B d(d\Phi.\Gamma_1).\Gamma_2 - \frac{1}{6} A B \Sigma d\Psi_1.\Gamma_1 \\ &\quad - \frac{1}{6} B d(d\Phi.\Gamma_2).\Gamma_1 + \frac{1}{6} B D_2 \Sigma \Psi_1 - \frac{1}{6} B \Sigma d(d\Psi_1.\Gamma_1).\Gamma_1 \\ &= B \Sigma \Psi_3 + \frac{1}{4} B_2 \Psi_2 - \frac{1}{6} A B \Psi_2 - \frac{1}{6} B d(d\Phi.\Gamma_1).\Gamma_2 - \frac{1}{6} B d(d\Phi.\Gamma_2).\Gamma_1 + \frac{1}{6} B D_2 \Sigma \Psi_1 \\ &\quad - \frac{1}{6} B \Sigma d(d\Psi_1.\Gamma_1).\Gamma_1\end{aligned}$$

because  $-\Psi_2 = D \Sigma \Psi_1 - d\Phi.\Gamma_2 - \Sigma d\Psi_1.\Gamma_1$ . Then we have

$$\begin{cases} \Gamma_4(W) = B \Sigma \Psi_3 + \frac{1}{4} B_2 \Psi_2 + \frac{1}{6} B D_2 \Sigma \Psi_1 - \frac{1}{6} A B \Psi_2 \\ \quad - \frac{1}{6} B (d(d\Phi.\Gamma_1).\Gamma_2 + d(d\Phi.\Gamma_2).\Gamma_1) - \frac{1}{6} B \Sigma d(d\Psi_1.\Gamma_1).\Gamma_1. \end{cases}$$

In this way, the second relation of (28) is established.  $\square$

## 8) Survey and conclusions

In this contribution, we have considered the exponentiation of differential operators, classical for the BGK variant of lattice Boltzmann schemes. We have also used an exponential iteration of the multi-resolution times lattice Boltzmann schemes. Then the ‘‘ABCD’’ block decomposition of the moment-velocity operator matrix allows one to formulate in a compact way the asymptotic expansion of the lattice Boltzmann schemes that give rise to the equivalent nonlinear partial differential equations of the conserved moments. We have calculated the coefficients of the expansion up to order four, with recursive formulas containing less than seven terms. To do this, the intensive use of differential calculus is mandatory, but the calculation has been systematized and the results have followed in an automatic way.

We have applied this expansion at second order for the compressible Navier-Stokes equations. We have proposed in [13] various lattice Boltzmann schemes in two and three spatial dimensions with a single particle distribution. The higher-order expressions can be useful to set initial conditions in simulations, in particular when studying behaviours of a given symmetry. Third-order precision is also very interesting to avoid some defects of lattice Boltzmann schemes, as studied in [28].

We hope that this work helps to standardize these kinds of calculations, since most all work in this field involves expansions to fourth order or less. At the same time, this work is meant to highlight the remarkable agreement between two so very different approaches, and to

remind practitioners that the problem of establishing this agreement to all orders is still a very open question.

## Acknowledgments

FD thanks the Centre National de la Recherche Scientifique for according a “Delegation” at the International Research Laboratory 3457 in the “Centre de Recherches Mathématiques” of the Université de Montréal during the period February-July 2021. A part of this contribution was done during this period.

## References

- [1] A. Augier, F. Dubois, L. Gouarin, B. Graille, “Linear lattice Boltzmann schemes for Acoustic: parameter choices and isotropy properties”, *Computers and Mathematics with Applications*, vol. **65**, p. 845-863, 2013.
- [2] P. L. Bhatnagar, E. P. Gross, M. Krook, “A model for collision processes in gases. I. Small amplitude processes in charged and neutral one-component systems”, *Physical Review*, vol. **94**, p. 511-525, 1954.
- [3] B. M. Boghosian, P. Coveney, “Inverse Chapman–Enskog derivation of the thermohydrodynamic lattice-BGK model for the ideal gas”, *International Journal of Modern Physics C*, 1998.
- [4] S. Chapman, T.G. Cowling, *The mathematical theory of non-uniform gases*, Cambridge University Press, 1939.
- [5] S. Chen, G. D. Doolen, “Lattice Boltzmann Method for Fluid Flows”, *Annual Review of Fluid Mechanics*, vol. **30**, p. 329-364, 1998.
- [6] C. Cohen-Tannoudji, B. Diu, F. Laloë, *Mécanique quantique, Quantum Mechanics*, John Wiley & Sons, New York, 1977.
- [7] F. Dubois, “Une introduction au schéma de Boltzmann sur réseau”, *ESAIM: Proceedings*, vol. **18**, p. 181-215, 2007.
- [8] F. Dubois, “Equivalent partial differential equations of a lattice Boltzmann scheme”, *Computers and Mathematics with Applications*, vol. **55**, p. 1441-1449, 2008.
- [9] F. Dubois, “Third order equivalent equation of lattice Boltzmann scheme”, *Discrete and Continuous Dynamical Systems, A*, vol. **23**, p. 221-248, 2009.
- [10] F. Dubois, “Nonlinear fourth-order Taylor expansion of lattice Boltzmann schemes”, *Asymptotic Analysis*, vol. **127**, p. 297-337, 2022.
- [11] F. Dubois, P. Lallemand, “Towards higher order lattice Boltzmann schemes”, *Journal of Statistical Mechanics, Theory and Experiment*, P06006, 2009.
- [12] F. Dubois, P. Lallemand, “Quartic Parameters for Acoustic Applications of Lattice Boltzmann Scheme”, *Computers and Mathematics with Applications*, vol. **61**, p. 3404-3416, 2011.

- [13] F. Dubois, P. Lallemand, “On single distribution lattice Boltzmann schemes for the approximation of Navier Stokes equations”, *hal archives ouvertes*, hal-03702835 and *arxiv.org*, 2206.13261, 2022, to appear.
- [14] F. Dubois, P. Lallemand, M. Tekitek, “On a superconvergent lattice Boltzmann boundary scheme”, *Computers and Mathematics with Applications*, vol. **59**, p. 2141-2149, 2010.
- [15] F. Dubois, P. Lallemand, M. Tekitek, “Generalized bounce back boundary condition for the nine velocities two-dimensional lattice Boltzmann scheme”, *Computers and Fluids*, vol. **193**, 103534, 2019.
- [16] F. Dubois, P. Lallemand, M. Tekitek, “On anti bounce back boundary condition”, *Computers and Mathematics with Applications*, vol. **79**, p. 555-575, 2020.
- [17] U. Frisch, B. Hasslacher, Y. Pomeau, “Lattice-gas automata for the Navier-Stokes equation”, *Physical Review Letters*, vol. **56**, p. 1505-1508, 1986.
- [18] Z. Guo, C. Shu, *Lattice Boltzmann method and its applications in engineering*, World Scientific, 2013.
- [19] J. Hardy, O. de Pazzis, Y. Pomeau, “Molecular dynamics of a classical lattice gas: Transport properties and time correlation functions”, *Physical Review A*, vol. **13**, p. 1949-1961, 1976.
- [20] X. He, L.-S. Luo, “Theory of the lattice Boltzmann method: from the Boltzmann equation to the lattice Boltzmann equation”, *Physical Review E*, vol. **56**, p. 6811-6817, 1997.
- [21] M. Hénon, “Viscosity of a lattice gas”, *Complex systems*, vol. **1**, p. 763-789, 1987.
- [22] F. J. Higuera, J. Jiménez, “Boltzmann approach to lattice gas simulations”, *Europhysics Letters*, vol. **9**, p. 663-668, 1989.
- [23] F. J. Higuera, S. Succi, R. Benzi, “Lattice gas dynamics with enhanced collisions”, *Europhysics Letters*, vol. **9**, p. 345-349, 1989.
- [24] D. d’Humières, “Generalized lattice-Boltzmann equations”, in *Rarefied Gas Dynamics: Theory and Simulations*, vol. **159** of *AIAA Progress in Astronautics and Aeronautics*, p. 450-458, 1992.
- [25] S. G. Jennings, “The mean free path in air”, *Journal of Aerosol Science*, vol. **19**, p. 159-166, 1988.
- [26] T. Krüger, H. Kusumaatmaja, A. Kuzmin, O. Shardt, G. Silva, E.M. Viggien, *The lattice Boltzmann method: principles and practice*, Springer Verlag, New York, 2017.
- [27] P. Lallemand, F. Dubois, “Comparison of simulations of convective flows”, *Communications in Computational Physics*, vol. **17**, p.1169-1184, mai 2015.
- [28] P. Lallemand, F. Dubois, L.-S. Luo, “Anomalous advection in LBE simulations”, International Conference for Mesoscopic Methods in Engineering and Science, Hamburg (Germany), 18-22 july 2016, to appear.
- [29] P. Lallemand, L.-S. Luo, “Theory of the lattice Boltzmann method: dispersion, dissipation, isotropy, galilean invariance, and stability”, *Physical Review E*, vol. **61**, p. 6546-6542, 2000.



- [30] A. Lerat, R. Peyret, “Sur l’origine des oscillations apparaissant dans les profils de choc calculés par des méthodes aux différences”, *Comptes Rendus de l’Académie des Sciences de Paris, série A*, vol. **276**, p. 759–762, 1973.
- [31] Y. H. Qian, D. d’Humières, P. Lallemand, “Lattice BGK models for Navier-Stokes equation”, *Europhysics Letters*, vol. **17**, p. 479-484, 1992.
- [32] Y. H. Qian, Y. Zhou, “On higher order dynamics in lattice-based models using Chapman-Enskog method”, *Physical Review E*, vol. **61**, p. 2103-2106, 2000.
- [33] Y. I. Shokin and N. N. Yanenko, “On the relation between the correctness of the first differential approximation and the stability of difference schemes for hyperbolic equation systems”, *Matematicheskie Zametki* (in russian), vol. **4**, p. 493–502, 1968.
- [34] S. Succi, *The Lattice Boltzmann equation for fluid dynamics and beyond*, Oxford University Press, Oxford, 2001.
- [35] W.F. Warming, B.J. Hyett, “The modified equation approach to the stability and accuracy analysis of finite difference methods”, *Journal of Computational Physics*, vol. **14**, p. 159–179, 1974.