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# GRAVITATIONAL MONOPOLES AND THE SIGN OF THE COSMOLOGICAL CONSTANT.

GOURAB BHATTACHARYA

ABSTRACT. We show for a Gravitational monopole, the cosmological constant is strictly negative, that is  $\lambda < 0$ .

## 1. INTRODUCTION

Let  $M^4$  be a smooth, compact, spin manifold with a Riemannian metric  $g$ . Let  $P \rightarrow M$  be a smooth principal bundle with the structure group  $G$ . We choose  $G = SU(2)$ . Let  $\gamma$  be a soldering form, which is a form on  $P$  such that it is a horizontal and  $G$ -equivariant differential 1-form on  $P$  with values in a linear representation  $V$  of  $G$  such that the associated bundle map from the tangent bundle  $TM$  to the associated bundle  $P \times_G V$  is a bundle isomorphism, one concludes  $\dim V = \dim M$ . We shall mainly deal locally over a suitable open set  $U \subset M$ , therefore assume,

$$(1.1) \quad g_{ab} = \gamma_a^{AA'} \gamma_{bAA'}.$$

Let  $A$  be a connection 1-form on  $P$ . Locally represented as  $A_a^{AB}$ , the corresponding curvature 2-form  $F$  is [cf. 2]

$$(1.2) \quad F_{ab}^{AB} = 2 (\partial_{[a} A_{b]}^{AB} + A_{[a}^{AC} A_{b]C}^B),$$

we use the physicist's convention  $[\dots]$  denotes antisymmetrization of the indices and  $(\dots)$  the symmetrization of the indices.

We define a self-dual 2-form in the following way [cf. 3], [cf. 2] :

$$(1.3) \quad \Sigma_{ab}^{AB} : 2\gamma_{[a}^{AA'} \gamma_{b]A'}^B,$$

then the vacuum Einstein equations with cosmological constant  $\lambda$

$$(1.4) \quad R_{ab} = \lambda g_{ab}$$

is written in the following form

$$(1.5) \quad \begin{aligned} D_{[a} \Sigma_{bc]}^{AB} &= 0, \\ \gamma_{[a}^{AA'} F_{bc]A}^B + \frac{1}{6} \lambda \gamma_{[a}^{AA'} \Sigma_{bc]A}^B &= 0, \end{aligned}$$

with  $D = d + A$ . One can show that (1.5) can be solved if the curvature 2-form satisfies the following equivalent condition [cf. 2]

$$(1.6) \quad \frac{1}{4} F_{[ab}^{AB} F_{cd]}^{CD} = 0,$$

and

$$(1.7) \quad F_{ab}^{AB} = -\frac{1}{6} \lambda \Sigma_{ab}^{AB}$$

where  $\lambda$  has dimension  $[\lambda] = (\text{length})^{-2}$ . We assume that the soldering form is nondegenerate and yields a metric (1.1) of Euclidean signature, then one can verify that (1.6) and (1.7) provide a solution of (1.5). Since the 2-form  $\Sigma$  is self-dual with respect to the metric (1.1) via the soldering form  $\gamma$ , therefore the corresponding  $SU(2)$  curvature forms are self-dual also; it can be shown that the solutions constructed in this manner have

self-dual Weyl tensor. We call (1.6), as the instanton equation, and it leads to conformally anti-self-dual Einstein spaces.

## 2. LINEARIZATION OF THE INSTANTON EQUATION.

We want to study the moduli space of solutions of the equation (1.7) about a given solution. Let  $A$  be a given solution and  $A + C$  be a perturbed solution with perturbation  $C$ . In terms of indices, the replacement

$$(2.1) \quad A_a{}^{AB} \mapsto A_a{}^{AB} + C_a{}^{AB}$$

to first order in  $C_a$ , (1.7) becomes

$$(2.2) \quad F_{[ab]}{}^{(AB} D_c C_{d]}^{CD)} = 0.$$

Explicitly, the equations (2.2) represent a collection of five first-order, linear partial differential equations for the perturbation and are independent of the metric  $g$ .

The idea is to interpret (2.2) as [cf. 2]

$$(2.3) \quad \mathbb{D}_1 C = F_{[ab]}{}^{(AB} D_c C_{d]}^{CD)} = 0$$

of a chain complex.

As there is a diffeomorphism group acting on the perturbation space,  $A + C$  may be diffeomorphically equivalent to the solution  $A$ , and we identify them as the single solution, the same applies to the action of the group  $SU(2)$  on the space of perturbations. These directions in the moduli space are *degenerate* and modulo these degeneracies, we choose two lie algebra-valued matrices  $N, M \in \mathfrak{su}(2)$ , a real-valued smooth function  $f$ , then any perturbation is of the form [cf. 2]

$$(2.4) \quad C_a = D_a N + (\nabla^b f) F_{ba} + [D^b M, F_{ba}].$$

The first term in (2.4) represents  $SU(2)$  transformations, the next two terms are associated with diffeomorphism covariance,  $[\cdot, \cdot]$  is the lie-bracket, these terms correspond to the action of an infinitesimal diffeomorphism; we can check by direct substitution that  $C_a$  is satisfied by equation (2.2) when (1.7) is satisfied.

We can interpret (2.4) as

$$(2.5) \quad C = \mathbb{D}_0(M, N, f)$$

of the same chain complex (2.3).

We now recall the notion of the principal symbol and symbol map [cf. 2]. It is obtained by replacing the highest-order partial derivatives with covectors, which we shall denote  $dx_a = k_a$ , and setting any lower-order terms to zero. The symbol of the operator  $\mathbb{D}_1$  in (2.2) is thus a linear map from the vector space of perturbations at a point to the vector-space of totally symmetric valence-four spinor-valued four-forms at the same point. If this map were injective for every choice of (nonvanishing)  $k_a$  then the corresponding differential operator would be elliptic)

The symbol map  $\xi(\mathbb{D}_1)$  is given by [cf. 2],

$$(2.6) \quad \xi(\mathbb{D}_1) \cdot C = F_{[ab]}{}^{(AB} k_c C_{d]}^{CD)}.$$

When interpreted in terms of symbols, the degenerate directions correspond to perturbations which makes  $\xi(\mathbb{D}_1) \cdot C = 0$ , this however means, the perturbed directions belong to the kernel of  $\xi(\mathbb{D}_1)$ . So, using (1.7), we get, [cf. 2]

$$(2.7) \quad C_a \in \ker \xi(\mathbb{D}_1) \iff \Sigma^{ab}{}^{AB} k_a C_b^{CD)} = 0.$$

One actually can show that the operator  $\mathbb{D}_1$  is not elliptic, it is due to the existence of the gauge transformation (2.4). One can argue in the following way: one chooses a 3-dimensional foliation at  $k^a$  of  $T^*M$ , that is leaves are 3-dimensional. Choose a  $SU(2)$ -soldering 1-form  $\sigma_a^{AB}$  then (2.7) gives [cf. 2]

$$(2.8) \quad \sigma^a{}^{(AB} C_a^{CD)} = 0.$$

Since  $\sigma_a^{AB}$  is a  $SU(2)$ -soldering 1-form for a 3-dimensional space inside  $T^*M$ , one readily deduce that,

$$(2.9) \quad C_a = Nk_a$$

satisfies (2.7), with  $N \in \mathfrak{su}(2)$ . Now one counts remaining independent solutions of (2.7), they are orthogonal to  $k_a$ ; more explicitly, if we assume [cf. 2]

$$(2.10) \quad C_a \perp k_a$$

and remove the symmetrization bracket of the spinor indices in (2.8), then what we get is the isomorphism between  $\mathfrak{su}(2)$ -valued three-dimensional covectors into  $\mathfrak{su}(2)$ -valued  $SU(2)$ -spinors. Thus these degeneracies come from the relationship between multitude of traces on spinor indices that gets lost when the indices are symmetrized; in particular, at a given point, when  $C_a \perp k_a$  holds, then the perturbation has nine independent components while the symmetrized product in (2.8) contains only five independent components. The remaining four componenets satisfy [cf. 2]

$$(2.11) \quad C_a = fk^b F_{ba}$$

and

$$(2.12) \quad C_a = k^b [M, F_{ba}],$$

where  $f$  is real-valued smooth function and  $M \in \mathfrak{su}(2)$ . Keeping in mind (2.9), (2.10), (2.11), and (2.12) the degenerate directions for  $\xi(\mathbb{D}_1)$  correspond to the symbols of the differential operators that correspond to (2.4). We conclude that the existence of the gauge transformation (2.4) destroys the injectivity of  $\xi(\mathbb{D}_1)$  to the vector-space of totally symmetric valence-four spinor-valued four-forms, therefore the operator  $\mathbb{D}_1$  is not elliptic [cf. 2].

### 3. CONSTRUCTION OF THE MODULI SPACES OF SOLUTIONS

The solutions to the linearized instanton equation are elements of the kernel of a linear map  $\mathbb{D}_1$  which transforms  $\mathfrak{su}(2)$ -valued one-forms into to the vector-space of totally symmetric valence-four spinor-valued four-forms, namely [cf. 2]

$$(3.1) \quad \mathbb{D}_1 : \wedge^1 \rightarrow \wedge^4$$

with

$$(3.2) \quad \mathbb{D}_1 C = F_{[ab}^{(AB} D_c C_{d]}^{CD)}.$$

Physically trivial perturbations are generated by another linear differential operator  $\mathbb{D}_0$ ; the domain of  $\mathbb{D}_0$  is a tensor product of vector spaces consisting of real-valued and  $\mathfrak{su}(2)$ -valued functions, while the range of  $\mathbb{D}_0$  consists of  $\mathfrak{su}(2)$ -valued one-forms [cf. 2]:

$$(3.3) \quad \mathbb{D}_0 : \wedge^0 \otimes \wedge^0 \otimes \wedge^0 \rightarrow \wedge^1,$$

with

$$(3.4) \quad \mathbb{D}_0(M, N, f) = C.$$

One therefore conclude that physically relevant perturbations are the equivalence classes  $[C]$  corresponding to the kernel of  $\mathbb{D}_1$  modulo the image of  $\mathbb{D}_0$  [cf. 2], therefore one has the following cohomology space  $H_{\mathbb{D}}^1$  with  $[C]$  as its member

$$(3.5) \quad H_{\mathbb{D}}^1 = \ker \mathbb{D}_1 / \text{Im } \mathbb{D}_0.$$

Since  $M$  is compact then  $H_{\mathbb{D}}^1$  is finite-dimensional.

One can rephrase the above direction in terms of the following chain complex

$$(3.6) \quad 0 \rightarrow \wedge^0 \otimes \wedge^0 \otimes \wedge^0 \xrightarrow{\mathbb{D}_0} \wedge^1 \xrightarrow{\mathbb{D}_1} \wedge^4$$

such that

$$(3.7) \quad \mathbb{D}_1 \mathbb{D}_0 = 0.$$

One notes

$$(3.8) \quad \ker \xi(\mathbb{D}_1) \equiv \text{Im } \xi(\mathbb{D}_0),$$

this however means that (3.6) is elliptic ( $\dim \Gamma(M, \wedge^0 \otimes \wedge^0 \otimes \wedge^0) + \dim \Gamma(M, \wedge^4) = \dim \Gamma(M, \wedge^1)$ ).

Let us now introduce an inner product to the fibres of the bundle  $\wedge^4 \rightarrow M$ , choose two sections  $C, \bar{C} \in \Gamma(M, \wedge^1)$  [cf. 2]

$$(3.9) \quad \langle C, \bar{C} \rangle = \int_M \sqrt{g} g^{ab} C_a{}^{AB} \bar{C}_b{}_{AB}.$$

The adjoint of  $\mathbb{D}$  is defined in the usual manner

$$(3.10) \quad \langle w, \mathbb{D}v \rangle = \langle \mathbb{D}^*w, v \rangle.$$

More explicitly [cf. 2]

$$(3.11) \quad \begin{aligned} \mathbb{D}_0^* : \wedge^1 &\rightarrow \wedge^0 \otimes \wedge^0 \otimes \wedge^0, \\ \mathbb{D}_0^* C &= (\text{Tr } F^{ab} D_a C_b; [D_a C_b, F^{ab}]; -D^a C_a), \\ \mathbb{D}_1^* : \wedge^4 &\rightarrow \wedge^1, \mathbb{D}_1^* \omega = F_{CD}^b D^b \omega_{abcd}^{ABCD} \end{aligned}$$

The covariant derivative also acts on tensors and can be assumed to be torsion free and is compatible with respect to the metric obtained from equation (1.6). One therefore can construct corresponding "Laplacians" that act on the sections of the bundles in question [cf. 2]

$$(3.12) \quad \begin{aligned} \Delta_0 : \wedge^0 \otimes \wedge^0 \otimes \wedge^0 &\rightarrow \wedge^0 \otimes \wedge^0 \otimes \wedge^0, \quad \Delta_0 := \mathbb{D}_0^* \mathbb{D}_0; \\ \Delta_1 : \wedge^1 &\rightarrow \wedge^1, \quad \Delta_1 := \mathbb{D}_1^* \mathbb{D}_1 + \mathbb{D}_0 \mathbb{D}_0^*; \\ \Delta_2 : \wedge^4 &\rightarrow \wedge^4, \quad \Delta_2 := \mathbb{D}_1 \mathbb{D}_1^*. \end{aligned}$$

According to the Fredholm alternative, these Laplacians have an orthogonal decomposition of the space of sections of each vector bundle, therefore [cf. 2]

$$(3.13) \quad \begin{aligned} \wedge^0 \otimes \wedge^0 \otimes \wedge^0 &\cong \text{range}(\mathbb{D}_0^*) \oplus \ker \Delta_0, \\ \wedge^1 &\cong \text{range}(\mathbb{D}_0) \oplus \text{range}(\mathbb{D}_1^*) \oplus \ker \Delta_1, \\ \wedge^4 &\cong \text{range}(\mathbb{D}_1) \oplus \ker \Delta_2 \end{aligned}$$

where the orthogonality of the decomposition is with respect to the inner products described above. One can show, as in the case of de Rham cohomology, that the equivalence class  $[C]$ , defined in (3.5), can be identified with the kernel of the Laplacian (3.12) on Lie-algebra valued one-forms [cf. 2]:

$$(3.14) \quad \ker \mathbb{D}_1 / \text{Im } \mathbb{D}_0 \cong \ker \Delta_1.$$

One uses the definition of the Laplacians as in (3.12), and verifies explicitly that the perturbations are elements of kernel of  $\Delta_1$  if and only if

$$(3.15) \quad \mathbb{D}_1 C = 0 = \mathbb{D}_0^* C.$$

these equations can be viewed as a combination of the linearized instanton equation and "gauge fixing" conditions [cf. 2].

Now the linearized instanton equation (2.2) is equivalent to the following equation [cf. 2]

$$(3.16) \quad \mathbb{D}_1^* \mathbb{D}_1 C = 0,$$

this implies,

$$(3.17) \quad D^b (F_{abAB} F^{cd(AB} D_c C_d{}^{CD)}) = 0.$$

From (3.15) we also get,

$$(3.18) \quad \begin{aligned} \text{Tr } F^{ab} D_a C_b &= 0, \\ [F^{ab}, D_a C_b] &= 0, \\ D^a C_a &= 0. \end{aligned}$$

First two equations of (3.18) implies [cf. 2]

$$(3.19) \quad F^{abAB} D_a C_b{}^{CD} = F^{ab(AB} D_a C_b{}^{CD)},$$

So, we remove the symmetrization in (3.17).

Since [cf. 2]

$$(3.20) \quad \Sigma_{ab}{}^{AB} \Sigma_{AB}{}^{cd} = 4(\delta_a^{[c} \delta_b{}^{d]} + \frac{1}{2} \epsilon_{ab}^{cd}).$$

we can replace (3.17) by

$$(3.21) \quad D^b [(\delta_a{}^{[c} \delta_b{}^{d]} + \frac{1}{2} \epsilon_{ab}^{cd}) D_c C_d] = 0.$$

One can expand the equation (3.21) and finds terms involving a gauge-co variant Laplacian, the Ricci tensor, the Riemann tensor, the  $SU(2)$  curvature, and the gradient of a divergence; the latter three of these can be made to vanish by using the cyclic identity for the Riemann tensor, the self-duality of the  $SU(2)$  curvature, and the last equation of (3.18), respectively [cf. 2]. One therefore can deduce that the equation (3.15) implies

$$(3.22) \quad -D^a D_a C_b + R_b{}^a C_a = 0,$$

where  $R_b{}^a$  is the Ricci tensor. We know, if  $M$  is Einstein, then

$$(3.23) \quad R_b{}^a = \lambda \delta_a{}^b.$$

One uses (3.22) and (3.23) to get

$$(3.24) \quad (-D^a D_a + \lambda) C_b = 0.$$

Equation (3.24) is an elliptic second-order differential equation [cf. 2] and, can be shown using the above arguments that the kernel of an elliptic operator on a compact manifold, as a vector space over the reals, is finite-dimensional, the physical instanton perturbations form a finite-dimensional subspace of all possible gravitational perturbations [cf. 2].

#### 4. THE GRAVITATIONAL MONOPOLE EQUATION

Let  $(X^4, g)$  be a Riemannian spin 4-manifold. Then the Clifford algebra bundle  $\mathcal{Cl}(X^4)$  is a vector bundle over  $X^4$  with fibre at  $x \in X^4$  is the Clifford algebra  $\mathcal{Cl}(T_x X)$ . With respect to the metric  $g$ , one identifies (isomorphism)  $\mathcal{Cl}(T_x X)$  with  $\mathcal{Cl}(T_x^* X)$ . Therefore, as a vector space, this is isomorphic to  $\wedge T_x^* X$ . Let us also assume  $E \rightarrow X$  is a Clifford module bundle with a covariant derivative  $\nabla^E$ . Then for each  $x \in X$  there is a Clifford action  $c : T_x^* X \otimes E_x \rightarrow E_x$  via  $c(\alpha \otimes s) = c(\alpha)s$ .

**Definition 4.1.** The twisted Dirac operator associated to  $(E, \nabla^E)$  is the operator,

$$(4.1) \quad \not{\nabla} := c \circ \nabla^E : C^\infty(X, E) \rightarrow C^\infty(X, E).$$

The equations we wish to consider are (sometimes we omit the mapping  $c$  and the dimension 4 for the convenience of computations),

$$(4.2) \quad \begin{aligned} \not{\nabla} \psi &= (d + d^*) \psi = 0, \\ c(W_g^+) &= \frac{1}{4} \langle e_i \cdot e_j \psi, \psi \rangle e^i \wedge e^j, \\ \text{or, } c((W_g^+)_{ijkl}) e^i \wedge e^j &= \frac{1}{4} \langle e_k \cdot e_l \psi, \psi \rangle. \end{aligned}$$

The Weitzenböck's formula [4]: the decomposition of the Laplace-Beltrami operator as a generalized Laplacian is, with the left-Clifford multiplication  $\epsilon_l = ext_l - int_l$ , and corresponding right Clifford action  $\epsilon_r$ ,

$$(4.3) \quad (d + d^*)^2 = \Delta^{\wedge T^* X} - \sum_{ijkl} R_{ijkl} \epsilon_l^i \epsilon_l^j \epsilon_r^k \epsilon_r^l + \frac{1}{4} s.$$

We keep in mind the Seiberg-Witten analysis, and analogously define and get the following definition of *Gravitational-Monopole functional*,

**Definition 4.2.** The Gravitational-Monopole functional of a pair  $(\psi, g)$  is given by,

$$(4.4) \quad \begin{aligned} S(g, \psi) &= \int_X \left( |\nabla \psi|^2 + |W^+ - \frac{1}{4} \langle e_i \cdot e_j \psi, \psi \rangle e^i \wedge e^j|^2 \right) d(vol)_g \\ \text{i.e., } S(g, \psi) &= \int_X \left( |(d + d^*)\psi|^2 + |W^+ - \frac{1}{4} \langle e_i \cdot e_j \psi, \psi \rangle e^i \wedge e^j|^2 \right) d(vol)_g. \end{aligned}$$

**Proposition 1.**

$$(4.5) \quad S(g, \psi) = \int_X \left( |\nabla \psi|^2 + |W^+|^2 + \frac{s}{4} |\psi|^2 + \frac{1}{8} |\psi|^4 \right) d(vol)_g$$

**Proposition 2.** As a direct consequence of (1), if the scalar curvature of  $X$  is non-negative, all solutions of (4.2) have  $\psi \equiv 0$ .

## 5. APPLICATION OF THE GRAVITATIONAL MONOPOLE CONSTRUCTION

As we saw in the previous section (4) Gravitational monopole equation implies scalar curvature is either zero or negative, and since the scalar curvature  $R = 24\lambda$ ,  $R \leq 0 \implies \lambda \leq 0$ . Now,  $\lambda = 0$  is ruled out as one calculates the volume of the space is determined by  $\lambda$  via the following equation [cf. 3]

$$(5.1) \quad V = 8\pi^2 c_1 / \lambda, \quad c_1 = \frac{1}{16\pi^2} \text{Tr} \int F \wedge F,$$

therefore  $\lambda = 0$  correspond to infinite action [cf. 3], so we necessarily have  $\lambda < 0$ .

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