

Convergence of the incremental projection method using conforming approximations

*R. Eymard and D. Maltese**

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Abstract

We prove the convergence of an incremental projection numerical scheme for the time-dependent incompressible Navier–Stokes equations, without any regularity assumption on the weak solution. The velocity and the pressure are discretised in conforming spaces, whose the compatibility is ensured by the existence of an interpolator for regular functions which preserves approximate divergence free properties. Owing to a priori estimates, we get the existence and uniqueness of the discrete approximation. Compactness properties are then proved, relying on a Lions-like lemma for time translate estimates. It is then possible to show the convergence of the approximate solution to a weak solution of the problem. The construction of the interpolator is detailed in the case of the lowest degree Taylor-Hood finite element.

1 Introduction

The Navier–Stokes equations for a homogeneous incompressible fluid can be written in a strong form as:

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} - \Delta \mathbf{u} + \nabla p = \mathbf{f} \text{ in } (0, T) \times \Omega, \quad (1a)$$

$$\operatorname{div} \mathbf{u} = 0 \text{ in } (0, T) \times \Omega, \quad (1b)$$

where the density and the viscosity are set to one for the sake of simplicity, and where

$$T > 0, \text{ and } \Omega \text{ is a connected, open and bounded subset of } \mathbb{R}^d, \ d \in \{2, 3\}, \quad (2)$$

with a Lipschitz boundary $\partial\Omega$.

The variables \mathbf{u} and p are respectively the velocity and the pressure of the fluid, and (1a) and (1b) respectively model the momentum conservation and the mass conservation of an incompressible fluid. This system is supplemented with the homogeneous Dirichlet boundary condition

$$\mathbf{u} = 0 \text{ on } (0, T) \times \partial\Omega, \quad (3)$$

and the initial condition

$$\mathbf{u}(0) = \mathbf{u}_0 \text{ in } \Omega. \quad (4)$$

The function \mathbf{u}_0 is the initial datum for the velocity and the function \mathbf{f} is the source term.

While this system of equations is coupling the velocity and the pressure, projection numerical schemes, introduced in [3] and [23], enable the successive resolution of decoupled elliptic equations for the velocity

*Université Gustave Eiffel, LAMA, (UMR 8050), UPEM, UPEC, CNRS, F-77454, Marne-la-Vallée (France), robert.eynard, david.maltese@univ-eiffel.fr

and the pressure. This leads to cheaper and smaller computations than those issued from a coupled approximation, and these methods are widely used (see [1], [11], [12], [18], [24]). Let us present the principle of the “incremental projection method” for the approximation of (1) by a continuous-in-space discrete-in-time scheme (see [10]):

- *Prediction step* : Find a function $\tilde{\mathbf{u}}^{n+1} \in H_0^1(\Omega)^d$, which is therefore regular in space and respects the boundary conditions but is not divergence-free, such that the following linearized momentum equation holds in the homogeneous Dirichlet weak sense:

$$\frac{1}{\delta t}(\tilde{\mathbf{u}}^{n+1} - \mathbf{u}^n) + (\mathbf{u}^n \cdot \nabla)\tilde{\mathbf{u}}^{n+1} - \Delta\tilde{\mathbf{u}}^{n+1} + \nabla p^n = \mathbf{f}^{n+1} \text{ in } \Omega, \quad (5a)$$

$$\tilde{\mathbf{u}}^{n+1} = 0 \text{ on } \partial\Omega. \quad (5b)$$

This step involves d decoupled resolution of elliptic problems for each of the components of the velocity, in the case of the Stokes problem (neglecting the term $(\mathbf{u}^n \cdot \nabla)\tilde{\mathbf{u}}^{n+1}$ in the small velocities approximation). In the Navier-Stokes problem, it leads to the resolution of an elliptic problem coupling all the components of the velocity.

- *Correction step* : Find the new pressure field $p^{n+1} \in H^1(\Omega) \cap L_0^2(\Omega)$ (denoting by $L_0^2(\Omega)$ the space of functions of $L^2(\Omega)$ with null average on Ω), and a corrected velocity $\mathbf{u}^{n+1} \in \mathbf{V}(\Omega)$ (denoting by $\mathbf{V}(\Omega)$ the space of L^2 -divergence-free functions and null normal trace, precisely defined by (10)) therefore with a lower regularity in space and a weaker boundary condition, such that, in the homogeneous Neumann weak sense for the unknown $p^{n+1} - p^n \in H^1(\Omega) \cap L_0^2(\Omega)$,

$$\frac{1}{\delta t}(\mathbf{u}^{n+1} - \tilde{\mathbf{u}}^{n+1}) + \nabla(p^{n+1} - p^n) = 0 \text{ in } \Omega, \quad (6a)$$

$$\operatorname{div}\mathbf{u}^{n+1} = 0 \text{ in } \Omega. \quad (6b)$$

This step involves the resolution of an elliptic problem for finding the new pressure.

Scheme (5)-(6) is called an “incremental projection scheme”, since it is obtained by introducing the previous pressure gradient in the prediction step and by solving the increment of the pressure in the correction step, which is a projection step of the predicted velocity on the divergence-free functions. Such a scheme seems to be much more efficient from a computational point of view than other projection schemes and has been the object of several error analysis, under some regularity assumptions on the solution of the continuous problem in the semi-discrete setting, see [14] and references therein. The non incremental schemes have been the object of some analyses in the fully discrete setting.

Some recent papers [19, 9] propose convergence proofs for fully discrete projection schemes. In [9], the incremental projection scheme is considered without any assumption of regularity assumptions on the exact solution. The proof of its convergence is done both for the semi-discrete scheme and for a fully discrete scheme using the Marker-And-Cell scheme (introduced in the seminal paper [17]). In this paper, we extend such a proof to the case where the spatial discretisation is done using conforming methods (in most of the cases, finite element methods or spectral methods). We refer to [16], [15], [13] using conforming finite element methods.

We consider, for any integer $N \geq 1$ defining the time step by $\delta t_N = T/N$, approximation spaces $\mathbf{X}_N \subset H_0^1(\Omega)^d$ for the predicted velocity, $M_N \subset H^1(\Omega) \cap L_0^2(\Omega)$ for the approximate pressure. Then the space \mathbf{V}_N , defined as the space of functions which are L^2 -orthogonal to the gradient of the elements of M_N (it is therefore not a subset of $\mathbf{V}(\Omega)$) is used for the approximation of the corrected velocity. The fully discrete incremental projection scheme is given in Section 2, and is shown to have a unique solution $(\tilde{\mathbf{u}}_N^n, p_N^n, \mathbf{u}_N^n)_{n=1, \dots, N} \in (\mathbf{X}_N \times M_N \times \mathbf{V}_N)^N$.

In order to prove the convergence of the incremental projection scheme, we need to prove the compactness of the sequence of the approximate velocities in L^2 . The main difficulty relies in the time translate

estimates; this difficulty is solved in the coupled case by the famous Lions' Lemma [21, Lemme 5.1 p.58], bounding the norm in L^2 by a combination of the norm in H^1 and of a semi-norm similar to a H^{-1} norm. In the case of the incremental projection method, the difficulty relies in the fact that there are two different approximations for the velocities. We handle this difficulty in Section 3.3, by closely following the method first introduced in [9]. We can then conclude the convergence proof to a solution, considered in the weak sense given below, in Section 3.4.

It is noticeable that, for this existence and uniqueness result, no compatibility condition between M_N and \mathbf{X}_N is required. This is no longer the case in Section 3, where the convergence of the method is proved. The compatibility condition is expressed through the existence of an interpolator $\Pi_N : \mathbf{W} \rightarrow \mathbf{X}_N \cap \mathbf{V}_N \cap L^\infty(\Omega)^d$, whose purpose is to build approximate values in the discrete velocity spaces of some regular functions. This interpolator replaces in this paper a Fortin interpolator (see [22, 6, 7, 8] for more properties on such operator). In the present paper, the requested properties are slightly different (see Section 3.1):

- The set \mathbf{W} is a dense subset of the set $H_0^1(\Omega)^d \cap \mathbf{V}(\Omega)$ for the $H_0^1(\Omega)^d$ norm (see (19)).
- For any $\varphi \in \mathbf{W}$, the convergence of $\Pi_N \varphi$ to φ is assumed to hold in $H_0^1(\Omega)^d$, while the L^∞ norm of $\Pi_N \varphi$ remains bounded (see (22)).

So the hypotheses which are done on the space discretisation in this paper are not shown to be equivalent to an inf-sup property. In Section 4, we consider the example of the lowest degree Taylor-Hood finite element. In this section we provide all the calculations that prove the convergence property which is requested on Π_N , since the properties which are studied in the literature are generally different. We show that constructing Π_N and checking its properties is much simplified by the fact that we only need to apply it on regular divergence free functions with compact support. It is in some way simpler than the verification of the inf-sup condition for these spaces. Note that the study done in this paper also applies to the semi-discrete scheme, since one can let $\mathbf{X}_N = H_0^1(\Omega)^d$ for the predicted velocity, $M_N = H^1(\Omega) \cap L_0^2(\Omega)$ for the approximate pressure (which yields $\mathbf{V}_N = \mathbf{V}$ for the corrected velocity), defining $\mathbf{W} = C_c^\infty(\Omega)^d \cap \mathbf{V}(\Omega)$ and $\Pi_N = \text{Id}$.

Throughout the paper, we shall assume that the data \mathbf{f} and \mathbf{u}_0 satisfy

$$\mathbf{f} \in L^2((0, T) \times \Omega)^d \text{ and } \mathbf{u}_0 \in L^2(\Omega)^d. \quad (7)$$

We denote in the whole paper

$$\forall \mathbf{v}, \mathbf{w} \in L^2(\Omega)^d, (\mathbf{v}, \mathbf{w}) := \int_{\Omega} \mathbf{v}(\mathbf{x}) \cdot \mathbf{w}(\mathbf{x}) \, d\mathbf{x}. \quad (8)$$

We recall that we denote by $L_0^2(\Omega)$ the space of L^2 functions with null average on Ω , which can be defined by

$$L_0^2(\Omega) = \{q \in L^2(\Omega) \text{ such that } \int_{\Omega} q(\mathbf{x}) \, d\mathbf{x} = 0\}, \quad (9)$$

and by $\mathbf{V}(\Omega)$ the space of L^2 -divergence-free functions, which can be defined by

$$\mathbf{V}(\Omega) = \{\mathbf{u} \in L^2(\Omega)^d \text{ such that } (\mathbf{u}, \nabla \xi) = 0 \text{ for any } \xi \in H^1(\Omega)\}. \quad (10)$$

Recall that we can write as well [2, Definition IV.3.2 p. 248]

$$\mathbf{V}(\Omega) = \{\mathbf{u} \in H_{\text{div}}(\Omega) \text{ such that } \text{div} \mathbf{u} = 0 \text{ and } \gamma_\nu \mathbf{u} = 0\},$$

denoting by γ_ν the normal trace of \mathbf{u} on $\partial\Omega$. Then $H_0^1(\Omega)^d \cap \mathbf{V}(\Omega)$ is the subset of $H_0^1(\Omega)^d$ of divergence-free functions. Let us define the weak solutions of Problem (1)-(4) in the sense of Leray [20].

Definition 1.1 (Weak solution): Under the assumptions (2) and (7), a function $\mathbf{u} \in L^2(0, T; H_0^1(\Omega)^d) \cap \mathbf{V}(\Omega) \cap L^\infty(0, T; L^2(\Omega)^d)$ is a weak solution of the problem (1)-(4) if

$$\begin{aligned} - \int_0^T (\mathbf{u}, \partial_t \mathbf{v}) \, dt + \int_0^T ((\mathbf{u} \cdot \nabla) \mathbf{u}, \mathbf{v}) \, dt + \int_0^T \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} \, dx \, dt \\ = (\mathbf{u}_0, \mathbf{v}(0, \cdot)) + \int_0^T (\mathbf{f}, \mathbf{v}) \, dt \end{aligned} \quad (11)$$

for any $\mathbf{v} \in C_c^\infty(\Omega \times [0, T])^d$, such that $\operatorname{div} \mathbf{v} = 0$ a.e. in $\Omega \times (0, T)$, where

2 The projection scheme using a conforming method

2.1 Space and time discretizations

We consider a partition of the time interval $[0, T]$, which we suppose uniform to alleviate the notations, so that the assumptions read (omitting in the whole paper to recall that N is assumed to be an integer):

$$N \geq 1, \quad \delta_N = \frac{T}{N}, \quad t_N^n = n \delta_N \text{ for } n \in \llbracket 0, N \rrbracket. \quad (12)$$

We consider a sequence of velocity-pressure approximations in $H_0^1(\Omega)^d$ and $H^1(\Omega) \cap L_0^2(\Omega)$ respectively. For the approximation of the predicted velocity, let

$$\begin{aligned} (\mathbf{X}_N)_{N \geq 1} \text{ be a sequence of} \\ \text{closed subspaces of } H_0^1(\Omega)^d. \end{aligned} \quad (13)$$

For the approximation of the pressure, let

$$\begin{aligned} (M_N)_{N \geq 1} \text{ be a sequence of} \\ \text{closed subspaces of } H^1(\Omega) \cap L_0^2(\Omega). \end{aligned} \quad (14)$$

For the approximation of the corrected velocity we define the space of weakly divergence free functions by

$$\mathbf{V}_N = \{ \mathbf{v} \in L^2(\Omega)^d \text{ such that } (\mathbf{v}, \nabla q) = 0 \text{ for any } q \in M_N \}. \quad (15)$$

We denote by $\mathcal{P}_{\mathbf{V}_N} : L^2(\Omega)^d \rightarrow \mathbf{V}_N$ the orthogonal projection in $L^2(\Omega)^d$ onto the space \mathbf{V}_N . We denote by $\mathcal{P}_{\mathbf{V}(\Omega)} : L^2(\Omega)^d \rightarrow \mathbf{V}(\Omega)$ the orthogonal projection in $L^2(\Omega)^d$ onto the space $\mathbf{V}(\Omega)$.

Remark 2.1: The choice $\mathbf{X}_N = H_0^1(\Omega)^d$, $M_N = H^1(\Omega) \cap L_0^2(\Omega)$ (which implies $\mathbf{V}_N = \mathbf{V}(\Omega)$) yields the continuous-in-space discrete-in-time scheme.

2.2 The projection scheme

Let $a : H_0^1(\Omega)^d \times H_0^1(\Omega)^d \rightarrow \mathbb{R}$ be the coercive bilinear form and let $b : H_0^1(\Omega)^d \times H_0^1(\Omega)^d \times H_0^1(\Omega)^d \rightarrow \mathbb{R}$ be the “skew-symmetric” trilinear form defined by

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) &= \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} \, dx, \\ b(\mathbf{u}, \mathbf{v}, \mathbf{w}) &= \int_\Omega (\mathbf{u} \cdot \nabla) \mathbf{v} \cdot \mathbf{w} \, dx + \frac{1}{2} \int_\Omega \operatorname{div} \mathbf{u} (\mathbf{v} \cdot \mathbf{w}) \, dx \\ &= \frac{1}{2} \int_\Omega \left((\mathbf{u} \cdot \nabla) \mathbf{v} \cdot \mathbf{w} - (\mathbf{u} \cdot \nabla) \mathbf{w} \cdot \mathbf{v} \right) \, dx. \end{aligned}$$

Using the above definitions, the first order time semi-discrete incremental projection scheme described in the introduction of this paper reads:

Initialization:

$$\text{Let } \mathbf{u}_N^0 = \mathcal{P}_{\mathbf{V}_N} \mathbf{u}_0, \tilde{\mathbf{u}}_N^n = 0 \text{ and } p_N^0 = 0. \quad (17a)$$

Solve for $0 \leq n \leq N-1$:

Prediction step:

$$\tilde{\mathbf{u}}_N^{n+1} \in \mathbf{X}_N, \frac{1}{\delta_N} \left((\tilde{\mathbf{u}}_N^{n+1}, \mathbf{v}) - (\mathbf{u}_N^n, \mathbf{v}) \right) + b(\tilde{\mathbf{u}}_N^n, \tilde{\mathbf{u}}_N^{n+1}, \mathbf{v}) \quad (17b)$$

$$+ a(\tilde{\mathbf{u}}_N^{n+1}, \mathbf{v}) + (\nabla p_N^n, \mathbf{v}) = (\mathbf{f}_N^{n+1}, \mathbf{v}), \text{ for any } \mathbf{v} \in \mathbf{X}_N,$$

Correction step:

$$(\nabla(p_N^{n+1} - p_N^n), \nabla q) = -\frac{1}{\delta_N} (q, \operatorname{div} \tilde{\mathbf{u}}_N^{n+1}), \text{ for any } q \in M_N. \quad (17c)$$

$$p_N^{n+1} \in M_N, \quad (17d)$$

$$\mathbf{u}_N^{n+1} = \tilde{\mathbf{u}}_N^{n+1} - \delta_N \nabla(p_N^{n+1} - p_N^n). \quad (17e)$$

The term $\mathbf{f}_N^{n+1} \in L^2(\Omega)^d$ is given by $\mathbf{f}_N^{n+1}(\mathbf{x}) = \frac{1}{\delta_N} \int_{t^n}^{t^{n+1}} \mathbf{f}(t, \mathbf{x}) dt$, for a.e. $\mathbf{x} \in \Omega$.

Remark 2.2: Note that \mathbf{u}_N^{n+1} is an element of \mathbf{V}_N . In particular we have $\tilde{\mathbf{u}}_N^{n+1} = \mathbf{u}_N^{n+1} + \delta_N \nabla(p_N^{n+1} - p_N^n)$ which provides a discrete Helmholtz-Leray decomposition for $\tilde{\mathbf{u}}_N^{n+1}$.

2.3 Existence of a solution to the projection scheme

The following existence result allows to define the approximate solutions obtained by the projection scheme (17).

Lemma 2.1 (Approximate solutions): Under Assumptions (2), (7), (12), (13), (14), for any $N \geq 1$, there exists a unique sequence $(\tilde{\mathbf{u}}_N^n, \mathbf{u}_N^n, p_N^n)_{n \in \llbracket 1, N \rrbracket} \subset \mathbf{X}_N \times \mathbf{V}_N \times M_N$ satisfying (17). We then define the functions $\mathbf{u}_N : (0, T) \rightarrow \mathbf{V}_N$ and $\tilde{\mathbf{u}}_N : (0, T) \rightarrow \mathbf{X}_N$ by

$$\mathbf{u}_N(t) = \sum_{n=0}^{N-1} \mathbb{1}_{(t_N^n, t_N^{n+1}]}(t) \mathbf{u}_N^n, \quad \tilde{\mathbf{u}}_N(t) = \sum_{n=0}^{N-1} \mathbb{1}_{(t_N^n, t_N^{n+1}]}(t) \tilde{\mathbf{u}}_N^{n+1}. \quad (18)$$

PROOF. For given functions $p_N^n \in M_N$, $\tilde{\mathbf{u}}_N^n \in \mathbf{X}_N$ and $\mathbf{u}_N^n \in \mathbf{V}_N$, Problem (17b) is under the form:

$$\tilde{\mathbf{u}}_N^{n+1} \in \mathbf{X}_N, \forall \mathbf{v} \in \mathbf{X}_N, a_N^n(\mathbf{u}_N^{n+1}, \mathbf{v}) = b_N(\mathbf{v}),$$

where, owing to the property $b(\cdot, \mathbf{v}, \mathbf{v}) = 0$, the continuous bilinear form a_N^n is such that

$$\forall \mathbf{v} \in \mathbf{X}_N, a_N^n(\mathbf{v}, \mathbf{v}) \geq \|\mathbf{v}\|_{H_0^1(\Omega)^d}^2.$$

Hence the existence and uniqueness of the solution $\tilde{\mathbf{u}}_N^{n+1} \in \mathbf{X}_N$ is given by the Lax-Milgram theorem. For given functions $p_N^n \in M_N$, $\tilde{\mathbf{u}}_N^{n+1} \in \mathbf{X}_N$, Problem (17c) is under the form:

$$p_N^{n+1} \in M_N, \forall q \in M_N, (\nabla p_N^{n+1}, \nabla q) = c_N(q).$$

Hence the existence and uniqueness of the solution $p_N^{n+1} \in M_N$ is given by the fact that the bilinear form $(p, q) \mapsto (\nabla p, \nabla q)$ is coercive on $H^1(\Omega) \cap L_0^2(\Omega)$. \square

3 Convergence study

3.1 Convergence assumptions

The following assumptions are done on the sequences $(\mathbf{X}_N)_{N \geq 1}$ and $(M_N)_{N \geq 1}$ in order to prove the convergence of the projection scheme as $N \rightarrow \infty$. These assumptions, provided in terms of the existence of a family of interpolators on some spaces $(\Pi_N)_{N \geq 1}$, replace the standard uniform inf-sup hypothesis. We assume that

$$\begin{aligned} \mathbf{W} \text{ is a dense subset of } \mathbf{V}(\Omega) \cap H_0^1(\Omega)^d \\ \text{for the norm of } H_0^1(\Omega)^d. \end{aligned} \quad (19)$$

We define

$$\forall N \geq 1, \mathbf{E}_N = \mathbf{X}_N \cap \mathbf{V}_N \cap L^\infty(\Omega)^d, \quad (20)$$

$$\forall N \geq 1, \forall \mathbf{u} \in \mathbf{E}_N, \|\mathbf{u}\|_{\mathbf{E}} = \|\nabla \mathbf{u}\|_{L^2(\Omega)^{d \times d}} + \|\mathbf{u}\|_{L^\infty(\Omega)^d}. \quad (21)$$

We assume that

$$\begin{aligned} \text{For any } N \geq 1 \text{ there exists a mapping } \Pi_N : \mathbf{W} \rightarrow \mathbf{E}_N \text{ such that,} \\ \text{for any } \boldsymbol{\varphi} \in \mathbf{W}, (\Pi_N \boldsymbol{\varphi})_{N \geq 1} \text{ converges to } \boldsymbol{\varphi} \text{ in } H_0^1(\Omega)^d \\ \text{and the sequence } (\|\Pi_N \boldsymbol{\varphi}\|_{\mathbf{E}})_{N \geq 1} \text{ is bounded.} \end{aligned} \quad (22)$$

The sequence $(M_N)_{N \geq 1}$ is assumed to be such that

$$\begin{aligned} (\mathcal{P}_{M_N}(p))_{N \geq 1} \text{ converges to } p \text{ in } H^1(\Omega), \\ \text{for any } p \in H^1(\Omega) \cap L_0^2(\Omega), \end{aligned} \quad (23)$$

where we denote by $\mathcal{P}_{M_N} : H^1(\Omega) \cap L_0^2(\Omega) \rightarrow M_N$ the orthogonal projection in $H^1(\Omega) \cap L_0^2(\Omega)$ onto the space M_N .

Remark 3.1: Note that (23) is equivalent to the fact that

$$\lim_{N \rightarrow +\infty} \inf_{q \in M_N} \|p - q\|_{H^1(\Omega)} = 0 \text{ for any } p \in H^1(\Omega) \cap L_0^2(\Omega),$$

Note that the sequence $(\mathbf{V}_N)_{N \geq 1}$ is an approximation of $\mathbf{V}(\Omega)$.

Remark 3.2: If, for all $N \geq 1$, we consider the semi-discrete case $\mathbf{X}_N = H_0^1(\Omega)^d$, $M_N = H^1(\Omega) \cap L_0^2(\Omega)$ (recall that this yields $\mathbf{V}_N = \mathbf{V}(\Omega)$), we let $\mathbf{W} = \mathbf{V}(\Omega) \cap C_c^\infty(\Omega)^d$ and $\Pi_N = \text{Id}$. Then all the assumptions of this section are satisfied.

Remark 3.3: We provide in Section 4 the construction of Π_N in the case of the lowest degree Taylor-Hood finite element. We show in Lemma 4.14 that it satisfies the assumptions given in this section, for a regular family of meshes in the standard sense.

3.2 Space estimate

Lemma 3.1: Under Assumptions (2), (7), (12), (13), (14), for any $N \geq 1$, the following relation holds for $n \in \llbracket 0, N-1 \rrbracket$:

$$\begin{aligned} \frac{1}{2\delta_N} \left(\|\mathbf{u}_N^{n+1}\|_{L^2(\Omega)^d}^2 - \|\mathbf{u}_N^n\|_{L^2(\Omega)^d}^2 \right) + \frac{\delta_N}{2} \left(\|\nabla p_N^{n+1}\|_{L^2(\Omega)^d}^2 - \|\nabla p_N^n\|_{L^2(\Omega)^d}^2 \right) \\ + \frac{1}{2\delta_N} \|\tilde{\mathbf{u}}_N^{n+1} - \mathbf{u}_N^n\|_{L^2(\Omega)^d}^2 + \|\tilde{\mathbf{u}}_N^{n+1}\|_{H_0^1(\Omega)^d}^2 = (\mathbf{f}_N^{n+1}, \tilde{\mathbf{u}}_N^{n+1}). \end{aligned} \quad (24)$$

Consequently, there exists C_1 depending only on $|\Omega|$, on the L^2 -norm of \mathbf{u}_0 and on the L^2 -norm of \mathbf{f} such that the functions $\tilde{\mathbf{u}}_N$ and \mathbf{u}_N defined by (18) satisfy:

$$\|\tilde{\mathbf{u}}_N\|_{L^2(0,T;H_0^1(\Omega)^d)} \leq C_1 \quad \text{and} \quad \|\mathbf{u}_N\|_{L^\infty(0,T;L^2(\Omega)^d)} \leq C_1. \quad (25)$$

$$\|\mathbf{u}_N - \tilde{\mathbf{u}}_N\|_{L^\infty(0,T;L^2(\Omega)^d)} \leq C_1 \quad \text{and} \quad \|\mathbf{u}_N - \tilde{\mathbf{u}}_N\|_{L^2(0,T;L^2(\Omega)^d)} \leq C_1 \sqrt{\delta_N}. \quad (26)$$

PROOF. We take $\tilde{\mathbf{u}}_N^{n+1}$ in (17b) and we obtain for $n \in \llbracket 0, N-1 \rrbracket$

$$\begin{aligned} \frac{1}{2\delta_N} \|\tilde{\mathbf{u}}_N^{n+1}\|_{L^2(\Omega)^d}^2 - \frac{1}{2\delta_N} \|\mathbf{u}_N^n\|_{L^2(\Omega)^d}^2 + \frac{1}{2\delta_N} \|\tilde{\mathbf{u}}_N^{n+1} - \mathbf{u}_N^n\|_{L^2(\Omega)^d}^2 \\ + \int_{\Omega} \nabla p_N^n \cdot \tilde{\mathbf{u}}_N^{n+1} \, d\mathbf{x} + \|\tilde{\mathbf{u}}_N^{n+1}\|_{H_0^1(\Omega)^d}^2 = (\mathbf{f}_N^{n+1}, \tilde{\mathbf{u}}_N^{n+1}). \end{aligned} \quad (27)$$

Squaring the relation $\tilde{\mathbf{u}}_N^{n+1} + \delta_N \nabla p_N^n = \mathbf{u}_N^{n+1} - \delta_N \nabla p_N^{n+1}$, integrating over Ω , and owing to $\mathbf{u}_N^{n+1} \in \mathbf{V}_N$, we get that for $n \in \llbracket 0, N-1 \rrbracket$,

$$\begin{aligned} \frac{1}{2\delta_N} \|\mathbf{u}_N^{n+1}\|_{L^2(\Omega)^d}^2 + \frac{\delta_N}{2} \|\nabla p_N^{n+1}\|_{L^2(\Omega)^d}^2 = \frac{1}{2\delta_N} \|\tilde{\mathbf{u}}_N^{n+1}\|_{L^2(\Omega)^d}^2 \\ + \frac{\delta_N}{2} \|\nabla p_N^n\|_{L^2(\Omega)^d}^2 - (\tilde{\mathbf{u}}_N^{n+1}, \nabla p_N^n). \end{aligned}$$

Summing the latter relation with (27) yields (24) for $n \in \llbracket 0, N-1 \rrbracket$. We then get Relations (25) by summing over the time steps, using the Cauchy-Schwarz and Poincaré inequalities. \square

Lemma 3.2: Under Assumptions (2), (13), (14), (19), (22), (23), let $(\mathbf{v}_N)_{N \geq 1}$ be a sequence of functions of $L^2(\Omega)^d$ such that $(\mathbf{v}_N)_{N \geq 1}$ converges to \mathbf{v} in $L^2(\Omega)^d$. Then the sequence $(\mathcal{P}_{\mathbf{V}_N(\Omega)} \mathbf{v}_N)_{N \geq 1}$ converges to $\mathcal{P}_{\mathbf{V}(\Omega)} \mathbf{v}$ in $L^2(\Omega)^d$.

PROOF. Using the fact that $(\mathbf{v}_N)_{N \geq 1}$ is bounded in $L^2(\Omega)^d$ we obtain that the sequence $(\mathcal{P}_{\mathbf{V}_N(\Omega)} \mathbf{v}_N)_{N \geq 1}$ is bounded in $L^2(\Omega)^d$. Hence there exists a subsequence still denoted by $(\mathcal{P}_{\mathbf{V}_N(\Omega)} \mathbf{v}_N)_{N \geq 1}$ that converges to a function $\tilde{\mathbf{v}}$ weakly in $L^2(\Omega)^d$. Let $\xi \in H^1(\Omega) \cap L_0^2(\Omega)$. Using the fact that for any $N \geq 1$ we have $\mathcal{P}_{\mathbf{V}_N(\Omega)} \mathbf{v}_N \in \mathbf{V}_N$ and $\mathcal{P}_{M_N} \xi \in M_N$ we obtain

$$(\mathcal{P}_{\mathbf{V}_N(\Omega)} \mathbf{v}_N, \nabla \mathcal{P}_{M_N} \xi) = 0, \quad \text{for any } N \geq 1.$$

Using the weak convergence of the sequence $(\mathcal{P}_{\mathbf{V}_N(\Omega)} \mathbf{v}_N)_{N \geq 1}$ in $L^2(\Omega)^d$ and the strong convergence of the sequence $(\mathcal{P}_{M_N} \xi)_{N \geq 1}$ in $H^1(\Omega)$ and passing to the limit in the previous identity gives

$$(\tilde{\mathbf{v}}, \nabla \xi) = 0, \quad \text{for any } \xi \in H^1(\Omega).$$

We then obtain that $\tilde{\mathbf{v}} \in \mathbf{V}(\Omega)$. Let $\varphi \in \mathbf{W}$ be given. Using the fact that $\Pi_N \varphi \in \mathbf{V}_N$ for any $N \geq 1$, we obtain

$$(\mathbf{v}_N, \Pi_N \varphi) = (\mathcal{P}_{\mathbf{V}_N(\Omega)} \mathbf{v}_N, \Pi_N \varphi), \quad \text{for any } N \geq 1.$$

Using the convergence to φ of the sequence $(\Pi_N \varphi)_{N \geq 1}$ in $H_0^1(\Omega)^d$, the weak convergence of the sequence $(\mathcal{P}_{\mathbf{V}_N(\Omega)} \mathbf{v}_N)_{N \geq 1}$ in $L^2(\Omega)^d$ and passing to the limit in the previous identity gives

$$(\mathbf{v}, \varphi) = (\tilde{\mathbf{v}}, \varphi) \quad \text{for any } \varphi \in \mathbf{W}.$$

We then obtain that $\tilde{\mathbf{v}} = \mathcal{P}_{\mathbf{V}} \mathbf{v}$ and the sequence $(\mathcal{P}_{\mathbf{V}_N(\Omega)} \mathbf{v}_N)_{N \geq 1}$ converges to $\mathcal{P}_{\mathbf{V}(\Omega)} \mathbf{v}$ weakly in $L^2(\Omega)^d$. We can write

$$\|\mathcal{P}_{\mathbf{V}_N(\Omega)} \mathbf{v}_N\|_{L^2(\Omega)^d}^2 = (\mathbf{v}_N, \mathcal{P}_{\mathbf{V}_N(\Omega)} \mathbf{v}_N), \quad \text{for any } N \geq 1.$$

Using the convergence of the sequence $(\mathbf{v}_N)_{N \geq 1}$ to \mathbf{v} in $L^2(\Omega)^d$ and the weak convergence of the sequence $(\mathcal{P}_{\mathbf{V}_N(\Omega)} \mathbf{v}_N)_{N \geq 1}$ to $\mathcal{P}_{\mathbf{V}(\Omega)} \mathbf{v}$ in $L^2(\Omega)^d$ we obtain

$$\lim_{N \rightarrow +\infty} \|\mathcal{P}_{\mathbf{V}_N(\Omega)} \mathbf{v}_N\|_{L^2(\Omega)^d}^2 = (\mathbf{v}, \mathcal{P}_{\mathbf{V}(\Omega)} \mathbf{v}) = \|\mathcal{P}_{\mathbf{V}(\Omega)} \mathbf{v}\|_{L^2(\Omega)^d}^2.$$

The weak convergence of the sequence $(\mathcal{P}_{\mathbf{V}_N(\Omega)} \mathbf{v}_N)_{N \geq 1}$ to $\mathcal{P}_{\mathbf{V}(\Omega)} \mathbf{v}$ in $L^2(\Omega)^d$ and the convergence of the sequence $(\|\mathcal{P}_{\mathbf{V}_N(\Omega)} \mathbf{v}_N\|_{L^2(\Omega)^d})_{N \geq 1}$ to $\|\mathcal{P}_{\mathbf{V}(\Omega)} \mathbf{v}\|_{L^2(\Omega)^d}$ give the expected result. \square

Lemma 3.3: Under Assumptions (2), (7), (12), (13), (14), (23), there exists $\mathbf{u} \in L^\infty(0, T; L^2(\Omega)^d) \cap L^2(0, T; H_0^1(\Omega)^d \cap \mathbf{V}(\Omega))$ such that, up to the extraction of a subsequence, the sequence $(\mathbf{u}_N)_{N \geq 1}$ weakly converges to \mathbf{u} in $L^2(0, T; H_0^1(\Omega)^d)$ and $(\tilde{\mathbf{u}}_N)_{N \geq 1}$ weakly converges to \mathbf{u} in $L^\infty(0, T; L^2(\Omega)^d)$ for the weak star topology.

PROOF. Owing to (25), we get the existence of $\tilde{\mathbf{u}} \in L^2(0, T; H_0^1(\Omega)^d)$ such that, up to the extraction of a subsequence, the sequence $(\tilde{\mathbf{u}}_N)_{N \geq 1}$ weakly converges to $\tilde{\mathbf{u}}$ in $L^2(0, T; H_0^1(\Omega)^d)$, and of $\mathbf{u} \in L^\infty(0, T; L^2(\Omega)^d)$ such that $(\mathbf{u}_N)_{N \geq 1}$ weakly converges to \mathbf{u} in $L^\infty(0, T; L^2(\Omega)^d)$ for the weak star topology. Let $\xi \in H^1(\Omega) \cap L_0^2(\Omega)$ and $\varphi \in C_c^\infty(]0, T[)$ be given. Using the fact that (17d) provides, for any $N \geq 1$, that

$$\int_0^T (\mathbf{u}_N, \nabla \mathcal{P}_{M_N} \xi) \varphi(t) dt = 0, \text{ for any } N \geq 1,$$

Using the convergence of the sequence $(\mathcal{P}_{M_N} \xi)_{N \geq 1}$ in $H^1(\Omega)$, the weak convergence of the sequence $(\mathbf{u}_N)_{N \geq 1}$ in $L^2((0, T) \times \Omega)^d$ and passing to the limit in the previous identity gives

$$\int_0^T (\mathbf{u}, \nabla \xi) \varphi(t) dt = 0, \text{ for any } \xi \in H^1(\Omega).$$

We then obtain that $\mathbf{u} \in L^2(0, T; H_0^1(\Omega)^d \cap \mathbf{V}(\Omega))$. Using (26), we get that $\tilde{\mathbf{u}} = \mathbf{u}$, which concludes the proof. \square

3.3 Time estimates

The weak convergence property given by Lemma 3.3 is not sufficient for passing to the limit in the scheme, owing to the presence of the nonlinear convection term. Hence we need some stronger compactness property on one of the subsequences $(\mathbf{u}_N)_{N \geq 1}$ or $(\tilde{\mathbf{u}}_N)_{N \geq 1}$. We will prove this compactness property in L^2 on $(\tilde{\mathbf{u}}_N)_{N \geq 1}$.

We introduce the semi-norm $|\cdot|_{*,N}$ defined for any $\mathbf{w} \in L^2(\Omega)^d$ by

$$|\mathbf{w}|_{*,N} = \sup\{(\mathbf{w}, \mathbf{v}), \mathbf{v} \in \mathbf{E}_N, \|\mathbf{v}\|_{\mathbf{E}} \leq 1\}. \quad (28)$$

Recall that \mathbf{E}_N and $\|\mathbf{v}\|_{\mathbf{E}}$ are defined by (20)-(21) and that the above definition remains meaningful even if $\mathbf{E}_N = \{0\}$.

Lemma 3.4 (A first estimate on the time translates): Under Assumptions (2), (7), (12), (13), (14), there exists C_2 only depending on $|\Omega|$, $\|\mathbf{u}_0\|_{L^2(\Omega)^d}$ and $\|\mathbf{f}\|_{L^2((0,T) \times \Omega)^d}$ such that for any $N \geq 1$ and for any $\tau \in (0, T)$,

$$\int_0^{T-\tau} |\tilde{\mathbf{u}}_N(t+\tau) - \tilde{\mathbf{u}}_N(t)|_{*,N}^2 dt \leq C_2 \tau (\tau + \delta_N).$$

PROOF. Let $N \geq 2$ and $\tau \in (0, T)$ (for $N = 1$ the quantity we have to estimate is zero). Let $(\chi_{N,\tau}^n)_{n \in \llbracket 1, N-1 \rrbracket}$ be the family of measurable functions defined for $n \in \llbracket 1, N-1 \rrbracket$ and $t \in \mathbb{R}$ by $\chi_{N,\tau}^n(t) = \mathbb{1}_{(t_N^n - \tau, t_N^n]}(t)$, then

$$\tilde{\mathbf{u}}_N(t + \tau) - \tilde{\mathbf{u}}_N(t) = \sum_{n=1}^{N-1} \chi_{N,\tau}^n(t) (\tilde{\mathbf{u}}_N^{n+1} - \tilde{\mathbf{u}}_N^n), \text{ for any } t \in (0, T - \tau). \quad (29)$$

Hence, owing to (17b), we have for any $\mathbf{v} \in X_N$ and for any $t \in (0, T - \tau)$ the following identity

$$\begin{aligned} (\tilde{\mathbf{u}}_N(t + \tau) - \tilde{\mathbf{u}}_N(t), \mathbf{v}) &= -\delta_N \sum_{n=1}^{N-1} \chi_{N,\tau}^n(t) a(\tilde{\mathbf{u}}_N^{n+1}, \mathbf{v}) \\ &\quad - \delta_N \sum_{n=1}^{N-1} \chi_{N,\tau}^n(t) b(\tilde{\mathbf{u}}_N^n, \tilde{\mathbf{u}}_N^{n+1}, \mathbf{v}) - \delta_N \sum_{n=1}^{N-1} \chi_{N,\tau}^n(t) (\nabla(2p_N^n - p_N^{n-1}), \mathbf{v}) \\ &\quad + \delta_N \sum_{n=1}^{N-1} \chi_{N,\tau}^n(t) (\mathbf{f}_N^{n+1}, \mathbf{v}). \end{aligned}$$

Let $\mathbf{v} \in \mathbf{E}_N$, and define $A(t) = (\tilde{\mathbf{u}}_N(t + \tau) - \tilde{\mathbf{u}}_N(t), \mathbf{v})$. Using the previous identity we obtain

$$A(t) = A_d(t) + A_c(t) + A_p(t) + A_f(t),$$

with

$$\begin{aligned} A_d(t) &= - \sum_{n=1}^{N-1} \chi_{N,\tau}^n(t) \delta_N \int_{\Omega} \nabla \tilde{\mathbf{u}}_N^{n+1} : \nabla \mathbf{v} \, d\mathbf{x}, \\ A_c(t) &= - \sum_{n=1}^{N-1} \chi_{N,\tau}^n(t) \delta_N b(\tilde{\mathbf{u}}_N^n, \tilde{\mathbf{u}}_N^{n+1}, \mathbf{v}), \\ A_p(t) &= \sum_{n=1}^{N-1} \chi_{N,\tau}^n(t) \delta_N \int_{\Omega} (2p_N^n - p_N^{n-1}) \operatorname{div} \mathbf{v} \, d\mathbf{x}, \\ A_f(t) &= \sum_{n=1}^{N-1} \chi_{N,\tau}^n(t) \delta_N (\mathbf{f}_N^{n+1}, \mathbf{v}). \end{aligned}$$

Using (21) we have

$$A_d(t) \leq \|\mathbf{v}\|_{\mathbf{E}} \sum_{n=1}^{N-1} \chi_{N,\tau}^n(t) \delta_N \|\tilde{\mathbf{u}}_N^{n+1}\|_{H_0^1(\Omega)^d}. \quad (30)$$

Using (21) and using the estimates (25)-(26) we have

$$\begin{aligned}
A_c(t) &= - \sum_{n=1}^{N-1} \chi_{N,\tau}^n(t) \delta_N ((\tilde{\mathbf{u}}_N^n - \mathbf{u}_N^{n-1}) \cdot \nabla) \tilde{\mathbf{u}}_N^{n+1}, \mathbf{v}) \\
&\quad - \sum_{n=1}^{N-1} \chi_{N,\tau}^n(t) \delta_N ((\mathbf{u}_N^{n-1} \cdot \nabla) \tilde{\mathbf{u}}_N^{n+1}, \mathbf{v}) \\
&\quad - \frac{1}{2} \sum_{n=1}^{N-1} \chi_{N,\tau}^n(t) \delta_N \int_{\Omega} \operatorname{div} \tilde{\mathbf{u}}_N^n (\tilde{\mathbf{u}}_N^{n+1} - \mathbf{u}_N^n) \cdot \mathbf{v} \, dx \\
&\quad - \frac{1}{2} \sum_{n=1}^{N-1} \chi_{N,\tau}^n(t) \delta_N \int_{\Omega} \operatorname{div} \tilde{\mathbf{u}}_N^n \mathbf{u}_N^n \cdot \mathbf{v} \, dx \\
&\leq C_1 \sum_{n=1}^{N-1} \chi_{N,\tau}^n(t) \delta_N (2 \|\tilde{\mathbf{u}}_N^{n+1}\|_{H_0^1(\Omega)^d} + \|\tilde{\mathbf{u}}_N^n\|_{H_0^1(\Omega)^d}) \|\mathbf{v}\|_{L^\infty(\Omega)^d} \\
&\leq C_1 \sum_{n=1}^{N-1} \chi_{N,\tau}^n(t) \delta_N (2 \|\tilde{\mathbf{u}}_N^{n+1}\|_{H_0^1(\Omega)^d} + \|\tilde{\mathbf{u}}_N^n\|_{H_0^1(\Omega)^d}) \|\mathbf{v}\|_{\mathbf{E}}. \quad (31)
\end{aligned}$$

We get from $\mathbf{E}_N \subset \mathbf{V}_N$ that $A_p(t) = 0$. Next, we note that

$$A_f(t) \leq \|\mathbf{v}\|_{\mathbf{E}} \sum_{n=1}^{N-1} \chi_{N,\tau}^n(t) \delta_N \|\mathbf{f}_N^{n+1}\|_{L^2(\Omega)^d}. \quad (32)$$

Summing Equations (30), (31), (32), we obtain

$$A(t) \leq C \|\mathbf{v}\|_{\mathbf{E}} \sum_{n=1}^{N-1} \chi_{N,\tau}^n(t) \delta_N (\|\tilde{\mathbf{u}}_N^{n+1}\|_{H_0^1(\Omega)^d} + \|\tilde{\mathbf{u}}_N^n\|_{H_0^1(\Omega)^d} + \|\mathbf{f}_N^{n+1}\|_{L^2(\Omega)^d})$$

where $C = 2 + 3C_1$. This implies

$$|\tilde{\mathbf{u}}_N(t + \tau) - \tilde{\mathbf{u}}_N(t)|_{*,N} \leq C \sum_{n=1}^{N-1} \chi_{N,\tau}^n(t) \delta_N (\|\tilde{\mathbf{u}}_N^{n+1}\|_{H_0^1(\Omega)^d} + \|\tilde{\mathbf{u}}_N^n\|_{H_0^1(\Omega)^d} + \|\mathbf{f}_N^{n+1}\|_{L^2(\Omega)^d}).$$

Since $\sum_{n=1}^{N-1} \chi_{N,\tau}^n(t) \delta_N \leq \tau + \delta_N$ for any $t \in (0, T - \tau)$ we then obtain

$$|\tilde{\mathbf{u}}_N(t + \tau) - \tilde{\mathbf{u}}_N(t)|_{*,N}^2 \leq 3C^2(\tau + \delta_N) \sum_{n=1}^{N-1} \chi_{N,\tau}^n(t) \delta_N (\|\tilde{\mathbf{u}}_N^{n+1}\|_{H_0^1(\Omega)^d}^2 + \|\tilde{\mathbf{u}}_N^n\|_{H_0^1(\Omega)^d}^2 + \|\mathbf{f}_N^{n+1}\|_{L^2(\Omega)^d}^2).$$

Noting that $\int_0^{T-\tau} \chi_{N,\tau}^n(t) \, dt \leq \tau$ for any $n \in \llbracket 1, N-1 \rrbracket$ yields

$$\begin{aligned}
&\int_0^{T-\tau} |\tilde{\mathbf{u}}_N(t + \tau) - \tilde{\mathbf{u}}_N(t)|_{*,N}^2 \, dt \\
&\leq 3C^2(\tau + \delta_N) \sum_{n=1}^{N-1} \delta_N (\|\tilde{\mathbf{u}}_N^{n+1}\|_{H_0^1(\Omega)^d}^2 + \|\tilde{\mathbf{u}}_N^n\|_{H_0^1(\Omega)^d}^2 + \|\mathbf{f}_N^{n+1}\|_{L^2(\Omega)^d}^2) \int_0^{T-\tau} \chi_{N,\tau}^n(t) \, dt \\
&\leq 3C^2(\tau + \delta_N) \tau (2 \|\tilde{\mathbf{u}}_N\|_{L^2(0,T;H_0^1(\Omega)^d)}^2 + \|\mathbf{f}\|_{L^2((0,T)\times\Omega)^d}^2) \leq C_2 \tau (\tau + \delta_N)
\end{aligned}$$

which gives the expected result. \square

Note that this is only an intermediate result since we seek an estimate on the time translates of the predicted velocity in the $L^2(L^2)$ norm.

Lemma 3.5 (Lions-like): Under Assumptions (2), (13), (14), (19) (22), (23), we have

$$\forall \varepsilon > 0, \exists C_\varepsilon > 0, \exists N_\varepsilon \geq 1, \forall N \geq N_\varepsilon, \forall \mathbf{w} \in \mathbf{X}_N$$

$$\|\mathcal{P}_{\mathbf{V}_N} \mathbf{w}\|_{L^2(\Omega)^d} \leq \varepsilon \|\mathbf{w}\|_{H_0^1(\Omega)^d} + C_\varepsilon |\mathbf{w}|_{*,N}. \quad (33)$$

PROOF. Let us assume that (33) does not hold. This means that there exists $\varepsilon > 0$ such that

$$\forall C \geq 1, \forall M \geq 1, \exists N \geq M, \exists \mathbf{w} \in \mathbf{X}_N$$

$$\|\mathcal{P}_{\mathbf{V}_N} \mathbf{w}\|_{L^2(\Omega)^d} > \varepsilon \|\mathbf{w}\|_{H_0^1(\Omega)^d} + C |\mathbf{w}|_{*,N}. \quad (34)$$

Let us set $\nu(0) = 0$, and let us build the infinite set $\mathcal{I} = \{\nu(n), n \in \mathbb{N}^*\} \subset \mathbb{N}^*$ and the sequence $(\mathbf{w}_N)_{N \in \mathcal{I}}$ by induction. For $n \in \mathbb{N}^*$, we select $C = n$ and $M = \nu(n-1) + 1$ in (34). We get the existence of a given $N \geq M$ and of a given $\mathbf{w} \in \mathbf{X}_N(\Omega)$ such that

$$\|\mathcal{P}_{\mathbf{V}_N} \mathbf{w}\|_{L^2(\Omega)^d} > \varepsilon \|\mathbf{w}\|_{H_0^1(\Omega)^d} + n |\mathbf{w}|_{*,N}.$$

We then define $\nu(n) = N$ and $\mathbf{w}_N = \mathbf{w} / \|\mathcal{P}_{\mathbf{V}_N} \mathbf{w}\|_{L^2(\Omega)^d}$. Let us denote $\mathcal{I} = \{\nu(n) \text{ such that } n \in \mathbb{N}^*\} \subset \mathbb{N}^*$. We denote by ν^{-1} the reciprocal function of $\nu : \mathbb{N}^* \rightarrow \mathcal{I}$. Since the mapping ν is strictly increasing, we then indeed get that \mathcal{I} is infinite. We can then write, for any $N \in \mathcal{I}$,

$$1 > \varepsilon \|\mathbf{w}_N\|_{H_0^1(\Omega)^d} + \nu^{-1}(N) |\mathbf{w}_N|_{*,N}, \text{ for any } N \in \mathcal{I}.$$

It then follows from the latter inequality that the sequence $(\mathbf{w}_N)_{N \in \mathcal{I}}$ is bounded in $H_0^1(\Omega)^d$ and that $|\mathbf{w}_N|_{*,N} \rightarrow 0$ as $N \rightarrow \infty$ with $N \in \mathcal{I}$. Hence there exists an infinite set $\mathcal{J} \subset \mathcal{I}$ such that $(\mathbf{w}_N)_{N \in \mathcal{J}}$ converges in $L^2(\Omega)^d$ to a function $\mathbf{w} \in H_0^1(\Omega)^d$ while $|\mathbf{w}_N|_{*,N} \rightarrow 0$ as $N \rightarrow \infty$ with $N \in \mathcal{J}$. We notice that $\|\mathcal{P}_{\mathbf{V}_N} \mathbf{w}_N\|_{L^2(\Omega)^d} = 1$ for all $N \in \mathcal{J}$.

Using Lemma 3.2 we have $\mathcal{P}_{\mathbf{V}_N} \mathbf{w}_N \rightarrow \mathcal{P}_{\mathbf{V}(\Omega)} \mathbf{w}$ in $L^2(\Omega)^d$ as $N \rightarrow \infty$ with $N \in \mathcal{J}$, which therefore implies $\|\mathcal{P}_{\mathbf{V}} \mathbf{w}\|_{L^2(\Omega)^d} = 1$.

Let $\varphi \in \mathbf{W}$ be given. For any $N \in \mathcal{J}$, by definition of $|\mathbf{w}_N|_{*,N}$, we have

$$(\mathbf{w}_N, \Pi_N \varphi) \leq |\mathbf{w}_N|_{*,N} \|\Pi_N \varphi\|_{\mathbf{E}}.$$

Using the fact that, for any $N \in \mathcal{J}$, $\Pi_N \varphi \in \mathbf{V}_N$, we then obtain

$$(\mathbf{w}_N, \Pi_N \varphi) = (\mathcal{P}_{\mathbf{V}_N} \mathbf{w}_N, \Pi_N \varphi) \leq |\mathbf{w}_N|_{*,N} \|\Pi_N \varphi\|_{\mathbf{E}}.$$

Since $\|\Pi_N \varphi\|_{\mathbf{E}}$ remains bounded by assumption (22), letting $N \rightarrow \infty$ with $N \in \mathcal{J}$ in this inequality yields that

$$(\mathcal{P}_{\mathbf{V}(\Omega)} \mathbf{w}, \varphi) = 0, \text{ for any } \varphi \in \mathbf{W}.$$

The density of $C_c^\infty(\Omega)^d \cap \mathbf{V}(\Omega)$ in $\mathbf{V}(\Omega)$ for the $L^2(\Omega)^d$ norm is proved in particular in [2, Lemma IV.3.5 p. 249]. Since the density of \mathbf{W} in $H_0^1(\Omega)^d \cap \mathbf{V}(\Omega)$ enables to approximate any element of $C_c^\infty(\Omega)^d \cap \mathbf{V}(\Omega)$ as closely as desired for the $L^2(\Omega)^d$ norm, we get that \mathbf{W} is dense as well in $\mathbf{V}(\Omega)$ for the $L^2(\Omega)^d$ norm. We can therefore let $\varphi \rightarrow \mathcal{P}_{\mathbf{V}(\Omega)} \mathbf{w}$ in $L^2(\Omega)^d$. This yields

$$\|\mathcal{P}_{\mathbf{V}(\Omega)} \mathbf{w}\|_{L^2(\Omega)^d}^2 = 0,$$

which contradicts $\|\mathcal{P}_{\mathbf{V}(\Omega)} \mathbf{w}\|_{L^2(\Omega)^d} = 1$. \square

Our aim is now to use Lemma 3.4 on the time translates of $(\tilde{\mathbf{u}}_N)_{N \geq 1}$ for the $L^2(| \cdot |_{*,N})$ semi-norm and (33) in the above lemma, in order to obtain an estimate on the time translates for the $L^2(0, T; L^2(\Omega)^d)$ norm, as stated by the next lemma.

Lemma 3.6 (L^2 estimate on the time translates): Under Assumptions (2), (7), (12), (13), (14), (19), (22), (23), the sequence $(\tilde{\mathbf{u}}_N)_{N \geq 1}$ satisfies

$$\int_0^{T-\tau} \|\tilde{\mathbf{u}}_N(t+\tau) - \tilde{\mathbf{u}}_N(t)\|_{L^2(\Omega)^d}^2 dt \rightarrow 0 \text{ as } \tau \rightarrow 0, \text{ uniformly with respect to } N, \quad (35)$$

and is therefore relatively compact in $L^2(0, T; L^2(\Omega)^d)$.

PROOF. Let us show that $A_N(\tau) \rightarrow 0$ as $\tau \rightarrow 0$ uniformly with respect to N , where we define for any $\tau \in (0, T)$

$$A_N(\tau) := \int_0^{T-\tau} \|\tilde{\mathbf{u}}_N(t+\tau) - \tilde{\mathbf{u}}_N(t)\|_2^2 dt.$$

For any $\mathbf{w} \in L^2(\Omega)^d$ and $\mathbf{v} \in \mathbf{V}_N$, we have

$$\mathbf{w} = \mathbf{w} - \mathcal{P}_{\mathbf{V}_N}(\mathbf{w}) + \mathcal{P}_{\mathbf{V}_N}(\mathbf{w}) = \mathbf{w} - \mathbf{v} - \mathcal{P}_{\mathbf{V}_N}(\mathbf{w} - \mathbf{v}) + \mathcal{P}_{\mathbf{V}_N}(\mathbf{w}).$$

Using the fact that $\mathcal{P}_{\mathbf{V}_N}$ is an orthogonal projection, we have

$$\|\mathbf{w} - \mathbf{v} - \mathcal{P}_{\mathbf{V}_N}(\mathbf{w} - \mathbf{v})\|_{L^2(\Omega)^d} \leq \|\mathbf{w} - \mathbf{v}\|_{L^2(\Omega)^d},$$

which leads to

$$\|\mathbf{w}\|_{L^2(\Omega)^d} \leq \|\mathbf{w} - \mathbf{v}\|_{L^2(\Omega)^d} + \|\mathcal{P}_{\mathbf{V}_N}(\mathbf{w})\|_{L^2(\Omega)^d},$$

giving

$$\forall \mathbf{w} \in L^2(\Omega)^d, \forall \mathbf{v} \in \mathbf{V}_N, \|\mathbf{w}\|_{L^2(\Omega)^d}^2 \leq 2\|\mathbf{w} - \mathbf{v}\|_{L^2(\Omega)^d}^2 + 2\|\mathcal{P}_{\mathbf{V}_N}(\mathbf{w})\|_{L^2(\Omega)^d}^2.$$

We apply the preceding inequality, letting $\mathbf{w} = \tilde{\mathbf{u}}_N(t+\tau) - \tilde{\mathbf{u}}_N(t)$ and $\mathbf{v} = \mathbf{u}_N(t+\tau) - \mathbf{u}_N(t)$ and we integrate on $t \in (0, T-\tau)$. Setting

$$B_N(\tau) = \int_0^{T-\tau} \|(\tilde{\mathbf{u}}_N - \mathbf{u}_N)(t+\tau) - (\tilde{\mathbf{u}}_N - \mathbf{u}_N)(t)\|_2^2 dt$$

we obtain

$$A_N(\tau) \leq 2B_N(\tau) + 2 \int_0^{T-\tau} \|\mathcal{P}_{\mathbf{V}_N}(\tilde{\mathbf{u}}_N(t+\tau) - \tilde{\mathbf{u}}_N(t))\|_{L^2(\Omega)^d}^2 dt. \quad (36)$$

Let $\zeta > 0$ be given.

From (26), we have $B_N(\tau) \leq 4C_1^2 \delta_N$. Let $N_B \geq 1$ be such that $4C_1^2 \delta_N \leq \zeta$ for all $N \geq N_B$. Since $B_N(\tau) \rightarrow 0$ as $\tau \rightarrow 0$ for $N = 1, \dots, N_B$, we can choose τ_B such that, for any $0 \leq \tau \leq \tau_B$ and $N = 1, \dots, N_B$, we have $B_N(\tau) \leq \zeta$.

This yields $B_N(\tau) \leq \zeta$ for all $N \geq 1$ and any $0 \leq \tau < \tau_B$.

Our aim is now to use (33) in Lemma 3.5, which implies

$$\forall \varepsilon > 0, \exists C_\varepsilon > 0, \exists N_\varepsilon \geq 1, \forall N \geq N_\varepsilon, \forall \mathbf{w} \in \mathbf{X}_N$$

$$\|\mathcal{P}_{\mathbf{V}_N}(\mathbf{w})\|_{L^2(\Omega)^d}^2 \leq 2\varepsilon^2 \|\mathbf{w}\|_{H_0^1(\Omega)^d}^2 + 2C_\varepsilon^2 |\mathbf{w}|_{*,N}^2,$$

letting $\mathbf{w} = \tilde{\mathbf{u}}_N(t+\tau) - \tilde{\mathbf{u}}_N(t)$.

From (25), we have

$$\int_0^{T-\tau} \|\tilde{\mathbf{u}}_N(t+\tau) - \tilde{\mathbf{u}}_N(t)\|_{H_0^1(\Omega)^d}^2 dt \leq 4C_1^2. \quad (37)$$

We then select ε such that

$$2\varepsilon^2 4C_1^2 = \zeta. \quad (38)$$

Then there exists $C_\zeta > 0$ and N_ζ such that for any $N \geq N_\zeta$ and for any $\tau \in (0, T)$ and for any $t \in (0, T - \tau)$,

$$\|\mathcal{P}_{\mathbf{V}_N}(\tilde{\mathbf{u}}_N(t + \tau) - \tilde{\mathbf{u}}_N(t))\|_{L^2(\Omega)^d}^2 \leq 2\varepsilon^2 \|\tilde{\mathbf{u}}_N(t + \tau) - \tilde{\mathbf{u}}_N(t)\|_{H_0^1(\Omega)^d}^2 + C_\zeta^2 |\tilde{\mathbf{u}}_N(t + \tau) - \tilde{\mathbf{u}}_N(t)|_{*,N}^2.$$

Integrating the previous relation provides, using (37) and (38),

$$\int_0^{T-\tau} \|\mathcal{P}_{\mathbf{V}_N}(\tilde{\mathbf{u}}_N(t + \tau) - \tilde{\mathbf{u}}_N(t))\|_{L^2(\Omega)^d}^2 dt \leq \zeta + 2C_\zeta^2 \int_0^{T-\tau} |\tilde{\mathbf{u}}_N(t + \tau) - \tilde{\mathbf{u}}_N(t)|_{*,N}^2 dt.$$

Thus, owing to lemma 3.6, we have for any $N \geq N_\zeta$ and for any $\tau \in (0, T)$

$$\int_0^{T-\tau} \|\mathcal{P}_{\mathbf{V}_N}(\tilde{\mathbf{u}}_N(t + \tau) - \tilde{\mathbf{u}}_N(t))\|_{L^2(\Omega)^d}^2 dt \leq \zeta + 2C_\zeta^2 C_2 \tau (\tau + \delta_N),$$

and therefore, for any $N \geq N_\zeta$ and for any $\tau \in (0, T)$, using (36)

$$A_N(\tau) \leq 2B_N(\tau) + 2\zeta + 4C_\zeta^2 C_2 \tau (\tau + \delta_N).$$

We now choose $\tau_C > 0$, such that, for any $0 \leq \tau \leq \tau_C$, $4C_\zeta^2 C_2 \tau (\tau + T) \leq \zeta$.

We then get, for any $N \geq N_\zeta$ and $0 \leq \tau \leq \min(\tau_B, \tau_C)$,

$$A_N(\tau) \leq 2\zeta + 2\zeta + \zeta = 5\zeta.$$

Since $A_N(\tau) \rightarrow 0$ as $\tau \rightarrow 0$ for $N = 1, \dots, N_\zeta$, we can choose τ_D such that, for any $0 \leq \tau \leq \tau_D$ and $N = 1, \dots, N_\zeta$, we have $A_N(\tau) \leq 5\zeta$.

We then obtain that $A_N(\tau) \leq 5\zeta$ for any $\tau \in [0, \min(\tau_B, \tau_C, \tau_D)]$ and $N \geq 1$. The proof of (35) is thus complete. \square

3.4 Convergence of the projection scheme to a weak solution

By Lemma 3.3, up to a subsequence, the sequence $(\tilde{\mathbf{u}}_N)_{N \geq 1}$ converges to some limit $\bar{\mathbf{u}}$ in $L^2(0, T; L^2(\Omega)^d)$, and owing to (26), so does the sequence $(\mathbf{u}_N)_{N \geq 1}$. There remains to check that $\bar{\mathbf{u}}$ is a weak solution to (1) in the sense of Definition 1.1.

Lemma 3.7: Under Assumptions (2), (7), (12), (13), (14), (19), (22), (23), there exist subsequences still denoted by $(\tilde{\mathbf{u}}_N)_{N \geq 1}$ and $(\mathbf{u}_N)_{N \geq 1}$ such that $(\tilde{\mathbf{u}}_N)_{N \geq 1}$ converges to \mathbf{u} strongly in $L^2(\Omega)^d$ and $(\tilde{\mathbf{u}}_N)_{N \geq 1}$ converges to \mathbf{u} strongly in $L^2(\Omega)^d$. Moreover the function $\bar{\mathbf{u}}$ is a weak solution to (1) in the sense of Definition 1.1.

PROOF. We proceed in several steps.

- *Step 1: compactness and convergence in L^2 .* Using Lemma 3.6 there exist subsequences, still denoted $(\mathbf{u}_N)_{N \geq 1}$ and $(\tilde{\mathbf{u}}_N)_{N \geq 1}$, that converge to $\bar{\mathbf{u}}$ in $L^2(0, T; L^2(\Omega)^d)$.
- *Step 2: convergence towards a weak solution.* There remains to show that $\bar{\mathbf{u}}$ is a weak solution in the sense of Definition 1.1 and in particular that $\bar{\mathbf{u}}$ satisfies (11). Let $\psi \in C_c^\infty([0, T])$ and $\boldsymbol{\varphi} \in \mathbf{W}$. Let \mathbf{v} be the function defined by $\mathbf{v}(t, \mathbf{x}) = \psi(t)\boldsymbol{\varphi}(\mathbf{x})$ for any $(t, \mathbf{x}) \in [0, T] \times \Omega$. Let $N \geq 1$. Let $(\mathbf{v}_N^n)_{n \in \llbracket 0, N \rrbracket}$ be the sequence of functions of \mathbf{X}_N defined by $\mathbf{v}_N^n = \psi(t_N^n)\Pi_N \boldsymbol{\varphi}$ for any $n \in \llbracket 0, N \rrbracket$. Let $\mathbf{v}_N : (0, T) \rightarrow H_0^1(\Omega)^d$ and $\mathbf{f}_N : (0, T) \rightarrow L^2(\Omega)^d$ be defined by

$$\mathbf{v}_N(t) = \sum_{n=0}^{N-1} \mathbf{1}_{(t_N^n, t_N^{n+1}]}(t) \mathbf{v}_N^{n+1}, \quad \mathbf{f}_N(t) = \sum_{n=0}^{N-1} \mathbf{1}_{(t_N^n, t_N^{n+1}]}(t) \mathbf{f}_N^{n+1}.$$

We remark that we have

$$\begin{aligned} \|\mathbf{v}_N - \mathbf{v}\|_{L^\infty(0,T;H_0^1(\Omega)^d)} &\leq \delta_N \|\psi'\|_{L^\infty(0,T)} \|\Pi_N \boldsymbol{\varphi}\|_{\mathcal{E}} \\ &\quad + \|\psi\|_{L^\infty(0,T)} \|\Pi_N \boldsymbol{\varphi} - \boldsymbol{\varphi}\|_{H_0^1(\Omega)^d}, \text{ for any } N \geq 1. \end{aligned} \quad (39)$$

Using (22) we obtain that the sequence $(\mathbf{v}_N)_{N \geq 1}$ converges to \mathbf{v} in $L^\infty(0,T;H_0^1(\Omega)^d)$. We have the sequence $(\mathbf{f}_N)_{N \geq 1}$ converges to \mathbf{f} in $L^2((0,T) \times \Omega)^d$. Multiplying (17b) by $\delta_N \mathbf{v}_N^{n+1}$, using the fact that $(\tilde{\mathbf{u}}_N^{n+1}, \mathbf{v}_N^{n+1}) = (\mathbf{u}_N^{n+1}, \mathbf{v}_N^{n+1})$ for any $n \in \llbracket 0, N-1 \rrbracket$ and summing over $n \in \llbracket 1, N-1 \rrbracket$ yields

$$\begin{aligned} \sum_{n=0}^{N-1} (\mathbf{u}_N^{n+1} - \mathbf{u}_N^n, \mathbf{v}_N^{n+1}) + \delta_N \sum_{n=0}^{N-1} b(\tilde{\mathbf{u}}_N^n, \tilde{\mathbf{u}}_N^{n+1}, \mathbf{v}_N^{n+1}) \\ + \int_0^T \int_\Omega \nabla \tilde{\mathbf{u}}_N : \nabla \mathbf{v}_N \, d\mathbf{x} \, dt = \int_0^T (\mathbf{f}_N, \boldsymbol{\varphi}_N) \, dt. \end{aligned} \quad (40)$$

Using the fact that $\boldsymbol{\varphi}_N^N = 0$ in Ω the first term of the left hand side reads

$$\begin{aligned} \sum_{n=0}^{N-1} (\mathbf{u}_N^{n+1} - \mathbf{u}_N^n, \mathbf{v}_N^{n+1}) &= - \sum_{n=0}^{N-1} (\mathbf{u}_N^n, \mathbf{v}_N^{n+1} - \mathbf{v}_N^n) - (\mathbf{u}_N^0, \mathbf{v}_N^0) \\ &= - \int_0^T \psi'(t) (\mathbf{u}_N(t), \Pi_N \boldsymbol{\varphi}) \, dt - \psi(0) (\mathcal{P}_{\mathbf{V}_N} \mathbf{u}_0, \Pi_N \boldsymbol{\varphi}) \end{aligned}$$

Since the sequence $(\mathbf{u}_N)_{N \geq 1}$ converges to $\bar{\mathbf{u}}$ in $L^2(0,T;L^2(\Omega)^d)$, the sequence $(\Pi_N(\boldsymbol{\varphi}))_{N \geq 1}$ converges to $\boldsymbol{\varphi}$ in $L^2(\Omega)^d$ and the sequence $(\mathcal{P}_{\mathbf{V}_N} \mathbf{u}_0)_{N \geq 1}$ converges to \mathbf{u}_0 in $L^2(\Omega)^d$, we obtain

$$\lim_{N \rightarrow +\infty} \sum_{n=0}^{N-1} (\mathbf{u}_N^{n+1} - \mathbf{u}_N^n, \mathbf{v}_N^{n+1}) = - \int_0^T (\bar{\mathbf{u}}, \partial_t \mathbf{v}) \, dt - (\mathbf{u}_0, \mathbf{v}(0, \cdot)). \quad (41)$$

Using the fact that the initial predicted velocity is zero the second term in the left hand-side reads

$$\begin{aligned} \delta_N \sum_{n=0}^{N-1} b(\tilde{\mathbf{u}}_N^n, \tilde{\mathbf{u}}_N^{n+1}, \mathbf{v}_N^{n+1}) &= \delta_N \sum_{n=1}^{N-1} b(\tilde{\mathbf{u}}_N^n, \tilde{\mathbf{u}}_N^{n+1}, \Pi_N \boldsymbol{\varphi}_N^{n+1}) \\ &= \int_{\delta_N}^T ((\tilde{\mathbf{u}}_N(t - \delta_N) \cdot \nabla) \tilde{\mathbf{u}}_N, \mathbf{v}_N) \, dt \\ &\quad + \int_{\delta_N}^T (\operatorname{div} \tilde{\mathbf{u}}_N(t - \delta_N) \tilde{\mathbf{u}}_N, \mathbf{v}_N) \, dt \end{aligned}$$

The convergence of the sequence $(\tilde{\mathbf{u}}_N)_{N \geq 1}$ in $L^2(0,T;L^4(\Omega)^d)$, the weak convergence of the sequence $(\tilde{\mathbf{u}}_N)_{N \geq 1}$ in $L^2(0,T;H_0^1(\Omega)^d)$, the convergence of the sequence $(\mathbf{v}_N)_{N \geq 1}$ in $L^\infty(0,T;L^4(\Omega)^d)$ give

$$\lim_{N \rightarrow +\infty} \delta_N \sum_{n=0}^{N-1} b(\tilde{\mathbf{u}}_N^n, \tilde{\mathbf{u}}_N^{n+1}, \mathbf{v}_N^{n+1}) = \int_0^T ((\bar{\mathbf{u}} \cdot \nabla) \bar{\mathbf{u}}, \mathbf{v}) \, dt. \quad (42)$$

The weak convergence of the sequence $(\nabla \tilde{\mathbf{u}}_N)_{N \geq 1}$ in $L^2(0,T;L^2(\Omega)^d)$ and the convergence of the sequence $(\nabla \mathbf{v}_N)_{N \geq 1}$ in $L^2(0,T;L^2(\Omega)^d)$ give

$$\lim_{N \rightarrow +\infty} \int_0^T \int_\Omega \nabla \tilde{\mathbf{u}}_N : \nabla \mathbf{v}_N \, d\mathbf{x} \, dt = \int_0^T \int_\Omega \nabla \bar{\mathbf{u}} : \nabla \mathbf{v} \, d\mathbf{x} \, dt. \quad (43)$$

The convergence of the sequence $(\mathbf{f}_N)_{N \geq 1}$ in $L^2(0, T; L^2(\Omega)^d)$ and the convergence of the sequence $(\mathbf{v}_N)_{N \geq 1}$ in $L^2(0, T; L^2(\Omega)^d)$ give

$$\lim_{N \rightarrow +\infty} \int_0^T (\mathbf{f}_N, \mathbf{v}_N) dt = \int_0^T (\mathbf{f}, \mathbf{v}) dt. \quad (44)$$

Using (41)-(44) and passing to the limit in (40) gives the weak sense (11) with \mathbf{v} as test function. We get that (11) holds for all test functions by density of the set of linear combinations of such functions.

□

4 Lowest-order Taylor-Hood finite element approximation

In this section, we show that the lowest-order Taylor-Hood element (continuous piecewise quadratic functions for the approximation of the velocity and continuous piecewise affine functions for the approximation of the pressure) satisfy the properties given in Section 2. This pair is of particular interest, since it is the lowest-order conforming stable element that ensures the same approximation order for velocity and pressure functions. Let us assume that the computational domain Ω is polygonal ($d = 2$) or polyhedral ($d = 3$). We define an interpolator Π_L with approximation properties and a divergence correcting linear operator Π_D to define the operator Π as

$$\Pi = \Pi_L + \Pi_D \circ (\text{Id} - \Pi_L).$$

The properties of Π_L are completely standard. We recall them by giving very short proofs. On the contrary, those of Π_D are much less known (see [5]), and the fact that we limit the application of Π to compact support functions enables to avoid difficult properties which must be shown on Fortin interpolators at the boundary. For this reason, we provide here the full computations concerning Π_D .

4.1 Regular simplicial mesh

We define a simplex in dimension $d \geq 1$ as the interior of the convex hull of a given set of $d + 1$ points (called its vertices) which are not all contained in the same hyperplane (a simplex is a triangle if $d = 2$ and a tetrahedron if $d = 3$).

We consider a regular simplicial mesh \mathcal{T} of Ω in the usual sense of the finite element literature [4], which means that:

- The family containing all the vertices of the elements of the mesh is denoted by $(\mathbf{y}_i)_{i \in \mathcal{N}}$. The set \mathcal{N} is partitioned into $\mathcal{N} = \mathcal{N}_{\text{int}} \cup \mathcal{N}_{\text{ext}}$, where $(\mathbf{y}_i)_{i \in \mathcal{N}_{\text{ext}}}$ is the set of exterior nodes and $(\mathbf{y}_i)_{i \in \mathcal{N}_{\text{int}}}$ is the set of interior nodes.
- For any $K \in \mathcal{T}$, the family of the vertices of K is denoted by $(\mathbf{y}_i)_{i \in \mathcal{N}_K}$ with $\mathcal{N}_K \subset \mathcal{N}$ contains $d + 1$ elements. For any $K \in \mathcal{T}$, we denote by $|K|$ the measure in \mathbb{R}^d of K , by h_K the diameter of K and by ρ_K the diameter of the largest ball included in K . Then, we define the mesh size $h_{\mathcal{T}}$ and the mesh regularity $\theta_{\mathcal{T}}$ by

$$h_{\mathcal{T}} = \max_{K \in \mathcal{T}} h_K, \quad \theta_{\mathcal{T}} = \max_{K \in \mathcal{T}} \frac{h_K}{\rho_K}.$$

- For any $K \in \mathcal{T}$, the set of the faces of K is denoted by \mathcal{T}_K . For any $F \in \mathcal{F}_K$, the family of the vertices of the simplex F is denoted by $(\mathbf{y}_i)_{i \in \mathcal{N}_F}$ (then $\mathcal{N}_F \subset \mathcal{N}_K$ contains d elements).

- The set \mathcal{E} of the edges of the elements contains all pairs $\{i, j\}$ such that there exists $K \in \mathcal{T}$ with $\{i, j\} \subset \mathcal{N}_K$. For any $\sigma = \{i, j\} \in \mathcal{E}$, we denote by $\mathbf{y}_\sigma = \frac{1}{2}(\mathbf{y}_i + \mathbf{y}_j)$. For all $K \in \mathcal{T}$, we denote by \mathcal{E}_K the subset of \mathcal{E} containing all the edges of K .

For $i \in \mathcal{N}$ (resp. $\{i, j\} \in \mathcal{E}$), we define \mathcal{T}_i (resp. \mathcal{T}_{ij}) as the set of all $K \in \mathcal{T}$ such that $i \in \mathcal{N}_K$ (resp. $\{i, j\} \in \mathcal{E}_K$) and we denote by

$$\omega_i = \bigcup_{K \in \mathcal{T}_i} \overline{K} \text{ and } \omega_{ij} = \bigcup_{K \in \mathcal{T}_{ij}} \overline{K}.$$

In the whole section 4, we denote by C_i , with $i \in \mathbb{N}$, non negative real values which may depend on d and on Ω , and increasingly on $\theta_{\mathcal{T}}$. The following lemma is a direct consequence of the definition of $\theta_{\mathcal{T}}$.

Lemma 4.1: There exists C_3 such that

$$\text{card}\mathcal{T}_{ij} \leq C_3,$$

$$\forall K \in \mathcal{T}_{ij}, \frac{1}{C_3} h_K^d \leq |\omega_{ij}| \leq C_3 h_K^d,$$

$$\forall K \in \mathcal{T}_{ij}, \text{diam}(\omega_{ij}) \leq C_3 h_K. \quad (45)$$

4.2 Discrete spaces and bases

We introduce the set $\mathbb{P}^k(K)$ of the polynomials over $K \in \mathcal{T}$ of degree less than or equal to k , regardless of the space dimension. We define

$$\mathbb{P}^k(\mathcal{T}) = \{p \in C(\overline{\Omega}, \mathbb{R}) \text{ such that } p|_K \in \mathbb{P}^k(K) \text{ for any } K \in \mathcal{T}\}.$$

We denote by $(\varphi_i)_{i \in \mathcal{N}}$ the nodal basis of $\mathbb{P}^1(\mathcal{T})$, that is, for any $i \in \mathcal{N}$, the element φ_i of $\mathbb{P}^1(\mathcal{T})$ such that $\varphi_i(\mathbf{y}_i) = 1$ and $\varphi_i(\mathbf{y}_j) = 0$ for all $j \in \mathcal{N} \setminus \{i\}$.

Using the properties of the family $(\varphi_i)_{i \in \mathcal{N}}$ we have

$$\overline{\{\mathbf{x} \in \Omega \text{ such that } \varphi_i(\mathbf{x}) \neq 0\}} = \omega_i \text{ for any } i \in \mathcal{N}. \quad (46)$$

Then the nodal basis of $\mathbb{P}^2(\mathcal{T})$ is given by the family $\left((\varphi_i(2\varphi_i - 1))_{i \in \mathcal{N}}, (4\varphi_i\varphi_j)_{\{i, j\} \in \mathcal{E}} \right)$. Denoting this family by $(\phi_i)_{i \in \mathcal{N} \cup \mathcal{E}}$, for any $i \in \mathcal{N} \cup \mathcal{E}$, we have that $\phi_i(\mathbf{y}_i) = 1$ and $\phi_i(\mathbf{y}_j) = 0$ for all $j \in (\mathcal{N} \cup \mathcal{E}) \setminus \{i\}$. Using (46) and using [7, Proposition 11.6] we can state the following lemma.

Lemma 4.2: There exists C_4 such that

$$\forall K \in \mathcal{T}, \max_{i \in \mathcal{N}_K} \|\nabla \varphi_i\|_{L^\infty(\omega_i)^d} \leq \frac{C_4}{h_K} \quad (47)$$

which leads that there exists C_5

$$\forall K \in \mathcal{T}, \forall \mathbf{x} \in K, \forall i \in \mathcal{N}_K \cup \mathcal{E}_K, |\nabla \phi_i(\mathbf{x})| \leq \frac{C_5}{h_K}. \quad (48)$$

The spaces of approximation for the velocity and the pressure are defined by

$$\mathbf{X} = \mathbb{P}^2(\mathcal{T})^d \cap H_0^1(\Omega)^d \quad \text{and} \quad M(\Omega) = \mathbb{P}^1(\mathcal{T}) \cap L_0^2(\Omega). \quad (49)$$

4.3 The $\mathbb{P}^2(\mathcal{T})$ Lagrange interpolator

The Lagrange interpolator, denoted by $\Pi_L : C_c^\infty(\Omega)^d \rightarrow \mathbf{X}$, is defined by

$$\forall \mathbf{v} = (v_k)_{k=1,\dots,d} \in C_c^\infty(\Omega)^d, \quad (\Pi_L \mathbf{v})_k = \sum_{i \in \mathcal{N} \cup \mathcal{E}} v_k(\mathbf{y}_i) \phi_i, \quad k = 1, \dots, d. \quad (50)$$

Lemma 4.3: There exists C_6 such that for any $\mathbf{v} \in C_c^\infty(\Omega)^d$, we have

$$\|\Pi_L \mathbf{v}\|_{L^\infty(\Omega)^d} \leq C_6 \|\nabla \mathbf{v}\|_{L^\infty(\Omega)^{d \times d}}. \quad (51)$$

PROOF. Let $\underline{x}_1, \bar{x}_1 \in \mathbb{R}$ such that, for any $\mathbf{x} = (x_1, \dots, x_d) \in \Omega$, $\underline{x}_1 \leq x_1 \leq \bar{x}_1$. Writing, for $k = 1, \dots, d$,

$$v_k(\mathbf{y}_i) = \int_{\underline{x}_1}^{(y_i)_1} \partial_1 v_k(s, ((y_i)_\ell)_{\ell=2,\dots,d}) ds,$$

we get

$$|v_k(\mathbf{y}_i)| \leq (\bar{x}_1 - \underline{x}_1) \|\nabla \mathbf{v}\|_{L^\infty(\Omega)^{d \times d}}.$$

Since $|\phi_i(\mathbf{x})| \leq 1$ for all $\mathbf{x} \in \Omega$, and using that the maximum number of $i \in \mathcal{N} \cup \mathcal{E}$ such that $\phi_i(\mathbf{x}) \neq 0$ is $(d+1)(d+2)/2$, we get from (50) that

$$|v_k(\mathbf{x})| \leq \frac{(d+1)(d+2)}{2} (\bar{x}_1 - \underline{x}_1) \|\nabla \mathbf{v}\|_{L^\infty(\Omega)^{d \times d}},$$

hence concluding the proof. \square

We have the following results.

Lemma 4.4: There exists C_7 such that for any $\mathbf{v} = (v_k)_{k=1,\dots,d} \in C_c^\infty(\Omega)^d$, it holds, for any $K \in \mathcal{T}$,

$$\|\mathbf{v} - \Pi_L \mathbf{v}\|_{L^\infty(K)^d} \leq C_7 h_K \|\nabla \mathbf{v}\|_{L^\infty(\Omega)^{d \times d}}, \quad (52)$$

$$\|\nabla \Pi_L \mathbf{v}\|_{L^\infty(K)^{d \times d}} \leq C_7 \|\nabla \mathbf{v}\|_{L^\infty(\Omega)^{d \times d}}, \quad (53)$$

$$\|\mathbf{v} - \Pi_L \mathbf{v}\|_{L^\infty(K)^d} \leq C_7 h_K^2 \|\mathbf{v}\|_{W^{2,\infty}(\Omega)}, \quad (54)$$

and

$$\|\nabla \mathbf{v} - \nabla \Pi_L \mathbf{v}\|_{L^\infty(K)^{d \times d}} \leq C_7 h_K \|\mathbf{v}\|_{W^{2,\infty}(\Omega)}. \quad (55)$$

PROOF. Let $\mathbf{v} = (v_k)_{k=1,\dots,d} \in C_c^\infty(\Omega)^d$. For $k = 1, \dots, d$, the mean value theorem implies that there exists \mathbf{x}_i belonging to the segment $[\mathbf{y}_i, \mathbf{x}]$ such that

$$v_k(\mathbf{y}_i) - v_k(\mathbf{x}) = (\mathbf{y}_i - \mathbf{x}) \cdot \nabla v_k(\mathbf{x}_i),$$

which implies

$$|v_k(\mathbf{y}_i) - v_k(\mathbf{x})| \leq h_K \|\nabla \mathbf{v}\|_{L^\infty(\Omega)^{d \times d}}.$$

Noticing that, for any $K \in \mathcal{T}$ and for all $\mathbf{x} \in K$, we have $\sum_{i \in \mathcal{N}_K \cup \mathcal{E}_K} \phi_i(\mathbf{x}) = 1$, we get

$$|\Pi_L v_k(\mathbf{x}) - v_k(\mathbf{x})| = \left| \sum_{i \in \mathcal{N}_K \cup \mathcal{E}_K} (v_k(\mathbf{y}_i) - v_k(\mathbf{x})) \phi_i(\mathbf{x}) \right| \leq \frac{(d+1)(d+1)}{2} h_K \|\nabla \mathbf{v}\|_{L^\infty(\Omega)^{d \times d}}.$$

This yields (52) as well as

$$\nabla v_k(\mathbf{x}) = \sum_{i \in \mathcal{N}_K \cup \mathcal{E}_K} (v_k(\mathbf{y}_i) - v_k(\mathbf{x})) \nabla \phi_i(\mathbf{x}) = \sum_{i \in \mathcal{N}_K \cup \mathcal{E}_K} (\mathbf{y}_i - \mathbf{x}) \cdot \nabla v_k(\mathbf{x}_i) \nabla \phi_i(\mathbf{x}).$$

Applying (48), we conclude that (53) holds. Writing, for $k = 1, \dots, d$ and any $j \in \mathcal{N}_K \cup \mathcal{E}_K$, the Taylor expansion

$$v_k(\mathbf{y}_j) = v_k(\mathbf{x}) + (\mathbf{y}_j - \mathbf{x}) \cdot \nabla v_k(\mathbf{x}) + \frac{1}{2}(\mathbf{y}_j - \mathbf{x})^t D_2 v_k(\mathbf{x}_j)(\mathbf{y}_j - \mathbf{x}),$$

for some point \mathbf{x}_j in the segment $[\mathbf{y}_j, \mathbf{x}]$, and using (50) in addition to

$$\sum_{j \in \mathcal{N}_K \cup \mathcal{E}_K} \phi_j(\mathbf{x}) = 1 \text{ and } \sum_{j \in \mathcal{N}_K \cup \mathcal{E}_K} \phi_j(\mathbf{x})(\mathbf{y}_j - \mathbf{x}) = 0,$$

we obtain (54). We deduce, from the preceding Taylor expansion, that

$$\begin{aligned} \sum_{j \in \mathcal{N}_K \cup \mathcal{E}_K} v_k(\mathbf{y}_j) \nabla \phi_i(\mathbf{x}) &= \sum_{j \in \mathcal{N}_K \cup \mathcal{E}_K} v_k(\mathbf{x}) \nabla \phi_i(\mathbf{x}) + \sum_{j \in \mathcal{N}_K \cup \mathcal{E}_K} (\mathbf{y}_j - \mathbf{x}) \cdot \nabla v_k(\mathbf{x}) \nabla \phi_i(\mathbf{x}) \\ &\quad + \sum_{j \in \mathcal{N}_K \cup \mathcal{E}_K} \frac{1}{2}(\mathbf{y}_j - \mathbf{x})^t D_2 v_k(\mathbf{x}_j)(\mathbf{y}_j - \mathbf{x}) \nabla \phi_i(\mathbf{x}). \end{aligned}$$

We notice that, since the interpolation of first order polynomials is exact,

$$\sum_{j \in \mathcal{N}_K \cup \mathcal{E}_K} v_k(\mathbf{x}) \nabla \phi_i(\mathbf{x}) = 0 \text{ and } \sum_{j \in \mathcal{N}_K \cup \mathcal{E}_K} (\mathbf{y}_j - \mathbf{x}) \cdot \nabla v_k(\mathbf{x}) \nabla \phi_i(\mathbf{x}) = \nabla v_k(\mathbf{x}),$$

and

$$\left| \sum_{j \in \mathcal{N}_K \cup \mathcal{E}_K} \frac{1}{2}(\mathbf{y}_j - \mathbf{x})^t D_2 v_k(\mathbf{x}_j)(\mathbf{y}_j - \mathbf{x}) \nabla \phi_i(\mathbf{x}) \right| \leq C_8 h_K \|\mathbf{v}\|_{W^{2,\infty}(\Omega)},$$

which allows to conclude (55). \square

4.4 Divergence correcting operator

Using (46) we have

$$\overline{\{\mathbf{x} \in \Omega \text{ such that } \varphi_i(\mathbf{x})\varphi_j(\mathbf{x}) \neq 0\}} = \omega_{ij} \text{ for any } \{i, j\} \in \mathcal{E}. \quad (56)$$

We define the family of normalized tangential edge bubble function

$$\forall \mathbf{x} \in \Omega, \mathbf{b}_{i,j}(\mathbf{x}) = \frac{(d+2)!}{d!|\omega_{i,j}|} \varphi_i(\mathbf{x})\varphi_j(\mathbf{x})(\mathbf{y}_j - \mathbf{y}_i) \text{ for any } \{i, j\} \in \mathcal{E}.$$

We observe that, for any $\{i, j\} \in \mathcal{E}$, we have $\mathbf{b}_{i,j} \in \mathbb{P}^2(\mathcal{T})$, and, in the case where the segment $]\mathbf{y}_i, \mathbf{y}_j[\subset \Omega$, we additionally have $\mathbf{b}_{i,j} \in \mathbf{X}$ (this property holds even if $\{i, j\} \in \mathcal{E}_{\text{ext}}$).

We define the divergence correcting operator $\Pi_D : W_0^{1,\infty}(\Omega)^d \rightarrow \mathbb{P}^2(\mathcal{T})^d$ for any $\mathbf{v} \in W_0^{1,\infty}(\Omega)^d$ by

$$\forall \mathbf{x} \in \Omega, \Pi_D \mathbf{v}(\mathbf{x}) = \sum_{i \in \mathcal{N}} \Pi_{D,i} \mathbf{v}(\mathbf{x}) \quad \text{with} \quad \Pi_{D,i} \mathbf{v}(\mathbf{x}) = \sum_{\substack{j \in \mathcal{N} \\ \{i,j\} \in \mathcal{E}}} (\text{div}(\varphi_i \mathbf{v}), \varphi_j) \mathbf{b}_{j,i}(\mathbf{x}). \quad (57)$$

Using the fact that $\mathbf{b}_{i,j} = -\mathbf{b}_{j,i}$ for any $\{i, j\} \in \mathcal{E}$ we remark that this operator satisfies

$$\forall \mathbf{v} \in W_0^{1,\infty}(\Omega)^d, \forall \mathbf{x} \in \Omega, \Pi_D \mathbf{v}(\mathbf{x}) = \sum_{\{i,j\} \in \mathcal{E}} (\mathbf{v}, -\varphi_i \nabla \varphi_j + \varphi_j \nabla \varphi_i) \mathbf{b}_{j,i}(\mathbf{x}). \quad (58)$$

Lemma 4.5: For any $\mathbf{v} \in W_0^{1,\infty}(\Omega)^d$ and for any $\mathbf{x} \in \Omega$, if $\Pi_D \mathbf{v}(\mathbf{x}) \neq 0$, then there exists $\mathbf{y} \in B(\mathbf{x}, C_3 h_T)$ such that $\mathbf{v}(\mathbf{y}) \neq 0$ (recall that C_3 is introduced in Lemma 4.1).

PROOF. If $\Pi_D \mathbf{v}(\mathbf{x}) \neq 0$, there exists $\{i, j\} \in \mathcal{E}$ such that $(\mathbf{v}, -\varphi_i \nabla \varphi_j + \varphi_j \nabla \varphi_i) \mathbf{b}_{j,i}(\mathbf{x}) \neq 0$, which implies $\mathbf{x} \in \omega_{ij}$, which yields that there exists $K \in \mathcal{T}_{ij}$ such that $\mathbf{x} \in \overline{K}$. This also implies that

$$\int_{\omega_{ij}} |\mathbf{v}(\mathbf{y})| d\mathbf{y} > 0.$$

Therefore there exists $\mathbf{y} \in \omega_{ij}$ such that $\mathbf{v}(\mathbf{y}) \neq 0$. Remarking that (45) implies that $\text{diam}(\omega_{ij}) \leq C_3 h_K$. Hence $|\mathbf{x} - \mathbf{y}| \leq C_3 h_T$. \square

Lemma 4.6: For any $\{i, j\} \in \mathcal{E}$ and for any $k \in \mathcal{N}$ we have the following relation

$$(\text{div} \mathbf{b}_{i,j}, \varphi_k) = \delta_{i,k} - \delta_{j,k}.$$

PROOF. Let $\{i, j\} \in \mathcal{E}$ and let $k \in \mathcal{N}$. Using (56) We have

$$(\text{div} \mathbf{b}_{i,j}, \varphi_k) = \int_{\omega_{ij}} \text{div} \mathbf{b}_{i,j} \varphi_k d\mathbf{x}.$$

- The element k is not an element of $\cup_{K \in \mathcal{T}_{ij}} \mathcal{N}_K$. Using (46) we have $\varphi_k(\mathbf{x}) = 0$ for any $\mathbf{x} \in \omega_{ij}$ and hence the expression vanishes.
- The element k is an element of $\cup_{K \in \mathcal{T}_{ij}} \mathcal{N}_K$. We have

$$\begin{aligned} (\text{div} \mathbf{b}_{i,j}, \varphi_k) &= -\frac{(d+2)!}{d! |w_{i,j}|} \sum_{K \in \mathcal{T}_{ij}} \int_K \varphi_i \varphi_j (\mathbf{y}_j - \mathbf{y}_i) \cdot \nabla \varphi_k d\mathbf{x} \\ &\quad + \frac{(d+2)!}{d! |w_{i,j}|} \sum_{K \in \mathcal{T}_{ij}} \sum_{F \in \mathcal{F}_K} \int_F \varphi_k \varphi_i \varphi_j (\mathbf{y}_j - \mathbf{y}_i) \cdot \mathbf{n}_F d\gamma(\mathbf{x}) \end{aligned}$$

For any $K \in \mathcal{T}_{ij}$ and $F \in \mathcal{F}_K$ we remark that $(\mathbf{y}_j - \mathbf{y}_i) \cdot \mathbf{n}_F = 0$ in the case $\{i, j\} \subset \mathcal{N}_F$ and $\varphi_i(\mathbf{x}) \varphi_j(\mathbf{x}) = 0$ for any $\mathbf{x} \in F$ in the case $\{i, j\} \not\subset \mathcal{N}_F$.

We then obtain for any $K \in \mathcal{T}_{ij}$ and for any $F \in \mathcal{F}_K$

$$\int_F \varphi_k \varphi_i \varphi_j (\mathbf{y}_j - \mathbf{y}_i) \cdot \mathbf{n}_F d\gamma(\mathbf{x}) = 0$$

which gives the following identity

$$(\text{div} \mathbf{b}_{i,j}, \varphi_k) = -\frac{(d+2)!}{d! |w_{i,j}|} \sum_{K \in \mathcal{T}_{ij}} \int_K \varphi_i \varphi_j (\mathbf{y}_j - \mathbf{y}_i) \cdot \nabla \varphi_k d\mathbf{x}.$$

For $k \notin \{i, j\}$ using $\varphi_k(\mathbf{y}_i) = 0$ and $\varphi_k(\mathbf{y}_j) = 0$ we obtain that the piecewise constant function $\nabla \varphi_k$ is orthogonal to the edge vector $(\mathbf{y}_j - \mathbf{y}_i)$ on each $K \in \mathcal{T}_{ij}$. Thus, the integrals vanishes and the claim follows. For $k \in \{i, j\}$ using $\varphi_k(\mathbf{y}_k) = 1$ we obtain for any $K \in \mathcal{T}_{ij}$ the following identity

$$\nabla(\varphi_k)|_K \cdot (\mathbf{y}_j - \mathbf{y}_i) = \begin{cases} -1, & \text{for } k = i, \\ 1, & \text{for } k = j. \end{cases}$$

For $k = i$ we obtain

$$(\operatorname{div} \mathbf{b}_{i,j}, \varphi_k) = \frac{(d+2)!}{d!|w_{i,j}|} \sum_{K \in \mathcal{T}_{ij}} \int_K \varphi_i \varphi_j \, d\mathbf{x} = 1,$$

and for $k = j$ we obtain

$$(\operatorname{div} \mathbf{b}_{i,j}, \varphi_k) = -\frac{(d+2)!}{d!|w_{i,j}|} \sum_{K \in \mathcal{T}_{ij}} \int_K \varphi_i \varphi_j \, d\mathbf{x} = -1,$$

which gives the expected result.

□

We can now prove that this operator preserves the divergence in the following sense.

Lemma 4.7: For any $\mathbf{v} \in W_0^{1,\infty}(\Omega)^d$ and any $\varphi \in \mathbb{P}^1(\mathcal{T})$ we have

$$(\operatorname{div} \Pi_D \mathbf{v}, \varphi) = (\operatorname{div} \mathbf{v}, \varphi).$$

PROOF. Using Lemma 4.6 we have, for any $k \in \mathcal{N}$,

$$\forall i \in \mathcal{N}, (\operatorname{div} \Pi_{D,i} \mathbf{v}, \varphi_k) = \sum_{\substack{j \in \mathcal{N} \\ \{i,j\} \in \mathcal{E}}} (\operatorname{div}(\varphi_i \mathbf{v}), \varphi_j) (\delta_{j,k} - \delta_{i,k}).$$

Hence we get $(\operatorname{div} \Pi_D \mathbf{v}, \varphi_k) = T_1 + T_2$, with

$$T_1 = \sum_{i \in \mathcal{N}} \sum_{\substack{j \in \mathcal{N} \\ \{i,j\} \in \mathcal{E}}} (\operatorname{div}(\varphi_i \mathbf{v}), \varphi_j) \delta_{j,k} \quad \text{and} \quad T_2 = - \sum_{i \in \mathcal{N}} \sum_{\substack{j \in \mathcal{N} \\ \{i,j\} \in \mathcal{E}}} (\operatorname{div}(\varphi_i \mathbf{v}), \varphi_j) \delta_{i,k}.$$

Remarking that, for any quantity $Q_{i,j}$,

$$\sum_{i \in \mathcal{N}} \sum_{\substack{j \in \mathcal{N} \\ \{i,j\} \in \mathcal{E}}} Q_{i,j} = \sum_{\{i,j\} \in \mathcal{E}} (Q_{i,j} + Q_{j,i}) = \sum_{j \in \mathcal{N}} \sum_{\substack{i \in \mathcal{N} \\ \{i,j\} \in \mathcal{E}}} Q_{i,j},$$

we obtain

$$T_1 = \sum_{j \in \mathcal{N}} \sum_{\substack{i \in \mathcal{N} \\ \{i,j\} \in \mathcal{E}}} (\operatorname{div}(\varphi_i \mathbf{v}), \varphi_j) \delta_{j,k} = (\operatorname{div}(\sum_{\substack{i \in \mathcal{N} \\ \{i,k\} \in \mathcal{E}}} \varphi_i \mathbf{v}), \varphi_k).$$

For any $\mathbf{x} \in \omega_k$, we have

$$\sum_{\substack{i \in \mathcal{N} \\ \{i,k\} \in \mathcal{E}}} \varphi_i(\mathbf{x}) = 1 - \varphi_k(\mathbf{x}),$$

which leads to

$$T_1 = (\operatorname{div}((1 - \varphi_k) \mathbf{v}), \varphi_k).$$

Next we have

$$T_2 = - \sum_{\substack{j \in \mathcal{N} \\ \{k,j\} \in \mathcal{E}}} (\operatorname{div}(\varphi_k \mathbf{v}), \varphi_j) = -(\operatorname{div}(\varphi_k \mathbf{v}), \sum_{\substack{j \in \mathcal{N} \\ \{k,j\} \in \mathcal{E}}} \varphi_j).$$

For any $\mathbf{x} \in \omega_k$, we have

$$\sum_{\substack{j \in \mathcal{N} \\ \{k,j\} \in \mathcal{E}}} \varphi_j(\mathbf{x}) = 1 - \varphi_k(\mathbf{x}),$$

which implies, since $(\operatorname{div}(\varphi_k \mathbf{v}), 1) = 0$ (recall that \mathbf{v} vanishes on the boundary of Ω),

$$T_2 = -(\operatorname{div}(\varphi_k \mathbf{v}), -\varphi_k) = (\operatorname{div}(\varphi_k \mathbf{v}), \varphi_k).$$

This yields

$$T_1 + T_2 = (\operatorname{div}(\mathbf{v}), \varphi_k),$$

which concludes the proof. \square

Lemma 4.8: There exists C_9 such that

$$\forall K \in \mathcal{T}, \forall \{i, j\} \in \mathcal{E}_K, \|\mathbf{b}_{ij}\|_{L^\infty(K)^d} \leq C_9 h_K^{1-d}, \quad (59)$$

and

$$\forall K \in \mathcal{T}, \forall \{i, j\} \in \mathcal{E}_K, \|\nabla \mathbf{b}_{ij}\|_{L^\infty(K)^d} \leq C_9 h_K^{-d}. \quad (60)$$

PROOF. We have, for $\{i, j\} \subset \mathcal{N}_K$,

$$|\varphi_i(\mathbf{x})\varphi_j(\mathbf{x})| \leq 1$$

and

$$|w_{i,j}| \geq C_{10} h_K^d,$$

which leads to (59) since $|\mathbf{y}_j - \mathbf{y}_i| \leq h_K$. From (47), we get

$$|\nabla(\varphi_i \varphi_j)(\mathbf{x})| \leq 2 \frac{C_4}{h_K}.$$

Combining with the preceding inequalities, we deduce (60). \square

Lemma 4.9: There exists C_{11} such that, for any $\mathbf{v} \in W_0^{1,\infty}(\Omega)^d$, we have

$$\|\Pi_D \mathbf{v}\|_{L^\infty(\Omega)^d} \leq C_{11} \|\mathbf{v}\|_{L^\infty(\Omega)^d}.$$

PROOF. Using (58) and the definition of the tangential bubble functions we have, for $\mathbf{x} \in K$ with $K \in \mathcal{T}$,

$$|\Pi_D \mathbf{v}(\mathbf{x})| \leq \sum_{\{i,j\} \in \mathcal{E}} |(\mathbf{v}, -\varphi_i \nabla \varphi_j + \varphi_j \nabla \varphi_i)| |\mathbf{b}_{i,j}(\mathbf{x})|.$$

If $K \subset \omega_{ij}$, then there exists C_{12} such that $\omega_{ij} \subset B(\mathbf{x}, C_{12} h_K)$. Since $m(\omega_{ij}) \geq C_{13} h_K^d$, we get the existence of C_{14} such that the number of $\{i, j\} \in \mathcal{E}$ with $\mathbf{b}_{i,j}(\mathbf{x}) \neq 0$ is bounded by C_{14} . Using (59) and (47), we obtain

$$|\Pi_D \mathbf{v}(\mathbf{x})| \leq \|\mathbf{v}\|_{L^\infty(\Omega)^d} C_{14} C_9 h_K \frac{C_4}{h_K},$$

which concludes the proof. \square

Lemma 4.10: There exists C_{15} such that

$$\forall \mathbf{v} \in W_0^{1,\infty}(\Omega)^d, \|\nabla \Pi_D \mathbf{v}\|_{L^\infty(\Omega)^{d \times d}} \leq C_{15} \max_{K \in \mathcal{T}} \frac{\|\mathbf{v}\|_{L^\infty(K)^d}}{h_K}. \quad (61)$$

PROOF. Let us denote, for a given $K \in \mathcal{T}$,

$$C_{\mathbf{v}} = \max_{K \in \mathcal{T}} \frac{\|\mathbf{v}\|_{L^\infty(K)^d}}{h_K}. \quad (62)$$

Using (58), we have, for $\mathbf{x} \in K$,

$$\nabla \Pi_D v(\mathbf{x}) = \sum_{\{i,j\} \in \mathcal{E}} (\mathbf{v}, -\varphi_i \nabla \varphi_j + \varphi_j \nabla \varphi_i) \nabla \mathbf{b}_{i,j}(\mathbf{x}).$$

We remark that, using (62) and (47) for $K \subset \omega_{ij}$ and $\mathbf{x} \in K$,

$$|\mathbf{v}(\mathbf{x})(-\varphi_i(\mathbf{x}) \nabla \varphi_j(\mathbf{x}) + \varphi_j(\mathbf{x}) \nabla \varphi_i(\mathbf{x}))| \leq \|\mathbf{v}\|_{L^\infty(K)} 2 \frac{C_4}{h_K}.$$

This yields

$$|(\mathbf{v}, -\varphi_i \nabla \varphi_j + \varphi_j \nabla \varphi_i)| \leq 2|\omega_{ij}| C_v C_4,$$

and therefore, using (60), we conclude (61). \square

4.5 The operator Π

We let $\mathbf{W} = C_c^\infty(\Omega)^d \cap \mathbf{V}(\Omega)$. We notice that, for any $\mathbf{v} \in \mathbf{W}$, then $\Pi_L \mathbf{v} \in W_0^{1,\infty}(\Omega)$. One can then consider the function $\Pi_D(\mathbf{v} - \Pi_L \mathbf{v}) \in \mathbb{P}^2(\mathcal{T})$. We notice that this function is not necessarily null on the boundary of the domain, and in this case, it is not an element of \mathbf{X} .

We then define $\Pi : \mathbf{W} \rightarrow \mathbf{E}_N = \mathbf{X}_N \cap \mathbf{V}_N \cap L^\infty(\Omega)^d$, for any $\mathbf{v} \in \mathbf{W}$, by:

$$\text{if } \Pi_D(\mathbf{v} - \Pi_L \mathbf{v}) \in H_0^1(\Omega), \text{ then } \Pi \mathbf{v} = \Pi_L \mathbf{v} + \Pi_D(\mathbf{v} - \Pi_L \mathbf{v}) \text{ else } \Pi \mathbf{v} = 0. \quad (63)$$

Lemma 4.11: For any $\mathbf{v} \in C_c^\infty(\Omega)^d \cap \mathbf{V}(\Omega)$, we have

$$\forall \varphi \in \mathbb{P}^1(\mathcal{T}), \quad (\text{div} \Pi \mathbf{v}, \varphi) = 0. \quad (64)$$

PROOF. If $\Pi_D(\mathbf{v} - \Pi_L \mathbf{v}) \in H_0^1(\Omega)$, we get from Lemma 4.7

$$(\text{div} \Pi_D(\mathbf{v} - \Pi_L \mathbf{v}), \varphi) = (\text{div}(\mathbf{v} - \Pi_L \mathbf{v}), \varphi) = -(\text{div} \Pi_L \mathbf{v}, \varphi),$$

which provides (64). Otherwise, we get (64) since $\Pi \mathbf{v} = 0$. \square

Lemma 4.12: There exists C_{16} such that for any $\mathbf{v} \in \mathbf{W}$, we have

$$\|\Pi \mathbf{v}\|_{L^\infty(\Omega)^d} \leq C_{16} \|\nabla \mathbf{v}\|_{L^\infty(\Omega)^{d \times d}}. \quad (65)$$

and

$$\|\nabla \Pi \mathbf{v}\|_{L^\infty(\Omega)^d} \leq C_{16} \|\nabla \mathbf{v}\|_{L^\infty(\Omega)^{d \times d}}. \quad (66)$$

PROOF. It suffices to consider the case $\Pi_D(\mathbf{v} - \Pi_L \mathbf{v}) \in H_0^1(\Omega)$ (otherwise the conclusion is straightforward). We apply Lemmas 4.3 and 4.9. We then get (65).

Now applying Lemmas (52) in Lemma 4.4 and Lemma 4.10, we obtain (66). \square

Lemma 4.13: There exists C_{17} such that, for any $\mathbf{v} \in \mathbf{W}$ such that $\Pi_D(\mathbf{v} - \Pi_L \mathbf{v}) \in H_0^1(\Omega)$, then

$$\|\Pi \mathbf{v} - \mathbf{v}\|_{W^{1,\infty}(\Omega)^d} \leq C_{17} h_{\mathcal{T}} \|\mathbf{v}\|_{W^{2,\infty}(\Omega)}. \quad (67)$$

PROOF. We first remark that, from (54) and (55) in Lemma 4.4, we get

$$\|\Pi_L \mathbf{v} - \mathbf{v}\|_{W^{1,\infty}(\Omega)^d} \leq C_{18} h_{\mathcal{T}} \|\mathbf{v}\|_{W^{2,\infty}(\Omega)}.$$

Again applying (54) in the same Lemma, and using Lemma 4.10, we get that

$$\|\Pi_D(\mathbf{v} - \Pi_L \mathbf{v})\|_{W^{1,\infty}(\Omega)^d} \leq C_{15} C_7 h_{\mathcal{T}} \|\mathbf{v}\|_{W^{2,\infty}(\Omega)},$$

which concludes the proof of (67). \square

Lemma 4.14: Let $\mathbf{W} = C_c^\infty(\Omega) \cap \mathbf{V}(\Omega)$. Let $(\mathcal{T}_N)_{N \geq 1}$ be a sequence of simplicial meshes in the sense of this section, such that $(h_{\mathcal{T}_N})_{N \geq 1}$ converges to zero while $(\theta_{\mathcal{T}_N})_{N \geq 1}$ remains bounded. We then consider the sequence $(\mathbf{X}_N, M_N, \Pi_N)_{N \geq 1}$ such that \mathbf{X}_N and M_N are the spaces \mathbf{X} and M defined by (49), Π_N is defined by (63), in the case where $\mathcal{T} = \mathcal{T}_N$.

Then the family $(\mathbf{W}, (\mathbf{X}_N, M_N, \Pi_N)_{N \geq 1})$ satisfies the convergence assumptions (19)-(23) in Section 3.1.

PROOF. The density of $C_c^\infty(\Omega)^d \cap \mathbf{V}(\Omega)$ in $H_0^1(\Omega)^d \cap \mathbf{V}(\Omega)$ for the $H_0^1(\Omega)^d$ norm is proved in particular in [2, Lemma IV.3.4 p. 249].

Let $\boldsymbol{\varphi} \in \mathbf{W}$. Applying (63), we have that $\Pi_N \boldsymbol{\varphi} \in \mathbf{E}_N$.

Indeed, if $\Pi_D(\mathbf{v} - \Pi_L \mathbf{v}) \in H_0^1(\Omega)$, then $\Pi_D(\mathbf{v} - \Pi_L \mathbf{v}) \in \mathbf{X}_N$ as well as $\Pi_L \mathbf{v} \in \mathbf{X}_N$. In this case, the application of Lemma 4.11 yields (19).

Otherwise, if $\Pi_D(\mathbf{v} - \Pi_L \mathbf{v}) \notin H_0^1(\Omega)$, then $\Pi_N \boldsymbol{\varphi} = 0 \in \mathbf{E}_N$ and (19) is also satisfied.

Let $\mathbf{v} \in C_c^\infty(\Omega) \cap \mathbf{V}(\Omega)$. Since the support of \mathbf{v} is compact, there exists $a > 0$ such that, for any $\mathbf{x} \in \partial\Omega$, $\mathbf{v} = 0$ on the ball $B(\mathbf{x}, a)$. Let $N_0 \geq 1$ be such that, for all $N \geq N_0$, $h_{\mathcal{T}_N} \leq a/(C_3 + 2)$. Let $N \geq N_0$, and let us consider Π_L defined by (50) in the case where $\mathcal{T} = \mathcal{T}_N$. Then, for any $\mathbf{x} \in \partial\Omega$, $\mathbf{v} - \Pi_L \mathbf{v} = 0$ on the ball $B(\mathbf{x}, a - h_{\mathcal{T}_N})$.

We then deduce from Lemma 4.5 that, for any $\mathbf{x} \in \Omega$ with $B(\mathbf{x}, h_{\mathcal{T}_N}) \cap \partial\Omega \neq \emptyset$, then $\mathbf{v} - \Pi_L \mathbf{v} = 0$ on the ball $B(\mathbf{x}, C_3 h_{\mathcal{T}_N})$, which leads to $\Pi_D(\mathbf{v} - \Pi_L \mathbf{v}) \in H_0^1(\Omega)$ for all $N \geq N_0$ in the case where $\mathcal{T} = \mathcal{T}_N$. Therefore, $\Pi_D(\mathbf{v} - \Pi_L \mathbf{v}) \in H_0^1(\Omega)$ for all $N \geq N_0$ in the case where $\mathcal{T} = \mathcal{T}_N$. This yields that, for any $N \geq N_0$, the conclusion of Lemma 4.13 holds. This proves (22).

Finally, (23) is a consequence of the convergence of \mathbb{P}^1 finite elements for the Neumann problem with null average.

□

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