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► **To cite this version:**

Herold G. Dehling, Davide Giraudo, Sara K. Schmidt. U-statistics of local sample moments under weak dependence. 2023. hal-03995213v2

**HAL Id: hal-03995213**

**<https://cnrs.hal.science/hal-03995213v2>**

Preprint submitted on 8 Sep 2023

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September 8, 2023

# U-STATISTICS OF LOCAL SAMPLE MOMENTS UNDER WEAK DEPENDENCE

HEROLD DEHLING, DAVIDE GIRAUDO, AND SARA K. SCHMIDT

ABSTRACT. In this paper, we study the asymptotic distribution of some U-statistics whose entries are functions of empirical moments computed from non-overlapping consecutive blocks of an underlying weakly dependent process. The length of these blocks converges to infinity, and thus we consider U-statistics of triangular arrays. We establish asymptotic normality of such U-statistics. The results can be used to construct tests for changes of higher order moments.

## 1. INTRODUCTION

Given some real-valued data  $Y_1, \dots, Y_n$  and a symmetric measurable function  $h : \mathbb{R}^m \rightarrow \mathbb{R}$ , we define the U-statistic with kernel  $h$  as

$$U_n := U_n(h) = \frac{1}{\binom{n}{m}} \sum_{1 \leq i_1 < i_2 < \dots < i_m \leq n} h(Y_{i_1}, \dots, Y_{i_m}).$$

U-statistics play an important role in nonparametric statistics, as many sample statistics can be expressed in this way, at least asymptotically. Well-known examples include the sample variance, Gini's mean difference, the Cramér-von Mises test statistic and the  $\chi^2$ -test statistic for goodness of fit. For details and further examples see e.g. [Serfling \[1980\]](#) and [Dehling \[2006\]](#). U-statistics have been introduced independently by [Halmos \[1946\]](#) and [Hoeffding \[1948\]](#). [Halmos \[1946\]](#) showed that for i.i.d. data,  $U_n(h)$  is an unbiased estimator of the parameter  $\theta = \mathbb{E}[h(Y_1, \dots, Y_m)]$ , and that it is minimum variance unbiased in nonparametric models. [Hoeffding \[1948\]](#) proved that, again for i.i.d. data and for square integrable kernels, the U-statistic is asymptotically normal. More precisely,

$$\sqrt{n} (U_n(h) - \mathbb{E}[h(Y_1, \dots, Y_m)]) \longrightarrow N(0, 4\gamma_h^2)$$

in distribution, where  $\gamma_h^2 := \text{Var}(\mathbb{E}[h(Y_1, \dots, Y_m)|Y_2, \dots, Y_m])$ . In the so-called degenerate case  $\gamma_h^2 = 0$ , a different normalization is required to get a non-trivial limit, which will be a non-normal distribution; see [Serfling \[1980\]](#) for further details. Most of the results for i.i.d. data can be extended to weakly dependent stationary processes  $(Y_i)_{i \geq 1}$ ; see e.g. [Dehling \[2006\]](#) for a survey.

In this paper, we study the asymptotic distribution of certain U-statistics whose entries are local summary statistics of an underlying weakly dependent process  $(X_t)_{t \in \mathbb{Z}}$ . More precisely, we consider local statistics  $g\left(\frac{1}{\ell_n} \sum_{t \in B_{n,j}} X_t, \frac{1}{\ell_n} \sum_{t \in B_{n,j}} X_t^2, \dots, \frac{1}{\ell_n} \sum_{t \in B_{n,j}} X_t^m\right)$ ,  $1 \leq j \leq b_n$ , which can be expressed as a function of the first  $m$  empirical moments of the consecutive non-overlapping blocks

$$(1.1) \quad B_{n,j} := \{(j-1)\ell_n + 1, \dots, j\ell_n\}$$

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*Key words and phrases.* U-Statistic, Triangular array, Central limit theorem.

Supported in part by the Collaborative Research centre 823, *Statistical Modelling of Nonlinear Dynamic Processes*, of the German Research Foundation. S. K. Schmidt was additionally supported by the Friedrich-Ebert-Stiftung.

for  $1 \leq j \leq b_n$ . We assume the block length  $\ell_n$  to converge to infinity. Given an appropriate scaling factor  $\sqrt{\ell_n}$  and certain regularity assumptions on  $g: \mathbb{R}^m \rightarrow \mathbb{R}$ , we will show that the statistics

$$(1.2) \quad W_{n,j} := \sqrt{\ell_n} g \left( \frac{1}{\ell_n} \sum_{t \in B_{n,j}} X_t, \frac{1}{\ell_n} \sum_{t \in B_{n,j}} X_t^2, \dots, \frac{1}{\ell_n} \sum_{t \in B_{n,j}} X_t^m \right)$$

are each asymptotically normal. We are then interested in U-statistics of the type

$$(1.3) \quad U_n := \frac{1}{b_n(b_n - 1)} \sum_{1 \leq j \neq k \leq b_n} h(W_{n,j}, W_{n,k}).$$

Such U-statistics arise naturally in nonparametric tests for the constancy of parameters of the underlying process  $(X_t)_{t \in \mathbb{Z}}$ . [Schmidt, Wornowizki, Fried, and Dehling \[2021\]](#) test for the constancy of the variance by analysing Gini's mean difference of the logarithmic local sample variances, i.e. they choose  $h(x, y) = |x - y|$  and  $W_{n,j} = \sqrt{\ell_n} \log(\frac{1}{\ell_n} \sum_{t=(j-1)\ell_n+1}^{j\ell_n} (X_t - \frac{1}{\ell_n} \sum_{r=(j-1)\ell_n+1}^{j\ell_n} X_r)^2)$ . [Schmidt \[2021\]](#) tests for changes in the mean by considering Gini's mean difference of the local sample means  $W_{n,j} = \frac{1}{\sqrt{\ell_n}} \sum_{t \in B_{n,j}} X_t$ . In both works, the behaviour of the test statistic under the hypothesis is determined by deriving a central limit theorem for  $U_n$ . The setup considered in the present paper allows for testing for constancy of higher order characteristics of the distribution of  $X_t$ , such as the skewness or kurtosis; see [Example 2.1](#) below.

Note that the entries of  $U_n$  from [\(1.3\)](#) stem from a triangular array  $(W_{n,j})_{1 \leq j \leq b_n, n \geq 1}$  and that under certain regularity assumptions made in this paper, they each converge to a normal law as  $\ell_n \rightarrow \infty$ . Moreover, assuming the number  $b_n$  of blocks to converge to infinity as well, there additionally holds (under appropriate assumptions) a central limit theorem for the U-statistic itself. The limit distribution of  $U_n$  is hence determined by the double asymptotics of the U-statistic and its entries. It is the goal of this paper to investigate more systematically such structures and to find minimal conditions that guarantee asymptotic normality of the resulting U-statistics of type [\(1.3\)](#).

## 2. MAIN RESULTS

We are interested in U-statistics of triangular arrays of the form [\(1.3\)](#), where  $h: \mathbb{R}^2 \rightarrow \mathbb{R}$  denotes a symmetric kernel function. We will henceforth always assume that the kernel fulfils

$$(2.1) \quad |h(x, y)| \leq C(1 + |x| + |y|)$$

for all  $x, y \in \mathbb{R}$  and some constant  $C$ . For the results under dependence later on, we will require the stronger assumption of Lipschitz-continuity.

Our first result is a central limit theorem for  $U_n$ , given the triangular array  $(W_{n,j})_{1 \leq j \leq b_n, n \geq 1}$  is row-wise i.i.d. with a very mild assumption on the distribution of the random variables  $W_{n,j}$ .

**Theorem 1.** *Let  $(W_{n,j})_{1 \leq j \leq b_n, n \geq 1}$  be a row-wise i.i.d. triangular array such that  $b_n \rightarrow \infty$  as  $n \rightarrow \infty$ . Assume that [\(2.1\)](#) holds, that  $\gamma_n^2 := \text{Cov}(h(W_{n,1}, W_{n,2}), h(W_{n,2}, W_{n,3})) > 0$ , and that the sequence  $(W_{n,1}^2/\gamma_n^2)_{n \geq 1}$  is uniformly integrable. Then the following convergence in distribution holds*

$$(2.2) \quad \frac{\sqrt{b_n}}{\gamma_n} \left( \frac{1}{b_n(b_n - 1)} \sum_{1 \leq j \neq k \leq b_n} h(W_{n,j}, W_{n,k}) - \mathbb{E}[h(W_{n,1}, W_{n,2})] \right) \rightarrow N(0, 4).$$

A related result was obtained in Löwe and Terveer [2021] for incomplete U-statistics of independent data.

The above theorem lays the groundwork for the more specific problems we investigate in this paper. As opposed to Theorem 1, the triangular array  $(W_{n,j})_{1 \leq j \leq b_n, n \geq 1}$  we consider from now on is in general not row-wise independent as the underlying process  $(X_t)_{t \in \mathbb{Z}}$  is weakly dependent. More specifically, we assume the stationary sequence  $(X_t)_{t \in \mathbb{Z}}$  to be expressible as a functional of an i.i.d. process. Thus, we can write  $X_t := f((\varepsilon_{t-u})_{u \in \mathbb{Z}})$ , where  $f: \mathbb{R}^{\mathbb{Z}} \rightarrow \mathbb{R}$  is measurable and  $(\varepsilon_u)_{u \in \mathbb{Z}}$  is i.i.d. In order to quantify the dependence, let  $(\varepsilon'_u)_{u \in \mathbb{Z}}$  be an independent copy of  $(\varepsilon_u)_{u \in \mathbb{Z}}$  and define

$$(2.3) \quad \delta_i((X_t)_{t \in \mathbb{Z}}) := \|X_0 - X_0^{*,i}\|_2,$$

where  $X_0^{*,i} = f((\varepsilon_{-u}^{*,i})_{u \in \mathbb{Z}})$  and  $\varepsilon_v^{*,i} = \varepsilon'_i$  if  $v = i$  and  $\varepsilon_v^{*,i} = \varepsilon_v$  otherwise. We thus measure the contribution of  $\varepsilon_i$  to  $X_0$  by looking at the difference between  $X_0$  and a coupled version  $X_0^{*,i}$  for which  $\varepsilon_i$  is replaced by an independent copy. This weak dependence concept was introduced by Wu [2005] under the term *physical dependence measure* and is now frequently used in statistical applications (see, e.g., El Machkouri [2014], Liu, Xiao, and Wu [2013], Wu [2008] and Wu [2011]).

In the following, the triangular array  $(W_{n,j})_{1 \leq j \leq b_n, n \geq 1}$  is assumed to be of the form (1.2). Example 2.1 presents some problems that are covered by this structure.

**Example 2.1.** (1) *Schmidt et al. [2021] propose a test for constancy of the variance based on the test statistic*

$$U_n = \frac{1}{b_n(b_n - 1)} \sum_{1 \leq j, k \leq b_n} \sqrt{\ell_n} |\log s_{n,j}^2 - \log s_{n,k}^2|,$$

where  $s_{n,j}^2 := \sum_{t \in B_{n,j}} (X_t - \frac{1}{\ell_n} \sum_{r \in B_{n,j}} X_r)^2$ . In our setting, this corresponds to  $m = 2$ ,  $g(x_1, x_2) = \log(x_2 - x_1^2)$  and  $h(x, y) = |x - y|$ .

(2) *Considering higher moments, one can construct a test for the constancy of the skewness or the kurtosis in a similar fashion to (1) by considering Gini's mean difference (that is,  $h(x, y) = |x - y|$ ) of the blockwise estimates  $\hat{\gamma}_{n,j}$ ,  $j = 1, \dots, b_n$ , or  $\hat{\kappa}_{n,j}$ ,  $j = 1, \dots, b_n$ , respectively. Note that an empirical version of the skewness is given by*

$$\begin{aligned} \hat{\gamma}_{n,j} &= \frac{\frac{1}{\ell_n} \sum_{t \in B_{n,j}} (X_t - \frac{1}{\ell_n} \sum_{r \in B_{n,j}} X_r)^3}{\left(\frac{1}{\ell_n} \sum_{t \in B_{n,j}} (X_t - \frac{1}{\ell_n} \sum_{r \in B_{n,j}} X_r)^2\right)^{3/2}} \\ &= \frac{\frac{1}{\ell_n} \sum_{t \in B_{n,j}} X_t^3 - 3\left(\frac{1}{\ell_n} \sum_{t \in B_{n,j}} X_t^2\right)\left(\frac{1}{\ell_n} \sum_{t \in B_{n,j}} X_t\right) + 2\left(\frac{1}{\ell_n} \sum_{t \in B_{n,j}} X_t\right)^3}{\left(\frac{1}{\ell_n} \sum_{t \in B_{n,j}} X_t^2 - \left(\frac{1}{\ell_n} \sum_{t \in B_{n,j}} X_t\right)^2\right)^{3/2}}, \end{aligned}$$

which is covered in our setting via the function

$$g(x_1, x_2, x_3) = \frac{x_3 - 3x_1x_2 + 2x_1^3}{(x_2 - x_1^2)^{3/2}}.$$

An empirical version of the kurtosis is given by

$$\hat{\kappa}_{n,j} = \frac{\frac{1}{\ell_n} \sum_{t \in B_{n,j}} (X_t - \frac{1}{\ell_n} \sum_{r \in B_{n,j}} X_r)^4}{\left(\frac{1}{\ell_n} \sum_{t \in B_{n,j}} (X_t - \frac{1}{\ell_n} \sum_{r \in B_{n,j}} X_r)^2\right)^2}$$

$$= \left( \frac{1}{\ell_n} \sum_{t \in B_{n,j}} X_t^2 - \left( \frac{1}{\ell_n} \sum_{t \in B_{n,j}} X_t \right)^2 \right)^{-2} \cdot \left( \frac{1}{\ell_n} \sum_{t \in B_{n,j}} X_t^4 - 4 \left( \frac{1}{\ell_n} \sum_{t \in B_{n,j}} X_t \right) \left( \frac{1}{\ell_n} \sum_{t \in B_{n,j}} X_t^3 \right) \right. \\ \left. + 6 \left( \frac{1}{\ell_n} \sum_{t \in B_{n,j}} X_t \right)^2 \left( \frac{1}{\ell_n} \sum_{t \in B_{n,j}} X_t^2 \right) - 3 \left( \frac{1}{\ell_n} \sum_{t \in B_{n,j}} X_t \right)^4 \right),$$

which corresponds to the function

$$g(x_1, x_2, x_3, x_4) = \frac{x_4 - 4x_1x_3 + 6x_1^2x_2 - 3x_1^4}{(x_2 - x_1^2)^2}.$$

We now state a central limit theorem for U-statistics of this more concrete type of triangular array (1.2). The question of the central limit theorem for U-statistics whose entries are Bernoulli shifts has been addressed in [Hsing and Wu \[2004\]](#) and [Giraudo \[2021\]](#), but these results do not treat the case of arrays.

**Theorem 2.** *Assume that the following conditions are satisfied.*

- (1) *The function  $h$  is Lipschitz-continuous.*
- (2)  $\mathbb{E}[X_1^{2m}] < \infty$  and

$$(2.4) \quad \sum_{i \in \mathbb{Z}} \sum_{k=1}^m i^2 \delta_i \left( (X_t^k)_{t \in \mathbb{Z}} \right) < \infty.$$

- (3) *The function  $g$  satisfies  $g(v_0) = 0$ , where  $v_0 = (\mathbb{E}[X_1^k])_{k=1}^m \in \mathbb{R}^m$ . There exists an  $a > 0$  such that  $g$  is differentiable at each point of  $\prod_{k=1}^m (\mathbb{E}[X_1^k] - 2a, \mathbb{E}[X_1^k] + 2a)$  and the gradient of  $g$  is bounded on  $\prod_{k=1}^m (\mathbb{E}[X_1^k] - 2a, \mathbb{E}[X_1^k] + 2a)$ .*
- (4) *The sequences  $(b_n)_{n \geq 1}$  and  $(\ell_n)_{n \geq 1}$  go to infinity as  $n$  goes to infinity. Moreover,  $\lim_{n \rightarrow +\infty} b_n / \ell_n = 0$ .*

Let  $\eta: \mathbb{R}^m \rightarrow [0, 1]$  be a smooth function with  $\eta(x) = 1$  if  $\|x - v_0\|_2 \leq a$  and  $\eta(x) = 0$  if  $\|x - v_0\|_2 > 2a$ , and define

$$W_{n,j}^{(\eta)} := \sqrt{\ell_n} (g \cdot \eta) \left( \left( \frac{1}{\ell_n} \sum_{t \in B_{n,j}} X_t^k \right)_{k=1}^m \right).$$

If

$$(2.5) \quad \sigma^2 := \sum_{t \in \mathbb{Z}} \text{Cov} \left( \sum_{k=1}^m \frac{\partial g}{\partial x_k} (v_0) X_0^k, \sum_{k=1}^m \frac{\partial g}{\partial x_k} (v_0) X_t^k \right) > 0,$$

the following convergence in distribution holds

$$(2.6) \quad \sqrt{b_n} \left( U_n - \frac{1}{b_n(b_n - 1)} \sum_{1 \leq j \neq k \leq b_n} \mathbb{E} \left[ h \left( W_{n,j}^{(\eta)}, W_{n,k}^{(\eta)} \right) \right] \right) \rightarrow N(0, 4\gamma^2),$$

where

$$\gamma^2 := \text{Cov} \left( h(\sigma N, \sigma N'), h(\sigma N', \sigma N'') \right)$$

for independent standard normally distributed random variables  $N$ ,  $N'$  and  $N''$ .

**Remark 2.2.** Note that the partial derivatives in (2.5) arise as a consequence of the delta method.

In the examples below, we provide some classes of processes  $(X_t)_{t \in \mathbb{Z}}$  that fulfil the above condition (2.4). For more details, we refer to Section 3.4.

**Example 2.3.** Let  $X_t$  be a Hölder continuous function of a linear process as considered, for instance, in *Ho and Hsing [1997]* and *Wu [2006]*. More precisely, define  $X_t$  as

$$X_t := \varphi \left( \sum_{j \in \mathbb{Z}} a_j \varepsilon_{t-j} \right),$$

where  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  is  $\gamma$ -Hölder continuous for some  $\gamma \in (0, 1]$ ,  $(a_j)_{j \in \mathbb{Z}}$  is a sequence of real numbers such that  $\sum_{j \in \mathbb{Z}} j^2 |a_j|^\gamma < \infty$  and  $(\varepsilon_u)_{u \in \mathbb{Z}}$  is an i.i.d. sequence such that  $\mathbb{E}[|\varepsilon_0|^{2m\gamma}] < \infty$ . Then  $(X_t)_{t \in \mathbb{Z}}$  satisfies condition (2.4).

**Example 2.4.** Assume  $X_t$  can be written as a function of a Gaussian linear process: Let  $X_t = \varphi(Y_t)$  with

$$Y_t := \sum_{j \in \mathbb{Z}} a_j \varepsilon_{t-j},$$

where  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ ,  $(\varepsilon_u)_{u \in \mathbb{Z}}$  is an i.i.d. sequence,  $\varepsilon_0$  has a standard normal distribution and  $a_j \in \mathbb{R}$  for all  $j \in \mathbb{Z}$  with  $\sum_{j \in \mathbb{Z}} |a_j| < \infty$  as well as  $\sum_{j \in \mathbb{Z}} a_j^2 = 1$ . Such processes were considered, e.g., in *Nualart [2009]*. Given that  $\varphi(Y_0) \in \mathbb{L}^2$  and  $\mathbb{E}[\varphi(Y_0)] = 0$ , the following expansion holds:

$$\varphi(Y_t) = \sum_{q=1}^{\infty} c_q(\varphi) H_q(Y_t),$$

where the  $q$ -th Hermite polynomial is defined by

$$H_q(x) := (-1)^q \exp\left(\frac{x^2}{2}\right) \frac{d^q}{dx^q} \exp\left(-\frac{x^2}{2}\right)$$

and

$$c_q(\varphi) := \frac{1}{q!} \mathbb{E}[\varphi(Y_0) H_q(Y_0)],$$

provided that  $\sum_{q=1}^{\infty} q! c_q(\varphi)^2$  converges. Then condition (2.4) is met if  $\sum_{j \in \mathbb{Z}} j^2 |a_j| < \infty$  and

$$\sum_{k=1}^m \sum_{q=1}^{\infty} \sqrt{q \cdot q!} |c_q(\varphi^k - \mathbb{E}[\varphi^k(Y_0)])| < \infty.$$

**Example 2.5.** Let  $(X_t)_{t \in \mathbb{Z}}$  be a Volterra process, i.e. let

$$X_t := \sum_{j, j' \in \mathbb{Z}, j \neq j'} a_{j, j'} \varepsilon_{t-j} \varepsilon_{t-j'},$$

where  $(\varepsilon_u)_{u \in \mathbb{Z}}$  is i.i.d. centred,  $\mathbb{E}[\varepsilon_0^2] < \infty$  and  $\sum_{j, j' \in \mathbb{Z}, j \neq j'} a_{j, j'}^2 < \infty$ . Such processes and extensions to Volterra series are considered in *Rugh [1981]* and *Priestley [1988]*. If  $\mathbb{E}[\varepsilon_0^{2m}] < \infty$  as well as

$$\sum_{j \in \mathbb{Z}} j^2 \sqrt{\sum_{j' \in \mathbb{Z}, j' \neq j} (a_{j, j'}^2 + a_{j', j}^2)} < \infty,$$

then  $(X_t)_{t \in \mathbb{Z}}$  satisfies condition (2.4).

It would be more natural to centre  $U_n$  in (2.6) by  $\sum_{1 \leq j \neq k \leq b_n} \mathbb{E}[h(W_{n,j}, W_{n,k})]$  rather than by the truncated version  $\sum_{1 \leq j \neq k \leq b_n} \mathbb{E}\left[h\left(W_{n,j}^{(n)}, W_{n,k}^{(n)}\right)\right]$ . However, the conditions of Theorem 2 do not guarantee that  $\mathbb{E}[|h(W_{n,j}, W_{n,k})|]$  exists, as the following example shows.

**Example 2.6.** Consider the case  $m = 2$ ,  $g(x_1, x_2) = \log x_2 \mathbf{1}_{x_2 > 0}$  and  $h(x, y) = |x - y|$  with i.i.d. observations  $(X_t)_{t \in \mathbb{Z}}$ . Since  $W_{n,1}$  and  $W_{n,2}$  are consequently likewise independent and identically distributed, finiteness of  $\mathbb{E}[|W_{n,1} - W_{n,2}|]$  is equivalent to the finiteness of  $\mathbb{E}[|W_{n,1}|]$ . Now, it suffices to find an i.i.d. sequence  $(X_t)_{t \in \mathbb{Z}}$  such that  $\mathbb{E}[X_1^4] < \infty$  and

$$\mathbb{E} \left[ \left| \log \left( \frac{1}{\ell_n} \sum_{t=1}^{\ell_n} X_t^2 \right) \right| \right] = \infty.$$

By choosing the distribution of  $X_1$  as

$$\mathbb{P}(X_1^2 = \exp(-\exp(\exp(k)))) = 2^{-k}$$

for  $k \geq 1$ , it follows

$$\begin{aligned} \left| \log \left( \frac{1}{\ell_n} \sum_{t=1}^{\ell_n} X_t^2 \right) \right| &\geq \sum_{k \geq 1} \left| \log \left( \frac{1}{\ell_n} \sum_{t=1}^{\ell_n} \exp(-\exp(\exp(k))) \right) \right| \mathbf{1}_{\cap_{t=1}^{\ell_n} \{X_t^2 = \exp(-\exp(\exp(k)))\}} \\ &= \sum_{k \geq 1} \exp(\exp(k)) \mathbf{1}_{\cap_{t=1}^{\ell_n} \{X_t^2 = \exp(-\exp(\exp(k)))\}}. \end{aligned}$$

Taking expectations and using independence leads to

$$\mathbb{E} \left[ \left| \log \left( \frac{1}{\ell_n} \sum_{t=1}^{\ell_n} X_t^2 \right) \right| \right] \geq \sum_{k \geq 1} \exp(\exp(k)) 2^{-k \ell_n} = \infty.$$

By imposing additional assumptions on the function  $g$ , the dependence coefficients as well as the sequences  $(b_n)_{n \geq 1}$  and  $(\ell_n)_{n \geq 1}$ , we are able to replace the centring term in (2.6) computed from the  $W_{n,j}^{(\eta)}$ 's by an expression that does not require truncation.

**Proposition 2.7.** Assume that the assumptions of Theorem 2 hold. If additionally all the second order partial derivatives of  $g$  at  $v_0$  exist and if there exists a  $\kappa \in (0, 1)$  such that  $\sum_{i \in \mathbb{Z}} \sum_{k=1}^m |i|^{\frac{1}{2} + \frac{1}{2\kappa}} \delta_i \left( (X_t^k)_{t \in \mathbb{Z}} \right) < \infty$  and  $b_n / \ell_n^{1-\kappa} \rightarrow 0$ , then the following convergence in distribution holds

$$\sqrt{b_n} (U_n - \mathbb{E}[h(Z_n, Z'_n)]) \rightarrow N(0, 4\gamma^2),$$

where

$$Z_n := \frac{1}{\sqrt{\ell_n}} \sum_{k=1}^m \frac{\partial g}{\partial x_k} (v_0) \sum_{t=1}^{\ell_n} (X_t^k - \mathbb{E}[X_1^k])$$

and  $Z'_n$  is an independent copy of  $Z_n$ .

**Remark 2.8.** The above proposition introduces a centring term which is easier to handle than the original one in Theorem 2. However, if we want to use  $U_n$  as a test statistic for testing for a constant value of the parameter  $g(\mathbb{E}[X_t], \mathbb{E}[X_t^2], \dots, \mathbb{E}[X_t^m])$ , we need to be able to explicitly calculate the centring term. This is achieved by the following corollary, where we show that, for the important example when  $h(x, y) = |x - y|$ , we can replace  $\mathbb{E}[|Z_n - Z'_n|]$  by  $\sigma \mathbb{E}[|N - N'|] = 2\sigma/\sqrt{\pi}$ . The remaining parameter  $\sigma$  can be estimated by standard procedures for estimating long-run variances.

**Corollary 2.9.** Suppose that the time series  $(X_t)_{t \in \mathbb{Z}}$  can be written as a one-sided Bernoulli shift, that is,  $X_t = f((\varepsilon_{t-u})_{u \geq 0})$ . We assume moreover that there exists a  $2 < p \leq 3$  such that

(1)  $\mathbb{E}[|X_t|^{p^m}] < \infty$  and

$$(2.7) \quad \sum_{i \geq 0} \sum_{k=1}^m (i^2 \delta_{i,p}((X_t^k)_{t \in \mathbb{Z}}) + i^{5/2} \delta_{i,2}((X_t^k)_{t \in \mathbb{Z}})) < \infty,$$

where  $\delta_{i,p}((X_t)_{t \in \mathbb{Z}}) := \|X_0 - X_0^{*,i}\|_p$ , and

$$(2) \quad \sqrt{b_n} \ell_n^{1-p/2} (\log \ell_n)^{p/2} + \sqrt{b_n/\ell_n} \rightarrow 0.$$

Denote

$$U_n := \frac{1}{b_n(b_n - 1)} \sum_{1 \leq j \neq k \leq b_n} |W_{n,j} - W_{n,k}|,$$

where  $W_{n,j}$  is defined as in (1.2). Then the following convergence in distribution holds

$$\sqrt{b_n} \left( U_n - \frac{2\sigma}{\sqrt{\pi}} \right) \rightarrow N(0, 4\gamma^2),$$

where  $\sigma$  and  $\gamma^2$  are defined as in Theorem 2.

### 3. PROOFS

**3.1. Proof of Theorem 1.** We use the Hoeffding decomposition of the kernel function  $h$  and define

$$\begin{aligned} \theta_n &:= \mathbb{E}[h(W_{n,1}, W_{n,2})], \\ h_{1,n}(x) &:= \mathbb{E}[h(x, W_{n,2})] - \theta_n, \\ h_{2,n}(x, y) &:= h(x, y) - h_{1,n}(x) - h_{1,n}(y) - \theta_n. \end{aligned}$$

At the level of the U-statistic, we then obtain

$$\begin{aligned} U(n) - \mathbb{E}[U(n)] &= \frac{1}{b_n(b_n - 1)} \sum_{1 \leq j \neq k \leq b_n} h(W_{n,j}, W_{n,k}) - \theta_n \\ &= \frac{2}{b_n} \sum_{j=1}^{b_n} h_{1,n}(W_{n,j}) + \frac{1}{b_n(b_n - 1)} \sum_{1 \leq j \neq k \leq b_n} h_{2,n}(W_{n,j}, W_{n,k}). \end{aligned}$$

In the following, we will prove the convergence in distribution

$$(3.1) \quad \frac{2}{\sqrt{b_n} \gamma_n} \sum_{j=1}^{b_n} h_{1,n}(W_{n,j}) \rightarrow N(0, 4),$$

and the convergence in probability

$$(3.2) \quad \frac{1}{b_n^{3/2} \gamma_n} \sum_{1 \leq j \neq k \leq b_n} h_{2,n}(W_{n,j}, W_{n,k}) \rightarrow 0,$$

where  $\gamma_n^2 = \text{Cov}(h(W_{n,1}, W_{n,2}), h(W_{n,2}, W_{n,3}))$ . The assertion then follows by an application of Slutsky's Lemma.

Starting with (3.1), we will apply Lindeberg's central limit theorem to the triangular array  $Y_{n,j} := h_{1,n}(W_{n,j}) / (\gamma_n \sqrt{b_n})$ . Note that by construction, the  $Y_{n,j}$ 's are identically distributed with  $\mathbb{E}[Y_{n,1}] = 0$ . Moreover, it holds  $\text{Var}(Y_{n,1}) = 1/b_n$  since by independence

$$\begin{aligned} \text{Var}(h_{1,n}(W_{n,1})) &= \text{Var}(\mathbb{E}[h(W_{n,1}, W_{n,2}) | W_{n,1}]) \\ &= \mathbb{E}[\mathbb{E}[h(W_{n,1}, W_{n,2}) | W_{n,1}]^2] - (\mathbb{E}[\mathbb{E}[h(W_{n,1}, W_{n,2}) | W_{n,1}]])^2 \\ &= \int_{\mathbb{R}} \mathbb{E}[h(x, W_{n,2}) h(x, W_{n,3})] d\mathbb{P}_{W_{n,1}}(x) - (\mathbb{E}[h(W_{n,1}, W_{n,2})])^2 \end{aligned}$$



$$\begin{aligned}
&= \mathbb{E} [h(W_{n,1}, W_{n,2}) \cdot h(W_{n,1}, W_{n,3})] - (\mathbb{E} [h(W_{n,1}, W_{n,2})])^2 \\
&= \text{Cov}(h(W_{n,1}, W_{n,2}), h(W_{n,2}, W_{n,3})) = \gamma_n^2.
\end{aligned}$$

It thus remains to verify the Lindeberg condition, that is, to show that for all  $\varepsilon > 0$ ,

$$\sum_{j=1}^{b_n} \mathbb{E} [Y_{n,j}^2 \mathbf{1}_{\{|Y_{n,j}| > \varepsilon\}}] \rightarrow 0.$$

Since the random variables  $Y_{n,j}$ ,  $1 \leq j \leq b_n$ , are identically distributed, Lindeberg's condition reduces to

$$\sum_{j=1}^{b_n} \mathbb{E} [Y_{n,j}^2 \mathbf{1}_{\{|Y_{n,j}| > \varepsilon\}}] = b_n \mathbb{E} [Y_{n,1}^2 \mathbf{1}_{\{|Y_{n,1}| > \varepsilon\}}] = \mathbb{E} \left[ \frac{h_{1,n}^2(W_{n,1})}{\gamma_n^2} \mathbf{1}_{\left\{ \left| \frac{h_{1,n}(W_{n,1})}{\gamma_n} \right| > \varepsilon \sqrt{b_n} \right\}} \right] \rightarrow 0.$$

Observe that by property (2.1) of the kernel function  $h$ , we have  $|h_{1,n}(x)| \leq 3C\mathbb{E}[|W_{n,1}|] + C|x|$  and hence  $h_{1,n}^2(x) \leq 18C^2\mathbb{E}[W_{n,1}^2] + 2C^2x^2$ , from which it follows that

$$\frac{h_{1,n}^2(W_{n,1})}{\gamma_n^2} \leq 18C^2 + 2C^2 \frac{W_{n,1}^2}{\gamma_n^2}.$$

Consequently, the uniform integrability of the sequence  $(h_{1,n}^2(W_{n,1})/\gamma_n^2)_{n \geq 1}$  follows from that of the sequence  $(W_{n,1}^2/\gamma_n^2)_{n \geq 1}$ , and thus Lindeberg's condition is met.

It remains to verify (3.2). Since  $\mathbb{E}[h_{2,n}(W_{n,j}, W_{n,k})h_{2,n}(W_{n,j'}, W_{n,k'})] = 0$  if  $\{j, k\} \neq \{j', k'\}$ , we obtain

$$(3.3) \quad \mathbb{E} \left[ \left( \frac{1}{b_n^{3/2} \gamma_n} \sum_{1 \leq j \neq k \leq b_n} h_{2,n}(W_{n,j}, W_{n,k}) \right)^2 \right] = \frac{2(b_n - 1)}{\gamma_n^2 b_n^2} \mathbb{E} [h_{2,n}^2(W_{n,1}, W_{n,2})].$$

By the properties of the kernel function  $h$  and the definition of  $h_{2,n}$ , there exists a constant  $C$  independent of  $n$ ,  $x$  and  $y$  such that  $|h_{2,n}(x, y)| \leq C(1 + |x| + |y| + \mathbb{E}[|W_{n,1}|])$  and hence  $\mathbb{E}[h_{2,n}^2(W_{n,1}, W_{n,2})] \leq C'\mathbb{E}[W_{n,1}^2]$  for a constant  $C'$  independent of  $n$ . Combining this with (3.3), we get that

$$\mathbb{E} \left[ \left( \frac{1}{b_n^{3/2} \gamma_n} \sum_{1 \leq j \neq k \leq b_n} h_{2,n}(W_{n,j}, W_{n,k}) \right)^2 \right] \leq \frac{2C}{b_n} \sup_{n \geq 1} \mathbb{E} \left[ \frac{W_{n,1}^2}{\gamma_n^2} \right]$$

and the uniform integrability of  $(W_{n,1}^2/\gamma_n^2)_{n \geq 1}$  guarantees the finiteness of the above supremum. This concludes the proof of (3.2) and that of Theorem 1.

**3.2. Sketch of proof for Theorem 2.** This section outlines the proof ideas for Theorem 2, while all details can be found in the next section. First, we reduce the problem to the case where in  $U_n$ , the term  $W_{n,j}$  is replaced by  $W_{n,j}^{(\eta)} = \sqrt{\ell_n}(g \cdot \eta) \left( \left( \frac{1}{\ell_n} \sum_{t \in B_{n,j}} X_t^k \right)_{k=1}^m \right)$ . We thus define

$$U_n^{(\eta)} := \frac{1}{b_n(b_n - 1)} \sum_{1 \leq j \neq k \leq b_n} h(W_{n,j}^{(\eta)}, W_{n,k}^{(\eta)}).$$

The next lemma shows that we can replace  $U_n$  in the central limit theorem by  $U_n^{(\eta)}$ .

**Lemma 3.1.** *Let the assumptions of Theorem 2 hold. Then  $\mathbb{P}(U_n \neq U_n^{(\eta)}) \rightarrow 0$ . In particular,  $(\sqrt{b_n}(U_n - U_n^{(\eta)}))_{n \geq 1}$  converges in probability to zero.*

It thus suffices to prove the convergence in distribution

$$(3.4) \quad \sqrt{b_n} \left( U_n^{(\eta)} - \frac{1}{b_n(b_n-1)} \sum_{1 \leq j \neq k \leq b_n} \mathbb{E} \left[ h \left( W_{n,j}^{(\eta)}, W_{n,k}^{(\eta)} \right) \right] \right) \rightarrow N(0, 4\gamma^2).$$

To do so, we use a second approximation step and replace the  $W_{n,j}^{(\eta)}$ 's by

$$W_{n,j}^{(M)} := \sqrt{\ell_n} (g \cdot \eta) \left( \left( \frac{1}{\ell_n} \sum_{t \in B_{n,j}} \mathbb{E} [X_t^k | \mathcal{F}_{t-M}^{t+M}] \right)_{k=1}^m \right),$$

where  $\mathcal{F}_M^N := \sigma(\varepsilon_u, M \leq u \leq N)$  for  $M, N \in \mathbb{Z}$  with  $M \leq N$ . The random variables  $X_t$  are thus replaced by random variables depending only on those  $\varepsilon_{t-u}$  with  $|t-u| \leq M$ . Note that this way, the entries of the U-statistic become almost independent (up to some small overlap in consecutive blocks). We define

$$U_n^{(M)} := \frac{1}{b_n(b_n-1)} \sum_{1 \leq j \neq k \leq b_n} h \left( W_{n,j}^{(M)}, W_{n,k}^{(M)} \right).$$

We can now decompose the expression on the left hand side of (3.4) for each fixed  $M \geq 1$  via

$$(3.5) \quad \sqrt{b_n} (U_n^{(M)} - \mathbb{E} [U_n^{(M)}]) + R_{n,M},$$

where the remainder term is given by the telescoping sum

$$R_{n,M} := \frac{1}{\sqrt{b_n}(b_n-1)} \sum_{N \geq M} \sum_{1 \leq j \neq k \leq b_n} \left( h \left( W_{n,j}^{(N+1)}, W_{n,k}^{(N+1)} \right) - \mathbb{E} \left[ h \left( W_{n,j}^{(N+1)}, W_{n,k}^{(N+1)} \right) \right] \right. \\ \left. - \left( h \left( W_{n,j}^{(N)}, W_{n,k}^{(N)} \right) - \mathbb{E} \left[ h \left( W_{n,j}^{(N)}, W_{n,k}^{(N)} \right) \right] \right) \right).$$

The following three lemmas show that the first term in (3.5) converges to the desired normal distribution, while the continuity of  $g \cdot \eta$  will guarantee that the remainder term  $R_{n,M}$  becomes asymptotically negligible.

**Lemma 3.2.** *Let the assumptions of Theorem 2 hold. Then, there exists an  $M_0 \in \mathbb{N}$  such that for all fixed  $M \geq M_0$ , the sequence  $\left( \sqrt{b_n} (U_n^{(M)} - \mathbb{E} [U_n^{(M)}]) \right)_{n \geq 1}$  converges in distribution to a centred normally distributed random variable with variance*

$$4\gamma_M^2 := 4 \operatorname{Cov} \left( h(\sigma_M N, \sigma_M N'), h(\sigma_M N, \sigma_M N'') \right),$$

where  $N, N'$  and  $N''$  are three independent standard normal random variables and

$$\sigma_M^2 := \sum_{t=-M-1}^{M+1} \operatorname{Cov} \left( \mathbb{E} \left[ \sum_{k=1}^m \frac{\partial g}{\partial x_k} (v_0) X_0^k \middle| \mathcal{F}_{-M}^M \right], \mathbb{E} \left[ \sum_{k=1}^m \frac{\partial g}{\partial x_k} (v_0) X_t^k \middle| \mathcal{F}_{t-M}^{t+M} \right] \right).$$

The central ingredient in the proof of Lemma 3.2 is Theorem 1. To meet its conditions, we approximate  $U_n^{(M)}$  by yet another U-statistic, which has independent entries (details are given in the next section).

**Lemma 3.3.** *Let the assumptions of Theorem 2 hold. Then the sequence  $(4\gamma_M^2)_{M \geq 1}$  converges to  $4\gamma^2 = 4 \operatorname{Cov} (h(\sigma N, \sigma N'), h(\sigma N', \sigma N''))$ .*

**Lemma 3.4.** *Let the assumptions of Theorem 2 hold. Then, for each  $\varepsilon > 0$ , it holds*

$$\lim_{M \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}(|R_{n,M}| > \varepsilon) = 0.$$

In the final step, we apply Theorem 4.2 of Billingsley [1968]. This theorem states that for stochastic processes  $(Y_{m,n})_{m,n \geq 1}$ ,  $(Y'_n)_{n \geq 1}$ ,  $(Z_m)_{m \geq 1}$ , and a random variable  $Z$ , satisfying

$$\begin{aligned} Y_{m,n} &\longrightarrow Z_m \text{ in distribution, as } n \rightarrow \infty, \text{ for each } m, \\ Z_m &\longrightarrow Z \text{ in distribution, as } m \rightarrow \infty, \\ \lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}(|Y'_n - Y_{m,n}| > \varepsilon) &= 0, \text{ for all } \varepsilon > 0, \end{aligned}$$

we may conclude that  $Y'_n \rightarrow Z$  in distribution. A combination of Lemmas 3.2, 3.3 and 3.4 thus yields the desired convergence in (3.4) and thus finishes the proof.

### 3.3. Proof details.

*Proof of Lemma 3.1.* We start by noting the following inclusions

$$\{U_n \neq U_n^{(\eta)}\} \subset \bigcup_{j=1}^{b_n} \{W_{n,j} \neq W_{n,j}^{(\eta)}\} \subset \bigcup_{j=1}^{b_n} \bigcup_{k=1}^m \left\{ \left| \left( \frac{1}{\ell_n} \sum_{t \in B_{n,j}} X_t^k \right) - \mathbb{E}[X_1^k] \right| > 2a/m \right\}.$$

Hence, using the fact that for each  $k$ , the random variables  $\left(\frac{1}{\ell_n} \sum_{t \in B_{n,j}} X_t^k - \mathbb{E}[X_1^k]\right)$ ,  $j = 1, \dots, b_n$ , have the same distribution, and by an application of Chebychev's inequality, we obtain

$$\mathbb{P}(U_n \neq U_n^{(\eta)}) \leq m^2 \frac{b_n}{4a^2 \ell_n} \sum_{k=1}^m \text{Var} \left( \frac{1}{\sqrt{\ell_n}} \sum_{t=1}^{\ell_n} X_t^k \right).$$

By Lemma A.1, it holds  $\text{Var} \left( \frac{1}{\sqrt{\ell_n}} \sum_{t=1}^{\ell_n} X_t^k \right) \leq \left( \sum_{i \in \mathbb{Z}} \delta_i((X_t^k)_{t \in \mathbb{Z}}) \right)^2$  and thus, we have  $\mathbb{P}(U_n \neq U_n^{(\eta)}) \leq C b_n / \ell_n$ , which converges to zero by assumption.  $\square$

*Proof of Lemma 3.2.* Let  $M \geq 1$  be fixed. We intend to approximate  $U_n^{(M)}$  once more to obtain a U-statistic  $\tilde{U}_n^{(M)}$  with independent entries. Thus, we define

$$\tilde{W}_{n,j}^{(M)} := \sqrt{\ell_n} (g \cdot \eta) \left( \left( \frac{1}{\ell_n} \sum_{t=(j-1)\ell_n+M+1}^{j\ell_n-M-1} \mathbb{E}[X_t^k | \mathcal{F}_{t-M}^{t+M}] \right)_{k=1}^m \right)$$

for  $j = 1, \dots, b_n$ , assuming that  $n$  is large enough such that  $2M \leq \ell_n$ . Due to the shortening of the blocks  $B_{n,j}$  by  $M$  observations on each side, the  $\tilde{W}_{n,j}^{(M)}$ 's are independent. Next, we define

$$\tilde{U}_n^{(M)} := \frac{1}{b_n (b_n - 1)} \sum_{1 \leq j \neq k \leq b_n} h \left( \tilde{W}_{n,j}^{(M)}, \tilde{W}_{n,k}^{(M)} \right).$$

Since the function  $g \cdot \eta$  is Lipschitz-continuous, it holds

$$\left| W_{n,j}^{(M)} - \tilde{W}_{n,j}^{(M)} \right| \leq C \sum_{k=1}^m \frac{1}{\sqrt{\ell_n}} \sum_{t=(j-1)\ell_n+1}^{(j-1)\ell_n+M} \mathbb{E} \left[ |X_t^k| \middle| \mathcal{F}_{t-M}^{t+M} \right] + C \sum_{k=1}^m \frac{1}{\sqrt{\ell_n}} \sum_{t=j\ell_n-M}^{j\ell_n} \mathbb{E} \left[ |X_t^k| \middle| \mathcal{F}_{t-M}^{t+M} \right].$$

Taking expectations, we obtain

$$\mathbb{E} \left[ \left| W_{n,j}^{(M)} - \tilde{W}_{n,j}^{(M)} \right| \right] \leq C \frac{M}{\sqrt{\ell_n}}$$

since  $\mathbb{E}[X_1^m] < \infty$  by assumption. Thus, for the U-statistic, we have

$$\sqrt{b_n} \mathbb{E} \left[ \left| (U_n^{(M)} - \mathbb{E}[U_n^{(M)}]) - (\tilde{U}_n^{(M)} - \mathbb{E}[\tilde{U}_n^{(M)}]) \right| \right]$$

$$\begin{aligned} &\leq 2 \frac{\sqrt{b_n}}{b_n(b_n-1)} \sum_{1 \leq j \neq k \leq b_n} \mathbb{E} \left[ \left| h \left( W_{n,j}^{(M)}, W_{n,k}^{(M)} \right) - h \left( \widetilde{W}_{n,j}^{(M)}, \widetilde{W}_{n,k}^{(M)} \right) \right| \right] \\ &\leq C \sqrt{b_n} \mathbb{E} \left[ \left| W_{n,1}^{(M)} - \widetilde{W}_{n,1}^{(M)} \right| \right] \leq CM \frac{\sqrt{b_n}}{\sqrt{\ell_n}}, \end{aligned}$$

where the last but one inequality follows by the Lipschitz-continuity of the kernel  $h$  and the stationarity of the  $W_{n,j}^{(M)}$ 's and of the  $\widetilde{W}_{n,j}^{(M)}$ 's.

In the following, it hence suffices to prove that  $\sqrt{b_n} \left( \widetilde{U}_n^{(M)} - \mathbb{E} \left[ \widetilde{U}_n^{(M)} \right] \right)$  converges in distribution to a centred normal random variable  $N_M$  with variance  $4\gamma_M^2$ . We first show that

$$\gamma_{M,n}^2 := \text{Cov} \left( h \left( \widetilde{W}_{n,1}^{(M)}, \widetilde{W}_{n,2}^{(M)} \right), h \left( \widetilde{W}_{n,1}^{(M)}, \widetilde{W}_{n,3}^{(M)} \right) \right) \rightarrow \gamma_M^2$$

and afterwards the convergence in distribution

$$\frac{\sqrt{b_n}}{\gamma_{M,n}} \left( \widetilde{U}_n^{(M)} - \mathbb{E} \left[ \widetilde{U}_n^{(M)} \right] \right) \rightarrow N(0, 4),$$

which combined yield the assertion.

We start by verifying  $\gamma_{M,n}^2 \rightarrow \gamma_M^2$  by means of Lemma A.6. To apply Lemma A.6, we have to check that the sequence  $\left( (\widetilde{W}_{n,1}^{(M)})^2 \right)_{n \geq 1}$  is uniformly integrable. By the assumptions on  $g$  and  $\eta$ , the gradient of  $g \cdot \eta$  is uniformly bounded over  $\mathbb{R}^m$ , such that

$$\begin{aligned} \left( \widetilde{W}_{n,1}^{(M)} \right)^2 &= \ell_n (g^2 \cdot \eta^2) \left( v_0 + \left( \frac{1}{\ell_n} \sum_{t=M+1}^{\ell_n-M-1} \mathbb{E} [X_t^k | \mathcal{F}_{t-M}^{t+M}] \right)_{k=1}^m - v_0 \right) \\ &\leq \ell_n C^2 \sum_{k=1}^m \left( \frac{1}{\ell_n} \sum_{t=M+1}^{\ell_n-M-1} \mathbb{E} [X_t^k | \mathcal{F}_{t-M}^{t+M}] - \mathbb{E} [X_1^k] \right)^2 \end{aligned}$$

and uniform integrability follows from Lemma A.2. We additionally have to show that  $\widetilde{W}_{n,1}^{(M)} \rightarrow N(0, \sigma_M^2)$  in distribution to apply Lemma A.6. To do so, we will use the differentiability of  $g \cdot \eta$  at  $v_0$  and the fact that  $(g \cdot \eta)(v_0) = 0$  in order to write

$$(g \cdot \eta)(v_0 + z) = \sum_{k=1}^m z_k \frac{\partial g}{\partial x_k}(v_0) + \varepsilon(z),$$

where  $\varepsilon(z) / \|z\|_2 \rightarrow 0$  as  $\|z\|_2 \rightarrow 0$ . Setting  $z = \left( \frac{1}{\ell_n} \sum_{t=M+1}^{\ell_n-M-1} (\mathbb{E} [X_t^k | \mathcal{F}_{t-M}^{t+M}] - \mathbb{E} [X_1^k]) \right)_{k=1}^m$ , we thus obtain

$$\begin{aligned} \widetilde{W}_{n,1}^{(M)} &= \frac{1}{\sqrt{\ell_n}} \sum_{t=M+1}^{\ell_n-M-1} \sum_{k=1}^m \frac{\partial g}{\partial x_k}(v_0) (\mathbb{E} [X_t^k | \mathcal{F}_{t-M}^{t+M}] - \mathbb{E} [X_1^k]) \\ &\quad + \sqrt{\ell_n} \varepsilon \left( \left( \frac{1}{\ell_n} \sum_{t=M+1}^{\ell_n-M-1} (\mathbb{E} [X_t^k | \mathcal{F}_{t-M}^{t+M}] - \mathbb{E} [X_1^k]) \right)_{k=1}^m \right). \end{aligned}$$

By the central limit theorem (see Lemma A.3 in the appendix), the first of the above terms converges to a centred normal distribution with variance  $\sigma_M^2$ . Moreover, due to the properties of the function  $\varepsilon$  and the strong law of large numbers, the second of the above terms converges in probability to 0. Indeed,

$$\sqrt{\ell_n} \varepsilon \left( \left( \frac{1}{\ell_n} \sum_{t=M+1}^{\ell_n-M-1} (\mathbb{E} [X_t^k | \mathcal{F}_{t-M}^{t+M}] - \mathbb{E} [X_1^k]) \right)_{k=1}^m \right)$$

$$\begin{aligned}
& \varepsilon \left( \left( \frac{1}{\ell_n} \sum_{t=M+1}^{\ell_n-M-1} (\mathbb{E} [X_t^k | \mathcal{F}_{t-M}^{t+M}] - \mathbb{E} [X_1^k]) \right)_{k=1}^m \right) \\
&= \frac{\varepsilon \left( \left( \frac{1}{\ell_n} \sum_{t=M+1}^{\ell_n-M-1} (\mathbb{E} [X_t^k | \mathcal{F}_{t-M}^{t+M}] - \mathbb{E} [X_1^k]) \right)_{k=1}^m \right)}{\sqrt{\sum_{k=1}^m \frac{1}{\ell_n} \left( \sum_{t=M+1}^{\ell_n-M-1} (\mathbb{E} [X_t^k | \mathcal{F}_{t-M}^{t+M}] - \mathbb{E} [X_1^k]) \right)^2}} \\
&\quad \cdot \sqrt{\sum_{k=1}^m \frac{1}{\ell_n} \left( \sum_{t=M+1}^{\ell_n-M-1} (\mathbb{E} [X_t^k | \mathcal{F}_{t-M}^{t+M}] - \mathbb{E} [X_1^k]) \right)^2},
\end{aligned}$$

which is a product of the form  $A_n B_n$ , where  $(A_n)_{n \geq 1}$  converges in probability to 0 and  $(B_n)_{n \geq 1}$  is tight. We hence have  $\widetilde{W}_{n,1}^{(M)} \rightarrow N(0, \sigma_M^2)$  in distribution and Lemma A.6 implies  $\gamma_{M,n}^2 \rightarrow \gamma_M^2$ .

Turning towards showing  $\sqrt{b_n}(\widetilde{U}_n^{(M)} - \mathbb{E}[\widetilde{U}_n^{(M)}])/\gamma_{M,n} \rightarrow N(0, 4)$  in distribution, we have to check the assumptions of Theorem 1 and therefore have to prove the uniform integrability of  $((\widetilde{W}_{n,1}^{(M)}/\gamma_{M,n})^2)_{n \geq 1}$ . The uniform integrability of  $((\widetilde{W}_{n,1}^{(M)})^2)_{n \geq 1}$  has already been shown above. Moreover, since we will show in the proof of Lemma 3.3 below that  $\gamma_M^2 \rightarrow \gamma^2 > 0$ , there exists an  $M_0$  such that for all  $M \geq M_0$ , it holds  $\gamma_M^2 > 0$ . Since for fixed  $M \geq M_0$ , we have shown above that  $\gamma_{M,n}^2 \rightarrow \gamma_M^2$ , there exists an  $n_{0,M}$  such that  $\gamma_{M,n}^2 \geq \gamma_M^2/2 > 0$  for all  $n \geq n_{0,M}$ . Hence, the sequence  $(\gamma_{M,n}^{-2})_{n \geq n_{0,M}}$  is bounded and  $((\widetilde{W}_{n,1}^{(M)}/\gamma_{M,n})^2)_{n \geq n_{0,M}}$  is uniformly integrable, such that Theorem 1 yields the desired convergence.  $\square$

*Proof of Lemma 3.3.* It suffices to prove that  $\gamma_M^2 \rightarrow \gamma^2$ . Since  $h$  is Lipschitz-continuous, the mapping  $x \mapsto \text{Cov}(h(xN), h(xN'))$  is continuous, where  $N, N'$  and  $N''$  denote three independent standard normal random variables. Therefore, we only need to check that  $\sigma_M^2 \rightarrow \sigma^2$ , where

$$\sigma_M^2 = \sum_{t=-2M-1}^{2M+1} \text{Cov} \left( \mathbb{E} \left[ \sum_{k=1}^m \frac{\partial g}{\partial x_k} (v_0) X_0^k \middle| \mathcal{F}_{-M}^M \right], \mathbb{E} \left[ \sum_{k=1}^m \frac{\partial g}{\partial x_k} (v_0) X_t^k \middle| \mathcal{F}_{t-M}^{t+M} \right] \right)$$

and

$$\sigma^2 = \sum_{t \in \mathbb{Z}} \text{Cov} \left( \sum_{k=1}^m \frac{\partial g}{\partial x_k} (v_0) X_0^k, \sum_{k=1}^m \frac{\partial g}{\partial x_k} (v_0) X_t^k \right).$$

By construction,  $\sigma_M^2 = \sum_{t \in \mathbb{Z}} \text{Cov} \left( \mathbb{E} \left[ \sum_{k=1}^m \frac{\partial g}{\partial x_k} (v_0) X_0^k \middle| \mathcal{F}_{-M}^M \right], \mathbb{E} \left[ \sum_{k=1}^m \frac{\partial g}{\partial x_k} (v_0) X_t^k \middle| \mathcal{F}_{t-M}^{t+M} \right] \right)$  and a small calculation shows

$$\begin{aligned}
& |\sigma_M^2 - \sigma^2| \\
&= \left| \sum_{t \in \mathbb{Z}} \text{Cov} \left( \sum_{k=1}^m \frac{\partial g}{\partial x_k} (v_0) (X_0^k - \mathbb{E} [X_0^k | \mathcal{F}_{-M}^M]), \sum_{k=1}^m \frac{\partial g}{\partial x_k} (v_0) (X_t^k - \mathbb{E} [X_t^k | \mathcal{F}_{t-M}^{t+M}]) \right) \right|.
\end{aligned}$$

Note that

$$\begin{aligned}
& \sum_{i \in \mathbb{Z}} \delta_i \left( \left( \sum_{k=1}^m \frac{\partial g}{\partial x_k} (v_0) \left( X_t^k - \mathbb{E} [X_t^k | \mathcal{F}_{t-M}^{t+M}] \right) \right)_{t \in \mathbb{Z}} \right) \\
& \leq \sum_{i \in \mathbb{Z}} \sum_{k=1}^m \frac{\partial g}{\partial x_k} (v_0) \delta_i \left( \left( X_t^k - \mathbb{E} [X_t^k | \mathcal{F}_{t-M}^{t+M}] \right)_{t \in \mathbb{Z}} \right) \leq 2 \sum_{i \in \mathbb{Z}} \sum_{k=1}^m \frac{\partial g}{\partial x_k} (v_0) \delta_i((X_t^k)_{t \in \mathbb{Z}}),
\end{aligned}$$

which is finite by assumption (2.4). We can thus apply Lemma A.2 from the appendix to the above sum of covariances and obtain

$$\begin{aligned} |\sigma_M^2 - \sigma^2| &= \lim_{n \rightarrow \infty} \frac{1}{n} \cdot \mathbb{E} \left[ \left( \sum_{t=1}^n \sum_{k=1}^m \frac{\partial g}{\partial x_k} (v_0) \left( X_t^k - \mathbb{E} \left[ X_t^k \middle| \mathcal{F}_{t-M}^{t+M} \right] \right) \right)^2 \right] \\ &\leq \left( \sum_{i \in \mathbb{Z}} \delta_i \left( \left( \sum_{k=1}^m \frac{\partial g}{\partial x_k} (v_0) \left( X_t^k - \mathbb{E} \left[ X_t^k \middle| \mathcal{F}_{t-M}^{t+M} \right] \right) \right)_{t \in \mathbb{Z}} \right) \right)^2, \end{aligned}$$

where the last inequality follows by Lemma A.1. We have already shown above that the last expression is finite. Moreover, for each fixed  $i$ , it holds by the martingale convergence theorem that

$$\lim_{M \rightarrow \infty} \delta_i \left( \left( \sum_{k=1}^m \frac{\partial g}{\partial x_k} (v_0) \left( X_t^k - \mathbb{E} \left[ X_t^k \middle| \mathcal{F}_{t-M}^{t+M} \right] \right) \right)_{t \in \mathbb{Z}} \right) = 0$$

and by dominated convergence, it thus follows  $|\sigma_M^2 - \sigma^2| \rightarrow 0$ .  $\square$

*Proof of Lemma 3.4.* Due to Chebychev's inequality, it suffices to prove

$$\lim_{M \rightarrow \infty} \limsup_{n \rightarrow \infty} \|R_{n,M}\|_2 = 0.$$

By the definition of  $R_{n,M}$ , the above equality holds if the sum  $\sum_{N \geq 1} a_N$  converges, where

$$\begin{aligned} a_N := \sup_{n \geq 1} \frac{1}{b_n^{3/2}} \left\| \sum_{1 \leq j \neq k \leq b_n} \left( h \left( W_{n,j}^{(N+1)}, W_{n,k}^{(N+1)} \right) - \mathbb{E} \left[ h \left( W_{n,j}^{(N+1)}, W_{n,k}^{(N+1)} \right) \right] \right. \right. \\ \left. \left. - \left( h \left( W_{n,j}^{(N)}, W_{n,k}^{(N)} \right) - \mathbb{E} \left[ h \left( W_{n,j}^{(N)}, W_{n,k}^{(N)} \right) \right] \right) \right) \right\|_2. \end{aligned}$$

Splitting the supremum up into those cases of  $n$  for which  $\ell_n \geq 2(N+1)$  and those for which  $\ell_n < 2(N+1)$ , we have to prove the convergence of  $\sum_{N \geq 1} a_{N,1}$  and  $\sum_{N \geq 1} a_{N,2}$ , where

$$\begin{aligned} a_{N,1} := \sup_{n \geq 1, 2(N+1) \leq \ell_n} \frac{1}{b_n^{3/2}} \left\| \sum_{1 \leq j \neq k \leq b_n} \left( h \left( W_{n,j}^{(N+1)}, W_{n,k}^{(N+1)} \right) - \mathbb{E} \left[ h \left( W_{n,j}^{(N+1)}, W_{n,k}^{(N+1)} \right) \right] \right. \right. \\ \left. \left. - \left( h \left( W_{n,j}^{(N)}, W_{n,k}^{(N)} \right) - \mathbb{E} \left[ h \left( W_{n,j}^{(N)}, W_{n,k}^{(N)} \right) \right] \right) \right) \right\|_2, \end{aligned}$$

$$\begin{aligned} a_{N,2} := \sup_{n \geq 1, 2(N+1) > \ell_n} \frac{1}{b_n^{3/2}} \left\| \sum_{1 \leq j \neq k \leq b_n} \left( h \left( W_{n,j}^{(N+1)}, W_{n,k}^{(N+1)} \right) - \mathbb{E} \left[ h \left( W_{n,j}^{(N+1)}, W_{n,k}^{(N+1)} \right) \right] \right. \right. \\ \left. \left. - \left( h \left( W_{n,j}^{(N)}, W_{n,k}^{(N)} \right) - \mathbb{E} \left[ h \left( W_{n,j}^{(N)}, W_{n,k}^{(N)} \right) \right] \right) \right) \right\|_2. \end{aligned}$$

We bound  $a_{N,1}$  by an application of Lemma A.5 from the appendix to obtain

$$a_{N,1} \leq C \sup_{n \geq 1, 2(N+1) \leq \ell_n} \left\| W_{n,1}^{(N+1)} - W_{n,1}^{(N)} \right\|_2.$$

In order to bound  $a_{N,2}$ , we notice that by assumption,  $b_n < \ell_n \leq 2(N+1) \leq CN$  for  $n$  large enough. Consequently, by the Lipschitz-continuity of  $h$ ,

$$a_{N,2} \leq 2 \sup_{n \geq 1, 2(N+1) > \ell_n} \sqrt{b_n} \left\| W_{n,1}^{(N+1)} - W_{n,1}^{(N)} \right\|_2 \leq 2 \sup_{n \geq 1, 2(N+1) > \ell_n} \sqrt{N} \left\| W_{n,1}^{(N+1)} - W_{n,1}^{(N)} \right\|_2.$$

Using the Lipschitz-continuity of  $g \cdot \eta$  and Lemma A.1, we derive that

$$\left\| W_{n,1}^{(N+1)} - W_{n,1}^{(N)} \right\|_2 \leq C \sum_{k=1}^m \sum_{i \in \mathbb{Z}} \delta_i \left( \left( \mathbb{E} [X_t^k | \mathcal{F}_{t-N-1}^{t+N+1}] - \mathbb{E} [X_t^k | \mathcal{F}_{t-N}^{t+N}] \right)_{t \in \mathbb{Z}} \right).$$

The above summands are 0 for  $|i| \geq N+2$ , while for  $|i| < N+2$ , we get by similar arguments as in the proof of Lemma A.4 that

$$\delta_i \left( \left( \mathbb{E} [X_t^k | \mathcal{F}_{t-N-1}^{t+N+1}] - \mathbb{E} [X_t^k | \mathcal{F}_{t-N}^{t+N}] \right)_{t \in \mathbb{Z}} \right) \leq \delta_{N+1} \left( (X_t^k)_{t \in \mathbb{Z}} \right) + \delta_{-N-1} \left( (X_t^k)_{t \in \mathbb{Z}} \right).$$

We thus obtain the bounds

$$a_{N,1} \leq CN \sum_{k=1}^m \left( \delta_{N+1} \left( (X_t^k)_{t \in \mathbb{Z}} \right) + \delta_{-N-1} \left( (X_t^k)_{t \in \mathbb{Z}} \right) \right)$$

and

$$a_{N,2} \leq CN^{3/2} \sum_{k=1}^m \left( \delta_{N+1} \left( (X_t^k)_{t \in \mathbb{Z}} \right) + \delta_{-N-1} \left( (X_t^k)_{t \in \mathbb{Z}} \right) \right).$$

The convergence of  $\sum_{N \geq 1} a_{N,1}$  and  $\sum_{N \geq 1} a_{N,2}$  then follows from assumption (2.4).  $\square$

*Proof of Proposition 2.7.* The proof is divided into two parts. First, we show that

$$\lim_{n \rightarrow \infty} \sqrt{b_n} \left( \frac{1}{b_n(b_n-1)} \sum_{1 \leq j \neq k \leq b_n} \left( \mathbb{E} \left[ h \left( W_{n,j}^{(\eta)}, W_{n,k}^{(\eta)} \right) \right] - \mathbb{E} \left[ h \left( Z_{n,j}, Z_{n,k} \right) \right] \right) \right) = 0,$$

where

$$Z_{n,j} = \frac{1}{\sqrt{\ell_n}} \sum_{k=1}^m \frac{\partial g}{\partial x_k} (v_0) \sum_{t \in B_{n,j}} (X_t^k - \mathbb{E} [X_1^k]),$$

and afterwards that

$$\lim_{n \rightarrow \infty} \sqrt{b_n} \left( \frac{1}{b_n(b_n-1)} \sum_{1 \leq j \neq k \leq b_n} \mathbb{E} \left[ h \left( Z_{n,j}, Z_{n,k} \right) \right] - \mathbb{E} \left[ h \left( Z_n, Z'_n \right) \right] \right) = 0.$$

The assertion then follows from Theorem 2.

Starting with the first part, due to the Lipschitz-continuity of  $h$  and stationarity, it suffices to show that

$$\lim_{n \rightarrow \infty} \sqrt{b_n} \mathbb{E} \left[ \left| W_{n,1}^{(\eta)} - Z_{n,1} \right| \right] = 0.$$

By the assumptions imposed on  $g$  and  $\eta$ , there exists a constant  $C$  such that for each  $z = (z_k)_{k=1}^m \in \mathbb{R}^m$ , it holds

$$\left| (g \cdot \eta)(z) - \sum_{k=1}^m \frac{\partial g}{\partial x_k} (v_0) (z_k - \mathbb{E} [X_1^k]) \right| \leq C \|z - v_0\|_2.$$

With the choice  $z = \left( \frac{1}{\ell_n} \sum_{t \in B_{n,1}} X_t^k \right)_{k=1}^m$ , this yields

$$\sqrt{b_n} \mathbb{E} \left[ \left| W_{n,j}^{(\eta)} - Z_{n,1} \right| \right] \leq C \sqrt{b_n \ell_n} \sum_{k=1}^m \mathbb{E} \left[ \left[ \left( \frac{1}{\ell_n} \sum_{t \in B_{n,1}} (X_t^k - \mathbb{E} [X_1^k]) \right)^2 \right] \right].$$

Lemma A.1 from the appendix now implies that  $\sqrt{b_n} \mathbb{E} \left[ \left| W_{n,j}^{(\eta)} - Z_{n,1} \right| \right] \leq C \sqrt{b_n / \ell_n}$ , which converges to 0 by assumption.

Turning towards the second part, we define

$$A_{n,j} := \frac{1}{\sqrt{\ell_n}} \sum_{k=1}^m \frac{\partial g}{\partial x_k} (v_0) \sum_{t \in B_{n,j}} (\mathbb{E} [X_t^k \mid \sigma(\varepsilon_u, |u-t| \leq \ell_n^\kappa)] - \mathbb{E} [X_1^k])$$

and

$$A'_{n,j} := \frac{1}{\sqrt{\ell_n}} \sum_{k=1}^m \frac{\partial g}{\partial x_k} (v_0) \sum_{t \in B'_{n,j}} (\mathbb{E} [X_t^k \mid \sigma(\varepsilon_u, |u-t| \leq \ell_n^\kappa)] - \mathbb{E} [X_1^k]),$$

where

$$B'_{n,j} := \{k \in \mathbb{N}, (j-1)\ell_n + 1 + \ell_n^\kappa \leq k \leq j\ell_n - \ell_n^\kappa\}.$$

Note that by construction, the sequence of random variables  $(A'_{n,j})_{j \geq 1}$  is independent. Moreover, the following decomposition holds

$$\begin{aligned} & \sqrt{b_n} \left| \frac{1}{b_n(b_n-1)} \sum_{1 \leq j \neq k \leq b_n} \mathbb{E} [h(Z_{n,j}, Z_{n,k})] - \mathbb{E} [h(Z_n, Z'_n)] \right| \\ & \leq \sqrt{b_n} \frac{1}{b_n(b_n-1)} \sum_{1 \leq j \neq k \leq b_n} \mathbb{E} [|h(Z_{n,j}, Z_{n,k}) - h(A'_{n,j}, A'_{n,k})|] \\ & + \sqrt{b_n} \frac{1}{b_n(b_n-1)} \left| \sum_{1 \leq j \neq k \leq b_n} (\mathbb{E} [h(A'_{n,j}, A'_{n,k})] - \mathbb{E} [h(Z_n, Z'_n)]) \right|. \end{aligned}$$

By the Lipschitz-continuity of  $h$  combined with stationarity, we can bound the first of the above terms by

$$C\sqrt{b_n} \mathbb{E} [|Z_{n,1} - A'_{n,1}|] \leq C\sqrt{b_n} \mathbb{E} [|Z_{n,1} - A_{n,1}|] + C\sqrt{b_n} \mathbb{E} [|A_{n,1} - A'_{n,1}|].$$

An application of Lemma A.4 from the appendix yields

$$\begin{aligned} \sqrt{b_n} \mathbb{E} [|Z_{n,1} - A_{n,1}|] & \leq C\sqrt{b_n} \sum_{k=1}^m \ell_n^\kappa \sum_{i: |i| > \ell_n^\kappa} \delta_i \left( (X_t^k)_{t \in \mathbb{Z}} \right) \\ & \leq C\sqrt{b_n/\ell_n} \ell_n^{\kappa/2} \sum_{k=1}^m \sum_{i: |i| > \ell_n^\kappa} \ell_n^{\kappa/2+1/2} \delta_i \left( (X_t^k)_{t \in \mathbb{Z}} \right) \\ & \leq C\sqrt{b_n/\ell_n} \ell_n^{\kappa/2} \sum_{k=1}^m \sum_{i: |i| > \ell_n^\kappa} |i|^{1/2+1/(2\kappa)} \delta_i \left( (X_t^k)_{t \in \mathbb{Z}} \right), \end{aligned}$$

which tends to zero by the assumptions on the dependence coefficients and since  $b_n = o(\ell_n^{1-\kappa})$ . By Lemma A.1,

$$\begin{aligned} \sqrt{b_n} \mathbb{E} [|A_{n,1} - A'_{n,1}|] & \leq C \frac{\sqrt{b_n}}{\sqrt{\ell_n}} \sum_{k=1}^m \left\| \sum_{t=1}^{\ell_n^\kappa} \mathbb{E} [X_t^k \mid \sigma(\varepsilon_u, |u-t| \leq \ell_n^\kappa)] \right\|_2 \\ & \leq C \frac{\sqrt{b_n}}{\sqrt{\ell_n}} \sum_{k=1}^m \ell_n^{\kappa/2} \sum_{i \in \mathbb{Z}} \delta_i \left( (\mathbb{E} [X_t^k \mid \sigma(\varepsilon_u, |u-t| \leq \ell_n^\kappa)])_{t \in \mathbb{Z}} \right) \\ & \leq C \frac{\sqrt{b_n}}{\sqrt{\ell_n}} \sum_{k=1}^m \ell_n^{\kappa/2} \sum_{i: |i| \leq \ell_n^\kappa} \delta_i \left( (X_t^k)_{t \in \mathbb{Z}} \right), \end{aligned}$$

which likewise tends to zero.



For the second of the above terms, we once more use the Lipschitz-continuity of  $h$  and stationarity in order to obtain the upper bound  $\sqrt{b_n} \mathbb{E} [|A'_{n,1} - Z_n|]$ , whose convergence has already been shown above since  $Z_n = Z_{n,1}$ .  $\square$

*Proof of Corollary 2.9.* It holds

$$(3.6) \quad \begin{aligned} & \sqrt{b_n} (\mathbb{E} [|Z_n - Z'_n|] - \sigma \mathbb{E} [|N - N'|]) = \sqrt{2}\sigma \sqrt{b_n} \left( \frac{\mathbb{E} [|Z_n - Z'_n|]}{\sqrt{2}\sigma} - \mathbb{E} [|N|] \right) \\ & = \sqrt{b_n} \mathbb{E} [|Z_n - Z'_n|] \frac{\sqrt{\text{Var}(Z_n)} - \sigma}{\sqrt{\text{Var}(Z_n)}} + \sqrt{2}\sigma \sqrt{b_n} \left( \frac{\mathbb{E} [|Z_n - Z'_n|]}{\sqrt{2\text{Var}(Z_n)}} - \mathbb{E} [|N|] \right). \end{aligned}$$

For the first of these terms, it suffices to show that  $\sqrt{b_n} \left| \sqrt{\text{Var}(Z_n)} - \sigma \right| \rightarrow 0$  since  $\mathbb{E} [|Z_n - Z'_n|]$  can be bounded by a constant independent of  $n$  due to Lemma A.1 and since for  $\text{Var}(Z_n) \rightarrow \sigma^2 > 0$ , it holds  $\text{Var}(Z_n) > \sigma^2/2$  for all  $n$  large enough. Proceeding as in the proof of Proposition 2 in El Machkouri, Volný, and Wu [2013], we derive that for a centered time series  $(Y_t)_{t \in \mathbb{Z}}$  such that  $Y_t = f((\varepsilon_{t-u})_{u \geq 0})$ , one has

$$(3.7) \quad \begin{aligned} & \left| \text{Var} \left( \frac{1}{\sqrt{N}} \sum_{t=1}^N Y_t \right) - \sum_{t \in \mathbb{Z}} \text{Cov}(Y_0, Y_t) \right| \\ & \leq \sum_{j:|j|>N} \sum_{i \in \mathbb{Z}} \delta_i((Y_t)_{t \in \mathbb{Z}}) \delta_{i-j}((Y_t)_{t \in \mathbb{Z}}) + 2 \sum_{j=1}^N \frac{j}{N} \sum_{i \in \mathbb{Z}} \delta_i((Y_t)_{t \in \mathbb{Z}}) \delta_{i-j}((Y_t)_{t \in \mathbb{Z}}). \end{aligned}$$

This follows from the following arguments. First, we expand  $\text{Var} \left( \sum_{t=1}^N Y_t / \sqrt{N} \right)$  in terms of  $\text{Cov}(Y_0, Y_t)$  via a use of stationarity. Then we write

$$(3.8) \quad \text{Cov}(Y_0, Y_t) = \sum_{i \in \mathbb{Z}} \mathbb{E} [P_i(Y_0) P_i(Y_t)],$$

where  $P_i(Y) := \mathbb{E} [Y | \sigma(\varepsilon_u, u \leq i)] - \mathbb{E} [Y | \sigma(\varepsilon_u, u \leq i-1)]$ . Finally, we use the fact that  $|\mathbb{E} [P_i(Y_0) P_i(Y_t)]| \leq \|P_i(Y_0)\|_2 \|P_i(Y_t)\|_2$  due to the Cauchy-Schwarz inequality and Theorem 1 in Wu [2005]. Letting  $N = \ell_n$ ,  $Y_t = \sum_{k=1}^m \frac{\partial g}{\partial x_k}(v_0) (X_t^k - \mathbb{E} [X_1^k])$  and  $a_i := \max_{1 \leq k \leq m} \delta_i \left( (X_t^k)_{t \in \mathbb{Z}} \right)$ , we infer that for some constant  $C$  independent of  $n$ ,

$$\begin{aligned} \sqrt{b_n} |\text{Var}(Z_n) - \sigma^2| & \leq C \sqrt{b_n} \sum_{k=1}^m \sum_{i \geq 0} \sum_{j=1}^{\ell_n} \frac{j}{\ell_n} \delta_i \left( (X_t^k)_{t \in \mathbb{Z}} \right) \delta_{i-j} \left( (X_t^k)_{t \in \mathbb{Z}} \right) \mathbf{1}_{i \geq j} \\ & \quad + C \sqrt{b_n} \sum_{k=1}^m \sum_{i \geq 0} \sum_{j=\ell_n+1}^{\infty} \frac{j}{\ell_n} \delta_i \left( (X_t^k)_{t \in \mathbb{Z}} \right) \delta_{i-j} \left( (X_t^k)_{t \in \mathbb{Z}} \right) \mathbf{1}_{i \geq j}. \end{aligned}$$

Denote  $A := \sum_{i \geq 0} a_i$ . Then elementary computations give

$$\begin{aligned} \sum_{i \geq 0} \sum_{j=1}^{\ell_n} \frac{j}{\ell_n} a_i a_{i-j} \mathbf{1}_{i \geq j} & = \sum_{j=1}^{\ell_n} \frac{j}{\ell_n} \sum_{i \geq j} a_i a_{i-j} = \sum_{j=1}^{\ell_n} \frac{j}{\ell_n} \sum_{k \geq 0} a_{k+j} a_k \\ & \leq \sum_{j=1}^{\ell_n} \frac{j}{\ell_n} \sum_{k \geq 0} a_{k+j} A \leq A \sum_{j=1}^{\ell_n} \frac{j}{\ell_n} j^{-5/2} \sum_{k \geq 0} (k+j)^{5/2} a_{j+k} \leq C / \sqrt{\ell_n} \end{aligned}$$

and

$$\begin{aligned} \sum_{i \geq 0} \sum_{j=\ell_n+1}^{\infty} \frac{j}{\ell_n} a_i a_{i-j} \mathbf{1}_{i \geq j} &= \sum_{i \geq \ell_n+1} \sum_{j=\ell_n+1}^i \frac{j}{\ell_n} a_i a_{i-j} \leq \sum_{i \geq \ell_n+1} \frac{i}{\ell_n} \sum_{j=\ell_n+1}^i a_i a_{i-j} \\ &= \sum_{i \geq \ell_n+1} a_i \frac{i}{\ell_n} \sum_{k=0}^{i-\ell_n-1} a_k \leq A \sum_{i \geq \ell_n+1} a_i \frac{i}{\ell_n} \leq C \ell_n^{-5/2}. \end{aligned}$$

Consequently,

$$\sqrt{b_n} |\text{Var}(Z_n) - \sigma^2| \leq C \sqrt{b_n/\ell_n},$$

and the convergence of  $\sqrt{b_n} \left| \sqrt{\text{Var}(Z_n)} - \sigma \right| \rightarrow 0$  follows from

$$\sqrt{b_n} \left| \sqrt{\text{Var}(Z_n)} - \sigma \right| = \sqrt{b_n} |\text{Var}(Z_n) - \sigma^2| \frac{1}{\sqrt{\text{Var}(Z_n)} + \sigma}.$$

For the second term of (3.6), we can apply Corollary 2.6 in Jirak [2016] to obtain the bound

$$\sqrt{b_n} \left( \frac{\mathbb{E}[|Z_n - Z'_n|]}{\sqrt{2\text{Var}(Z_n)}} - \mathbb{E}[|N|] \right) \leq C \sqrt{b_n} \ell_n^{1-p/2} (\log \ell_n)^{p/2},$$

provided the assumptions stated in Corollary 2.9 hold. Indeed,  $Z_n - Z'_n$  can be expressed as

$$Z_n - Z'_n = \frac{1}{\sqrt{\ell_n}} \sum_{t=1}^{\ell_n} \phi \left( (\varepsilon_{t-u}, \varepsilon'_{t-u})_{u \in \mathbb{N}} \right),$$

where  $(\varepsilon'_u)_{u \in \mathbb{Z}}$  is an independent copy of  $(\varepsilon_u)_{u \in \mathbb{Z}}$  and

$$\phi \left( (x_u, y_u)_{u \in \mathbb{N}} \right) := \sum_{k=1}^m \frac{\partial g}{\partial x_k} (v_0) \left( f^k \left( (x_u)_{u \in \mathbb{N}} \right) - f^k \left( (y_u)_{u \in \mathbb{N}} \right) \right).$$

□

### 3.4. Verification of Examples.

*Details of Example 2.3.* Since  $\varphi$  is  $\gamma$ -Hölder continuous, there exists a constant  $C$  such that for each  $x, y \in \mathbb{R}$ ,  $|\varphi(x) - \varphi(y)| \leq C|x - y|^\gamma$ . In particular,  $|\varphi(x)| \leq C|x|^\gamma + |\varphi(0)|$  and it follows that

$$\begin{aligned} |\varphi^k(x) - \varphi^k(y)| &= |\varphi(x) - \varphi(y)| \left| \sum_{j=0}^{k-1} \varphi(x)^j \varphi(y)^{k-j-1} \right| \\ &\leq \max\{C, |\varphi(0)|\}^{k-1} |\varphi(x) - \varphi(y)| \sum_{j=0}^{k-1} (1 + |x|^\gamma)^j (1 + |y|^\gamma)^{k-j-1} \\ &\leq C' |x - y|^\gamma (1 + |x|^\gamma + |y|^\gamma)^{k-1}. \end{aligned}$$

For a fixed  $i$ , we thus have

$$\begin{aligned} \delta_i \left( (X_t^k)_{t \in \mathbb{Z}} \right) &= \left\| \varphi \left( \sum_{j \in \mathbb{Z}} a_j \varepsilon_{-j} \right)^k - \varphi \left( \sum_{j \in \mathbb{Z}} a_j \varepsilon_{-j}^{*,i} \right)^k \right\|_2 \\ &\leq C' \left\| |a_{-i}|^\gamma |\varepsilon_i - \varepsilon'_i|^\gamma \left( 1 + \left| \sum_{j \in \mathbb{Z}, j \neq -i} a_j \varepsilon_{-j} + a_{-i} \varepsilon_i \right|^\gamma + \left| \sum_{j \in \mathbb{Z}, j \neq -i} a_j \varepsilon_{-j} + a_{-i} \varepsilon'_i \right|^\gamma \right)^{k-1} \right\|_2 \end{aligned}$$

$$\begin{aligned}
&\leq C' \left\| |a_{-i}|^\gamma |\varepsilon_i - \varepsilon'_i|^\gamma \left(1 + |a_{-i}\varepsilon_i|^\gamma + |a_{-i}\varepsilon'_i|^\gamma\right)^{k-1} \right\|_2 \\
&+ C' \left\| |a_{-i}|^\gamma |\varepsilon_i - \varepsilon'_i|^\gamma \left(1 + \left| \sum_{j \in \mathbb{Z}, j \neq -i} a_j \varepsilon_{-j} \right|^\gamma + \left| \sum_{j \in \mathbb{Z}, j \neq -i} a_j \varepsilon'_{-j} \right|^\gamma\right)^{k-1} \right\|_2 \\
&\leq C' \left( \left\| |\varepsilon_0|^{k\gamma} \right\|_2 + \left\| |\varepsilon_0|^{(k-1)\gamma} \right\|_2 \cdot \left\| |\varepsilon_0|^\gamma \right\|_2 \right) |a_{-i}|^{k\gamma} + C' |a_{-i}|^\gamma \left\| |\varepsilon_0|^\gamma \right\|_2 \\
&+ C' \left\| |\varepsilon_0|^\gamma \right\|_2 |a_{-i}|^\gamma \left\| \left| \sum_{j \in \mathbb{Z}, j \neq -i} a_j \varepsilon_{-j} \right|^{(k-1)\gamma} \right\|_2,
\end{aligned}$$

where the second inequality follows by a combination of the triangle inequality for  $\gamma \in (0, 1]$  and the  $c_r$ -inequality, and the third one is due to the independence of  $\sum_{j \in \mathbb{Z}, j \neq -i} a_j \varepsilon_{-j}$  and  $(\varepsilon_i, \varepsilon'_i)$ .

Recall that we have to check that  $\sum_{i \in \mathbb{Z}} i^2 \delta_i \left( (X_t^k)_{t \in \mathbb{Z}} \right) < \infty$  for each  $k = 1, \dots, m$ . By assumption,  $\mathbb{E} \left[ |\varepsilon_0|^{2m\gamma} \right] < \infty$  and  $\sum_{i \in \mathbb{Z}} i^2 |a_i|^\gamma < \infty$ , such that it remains to show  $\mathbb{E} \left[ \left| \sum_{j \in \mathbb{Z}, j \neq -i} a_j \varepsilon_{-j} \right|^{2\gamma(k-1)} \right] < \infty$ . In case  $2\gamma(k-1) \leq 1$ , we can simply employ

$$\mathbb{E} \left[ \left| \sum_{j \in \mathbb{Z}, j \neq -i} a_j \varepsilon_{-j} \right|^{2(k-1)\gamma} \right] \leq \sum_{j \in \mathbb{Z}, j \neq 0} |a_j|^{2(k-1)\gamma} \mathbb{E} \left[ |\varepsilon_0|^{2(k-1)\gamma} \right],$$

which is finite by assumption. If  $2\gamma(k-1) \in (1, 2]$ , then the Von Bahr-Esseen inequality gives (up to a constant) the same upper bound. If  $2\gamma(k-1) > 2$ , Rosenthal's inequality yields

$$\mathbb{E} \left[ \left| \sum_{j \in \mathbb{Z}, j \neq -i} a_j \varepsilon_{-j} \right|^{2(k-1)\gamma} \right] \leq C \sum_{j \in \mathbb{Z}, j \neq 0} |a_j|^{2(k-1)\gamma} \mathbb{E} \left[ |\varepsilon_0|^{2(k-1)\gamma} \right] + C \left( \sum_{j \in \mathbb{Z}, j \neq 0} a_j^2 \mathbb{E} \left[ \varepsilon_0^2 \right] \right)^{(k-1)\gamma}.$$

□

*Details of Example 2.4.* This is a consequence of the estimation of the physical dependence measure in Example 3 on pages 5967-5968 of [Biermé and Durieu \[2014\]](#) applied separately to each function  $\varphi^k$  for  $1 \leq k \leq m$ . □

*Details of Example 2.5.* In order to give a bound on  $\delta_i((X_t^k)_{t \in \mathbb{Z}})$  for a fixed  $i$  and a  $k \in \{1, \dots, m\}$ , we decompose  $X_0$  as follows: Set  $X_0 = \varepsilon_i Y_i + Z_i$ , where

$$\begin{aligned}
Y_i &:= \sum_{j' \in \mathbb{Z}, j' \neq -i} a_{-i, j'} \varepsilon_{-j'} + \sum_{j \in \mathbb{Z}, j \neq -i} a_{j, -i} \varepsilon_{-j} \\
Z_i &:= \sum_{j, j' \in \mathbb{Z}, j \neq j', j \neq -i, j' \neq -i} a_{j, j'} \varepsilon_{-j} \varepsilon_{-j'}.
\end{aligned}$$

Thus,

$$X_0^k - (X_0^{*,i})^k = \sum_{\ell=0}^k \binom{k}{\ell} \left( \varepsilon_i^\ell Y_i^\ell Z_i^{k-\ell} - (\varepsilon'_i)^\ell Y_i^\ell Z_i^{k-\ell} \right)$$

and since the term with index 0 vanishes, we derive that

$$\delta_i((X_t^k)_{t \in \mathbb{Z}}) = \left\| X_0^k - (X_0^{*,i})^k \right\|_2 \leq \sum_{\ell=1}^k \binom{k}{\ell} \left\| (\varepsilon_i^\ell - (\varepsilon'_i)^\ell) Y_i^\ell Z_i^{k-\ell} \right\|_2$$

$$\leq 2 \sum_{\ell=1}^k \binom{k}{\ell} \|\varepsilon_0^\ell\|_2 \|Y_i^\ell Z_i^{k-\ell}\|_2 \leq 2 \sum_{\ell=1}^k \binom{k}{\ell} \|\varepsilon_0\|_{2\ell}^\ell \|Y_i\|_{2k}^\ell \|Z_i\|_{2k}^{k-\ell},$$

where the second inequality is due to the independence of  $(\varepsilon_i, \varepsilon'_i)$  and  $(Y_i, Z_i)$ , and the third inequality follows from an application of Hölder's inequality with conjugate exponents  $k/\ell$  and  $k/(k-\ell)$  for  $\ell \leq k-1$ . Following the arguments given on pages 2376-2377 in [Zhang, Reding, and Peligrad \[2020\]](#), we obtain

$$\|Z_i\|_{2k} \leq C \sqrt{\sum_{j,j' \in \mathbb{Z} \setminus \{-i\}, j \neq j'} a_{j,j'}^2} \|\varepsilon_0\|_{2k} \leq C \sqrt{\sum_{j,j' \in \mathbb{Z}, j \neq j'} a_{j,j'}^2} \|\varepsilon_0\|_{2k},$$

such that  $\|Z_i\|_{2k}$  can be bounded independently of  $i$ . Moreover, an application of Rosenthal's inequality yields

$$\|Y_i\|_{2k} \leq C \sqrt{\sum_{j,j' \in \mathbb{Z}, j \neq j'} a_{j,j'}^2} \cdot \sqrt{\sum_{j \in \mathbb{Z}, j \neq -i} (a_{-i,j}^2 + a_{j,-i}^2)} \|\varepsilon_0\|_{2k}.$$

Thus,  $\delta_i \left( (X_t^k)_{t \in \mathbb{Z}} \right) \leq C \sqrt{\sum_{j \in \mathbb{Z}, j \neq -i} (a_{-i,j}^2 + a_{j,-i}^2)}$  and the result follows.  $\square$

## A. APPENDIX

**A.1. Auxiliary results for functionals of i.i.d. sequences.** This appendix collects some auxiliary results for functionals of i.i.d. sequences, which we require for our proofs. We consider sequences  $(X_t)_{t \in \mathbb{Z}}$  of the form  $X_t := f((\varepsilon_{t-u})_{u \in \mathbb{Z}})$ , where  $f: \mathbb{R}^{\mathbb{Z}} \rightarrow \mathbb{R}$  is measurable,  $(\varepsilon_u)_{u \in \mathbb{Z}}$  is an i.i.d. sequence and  $\mathbb{E}[X_t] = 0$ . Denote  $\mathcal{F}_M^N := \sigma(\varepsilon_u, M \leq u \leq N)$ . To quantify the dependence, let  $(\varepsilon'_u)_{u \in \mathbb{Z}}$  denote an independent copy of  $(\varepsilon_u)_{u \in \mathbb{Z}}$  and define

$$\delta_i \left( (X_t)_{t \in \mathbb{Z}} \right) := \|X_0 - X_0^{*,i}\|_2,$$

where  $X_0^{*,i} = f((\varepsilon_{-u}^{*,i})_{u \in \mathbb{Z}})$  and  $\varepsilon_v^{*,i} = \varepsilon'_i$  if  $v = i$  and  $\varepsilon_v^{*,i} = \varepsilon_v$  otherwise.

We start by presenting a bound on the partial sum of  $(X_t)_{t \in \mathbb{Z}}$ , which is a special case of Proposition 1 in [El Machkouri et al. \[2013\]](#).

**Lemma A.1.** *The following inequality holds for all  $N \in \mathbb{N}$ :*

$$\left\| \sum_{t=1}^N X_t \right\|_2 \leq \sqrt{N} \sum_{i \in \mathbb{Z}} \delta_i \left( (X_t)_{t \in \mathbb{Z}} \right).$$

**Lemma A.2.** *Suppose that  $\sum_{i \in \mathbb{Z}} \delta_i \left( (X_t)_{t \in \mathbb{Z}} \right) < \infty$ . Then the sequence  $(\frac{1}{N} (\sum_{t=1}^N X_t)^2)_{N \geq 1}$  is uniformly integrable. Moreover, the series  $\sum_{t \in \mathbb{Z}} |\text{Cov}(X_0, X_t)|$  converges and*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \mathbb{E} \left[ \left( \sum_{t=1}^N X_t \right)^2 \right] = \sum_{t \in \mathbb{Z}} \text{Cov}(X_0, X_t).$$

*Proof.* The convergence of the series is established in Proposition 2 of [El Machkouri et al. \[2013\]](#). It remains to check the uniform integrability of  $(\frac{1}{N} (\sum_{t=1}^N X_t)^2)_{N \geq 1}$ . Since the sequence is bounded in  $\mathbb{L}^1$ , we only have to check that

$$\lim_{\delta \rightarrow 0} \sup_{A: \mathbb{P}(A) < \delta} \mathbb{E} \left[ \frac{1}{N} \left( \sum_{t=1}^N X_t \right)^2 \mathbf{1}_A \right] = 0.$$

To do so, let  $X_t^{(M)} := \mathbb{E} [X_t \mid \mathcal{F}_{t-M}^{t+M}]$ . Then

$$\begin{aligned} & \sup_{A: \mathbb{P}(A) < \delta} \mathbb{E} \left[ \frac{1}{N} \left( \sum_{t=1}^N X_t \right)^2 \mathbf{1}_A \right] \\ & \leq 2 \sup_{A: \mathbb{P}(A) < \delta} \mathbb{E} \left[ \frac{1}{N} \left( \sum_{t=1}^N X_t^{(M)} \right)^2 \mathbf{1}_A \right] + 2 \mathbb{E} \left[ \frac{1}{N} \left( \sum_{t=1}^N (X_t - X_t^{(M)}) \right)^2 \right] \end{aligned}$$

and using Lemma A.1, we obtain

$$\begin{aligned} \sup_{A: \mathbb{P}(A) < \delta} \mathbb{E} \left[ \frac{1}{N} \left( \sum_{t=1}^N X_t \right)^2 \mathbf{1}_A \right] & \leq 2 \sup_{A: \mathbb{P}(A) < \delta} \mathbb{E} \left[ \frac{1}{N} \left( \sum_{t=1}^N X_t^{(M)} \right)^2 \mathbf{1}_A \right] \\ & \quad + 2 \left( \sum_{i \in \mathbb{Z}} \delta_i \left( (X_t - \mathbb{E} [X_t \mid \mathcal{F}_{t-M}^{t+M}])_{t \in \mathbb{Z}} \right) \right)^2. \end{aligned}$$

Employing the uniform integrability of  $(\frac{1}{N} (\sum_{t=1}^N X_t^{(M)})^2)_{N \geq 1}$  for any fixed  $M$ , we derive that

$$\limsup_{\delta \rightarrow 0} \sup_{A: \mathbb{P}(A) < \delta} \mathbb{E} \left[ \frac{1}{N} \left( \sum_{t=1}^N X_t \right)^2 \mathbf{1}_A \right] \leq 2 \left( \sum_{i \in \mathbb{Z}} \delta_i \left( (X_t - \mathbb{E} [X_t \mid \mathcal{F}_{t-M}^{t+M}])_{t \in \mathbb{Z}} \right) \right)^2.$$

Since  $\delta_i \left( (X_t - \mathbb{E} [X_t \mid \mathcal{F}_{t-M}^{t+M}])_{t \in \mathbb{Z}} \right) \leq 2\delta_i \left( (X_t)_{t \in \mathbb{Z}} \right)$ , we conclude by an application of the dominated convergence theorem.  $\square$

There moreover holds a central limit theorem for  $(X_t)_{t \in \mathbb{Z}}$  (see Theorem 1 in [El Machkouri et al. \[2013\]](#)):

**Lemma A.3.** *Suppose that  $\sum_{i \in \mathbb{Z}} \delta_i \left( (X_t)_{t \in \mathbb{Z}} \right) < \infty$ . Then the following convergence in distribution holds*

$$\frac{1}{\sqrt{n}} \sum_{t=1}^n X_t \rightarrow N(0, \sigma^2),$$

where

$$\sigma^2 = \sum_{t \in \mathbb{Z}} \text{Cov}(X_0, X_t).$$

We also require an estimate on the  $\mathbb{L}^2$ -norm of partial sums of  $(X_t - \mathbb{E} [X_t \mid \mathcal{F}_{t-M}^{t+M}])_{t \geq 1}$  for some  $M \in \mathbb{N}$ .

**Lemma A.4.** *It holds*

$$\left\| \sum_{t=1}^N (X_t - \mathbb{E} [X_t \mid \mathcal{F}_{t-M}^{t+M}]) \right\|_2 \leq (4M + 3) \sqrt{N} \sum_{i: |i| \geq M} \delta_i \left( (X_t)_{t \in \mathbb{Z}} \right).$$

*Proof.* Lemma A.1 applied to  $X_t - \mathbb{E} [X_t \mid \mathcal{F}_{t-M}^{t+M}]$  yields

$$\left\| \sum_{t=1}^N (X_t - \mathbb{E} [X_t \mid \mathcal{F}_{t-M}^{t+M}]) \right\|_2 \leq \sqrt{N} \sum_{i \in \mathbb{Z}} \delta_i \left( (X_t - \mathbb{E} [X_t \mid \mathcal{F}_{t-M}^{t+M}])_{t \in \mathbb{Z}} \right).$$

For  $|i| \geq M + 1$ , we use  $\delta_i \left( (X_t - \mathbb{E}[X_t | \mathcal{F}_{t-M}^{t+M}])_{t \in \mathbb{Z}} \right) \leq \delta_i ((X_t)_{t \in \mathbb{Z}})$ , whereas for  $|i| \leq M$ , we employ  $\delta_i \left( (X_t - \mathbb{E}[X_t | \mathcal{F}_{t-M}^{t+M}])_{t \in \mathbb{Z}} \right) \leq 2 \|X_0 - \mathbb{E}[X_0 | \mathcal{F}_{-M}^M]\|_2$ , thus obtaining

$$(A.1) \quad \left\| \sum_{t=1}^N (X_t - \mathbb{E}[X_t | \mathcal{F}_{t-M}^{t+M}]) \right\|_2 \leq 2(2M+1) \sqrt{N} \|X_0 - \mathbb{E}[X_0 | \mathcal{F}_{-M}^M]\|_2 + \sqrt{N} \sum_{i \in \mathbb{Z}; |i| \geq M+1} \delta_i ((X_t)_{t \in \mathbb{Z}}).$$

In the following, we derive an upper bound for the first of the above terms. By the martingale convergence theorem, it holds

$$\|X_0 - \mathbb{E}[X_0 | \mathcal{F}_{-M}^M]\|_2^2 \leq \sum_{i \geq M+1} \left\| \mathbb{E}[X_0 | \mathcal{F}_{-i}^i] - \mathbb{E}[X_0 | \mathcal{F}_{-(i-1)}^{i-1}] \right\|_2^2.$$

Moreover,

$$\begin{aligned} & \left\| \mathbb{E}[X_0 | \mathcal{F}_{-i}^i] - \mathbb{E}[X_0 | \mathcal{F}_{-(i-1)}^{i-1}] \right\|_2 \\ & \leq \left\| \mathbb{E}[X_0 | \mathcal{F}_{-i}^i] - \mathbb{E}[X_0^{*,i} | \mathcal{F}_{-i}^i] \right\|_2 + \left\| \mathbb{E}[X_0^{*,i} | \mathcal{F}_{-i}^i] - \mathbb{E}[X_0 | \mathcal{F}_{-(i-1)}^{i-1}] \right\|_2 \\ & \leq \delta_i ((X_t)_{t \in \mathbb{Z}}) + \delta_{-i} ((X_t)_{t \in \mathbb{Z}}), \end{aligned}$$

where the second inequality follows from

$$\mathbb{E}[X_0^{*,i} | \mathcal{F}_{-i}^i] = \mathbb{E}[X_0^{*,i} | \mathcal{F}_{-i}^{i-1}] = \mathbb{E}[X_0 | \mathcal{F}_{-i}^{i-1}]$$

combined with

$$\mathbb{E}[X_0 | \mathcal{F}_{-(i-1)}^{i-1}] = \mathbb{E}[X_0^{*,i} | \mathcal{F}_{-(i-1)}^{i-1}] = \mathbb{E}[X_0^{*,i} | \mathcal{F}_{-i}^{i-1}].$$

By the comparison of the  $\ell^1$ - and  $\ell^2$ -norm, we thus have

$$\|X_0 - \mathbb{E}[X_0 | \mathcal{F}_{-M}^M]\|_2 \leq \sum_{i: |i| \geq M+1} \delta_i ((X_t)_{t \in \mathbb{Z}}).$$

Inserting the above bound into (A.1) concludes the proof.  $\square$

## A.2. A moment inequality for $U$ -statistics.

**Lemma A.5.** *Let  $(\varepsilon_u)_{u \in \mathbb{Z}}$  be an i.i.d. sequence. Let  $M \geq 0$  and  $\ell > 2M$  be integers. Define the random vectors  $V_j$  by  $V_j := (\varepsilon_u)_{u=j\ell+1-M}^{(j+1)\ell+M}$ . Let  $h: \mathbb{R}^2 \rightarrow \mathbb{R}$  be a Lipschitz-continuous function, let  $f_1, f_2: \mathbb{R}^{\ell+2M} \rightarrow \mathbb{R}$  be measurable functions and let  $U_N$  be defined by*

$$U_N := \sum_{1 \leq j < k \leq N} (h(f_1(V_j), f_1(V_k)) - h(f_2(V_j), f_2(V_k))).$$

Then the following inequality holds

$$N^{-3/2} \|U_N - \mathbb{E}[U_N]\|_2 \leq C \|f_1(V_0) - f_2(V_0)\|_2,$$

where  $C$  is a constant depending only on  $h$ .

*Proof.* The difficulty here lies in the fact that the vectors  $V_j, j \geq 1$ , are not independent. Denote

$$H_{j,k} := h(f_1(V_j), f_1(V_k)) - h(f_2(V_j), f_2(V_k)).$$

We will prove the inequality

$$\|U_N - \mathbb{E}[U_N]\|_2 \leq CN^{3/2} \sup_{k \geq 1} \|H_{0,k}\|_2,$$

from which the assertion then follows by the Lipschitz-continuity of  $h$ . To verify the above inequality, we will distinguish between the cases where  $N$  is even and those where  $N$  is odd. Let us first consider even values of  $N$ , in which case we can write  $2N$  instead of  $N$ . Denote by  $\mathcal{F}_k$  the  $\sigma$ -algebra generated by the random variables  $V_{k'}$  for  $k' \leq k$ . Then it holds

$$\begin{aligned} \|U_{2N} - \mathbb{E}[U_{2N}]\|_2 &= \left\| \sum_{k=2}^{2N} \sum_{j=1}^{k-1} (H_{j,k} - \mathbb{E}[H_{j,k} | \mathcal{F}_{k-2}]) + \sum_{k=2}^{2N} \sum_{j=1}^{k-1} (\mathbb{E}[H_{j,k} | \mathcal{F}_{k-2}] - \mathbb{E}[H_{j,k}]) \right\|_2 \\ &\leq \left\| \sum_{i=1}^N \sum_{j=1}^{2i-1} (H_{j,2i} - \mathbb{E}[H_{j,2i} | \mathcal{F}_{2i-2}]) \right\|_2 + \left\| \sum_{i=1}^{N-1} \sum_{j=1}^{2i} (H_{j,2i+1} - \mathbb{E}[H_{j,2i+1} | \mathcal{F}_{2i+1-2}]) \right\|_2 \\ &\quad + \left\| \sum_{k=2}^{2N} \sum_{j=1}^{k-2} (\mathbb{E}[H_{j,k} | \mathcal{F}_{k-2}] - \mathbb{E}[H_{j,k}]) \right\|_2 + \left\| \sum_{k=2}^{2N} (\mathbb{E}[H_{k-1,k} | \mathcal{F}_{k-2}] - \mathbb{E}[H_{k-1,k}]) \right\|_2. \end{aligned}$$

For the first two terms, we additionally define  $d_i := \sum_{j=1}^{2i-1} (H_{j,2i} - \mathbb{E}[H_{j,2i} | \mathcal{F}_{2i-2}])$  and  $d'_i := \sum_{j=1}^{2i} (H_{j,2i+1} - \mathbb{E}[H_{j,2i+1} | \mathcal{F}_{2i+1-2}])$ , such that the sequences  $(d_i, \mathcal{F}_{2i})_{i \geq 1}$  and  $(d'_i, \mathcal{F}_{2i+1})_{i \geq 1}$  are martingale differences. For the third term, we use the independence between  $V_k$  and  $\mathcal{F}_{k-2}$  to get that  $\mathbb{E}[H_{j,k} | \mathcal{F}_{k-2}] = \mathbb{E}[H_{j,-1} | V_j]$ , and we have to bound the moments of a two-dependent identically distributed centred sequence. For the fourth term, we simply use the triangle inequality. By orthogonality of  $(d_i)_{i \geq 1}$  and orthogonality of  $(d'_i)_{i \geq 1}$ , it follows

$$\begin{aligned} \|U_{2N} - \mathbb{E}[U_{2N}]\|_2 &\leq \left( \sum_{i=1}^N \left\| \sum_{j=1}^{2i-1} (H_{j,2i} - \mathbb{E}[H_{j,2i} | \mathcal{F}_{2i-2}]) \right\|_2 \right)^{1/2} \\ &\quad + \left( \sum_{i=1}^{N-1} \left\| \sum_{j=1}^{2i} (H_{j,2i+1} - \mathbb{E}[H_{j,2i+1} | \mathcal{F}_{2i+1-2}]) \right\|_2 \right)^{1/2} \\ &\quad + \left\| \sum_{k=2}^{2N} \sum_{j=1}^{k-2} (\mathbb{E}[H_{j,-1} | V_j] - \mathbb{E}[H_{j,-1}]) \right\|_2 + 4N \sup_{k \geq 1} \|H_{0,k}\|_2. \end{aligned}$$

The first of the above terms can be further bounded via

$$\begin{aligned} &\left( \sum_{i=1}^N \left\| \sum_{j=1}^{2i-1} (H_{j,2i} - \mathbb{E}[H_{j,2i} | \mathcal{F}_{2i-2}]) \right\|_2 \right)^{1/2} \leq \left( \sum_{i=1}^N \left( \sum_{j=1}^{2i-1} \|(H_{j,2i} - \mathbb{E}[H_{j,2i} | \mathcal{F}_{2i-2}])\|_2 \right)^2 \right)^{1/2} \\ &\leq \left( \sum_{i=1}^N 4 \left( \sum_{j=1}^{2i-1} \|H_{j,2i}\|_2 \right)^2 \right)^{1/2} \leq \left( \sum_{i=1}^N 16i^2 \sup_{k \geq 1} \|H_{0,k}\|_2^2 \right)^{1/2} \leq 4N^{3/2} \sup_{k \geq 1} \|H_{0,k}\|_2. \end{aligned}$$

The second term can be treated analogously. In order to bound the third term, we switch the sums over  $j$  and  $k$  to obtain

$$\begin{aligned} &\left\| \sum_{k=2}^{2N} \sum_{j=1}^{k-2} (\mathbb{E}[H_{j,-1} | V_j] - \mathbb{E}[H_{j,-1}]) \right\|_2 \\ &= \left\| \sum_{j=1}^{2N-2} (2N - j + 1) (\mathbb{E}[H_{j,-1} | V_j] - \mathbb{E}[H_{j,-1}]) \right\|_2 \end{aligned}$$

$$\begin{aligned}
&\leq \left\| \sum_{i=1}^{N-1} (2N - 2i + 1) (\mathbb{E}[H_{2i,-1} | V_{2i}] - \mathbb{E}[H_{2i,-1}]) \right\|_2 \\
&\quad + \left\| \sum_{i=1}^{N-1} (2N - (2i - 1) + 1) (\mathbb{E}[H_{2i-1,-1} | V_{2i-1}] - \mathbb{E}[H_{2i-1,-1}]) \right\|_2 \\
&= \sqrt{\sum_{i=1}^{N-1} (2N - 2i + 1)^2 \|\mathbb{E}[H_{0,-1} | V_0] - \mathbb{E}[H_{0,-1}]\|_2^2} \\
&\quad + \sqrt{\sum_{i=1}^{N-1} (2N - (2i - 1) + 1)^2 \|\mathbb{E}[H_{0,-1} | V_0] - \mathbb{E}[H_{0,-1}]\|_2^2} \\
&\leq CN^{3/2} \sup_{k \geq 1} \|H_{0,k}\|_2.
\end{aligned}$$

This proves  $\|U_{2N} - \mathbb{E}[U_{2N}]\|_2 \leq CN^{3/2} \sup_{k \geq 1} \|H_{0,k}\|_2$ . In order to show the corresponding inequality for the index  $2N + 1$  instead of  $2N$ , we note that  $U_{2N+1} - \mathbb{E}[U_{2N+1}]$  differs from  $U_{2N} - \mathbb{E}[U_{2N}]$  only by the term  $\sum_{j=1}^{2N} (H_{j,2N+1} - \mathbb{E}[H_{j,2N+1}])$ , whose  $\mathbb{L}^2$ -norm is smaller than  $4N \sup_{k \geq 1} \|H_{0,k}\|_2$ . This ends the proof of Lemma A.5.  $\square$

### A.3. Tools for the proof of Lemma 3.2.

**Lemma A.6.** *Let  $(Y_n)_{n \geq 1}$  be a sequence of random variables such that  $(Y_n^2)_{n \geq 1}$  is uniformly integrable and  $Y_n \rightarrow N(0, \sigma^2)$  in distribution with  $\sigma > 0$ . Let  $Y'_n$  and  $Y''_n$  be independent copies of  $Y_n$  and let  $h: \mathbb{R}^2 \rightarrow \mathbb{R}$  be a Lipschitz-continuous function. Then*

$$\lim_{n \rightarrow \infty} \text{Cov}(h(Y_n, Y'_n), h(Y_n, Y''_n)) = \text{Cov}(h(N, N'), h(N, N'')),$$

where  $N, N', N''$  are independent  $N(0, \sigma^2)$ -distributed random variables.

*Proof.* By independence, the sequence of random vectors  $(Y_n, Y'_n, Y''_n)$  converges in distribution to  $(N, N', N'')$ . By Skorohod's representation theorem, there exist a probability space  $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ , sequences of random variables  $(Z_n)_{n \geq 1}$ ,  $(Z'_n)_{n \geq 1}$  and  $(Z''_n)_{n \geq 1}$  and random variables  $Z, Z'$  and  $Z''$ , each defined on  $\tilde{\Omega}$ , such that for all  $n \geq 1$ , the vectors  $(Y_n, Y'_n, Y''_n)$  and  $(Z_n, Z'_n, Z''_n)$  have the same distribution,  $(Z, Z', Z'')$  has the same distribution as  $(N, N', N'')$ , and the sequence  $(Z_n)_{n \geq 1}$  (respectively  $(Z'_n)_{n \geq 1}$  and  $(Z''_n)_{n \geq 1}$ ) converges to  $Z$  (respectively  $Z'$  and  $Z''$ ) almost surely. Note that for each fixed  $n$ , it holds

$$\text{Cov}(h(Y_n, Y'_n), h(Y_n, Y''_n)) = \text{Cov}(h(Z_n, Z'_n), h(Z_n, Z''_n))$$

as well as

$$\text{Cov}(h(N, N'), h(N, N'')) = \text{Cov}(h(Z, Z'), h(Z, Z'')).$$

Due to the elementary fact that  $\text{Cov}(U_n, V_n) \rightarrow \text{Cov}(U, V)$  if  $U_n \rightarrow U$  and  $V_n \rightarrow V$  in  $\mathbb{L}^2$ , it hence suffices to show

$$\|h(Z_n, Z'_n) - h(Z, Z')\|_2 \rightarrow 0.$$

Since  $h$  is Lipschitz-continuous and the sequence  $(Z_n^2 + (Z'_n)^2 + Z^2 + (Z')^2)_{n \geq 1}$  is uniformly integrable, the sequence  $((h(Z_n, Z'_n) - h(Z, Z'))^2)_{n \geq 1}$  is uniformly integrable as well. By the continuity of  $h$ , the sequence  $((h(Z_n, Z'_n) - h(Z, Z'))^2)_{n \geq 1}$  converges to 0 almost surely. Combined, this yields the desired  $\mathbb{L}^2$ -convergence and finishes the proof.  $\square$

**Acknowledgment:** The authors would like to thank the referee for his/her comments that improved the presentation of the paper.



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