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# Multidimensional Stein method and quantitative asymptotic independence

Ciprian A. Tudor

CNRS, Université de Lille  
Laboratoire Paul Painlevé UMR 8524  
F-59655 Villeneuve d'Ascq, France.  
ciprian.tudor@univ-lille.fr

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## Abstract

Let  $(X_1, X_2, \dots, X_n)$  be a random vector and denote by  $P_{(X_1, X_2, \dots, X_n)}$  its probability distribution on  $\mathbb{R}^n$ . Inspired by [11], we develop a multidimensional Stein-Malliavin calculus which allows to measure the Wasserstein distance between the law  $P_{(X_1, X_2, \dots, X_n)}$  and the probability distribution  $P_Z \otimes P_{(X_2, \dots, X_n)}$ , where  $Z$  is a Gaussian random variable. That is, we give estimates, in terms of the Malliavin operators, for the distance between the law of the random vector  $(X_1, \dots, X_n)$  and the law of the vector  $(Z, X_2, \dots, X_n)$ , where  $Z$  is Gaussian and independent of  $(X_2, \dots, X_n)$ . Then we focus on the particular case of random vectors in Wiener chaos and we give an asymptotic version of this result. In this case, if we consider a  $n$ -dimensional sequence with components in Wiener chaos, such that the first component is asymptotically Gaussian and the rest of the vector converges to an arbitrary limit, we are able to evaluate the distance between this vector and its limit. To illustrate our method, we derive the rate of convergence for the Wasserstein distance for a two-dimensional sequence of multiple stochastic integrals, the first converging to a normal law and the second to a Rosenblatt distribution.

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**Key words:** Stein's method, Malliavin calculus, multiple stochastic integrals, asymptotic independence.

## 1 Introduction

The Stein's method constitutes a collection of mathematical techniques that allow to give quantitative bounds for the distance between the probability distributions of random variables. It has been initially introduced in the paper [13] and then developed by many authors. We refer, among many others to the monographs and surveys [3], [12], [14] for a detailed description of this method. Of

particular interest is the situation when one random variable is Gaussian, but the cases of other target distributions has been analyzed in the literature.

A more recent theory is the so-called Stein-Malliavin calculus which combines the Stein's method with the techniques of the Malliavin calculus. The first work in this direction is [4] (see [5] for a more detailed exposition) and since, numerous authors extended, refined or applied this theory. In this theory, the bounds obtained for the distance between the law of an arbitrary random variable and the target distribution are given in terms of the Malliavin operators.

The starting point of the Stein's method for normal approximation is the following observation:  $Z \sim N(0, \sigma^2)$  with  $\sigma > 0$  if and only if

$$\sigma^2 \mathbf{E}f'(Z) - \mathbf{E}Zf(Z) = 0$$

for every function  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that  $\mathbf{E}|f'(Z)| < \infty$ . Then, one can think that if a random variable  $X$  has the property that  $\sigma^2 \mathbf{E}f'(X) - \mathbf{E}Xf(X)$  is close to zero for a large class of functions  $f$ , then the probability distribution of  $X$  should be close to  $N(0, \sigma^2)$ . From this observation, the whole Stein's theory has been constructed, leading to various bounds for the distance between the probability law of the random variable  $X$  and the normal distribution  $N(0, \sigma^2)$ .

In this work, we deal with a variant of this method recently developed in the reference [11] that allows to measure the distance between the components of a random vector  $(X_1, X_2)$ , where  $X_1 \sim N(0, \sigma^2)$  and  $X_2$  has an arbitrary distribution. The nice observation made in [11] is that  $X_1 \sim N(0, \sigma^2)$  and  $X_1$  is independent of  $X_2$  if and only if

$$\sigma^2 \mathbf{E}\partial_{x_1} f(X_1, X_2) - \mathbf{E}X_1 f(X_1, X_2) = 0$$

for a large class of differentiable functions  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ . We denoted by  $\partial_{x_1} f$  the partial derivative of  $f$  with respect to its first variable. As in the standard Stein's method, one follows the intuition that if some random vector  $(X_1, X_2)$  satisfies that  $\sigma^2 \mathbf{E}\partial_{x_1} f(X_1, X_2) - \mathbf{E}X_1 f(X_1, X_2)$  is close to zero, then  $X_1$  should be close in law to  $N(0, \sigma^2)$  and  $P_{(X_1, X_2)}$  should be close to  $P_{X_1} \otimes P_{X_2}$ , where  $P_{(X_1, X_2)}$  is the law of  $(X_1, X_2)$  while  $P_X$  stands for the law of the random variable  $X$ . By combining this idea with Malliavin calculus, in [11] one gives bounds for the Wasserstein distance between  $P_{(X_1, X_2)}$  and  $P_{X_1} \otimes P_{X_2}$  in terms of the Malliavin operators.

Our purpose is, in a first step, to generalize the above idea by considering random vectors of arbitrary dimension. Then, we focus on the particular case of sequences of random variables belonging to a Wiener chaos and we give asymptotic-type results. We consider a random sequence  $(X_k, k \geq 1)$  that belongs to a Wiener chaos of fixed order which converges in law as  $k \rightarrow \infty$  to  $Z \sim N(0, \sigma^2)$  and a multidimensional sequence  $(Y_k, k \geq 1) = (Y_{1,k}, \dots, Y_{n,k}, k \geq 1)$  also with components in Wiener chaos which converges in law, as  $k \rightarrow \infty$ , to a random vector  $Y = (Y_1, \dots, Y_n)$  and we obtain explicit quantitative estimates for the Wasserstein distance between  $P_{(X_k, Y_k)}$  and  $P_Z \otimes P_Y$  for  $k$  large enough. In particular, if  $X_k$  belongs to the  $q$ th Wiener chaos and  $Y_{j,k}$  belongs to the  $q_j$ th Wiener chaos with  $q_j < q$  for all  $j = 1, 2, \dots, n$  then the sequences  $(X_k, k \geq 1)$  and  $(Y_k, k \geq 1) = (Y_{1,k}, \dots, Y_{n,k}, k \geq 1)$  are asymptotically independent, in the sense that their limits  $Z$  and  $(Y_1, \dots, Y_n)$  are independent. This fact is related to the main findings in [6] and [7]. As an application, we derive the rate of convergence for the Wasserstein distance for a two-dimensional sequence of multiple stochastic integrals, the first converging to a normal law and the second to a Rosenblatt distribution.

We organized our paper as follows. In Section 2 we introduce and analyze the main elements of the multidimensional Stein method: the Stein operator, the Stein equation and its solution. In Section 3 we combine the Stein method with the tools of Malliavin calculus in order to obtain estimates for the Wasserstein distance between random vectors with the same marginals but with different correlation structure between their components. In Section 4 we state and prove asymptotic versions of the these results, by focusing on random vectors with components in Wiener chaos. We illustrate our method by an example treated in Section 5, by regarding the situation of a two-dimensional random sequence whose first component converges to a Gaussian law and the second component satisfies a non-central limit theorem. Section 6 is the Appendix where we included the basic concepts of Wiener chaos and Malliavin calculus needed in our work.

## 2 Multidimensional Stein method

The basis of the Stein's method consists in the definition of the Stein's operator and of the Stein's equation. For the normal approximation, the standard operator is

$$\mathcal{L}f(x) = \sigma^2 f'(x) - xf(x), \quad x \in \mathbb{R},$$

which acts on suitable differentiable functions  $f : \mathbb{R} \rightarrow \mathbb{R}$ . This operator satisfies  $\mathbf{E}\mathcal{L}(Z) = 0$  if and only if  $Z \sim N(0, \sigma^2)$ . The corresponding Stein's equation is

$$\mathcal{L}f(x) = \mathbf{E}h(x) - \mathbf{E}h(Z), \quad x \in \mathbb{R},$$

where  $h : \mathbb{R} \rightarrow \mathbb{R}$  is a given function such that  $\mathbf{E}|h(Z)| < \infty$ . The idea of the Stein's method is to find a solution  $f_h$  to the Stein's equation with nice properties and to use it in order to obtain estimates for  $\mathbf{E}h(X) - \mathbf{E}h(Z)$  for an arbitrary random variable  $X$ .

We follow the same line in a multidimensional context. Now, the purpose is not the normal approximation but to quantify the distance between the probability distribution of a random vector  $(X_1, \dots, X_n)$  and the random vector  $(Z, X_2, \dots, X_n)$  where  $Z$  is a centered Gaussian random variable with variance  $\sigma^2$  and it is independent of  $(X_2, \dots, X_n)$ .

Let us consider the operator  $\mathcal{N}$  given by

$$\mathcal{N}f(x_1, \dots, x_n) = \sigma^2 \partial_{x_1} f(x_1, \dots, x_n) - x_1 f(x_1, \dots, x_n), \quad x_1, \dots, x_n \in \mathbb{R}, \quad (1)$$

where  $\partial_{x_1} f$  denotes the partial derivative of  $f$  with respect to its first variable. The operator  $\mathcal{N}$  acts on the set of differentiable functions  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ .

Recall that if  $X$  is a random variable, we denote by  $P_X$  its probability distribution while for a random vector  $(X_1, \dots, X_n)$ , we use the notation  $P_{(X_1, \dots, X_n)}$ . The following two lemmas show that the operator (1) characterizes the law of  $X_1$  and the independence of  $X_1$  and  $(X_2, \dots, X_n)$ . The material from this section is inspired from Section 5 in [11].

**Lemma 1.** *Assume  $X_1 \sim N(0, \sigma^2)$  and  $X_1$  is independent of the random vector  $(X_2, \dots, X_n)$ . Then  $\mathbf{E}\mathcal{N}(X_1, \dots, X_n) = 0$  for all  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  differentiable with  $\mathbf{E}|\partial_{x_1} f(X_1, \dots, X_n)| < \infty$ .*

**Proof:** By the standard Stein method, for all  $x_2, \dots, x_n \in \mathbb{R}$ ,

$$\sigma^2 \mathbf{E} \partial_{x_1} f(X_1, x_2, \dots, x_n) = \mathbf{E} X_1 f(X_1, x_2, \dots, x_n)$$

or

$$\sigma^2 \int_{\mathbb{R}} \partial_{x_1} f(x_1, \dots, x_n) dP_{X_1}(x_1) = \int_{\mathbb{R}} x_1 f(x_1, \dots, x_n) dP_{X_1}(x_1).$$

Let us integrate with respect to the probability measure  $P_{(X_2, \dots, X_n)}$ . We have

$$\begin{aligned} & \sigma^2 \int_{\mathbb{R}^{n-1}} \left( \int_{\mathbb{R}} \partial_{x_1} f(x_1, \dots, x_n) dP_{X_1}(x_1) \right) dP_{(X_2, \dots, X_n)}(x_2, \dots, x_n) \\ &= \sigma^2 \int_{\mathbb{R}^n} \partial_{x_1} f(x_1, \dots, x_n) dP_{X_1}(x_1) \otimes dP_{(X_2, \dots, X_n)}(x_2, \dots, x_n) \\ &= \sigma^2 \int_{\mathbb{R}^n} \partial_{x_1} f(x_1, \dots, x_n) dP_{(X_1, \dots, X_n)}(x_1, \dots, x_n) = \sigma^2 \mathbf{E} \partial_{x_1} f(X_1, \dots, X_n), \end{aligned}$$

where we used the independence of  $X_1$  and  $(X_2, \dots, X_n)$  for the first equality on the above line. Similarly,

$$\begin{aligned} & \int_{\mathbb{R}^{n-1}} \left( \int_{\mathbb{R}} x_1 f(x_1, \dots, x_n) dP_{X_1}(x_1) \right) dP_{(X_1, \dots, X_n)}(x_2, \dots, x_n) \\ &= \int_{\mathbb{R}^n} x_1 f(x_1, \dots, x_n) dP_{X_1}(x_1) \otimes P_{(X_2, \dots, X_n)}(x_1, \dots, x_n) = \int_{\mathbb{R}^n} x_1 f(x_1, \dots, x_n) dP_{(X_1, \dots, X_n)}(x_1, \dots, x_n) \\ &= \mathbf{E} X_1 f(X_1, \dots, X_n). \end{aligned}$$

■

We also have a lemma in the converse direction. By  $\|\cdot\|$  we denote the infinity norm on  $\mathbb{R}^n$ .

**Lemma 2.** Consider a random vector  $(X_1, \dots, X_n)$  with  $\mathbf{E}|X_1| < \infty$ . Assume that

$$\mathbf{E} \mathcal{N}(X_1, \dots, X_n) = 0 \tag{2}$$

for all differentiable functions  $f$  with  $\|\partial_{x_1} f\|_{\infty} < \infty$ . Then  $X_1 \sim N(0, \sigma^2)$  and  $X_1$  is independent of  $(X_2, \dots, X_n)$ .

**Proof:** Let  $\varphi$  be the characteristic function of the vector  $(X_1, \dots, X_n)$ , i.e.

$$\varphi(\lambda_1, \dots, \lambda_n) = \mathbf{E} \left( e^{i(\lambda_1 X_1 + \dots + \lambda_n X_n)} \right),$$

for  $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ . By applying (2) for the real and imaginary parts of  $\varphi$ , we get

$$\begin{aligned} \partial_{\lambda_1} \varphi(\lambda_1, \dots, \lambda_n) &= i \mathbf{E} \left( X_1 e^{i(\lambda_1 X_1 + \dots + \lambda_n X_n)} \right) \\ &= i \sigma^2 \mathbf{E} \left( \partial_{x_1} e^{i(\lambda_1 X_1 + \dots + \lambda_n X_n)} \right) = -\lambda_1 \sigma^2 \varphi(\lambda_1, \dots, \lambda_n). \end{aligned}$$

By noticing that for every  $\lambda_2, \dots, \lambda_n \in \mathbb{R}$ ,  $\varphi(0, \lambda_2, \dots, \lambda_n) = \varphi_{(X_2, \dots, X_n)}(\lambda_2, \dots, \lambda_n)$  (the characteristic function of the vector  $(X_2, \dots, X_n)$ ), we obtain

$$\varphi(\lambda_1, \dots, \lambda_n) = \varphi_{(X_2, \dots, X_n)}(\lambda_2, \dots, \lambda_n) e^{-\frac{\sigma^2 \lambda_1^2}{2}},$$

and this implies  $X_1 \sim N(0, \sigma^2)$  and  $X_1$  independent of  $(X_2, \dots, X_n)$ . ■

Let us now introduce the multidimensional Stein's equation

$$\mathcal{N}f(x_1, \dots, x_n) = h(x_1, \dots, x_n) - \mathbf{E}h(Z, x_2, \dots, x_n), \quad x_1, \dots, x_n \in \mathbb{R}, \quad (3)$$

where  $Z \sim N(0, \sigma^2)$ . In (3),  $h : \mathbb{R}^n \rightarrow \mathbb{R}$  is given and we assume that  $h$  is continuously differentiable with bounded partial derivatives. Let us show that (3) admits a solution with suitable properties.

**Proposition 1.** *Let  $h : \mathbb{R}^n \rightarrow \mathbb{R}$  be continuously differentiable with bounded partial derivatives. Then (3) admits a unique bounded solution which is given by*

$$f_h(x_1, \dots, x_n) = -\frac{1}{\sigma^2} \int_0^1 \frac{1}{2\sqrt{t(1-t)}} \mathbf{E} \left[ Zh \left( \sqrt{t}x_1 + \sqrt{1-t}Z, x_2, \dots, x_n \right) \right] dt. \quad (4)$$

Moreover, we have the following bounds:

1.

$$\|f_h\|_\infty \leq \|\partial_{x_1} h\|_\infty. \quad (5)$$

2.

$$\|\partial_{x_1} f_h\|_\infty \leq \frac{1}{\sigma} \sqrt{\frac{2}{\pi}} \|\partial_{x_1} h\|_\infty. \quad (6)$$

3. For  $j = 2, \dots, n$ ,

$$\|\partial_{x_j} f_h\|_\infty \leq \frac{1}{\sigma} \sqrt{\frac{\pi}{2}} \|\partial_{x_j} h\|_\infty. \quad (7)$$

**Proof:** By using the dominated convergence theorem, we get, by taking the derivative with respect to  $x_1$  in (4),

$$\partial_{x_1} f_h(x_1, \dots, x_n) = -\frac{1}{\sigma^2} \int_0^1 \frac{1}{2\sqrt{1-t}} \mathbf{E} \left[ Z \partial_{x_1} h \left( \sqrt{t}x_1 + \sqrt{1-t}Z, x_2, \dots, x_n \right) \right] dt. \quad (8)$$

Now, we apply the standard Stein identity to the function  $g(x) = h(\sqrt{t}x_1 + \sqrt{1-t}x, x_2, \dots, x_n)$  and we obtain

$$\begin{aligned} & \mathbf{E} \left[ Z \partial_{x_1} h \left( \sqrt{t}x_1 + \sqrt{1-t}Z, x_2, \dots, x_n \right) \right] \\ &= \mathbf{E}g'(Z) = \sigma^2 \sqrt{1-t} \mathbf{E} \left[ \partial_{x_1} h \left( \sqrt{t}x_1 + \sqrt{1-t}Z, x_2, \dots, x_n \right) \right]. \end{aligned} \quad (9)$$

By plugging (9) into (4), the function  $f_h$  can be written as

$$f_h(x_1, \dots, x_n) = -\int_0^1 \frac{1}{2\sqrt{t}} \mathbf{E} \left[ \partial_{x_1} h \left( \sqrt{t}x_1 + \sqrt{1-t}Z, x_2, \dots, x_n \right) \right] dt. \quad (10)$$

By (8) and (10), we can write

$$\begin{aligned}
& \partial_{x_1} f_h(x_1, \dots, x_n) - x_1 f_h(x_1, \dots, x_n) \\
&= \int_0^1 \mathbf{E} \left[ \left( -\frac{Z}{2\sqrt{1-t}} + \frac{x_1}{2\sqrt{t}} \right) \partial_{x_1} h \left( \sqrt{t}x_1 + \sqrt{1-t}Z, x_2, \dots, x_n \right) \right] \\
&= \mathbf{E} \int_0^1 \frac{d}{dt} h \left( \sqrt{t}z_1 + \sqrt{1-t}Z, x_2, \dots, x_n \right) dt = h(x_1, \dots, x_n) - \mathbf{E}h(Z, x_2, \dots, x_n).
\end{aligned}$$

Consequently,  $f_h$  given by (4) is a solution to (3). To prove (5), we use (10) to get

$$\|f_h\|_\infty \leq \int_0^1 \frac{1}{2\sqrt{t}} \|\partial_{x_1} h\|_\infty \leq \|\partial_{x_1} h\|_\infty$$

The bound (6) follows from (8) since

$$\|\partial_{x_1} f_h\|_\infty \leq \frac{\mathbf{E}|Z|}{\sigma^2} \|\partial_{x_1} h\|_\infty \leq \sigma^{-1} \sqrt{\frac{2}{\pi}} \|\partial_{x_1} h\|_\infty.$$

To prove (7), we differentiate with respect to  $x_j, j = 2, \dots, n$  in (4),

$$\partial_{x_j} f_h(x_1, \dots, x_n) = -\frac{1}{\sigma^2} \int_0^1 \frac{1}{2\sqrt{t(1-t)}} \mathbf{E} \left[ Z \partial_{x_j} h \left( \sqrt{t}x_1 + \sqrt{1-t}Z, x_2, \dots, x_n \right) \right] dt$$

and

$$\|\partial_{x_j} f_h\|_\infty \leq \frac{\mathbf{E}|Z|}{\sigma^2} \|\partial_{x_j} h\|_\infty \int_0^1 \frac{1}{2\sqrt{t(1-t)}} dt = \frac{1}{\sigma} \sqrt{\frac{\pi}{2}} \|\partial_{x_j} h\|_\infty.$$

To finish the proof, we notice that for any other solution  $g_h$  to (3), one has

$$\partial_{x_1} \left( e^{-\frac{x_1^2}{2\sigma^2}} (f_h(x_1, \dots, x_n) - g_h(x_1, \dots, x_n)) \right) = 0$$

so  $g_h(x_1, \dots, x_n) = f_h(x_1, \dots, x_n) + e^{\frac{x_1^2}{2\sigma^2}} c(x_2, \dots, x_n)$  so  $g_h$  is bounded if and only if  $c(x_2, \dots, x_n) = 0$ . ■

By Proposition 1, if  $f_h$  is the solution (4) to the Stein's equation (3), we have

$$\sigma^2 \partial_{x_1} f_h(x_1, \dots, x_n) - x_1 f_h(x_1, \dots, x_n) = h(x_1, \dots, x_n) - \mathbf{E}h(Z, x_2, \dots, x_n)$$

for any  $h$  differentiable with bounded partial derivatives. Let  $X_1, \dots, X_n$  be random variables with  $\mathbf{E}|X_1| < \infty$ . Let us integrate with respect to  $\theta := P_{(X_1, \dots, X_n)}$  in the above identity. We have

$$\int_{\mathbb{R}^n} h(x_1, \dots, x_n) d\theta(x_1, \dots, x_n) = \mathbf{E}h(X_1, \dots, X_n)$$

and

$$\begin{aligned}
& \int_{\mathbb{R}^n} \mathbf{E}h(Z, x_2, \dots, x_n) d\theta(x_1, \dots, x_n) \\
&= \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}} h(z, x_2, \dots, x_n) dP_Z(z) \right) d\theta(x_1, \dots, x_n) \\
&= \int_{\mathbb{R}^{n-1}} \left( \int_{\mathbb{R}} h(z, x_2, \dots, x_n) dP_Z(z) \right) dP_{(X_2, \dots, X_n)}(x_2, \dots, x_n) \\
&= \int_{\mathbb{R}^n} h(x_1, x_2, \dots, x_n) dP_Z \otimes P_{(X_2, \dots, X_n)}(x_2, \dots, x_n) = \int_{\mathbb{R}^n} h(x_1, x_2, \dots, x_n) d\eta(x_1, \dots, x_n),
\end{aligned}$$

with

$$\eta = P_Z \otimes P_{(X_2, \dots, X_n)}.$$

Therefore

$$\begin{aligned}
& \sigma^2 \mathbf{E} \partial_{X_1} f_h(X_1, \dots, X_n) - \mathbf{E} X_1 f_h(X_1, \dots, X_n) = \mathbf{E} h(X_1, \dots, X_n) - \mathbf{E} h(X'_1, X_2, \dots, X_n) \\
&= \int_{\mathbb{R}^n} h(x_1, x_2, \dots, x_n) d\theta(x_1, \dots, x_n) - \int_{\mathbb{R}^n} h(x_1, x_2, \dots, x_n) d\eta(x_1, \dots, x_n) \tag{11}
\end{aligned}$$

where  $X'_1$  has the same law as  $Z \sim N(0, \sigma^2)$  and  $X'_1$  is independent of  $(X_2, \dots, X_n)$ .

### 3 Stein method and Malliavin calculus

We use the ideas of the Stein method for normal approximation (see [5]) to prove the following result.

Let

$$\mathcal{A} = \{h : \mathbb{R}^n \rightarrow \mathbb{R}, h \text{ is Lipschitz continuous with } \|h\|_{Lip} \leq 1\}$$

and let  $F, G$  be two  $n$ -dimensional random vectors such that  $h(F), h(G) \in L^1(\Omega)$  for every  $h \in \mathcal{A}$ . Then the Wasserstein distance between the probability distributions of  $F$  and  $G$  is defined by

$$d_W(P_F, P_G) = \sup_{h \in \mathcal{A}} |\mathbf{E}h(F) - \mathbf{E}h(G)|. \tag{12}$$

We denote by  $\|h\|_{Lip}$  the Lipschitz norm of  $h$  given by

$$\|h\|_{Lip} = \sup_{x, y \in \mathbb{R}^n, x \neq y} \frac{|h(x) - h(y)|}{\|x - y\|},$$

with  $\|\cdot\|$  the Euclidean norm in  $\mathbb{R}^n$ . The operators  $D, L, \delta$  below are defined with respect to an isonormal process  $(W(h), h \in H)$ , see the Appendix. By  $\langle \cdot, \cdot \rangle$  we denote the scalar product in the Hilbert space  $H$ .

**Theorem 1.** *Let  $X_1, \dots, X_n$  be centered random variables in  $\mathbb{D}^{1,2}$ . Let  $\theta = P_{(X_1, \dots, X_n)}$  and  $\eta = P_Z \otimes P_{(X_2, \dots, X_n)}$ , where  $Z \sim N(0, \sigma^2)$ . Then*

$$d_W(\theta, \eta) \leq C \left( \mathbf{E} |\sigma^2 - \langle D(-L)^{-1} X_1, DX_1 \rangle| + \sum_{j=2}^n \mathbf{E} |\langle D(-L)^{-1} X_1, DX_j \rangle| \right). \tag{13}$$



**Proof:** Let  $h : \mathbb{R}^n \rightarrow \mathbb{R}$  be continuously differentiable with bounded derivatives and let  $f_h$  be the corresponding solution to the Stein's equation (3). By using the well-known formula  $X_1 = \delta D(-L)^{-1} X_1$  in (11), we obtain, by integrating by parts

$$\begin{aligned}
& \int_{\mathbb{R}^n} h(x_1, x_2, \dots, x_n) d\theta(x_1, \dots, x_n) - \int_{\mathbb{R}^n} h(x_1, x_2, \dots, x_n) d\eta(x_1, \dots, x_n) \\
&= \sigma^2 \mathbf{E} \partial_{x_1} f_h(X_1, \dots, X_n) - \mathbf{E} \delta D(-L)^{-1} X_1 f_h(X_1, \dots, X_n) \\
&= \sigma^2 \mathbf{E} \partial_{x_1} f_h(X_1, \dots, X_n) - \mathbf{E} \langle D(-L)^{-1} X_1, Df_h(X_1, \dots, X_n) \rangle \\
&= \sigma^2 \mathbf{E} \partial_{x_1} f_h(X_1, \dots, X_n) - \mathbf{E} \sum_{j=1}^n \partial_{x_j} f_h(X_1, \dots, X_n) \langle D(-L)^{-1} X_1, DX_j \rangle \\
&= \mathbf{E} \partial_{x_1} f_h(X_1, \dots, X_n) (\sigma^2 - \langle D(-L)^{-1} X_1, DX_1 \rangle) \\
&\quad - \mathbf{E} \sum_{j=2}^n \partial_{x_j} f_h(X_1, \dots, X_n) \langle D(-L)^{-1} X_1, DX_j \rangle.
\end{aligned}$$

Hence, by using inequalities (6) and (7) in Proposition 1,

$$\begin{aligned}
& \left| \int_{\mathbb{R}^n} h(x_1, x_2, \dots, x_n) d\theta(x_1, \dots, x_n) - \int_{\mathbb{R}^n} h(x_1, x_2, \dots, x_n) d\eta(x_1, \dots, x_n) \right| \\
&\leq C \left( \mathbf{E} |\sigma^2 - \langle D(-L)^{-1} X_1, DX_1 \rangle| + \sum_{j=2}^n \mathbf{E} |\langle D(-L)^{-1} X_1, DX_j \rangle| \right). \tag{14}
\end{aligned}$$

To finish the proof, we borrow again an argument from [11] (proof of Lemma 9 in this reference) to approximate a Lipschitz function by continuously differentiable functions with bounded derivatives. Indeed, if  $h \in \mathcal{A}$  and  $\varepsilon > 0$ , then consider

$$h_\varepsilon(x_1, \dots, x_n) = \mathbf{E} h(x + \sqrt{\varepsilon} N_1, \dots, x_n + \sqrt{\varepsilon} N_n),$$

where  $N_1, \dots, N_n$  are independent standard normal random variables. Then  $h_\varepsilon$  is differentiable and it satisfies

$$\|h_\varepsilon - h\|_\infty \rightarrow_{\varepsilon \rightarrow 0} 0, \quad \max_{j=1, \dots, n} \|\partial_{x_j} h_\varepsilon\|_\infty \leq \|h_\varepsilon\|_{Lip} \leq \|h\|_{Lip} \leq 1.$$

Therefore, by (14),

$$\begin{aligned}
& \left| \int_{\mathbb{R}^n} h(x_1, x_2, \dots, x_n) d\theta(x_1, \dots, x_n) - \int_{\mathbb{R}^n} h(x_1, x_2, \dots, x_n) d\eta(x_1, \dots, x_n) \right| \\
&\leq 2\|h_\varepsilon - h\|_\infty + \left| \int_{\mathbb{R}^n} h_\varepsilon(x_1, x_2, \dots, x_n) d\theta(x_1, \dots, x_n) - \int_{\mathbb{R}^n} h_\varepsilon(x_1, x_2, \dots, x_n) d\eta(x_1, \dots, x_n) \right| \\
&\leq 2\|h_\varepsilon - h\|_\infty + C \left( \mathbf{E} |\sigma^2 - \langle D(-L)^{-1} X_1, DX_1 \rangle| + \sum_{j=2}^n \mathbf{E} |\langle D(-L)^{-1} X_1, DX_j \rangle| \right)
\end{aligned}$$

and we conclude by letting  $\varepsilon \rightarrow 0$ . ■

The corollary below is used to deal with random vectors with components in Wiener chaos.

**Corollary 1.** *With the notation from Theorem 1, if  $X_1, \dots, X_n \in \mathbb{D}^{1,4}$ , then*

$$d_W(\theta, \eta) \leq C \left[ \left( \mathbf{E} \left| \sigma^2 - \langle D(-L)^{-1} X_1, DX_1 \rangle \right|^2 \right)^{\frac{1}{2}} + \sum_{j=2}^n \left( \mathbf{E} \left| \langle D(-L)^{-1} X_1, DX_j \rangle \right|^2 \right)^{\frac{1}{2}} \right].$$

**Proof:** The proof follows from Theorem 1, by using Cauchy-Schwarz's inequality in the right-hand side of (13) and by noticing that  $\langle D(-L)^{-1} X_i, DX_j \rangle$  belongs to  $L^2(\Omega)$  when  $X_i, X_j \in \mathbb{D}^{1,4}$ , for  $i, j = 1, 2, \dots, n$ .  $\blacksquare$

**Remark 1.** *As a particular case of relation (13) in Theorem 1, it follows that if  $X_1 \sim N(0, \sigma^2)$  and  $\langle DX_1, DX_2 \rangle = 0$  almost surely, then  $X_1$  is independent of  $X_2$ .*

## 4 The case of multiple stochastic integrals

By applying the results in previous section to the case of multiple stochastic integrals, we find pretty surprising results.

Let us recall the following facts. If  $F = I_q(f)$  and  $G = I_p(g)$  with  $f \in H^{\odot q}$  and  $g \in H^{\odot p}$ , then (see [5], Lemmas 5.2.4 and 6.2.1)

$$\begin{aligned} \mathbf{E} \left( \langle D(-L)^{-1} F, DF \rangle - \mathbf{E} F^2 \right)^2 &= \mathbf{E} \left( \frac{1}{q} \|DF\|^2 - \mathbf{E} F^2 \right)^2 \\ &= q^2 \sum_{r=1}^{q-1} (r-1)!^2 \left( C_{q-1}^{r-1} \right)^4 (2q-2r)! \|f \tilde{\otimes}_r f\|_{H^{\otimes(2q-2r)}}^2 \\ &=: \sum_{r=1}^{q-1} a(r, q) \|f \tilde{\otimes}_r f\|_{H^{\otimes(2q-2r)}}^2, \end{aligned} \quad (15)$$

with, for  $r = 1, \dots, q-1$ ,

$$a(r, q) = q^2 (r-1)!^2 \left( C_{q-1}^{r-1} \right)^4 (2q-2r)!. \quad (16)$$

Also, if  $q \geq p$ ,

$$\begin{aligned} \mathbf{E} \langle D(-L)^{-1} F, DG \rangle^2 &= \mathbf{E} \left( \frac{1}{q} \langle DF, DG \rangle \right)^2 \\ &= (\mathbf{E}(FG))^2 \mathbf{1}_{p=q} + p^2 \sum_{r=1}^{p-1} (r-1)!^2 \left( C_{q-1}^{r-1} \right)^2 \left( C_{p-1}^{r-1} \right)^2 (p+q-2r)! \|f \tilde{\otimes}_r g\|_{H^{\otimes p+q-2r}}^2 \\ &\quad + p^2 (p-1)!^2 \left( C_{q-1}^{p-1} \right)^2 (q-p)! \|f \tilde{\otimes}_p g\|_{H^{\otimes(q-p)}}^2 \mathbf{1}_{p < q} \\ &= (\mathbf{E}(FG))^2 \mathbf{1}_{p=q} + \sum_{r=1}^{p-1} c(r, p, q) \|f \tilde{\otimes}_r g\|_{H^{\otimes p+q-2r}}^2 + c(p, p, q) \|f \tilde{\otimes}_p g\|_{H^{\otimes(q-p)}}^2 \mathbf{1}_{p < q} \\ &= (\mathbf{E}(FG))^2 \mathbf{1}_{p=q} + \sum_{r=1}^p c(r, p, q) \|f \tilde{\otimes}_r g\|_{H^{\otimes p+q-2r}}^2, \end{aligned} \quad (17)$$

where, for  $r = 1, \dots, p-1$ ,

$$c(r, p, q) = p^2(r-1)!^2 \left(C_{q-1}^{r-1}\right)^2 \left(C_{p-1}^{r-1}\right)^2 (p+q-2r)! \quad (18)$$

and

$$c(p, p, q) = p^2(p-1)!^2 \left(C_{q-1}^{p-1}\right)^2 (q-p)! 1_{p < q}. \quad (19)$$

The proof of the below lemma is contained in e.g. [5]. We prefer to state it since it plays an important role in the sequel.

**Lemma 3.** *Let  $F = I_q(f)$  and  $G = I_p(g)$  with  $f \in H^{\odot q}$  and  $g \in H^{\odot p}$ . Assume  $q \geq p$ . Then*

$$\mathbf{E}\langle D(L)^{-1}F, DG \rangle^2 \leq (\mathbf{E}(FG))^2 1_{p=q} + \frac{\mathbf{E}G^2}{p!} \sum_{r=1}^p c(r, p, q) \|f \otimes_{q-r} f\|_{H^{\otimes 2r}}$$

with  $c(r, p, q)$  given by (18), (19).

**Proof:** We use the inequalities, for  $r = 1, \dots, p-1$

$$\begin{aligned} \|f \tilde{\otimes}_r g\|_{H^{\otimes(p+q-2r)}}^2 &\leq \|f \otimes g\|_{H^{\otimes(p+q-2r)}}^2 = \langle f \otimes_r g, f \otimes_r g \rangle_{H^{\otimes(p+q-2r)}} \\ &= \langle f \otimes_{q-r} f, g \otimes_{p-r} g \rangle_{H^{\otimes 2r}} \leq \|f \otimes_{q-r} f\|_{H^{\otimes 2r}} \|g \otimes_{p-r} g\|_{H^{\otimes 2r}} \\ &\leq \|g\|_{H^{\otimes q}}^2 \|f \otimes_{q-r} f\|_{H^{\otimes 2r}} = \frac{\mathbf{E}G^2}{p!} \|f \otimes_{q-r} f\|_{H^{\otimes 2r}}, \end{aligned}$$

and for  $r = p$ ,

$$\begin{aligned} \|f \tilde{\otimes}_p g\|_{H^{\otimes(q-p)}}^2 &\leq \|f \otimes g\|_{H^{\otimes(q-p)}}^2 \leq \|g\|_{H^{\otimes p}}^2 \|f \otimes_{q-p} f\|_{H^{\otimes 2p}} \\ &= \frac{\mathbf{E}G^2}{p!} \|f \otimes_{q-p} f\|_{H^{\otimes 2p}}. \end{aligned}$$

By combining the above bound with (17), we get the conclusion. ■

Let us state the main result of this section. It gives the asymptotic independence between the components of a random vector in Wiener chaos, which has only one components asymptotically Gaussian.

**Theorem 2.** *For  $j = 1, 2, \dots, n$ , let  $(X_{j,k} = I_{q_j}(f_{j,k}), k \geq 1)$  be  $n$  sequences of multiples stochastics integrals with  $f_{j,k} \in H^{\odot q_j}$  and  $q_1 \geq \max(q_2, \dots, q_n)$ . Assume*

$$X_{1,k} \xrightarrow[k \rightarrow \infty]{(d)} Z \sim N(0, \sigma^2) \quad (20)$$

and

$$(X_{2,k}, \dots, X_{n,k}) \xrightarrow[k \rightarrow \infty]{(d)} (U_2, \dots, U_n), \quad (21)$$

for some random variables  $U_2, \dots, U_n$ . Also assume that

$$\mathbf{E}X_{1,k}X_{j,k} \xrightarrow[k \rightarrow \infty]{} 0 \text{ for every } j = 2, \dots, n. \quad (22)$$

Then, if  $\theta_k = P_{(X_{1,k}, \dots, X_{n,k})}$ , and  $\eta = P_Z \otimes P_{(U_2, \dots, U_n)}$ , then

$$d_W(\theta_k, \eta) \leq C \left[ \left( \mathbf{E} |\sigma^2 - \langle D(-L)^{-1} X_{1,k}, DX_{1,k} \rangle|^2 \right)^{\frac{1}{2}} + \sum_{j=2}^n \left( \mathbf{E} |\langle D(-L)^{-1} X_{1,k}, DX_{j,k} \rangle|^2 \right)^{\frac{1}{2}} \right] + d_W \left( P_{(X_{2,k}, \dots, X_{n,k})}, P_{(U_2, \dots, U_n)} \right). \quad (23)$$

In particular,

$$(X_{1,k}, X_{2,k}, \dots, X_{n,k}) \xrightarrow[k \rightarrow \infty]{(d)} (Z', U_2, \dots, U_n), \quad (24)$$

where  $Z' \sim N(0, \sigma^2)$  and  $Z'$  is independent of  $(U_2, \dots, U_n)$ .

**Proof:** Let us denote, for  $k \geq 1$ ,

$$\eta_k = P_Z \otimes P_{(X_{2,k}, \dots, X_{n,k})}.$$

By Corollary 1, for every  $k \geq 1$ ,

$$d_W(\theta_k, \eta_k) \leq C \left[ \left( \mathbf{E} |\sigma^2 - \langle D(-L)^{-1} X_{1,k}, DX_{1,k} \rangle|^2 \right)^{\frac{1}{2}} + \sum_{j=2}^n \left( \mathbf{E} |\langle D(-L)^{-1} X_{1,k}, DX_{j,k} \rangle|^2 \right)^{\frac{1}{2}} \right]. \quad (25)$$

Also, by triangle's inequality,

$$\begin{aligned} d_W(\theta_k, \eta) &\leq d_W(\theta_k, \eta_k) + d_W(\eta_k, \eta) \\ &\leq C \left[ \left( \mathbf{E} |\sigma^2 - \langle D(-L)^{-1} X_{1,k}, DX_{1,k} \rangle|^2 \right)^{\frac{1}{2}} + \sum_{j=2}^n \left( \mathbf{E} |\langle D(-L)^{-1} X_{1,k}, DX_{j,k} \rangle|^2 \right)^{\frac{1}{2}} \right] \\ &\quad + d_W \left( P_{(X_{2,k}, \dots, X_{n,k})}, P_{(U_2, \dots, U_n)} \right) \end{aligned}$$

by (25) and the bound  $d_W(\eta_k, \eta) \leq d_W \left( P_{(X_{2,k}, \dots, X_{n,k})}, P_{(U_2, \dots, U_n)} \right)$ , which is an immediate consequence of the definition of the Wasserstein distance (12). So (23) is obtained.

Let us now show the limit theorem (24). The Fourth Moment Theorem (Theorem 3 point 4.) implies that

$$\mathbf{E} |\sigma^2 - \langle D(-L)^{-1} X_{1,k}, DX_{1,k} \rangle|^2 \xrightarrow[k \rightarrow \infty]{} 0$$

and (by point 3. in the same theorem), for every  $r = 1, \dots, q_1 - 1$ ,

$$\|f_{1,k} \otimes_r f_{1,k}\|_{H^{\otimes(2q-2r)}} \xrightarrow[k \rightarrow \infty]{} 0. \quad (26)$$

By (26), (22) and Lemma 3, for  $j = 2, \dots, n$ ,

$$\begin{aligned} \mathbf{E} |\langle D(-L)^{-1} X_{1,k}, DX_{j,k} \rangle|^2 &\leq C \left[ (\mathbf{E} X_{1,k} X_{j,k})^2 \mathbf{1}_{q_j=q_1} \right. \\ &\quad \left. + \sum_{r=1}^{q_j} c(r, q_j, q_1) \|f_{1,k} \otimes_{q_1-r} f_{1,k}\|_{H^{\otimes 2r}} \right] \xrightarrow[k \rightarrow \infty]{} 0. \end{aligned}$$

Thus, by (25),

$$d_W(\theta_k, \eta_k) \rightarrow_{k \rightarrow \infty} 0. \quad (27)$$

To get (24) we notice that for every  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  continuous and bounded,

$$\begin{aligned} & \left| \int_{\mathbb{R}^n} f d\theta_k - \int_{\mathbb{R}^n} f d\eta \right| \\ & \leq \left| \int_{\mathbb{R}^n} f d\theta_k - \int_{\mathbb{R}^n} f d\eta_k \right| + \left| \int_{\mathbb{R}^n} f d\eta_k - \int_{\mathbb{R}^n} f d\eta \right|. \end{aligned}$$

The first summand in the right-hand side above goes to zero as  $k \rightarrow \infty$  due to (27) while from the assumptions of the theorem, the sequence  $(\eta_k, k \geq 1)$  of probability measures on  $\mathbb{R}^n$  converges weakly to  $\eta$  and therefore the second term also goes as  $k \rightarrow \infty$ .  $\blacksquare$

**Remark 2.** • Condition (20) can be replaced by  $\mathbf{E}X_{1,k}^2 \rightarrow_{k \rightarrow \infty} \sigma^2$  and any of the assumptions from points 2.-4. of Theorem 3. On the other hand, condition (22) is clearly satisfied if  $q_1 > \max(q_2, \dots, q_n)$ .

- From Theorem 2, we can deduce a result in [10] concerning the normal convergence of sequences of vector-valued multiple stochastic integrals. That is, if  $(I_p(f_k), k \geq 1)$  and  $(I_q(g_k), k \geq 1)$  are two sequences of multiple stochastic integrals that converges in law, as  $k \rightarrow \infty$ , to  $Z_1 \sim N(0, \sigma_1^2)$  and  $Z_2 \sim N(0, \sigma_2^2)$  respectively and  $\mathbf{E}I_p(f_k)I_q(g_k) \rightarrow_{k \rightarrow \infty} 0$ , then the vector  $(I_p(f_k), I_q(g_k), k \geq 1)$  converges in distribution, as  $k \rightarrow \infty$ , to the centered Gaussian vector  $(Z_1, Z_2)$  with independent components and  $\mathbf{E}Z_1^2 = \sigma_1^2, \mathbf{E}Z_2^2 = \sigma_2^2$ . Moreover, the bound (23) corresponds to the estimate in Theorem 6.2.2 of [5].
- It is also possible to give a quantitative bound for the distance between  $\theta_k = P_{(X_{1,k}, \dots, X_{n,k})}$  and  $\eta = P_Z \otimes P_{(U_2, \dots, U_n)}$  in terms of the contractions of the kernels  $f_{j,k}$ . Indeed, by (15), (17) and Lemma 3,

$$\begin{aligned} d_W(\theta_k, \eta) & \leq C \left[ \sum_{r=1}^{q_1-1} a(r, q) \|f_{1,k} \tilde{\otimes}_r f_{1,k}\|_{H^{\otimes(2q-2r)}}^2 + \sum_{j=2}^n (\mathbf{E}(X_{1,k} X_{j,k}))^2 1_{q_1=q_j} \right. \\ & \quad \left. + \sum_{j=2}^n \frac{\mathbf{E}X_{j,k}^2}{q_j!} \sum_{r=1}^{q_j} c(r, p, q_j) \|f_{1,k} \tilde{\otimes}_{q_1-r} f_{1,k}\|_{H^{\otimes 2r}} \right], \end{aligned}$$

with  $a(r, q)$  defined by (16) and  $c(r, p, q)$  by (18), (19). However, this bound is not optimal (it can be seen from the example treated in Section 5), see Remark 3.

As a consequence of Theorem 2, we obtain that every random sequence in the  $q$ th Wiener chaos is asymptotically independent of any random variable belonging to the sum of the first  $q - 1$  Wiener chaoses.

**Corollary 2.** Let  $(X_k, k \geq 1) = (I_q(f_k), k \geq 1)$  with  $f_k \in H^{\odot q}$  for every  $k \geq 1$ . Assume  $X_k \xrightarrow[k \rightarrow \infty]{(d)} Z \sim N(0, \sigma^2)$ . Let  $G = \sum_{p=0}^{q-1} I_p(g_p)$  where  $g_p \in H^{\odot p}$  for  $p = 0, 1, \dots, q - 1$ . Then

$$d_W(P_{(X_k, G)}, P_Z \otimes P_G) \leq C \left[ \left( \mathbf{E}(\sigma^2 - \langle D(-L)^{-1} X_k, D X_k \rangle)^2 \right)^{\frac{1}{2}} + \left( \mathbf{E} \langle D(-L)^{-1} X_k, D G \rangle^2 \right)^{\frac{1}{2}} \right]. \quad (28)$$

In particular,

$$(X_k, G) \xrightarrow{d \rightarrow \infty} (Z', G)$$

where  $Z' \sim N(0, \sigma^2)$  is independent of  $G$ .

**Proof:** The bound (28) is a direct consequence of Corollary 1. By the Fourth Moment Theorem (Theorem 3),  $\mathbf{E}(\sigma^2 - \langle D(-L)^{-1}X_k, DX_k \rangle)^2 \rightarrow_{k \rightarrow \infty} 0$ . Now,

$$\langle D(-L)^{-1}X_k, DG \rangle = \sum_{p=1}^{q-1} \langle D(-L)^{-1}X_k, DI_p(g_p) \rangle$$

and then, with  $c(r, p, q)$  from (18), (19),

$$\begin{aligned} \mathbf{E} \langle D(-L)^{-1}X_k, DG \rangle^2 &\leq c_q \sum_{p=1}^q \mathbf{E} \langle D(-L)^{-1}X_k, DI_p(g_p) \rangle^2 \\ &\leq c_q \sum_{p=1}^{q-1} \sum_{r=1}^p c(r, p, q) \|f_k \otimes_{q-p} f_k\|_{H^{\otimes 2r}}^2 \end{aligned}$$

and again by the Fourth Moment Theorem (Theorem 3, point 3.),  $\|f_k \otimes_{q-p} f_k\|_{H^{\otimes 2r}} \rightarrow_{k \rightarrow \infty} 0$  for every  $r = 1, \dots, p$ . ■

## 5 Application: Quantitative bounds in a central-noncentral limit theorem

Our approach allows to give qualitative bounds for the multidimensional sequences of multiple stochastic integral when only one of these sequences converges to a normal distribution. Here we illustrate the method by treating a two-dimensional sequence in Wiener chaos, one component being asymptotically Gaussian and the second component satisfying a non-central limit theorem. Such estimates are new in the literature and they cannot be obtained via the standard Stein method. Let  $(B^H, t \geq 0)$  be a fractional Brownian motion with Hurst index  $H \in (0, 1)$ . For  $N \geq 1$ , define

$$V_N = q! \frac{1}{\sqrt{N}} \sum_{k=0}^N H_q(B_{k+1}^H - B_k^H), \quad (29)$$

where  $H_q$  is the Hermite polynomial of degree  $q$ . Then, the Breuer-Major theorem (see [1]) states that, if  $H \in (0, 1 - \frac{1}{2q})$  the sequence  $(V_N, N \geq 1)$  converges to a Gaussian random variable  $Z \sim N(0, \sigma_{q,H}^2)$ , where the variance  $\sigma_{q,H}^2$  is explicitly known.

On the other hand, the sequence  $(U_N, N \geq 1)$  given by

$$U_N = 2N^{1-2H} \sum_{i=0}^{N-1} H_2(B_{k+1}^H - B_k^H), \quad N \geq 1, \quad (30)$$

converges in distribution, for  $H \in (\frac{3}{4}, 1)$ , to  $c_{2,H}R^{(2H-1)}$  where  $R^{(2H-1)}$  is a Rosenblatt random variable with Hurst parameter  $2H - 1$  and again the constant  $c_{2,H} > 0$  is known.

Moreover, the random sequence  $(V_N, U_N)$  converges in law, as  $N \rightarrow \infty$ , to  $(Z, c_{2,H}R^{(2H-1)})$ , with  $Z$  independent of  $R^{(2H-1)}$ . This can be obtained from the main findings in [6] or [7] but it also follows from our Theorem 2. The purpose is to find the rate of convergence, under the Wasserstein distance, for this two-dimensional limit theorem.

Let us assume in the sequel

$$H \in \left(\frac{3}{4}, 1 - \frac{1}{2q}\right) \text{ and } q \geq 3. \quad (31)$$

By Theorem 2, we have

$$\begin{aligned} & d_W \left( P_{(V_N, U_N)}, P_Z \otimes P_{c_{2,H}R^{(2H-1)}} \right) \\ & \leq C \left[ \left( \mathbf{E} \left( \sigma^2 - \langle DV_N, D(-L)^{-1}V_N \rangle \right)^2 \right)^{\frac{1}{2}} + d_W(P_{U_N}, P_{c_{2,H}R^{(2H-1)}}) + \sqrt{\mathbf{E}(\langle DV_N, DU_N \rangle)^2} \right]. \end{aligned}$$

We know the rate of convergence to their limits for each of the sequences  $(V_N, N \geq 1)$  and  $(U_N, N \geq 1)$ . If one assumes (31), then (see Theorem 4.1 in [4])

$$\left( \mathbf{E} \left( \sigma^2 - \langle DV_N, D(-L)^{-1}V_N \rangle \right)^2 \right)^{\frac{1}{2}} \leq C_{H,q} \begin{cases} N^{H-1} & \text{if } H \in \left(\frac{3}{4}, \frac{2q-3}{2q-2}\right] \\ N^{qH-q+\frac{1}{2}} & \text{if } H \in \left(\frac{2q-3}{2q-2}, \frac{2q-1}{2q}\right). \end{cases} \quad (32)$$

Moreover, for any  $H$  satisfying (31) (see [2] or [5], relation (7.4.13))

$$d_W(U_N, c_{2,H}R^{(2H-1)}) \leq C_H N^{\frac{3}{2}-2H}. \quad (33)$$

In particular, if  $q = 3$ , it follows from (32) and (33) that

$$d_W(V_N, Z) + d_W(U_N, c_{2,H}R^{(2H-1)}) \leq C_H \left( N^{\frac{3}{2}-2H} + N^{3H-\frac{5}{2}} \right) \quad (34)$$

$$\leq \begin{cases} N^{\frac{3}{2}-2H} & \text{if } H \in \left(\frac{3}{4}, \frac{4}{5}\right) \\ N^{3H-\frac{5}{2}} & \text{if } H \in \left[\frac{4}{5}, \frac{5}{6}\right) \end{cases} \quad (35)$$

Let us estimate the quantity  $\sqrt{\mathbf{E}(\langle DV_N, DU_N \rangle)^2}$ . Denote by  $\mathcal{H}$  the canonical Hilbert space associated to the fractional Brownian motion, defined as the closure of the set of step functions on the positive real line with respect to the scalar product

$$\langle 1_{[0,t]}, 1_{[0,s]} \rangle_{\mathcal{H}} = \mathbf{E} B_t^H B_s^H = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

We can write, if  $I_q$  is the multiple stochastic integral with respect to the isonormal process generated by  $B^H$ ,

$$V_N = I_q(F_N) \text{ with } f_N = \frac{1}{\sqrt{N}} \sum_{k=1}^N h_k^{\otimes q}$$

and

$$U_N = I_2(g_N) \text{ with } g_N = N^{1-2H} \sum_{l=1}^N h_l^{\otimes 2},$$

where  $h_k = 1_{[k-1, k]}$  for  $k = 1, \dots, N$ . In particular  $\|h_k\|_{\mathcal{H}} = 1$  and

$$\langle h_k, h_l \rangle_{\mathcal{H}} = \rho_H(k-l) \text{ with } \rho_H(v) = \frac{1}{2} ((v+1)^{2H} + (v-1)^{2H} - 2v^{2H}), \quad v \geq 1. \quad (36)$$

We also have  $\rho_H(0) = 1$  and we define  $\rho_H$  on  $\mathbb{Z}$  by setting  $\rho_H(-v) = \rho_H(v)$  if  $v < 0$ . Thus

$$\begin{aligned} \langle DV_N, DU_N \rangle &= 2qN^{\frac{1}{2}-2H} \sum_{k,l=1}^N I_{q-1}(h_k^{\otimes(q-1)}) I_1(h_l) \langle h_k, h_l \rangle \\ &= 2qN^{\frac{1}{2}-2H} \sum_{k,l=1}^N \left[ I_q(h_k^{\otimes(q-1)} \otimes h_l) + (q-1) I_{q-2}(h_k^{\otimes(q-1)} \otimes_1 h_l) \right] \langle h_k, h_l \rangle \\ &= 2qN^{\frac{1}{2}-2H} \sum_{k,l=1}^N \left[ I_q(h_k^{\otimes(q-1)} \otimes h_l) + (q-1) I_{q-2}(h_k^{\otimes(q-2)}) \langle h_k, h_l \rangle \right] \langle h_k, h_l \rangle, \end{aligned}$$

where we applied the product formula (47). Consequently,

$$\begin{aligned} &\mathbf{E} \langle DV_N, DU_N \rangle^2 \\ &\leq c_q N^{1-4H} \left[ \sum_{i,j,k,l=1}^N \langle h_i^{\otimes(q-1)} \tilde{\otimes} h_j, h_k^{\otimes(q-1)} \tilde{\otimes} h_l \rangle \langle h_i, h_j \rangle \langle h_k, h_l \rangle + \langle h_i, h_k \rangle^{q-2} \langle h_i, h_j \rangle^2 \langle h_k, h_l \rangle^2 \right] \\ &\leq c_q N^{1-4H} \left[ \sum_{i,j,k,l=1}^N \langle h_i, h_k \rangle^{q-1} \langle h_i, h_j \rangle \langle h_k, h_l \rangle \langle h_j, h_l \rangle + \sum_{i,j,k,l=1}^N \langle h_i, h_k \rangle^{q-2} \langle h_i, h_j \rangle \langle h_k, h_l \rangle \langle h_i, h_l \rangle \langle h_j, h_k \rangle \right. \\ &\quad \left. + \sum_{i,j,k,l=1}^N \langle h_i, h_k \rangle^{q-2} \langle h_i, h_j \rangle^2 \langle h_k, h_l \rangle^2 \right] =: a_{1,N} + a_{2,N} + a_{3,N}. \end{aligned}$$

We used Lemma 4.5 in [15] in order to express the scalar product  $\langle h_i^{\otimes(q-1)} \tilde{\otimes} h_j, h_k^{\otimes(q-1)} \tilde{\otimes} h_l \rangle$ . Using the inequality

$$\langle h_i, h_j \rangle \langle h_k, h_l \rangle \langle h_i, h_l \rangle \langle h_j, h_k \rangle \leq \frac{1}{2} (\langle h_i, h_j \rangle^2 \langle h_k, h_l \rangle^2 + \langle h_i, h_l \rangle^2 \langle h_k, h_j \rangle^2),$$

we get  $a_{2,N} \leq a_{3,N}$  so we have to estimate  $a_{1,N}$  and  $a_{3,N}$ . Now, by (36),

$$\begin{aligned} a_{3,N} &= c_q N^{1-4H} \sum_{i,j,k,l=1}^N \rho_H(i-k)^{q-2} \rho_H(i-j)^2 \rho_H(k-l)^2 \\ &\leq c_q N^{1-4H} \sum_{i,k=1}^N \rho_H(i-k)^{q-2} \left( \sum_{a=-N}^N \rho_H(a)^2 \right)^2. \end{aligned}$$



By using the bound  $\sum_{a=-N}^N \rho_H(a)^2 \leq c_H N^{4H-3}$  we obtain

$$\begin{aligned} a_{3,N} &\leq c_{q,H} N^{4H-5} \sum_{i,k=1}^N \rho_H(i-k)^{q-2} \leq c_{q,H} N^{4H-4} \sum_{k \geq 1} k^{(2H-2)(q-2)} \\ &\leq c_{q,H} N^{4H-4} \begin{cases} 1, & \text{if } H < 1 - \frac{1}{2(q-2)} \\ \log(N) & \text{if } H = 1 - \frac{1}{2(q-2)} \\ N^{(2H-2)(q-2)+1} & \text{if } H \in \left(1 - \frac{1}{2(q-2)}, 1 - \frac{1}{2q}\right). \end{cases} \end{aligned}$$

For  $q = 3$ , we have for  $H \in \left(\frac{3}{4}, \frac{5}{6}\right)$ ,

$$a_{3,N} \leq c_H N^{6H-5} \quad (37)$$

Let us deal with

$$a_{1,N} = c_{q,H} N^{1-4H} \sum_{i,j,k,l=1}^N \rho_H(i-k)^{q-1} \rho_H(i-j) \rho_H(k-l) \rho_H(j-l).$$

This summand is the most complicated. Similar quantities (but not exactly the same!) have been treated in e.g. [4], proof of Theorem 4.1. We decompose the sum over  $(i, j, k, l) \in \{1, \dots, N\}^4$  upon the following cases:

1.  $(i = j = k = l)$ ,
2.  $((i = j = k, l \neq i), (i = j = l, k \neq i), (i = k = l, j \neq i), (j = k = l, i \neq j))$ ,
3.  $((i = j, k = l, k \neq i), (i = k, j = l, j \neq i), (i = l, j = k, j \neq i))$ ,
- 4.

$$\begin{aligned} &((i = j, k \neq i, k \neq l, l \neq i), (i = k, j \neq i, j \neq l, k \neq l), (i = l, k \neq i, k \neq j, j \neq i), \\ &(j = k, k \neq i, k \neq l, l \neq i), (j = l, k \neq i, k \neq l, j \neq i), (k = l, k \neq i, k \neq j, j \neq i)). \end{aligned}$$

5.  $i, j, k, l$  are all different.

We denote by  $a_{1,N}^{(j)}$ ,  $j = 1, 2, 3, 4, 5$  the sum of all the terms from the groups 1.-5. defined above. The first of these terms can be easily estimated since

$$a_{1,N}^{(1)} = c_{q,H} N^{1-4H} \sum_{i=1}^N \rho_H(0)^{q+2} = c_{q,H} N^{2-4H}. \quad (38)$$

For, the first sum from point 2.

$$c_{q,H} N^{1-4H} \sum_{i,l=1}^N \rho_H(i-l)^2 \leq c_{q,H} N^{2-4H} \sum_{i=1}^N i^{4H-4} \leq c_{q,H} N^{2-4H} N^{4H-3} = c_{q,H} N^{-1}$$

while the second from point 2.

$$c_{q,H}N^{1-4H} \sum_{i,k=1}^N \rho_H(i-k)^q \leq c_{q,H}N^{2-4H} \sum_{k \in \mathbb{Z}} \rho_H(k)^q \leq c_{q,H}N^{2-4H}.$$

So, by symmetry,

$$a_{2,N}^{(2)} \leq c_{q,H}(N^{-1} + N^{2-4H}) \leq c_{q,H}N^{-1}. \quad (39)$$

The sums from group 3. are similar to the those from group 2. and we get

$$a_{1,N}^{(3)} \leq c_{q,H}N^{-1}. \quad (40)$$

Let us with the summands corresponding to point 4. The first one in this set reads

$$\begin{aligned} & c_{q,H}N^{1-4H} \sum_{i \neq k \neq l \neq i} \rho_H(i-k)^{q-1} \rho_H(k-l) \rho_H(i-l) \\ & \leq c_{q,H}N^{2-4H} \sum_{a,b=-N}^N |\rho_H|(a-b)^{q-1} |\rho_H|(a) |\rho_H|(b) \leq c_{q,H}N^{2-4H} \sum_{a,b=-N}^N |\rho_H|(a-b)^{q-1} |\rho_H|(a)^2 \\ & \leq c_{q,H}N^{2-4H} \sum_{a=-N}^N |a|^{4H-4} \sum_{b=-2N}^{2N} |b|^{(2H-2)(q-1)}. \end{aligned}$$

It follows that this term is less than

$$c_{q,H} \begin{cases} N^{-1} & \text{if } H < 1 - \frac{1}{2(q-1)} \\ N^{-1} \log N & \text{if } H = 1 - \frac{1}{2(q-1)} \\ N^{(2H-2)(q-1)+2} & \text{if } H \in \left(1 - \frac{1}{2(q-1)}, 1 - \frac{1}{2q}\right). \end{cases}$$

Regarding the second summand in 4., we can bound as follows

$$\begin{aligned} & c_{q,H}N^{1-4H} \sum_{i \neq j \neq l \neq i} \rho_H(i-j) \rho_H(i-l) \rho_H(j-l) \\ & \leq c_{q,H}N^{1-4H} N^3 N^{6H-6} \frac{1}{N^3} \sum_{i \neq j \neq l \neq i} \left(\frac{|i-j|}{N}\right)^{2H-2} \left(\frac{|i-l|}{N}\right)^{2H-2} \left(\frac{|j-l|}{N}\right)^{2H-2} \\ & = c_{q,H}N^{2H-2} \frac{1}{N^3} \sum_{i \neq j \neq l \neq i} \left(\frac{|i-j|}{N}\right)^{2H-2} \left(\frac{|i-l|}{N}\right)^{2H-2} \left(\frac{|j-l|}{N}\right)^{2H-2} \leq c_{q,H}N^{2H-2}, \end{aligned}$$

since the quantity  $\frac{1}{N^3} \sum_{i \neq j \neq l \neq i} \left(\frac{|i-j|}{N}\right)^{2H-2} \left(\frac{|i-l|}{N}\right)^{2H-2} \left(\frac{|j-l|}{N}\right)^{2H-2}$  is a Riemann sum that converges to  $\int_{[0,1]^3} |x-y|^{2H-2} |y-z|^{2H-2} |z-x|^{2H-2} dx dy dz < \infty$ . We have similar bounds for the other terms and we get

$$a_{1,N}^{(4)} \leq c_{q,H}N^{2H-2}. \quad (41)$$

Notice that the estimation of the dominant term, the second in this group is sharp.

For the only summand in group 5., we separate its analysis uupon all the possible orders:  $i > j > k > l, i > j > l > k, \dots$ . The first summand is treated as follows

$$\begin{aligned}
& c_q N^{1-4H} \sum_{i>j>k>l} \rho_H(i-k)^{q-1} \rho_H(i-j) \rho_H(k-l) \rho_H(j-l) \\
& \leq c_{q,H} N^{1-4H} \sum_{i>j>k>l} |i-k|^{(2H-2)(q-1)} |i-j|^{2H-2} |k-l|^{2H-2} |j-l|^{2H-2} \\
& \leq c_{q,H} N^{1-4H} \sum_{i>j>k>l} |i-k|^{(2H-2)(q-1)} |i-j|^{2H-2} |k-l|^{4H-4} \\
& \leq c_{q,H} N^{1-4H} \sum_{i>j>k} |i-k|^{(2H-2)(q-1)} |i-j|^{2H-2} \sum_{l=-N}^N |l|^{4H-4} \\
& \leq c_{q,H} N^{-2} \sum_{i>j>k} |i-k|^{(2H-2)(q-1)} |i-j|^{2H-2} \\
& \leq c_{q,H} N^{-2} \sum_{i>k} |i-k|^{(2H-2)(q-1)} \sum_{j=-N}^N |j|^{2H-2} \leq c_{q,H} N^{2H-3} \sum_{i>k} |i-k|^{(2H-2)(q-1)} \\
& \leq c_{q,H} N^{2H-2} \sum_{k=1}^N k^{(2H-2)(q-1)}.
\end{aligned}$$

With analogous estimates for the other cases of point 5., we obtain

$$a_{1,N}^{(5)} \leq c_{q,H} \begin{cases} N^{2H-2} & \text{if } H < 1 - \frac{1}{2(q-1)} \\ N^{2H-2} \log N & \text{if } = 1 - \frac{1}{2(q-1)} \\ N^{(2H-2)q+1} & \text{if } H \in \left(1 - \frac{1}{2(q-1)}, 1 - \frac{1}{2q}\right). \end{cases} \quad (42)$$

So, by (38), (39), (40), (41) and (42)

$$a_{1,N} \leq c_{q,H} \begin{cases} N^{2H-2} & \text{if } H \in \left(\frac{3}{4}, 1 - \frac{1}{2(q-1)}\right) \\ N^{(2H-2)q+1} & \text{if } H \in \left(1 - \frac{1}{2(q-1)}, 1 - \frac{1}{2q}\right). \end{cases}$$

Thus

$$\mathbf{E}\langle DV_N, DU_N \rangle^2 \leq c_{q,H} \begin{cases} N^{2H-2} & \text{if } H \in \left(\frac{3}{4}, 1 - \frac{1}{2(q-1)}\right) \\ N^{(2H-2)q+1} & \text{if } H \in \left(1 - \frac{1}{2(q-1)}, 1 - \frac{1}{2q}\right), \end{cases}, \quad (43)$$

the bound on the first branch being immaterial for  $q = 3, 4$ . If  $q = 3$ , then

$$\mathbf{E}\langle DV_N, DU_N \rangle^2 \leq c_H N^{6H-5}. \quad (44)$$

We obtain

$$d_W \left( (V_N, U_N), (Z, c_{2,H} R^{(2H-1)}) \right) \leq c_{q,H} \begin{cases} N^{H-1} + N^{\frac{3}{2}-2H} & \text{for } H \in \left(\frac{3}{4}, 1 - \frac{1}{2(q-1)}\right) \\ N^{(H-1)q+\frac{1}{2}} + N^{\frac{3}{2}-2H} & \text{for } \left(1 - \frac{1}{2(q-1)}, 1 - \frac{1}{2q}\right) \end{cases}$$

For  $q = 3$ , we have from (34), (37) and (44),

$$d_W \left( (V_N, U_N), (Z, c_{2,H} R^{(2H-1)}) \right) \leq c_H \begin{cases} N^{\frac{3}{2}-2H} & \text{if } H \in \left( \frac{3}{4}, \frac{4}{5} \right) \\ N^{3H-\frac{5}{2}} & \text{if } H \in \left[ \frac{4}{5}, \frac{5}{6} \right). \end{cases}$$

**Remark 3.** It follows from the above calculation that the quantity  $(\mathbf{E} \langle DV_N, DU_N \rangle^2)^{\frac{1}{2}}$ , which somehow measures the correlation between  $V_N$  and  $U_N$  has the same size, for  $N$  large, as  $d_W(V_N, Z)$  (compare (32) and (43)).

## 6 Appendix: Wiener-Chaos and Malliavin derivatives

Here we describe the elements from stochastic analysis that we will need in the paper. Consider  $H$  a real separable Hilbert space and  $(W(h), h \in H)$  an isonormal Gaussian process on a probability space  $(\Omega, \mathcal{A}, P)$ , which is a centered Gaussian family of random variables such that  $\mathbf{E}[W(\varphi)W(\psi)] = \langle \varphi, \psi \rangle_H$ . Denote by  $I_n$  the multiple stochastic integral with respect to  $B$  (see [8]). This mapping  $I_n$  is actually an isometry between the Hilbert space  $H^{\odot n}$  (symmetric tensor product) equipped with the scaled norm  $\frac{1}{\sqrt{n!}} \|\cdot\|_{H^{\otimes n}}$  and the Wiener chaos of order  $n$  which is defined as the closed linear span of the random variables  $H_n(W(h))$  where  $h \in H, \|h\|_H = 1$  and  $H_n$  is the Hermite polynomial of degree  $n \in \mathbb{N}$

$$H_n(x) = \frac{(-1)^n}{n!} \exp\left(\frac{x^2}{2}\right) \frac{d^n}{dx^n} \left( \exp\left(-\frac{x^2}{2}\right) \right), \quad x \in \mathbb{R}.$$

The isometry of multiple integrals can be written as follows: for  $m, n$  positive integers,

$$\begin{aligned} \mathbf{E}(I_n(f)I_m(g)) &= n! \langle \tilde{f}, \tilde{g} \rangle_{H^{\otimes n}} \quad \text{if } m = n, \\ \mathbf{E}(I_n(f)I_m(g)) &= 0 \quad \text{if } m \neq n. \end{aligned} \tag{45}$$

It also holds that

$$I_n(f) = I_n(\tilde{f})$$

where  $\tilde{f}$  denotes the symmetrization of  $f$  defined by the formula

$$\tilde{f}(x_1, \dots, x_n) = \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} f(x_{\sigma(1)}, \dots, x_{\sigma(n)}).$$

We recall that any square integrable random variable which is measurable with respect to the  $\sigma$ -algebra generated by  $W$  can be expanded into an orthogonal sum of multiple stochastic integrals

$$F = \sum_{n=0}^{\infty} I_n(f_n) \tag{46}$$

where  $f_n \in H^{\odot n}$  are (uniquely determined) symmetric functions and  $I_0(f_0) = \mathbf{E}[F]$ .

Let  $L$  be the Ornstein-Uhlenbeck operator

$$LF = - \sum_{n \geq 0} n I_n(f_n)$$

if  $F$  is given by (46) and it is such that  $\sum_{n=1}^{\infty} n^2 n! \|f_n\|_{\mathcal{H}^{\otimes n}}^2 < \infty$ .

For  $p > 1$  and  $\alpha \in \mathbb{R}$  we introduce the Sobolev-Watanabe space  $\mathbb{D}^{\alpha,p}$  as the closure of the set of polynomial random variables with respect to the norm

$$\|F\|_{\alpha,p} = \|(I - L)^{\frac{\alpha}{2}} F\|_{L^p(\Omega)}$$

where  $I$  represents the identity. We denote by  $D$  the Malliavin derivative operator that acts on smooth functions of the form  $F = g(W(h_1), \dots, W(h_n))$  ( $g$  is a smooth function with compact support and  $h_i \in H$ )

$$DF = \sum_{i=1}^n \frac{\partial g}{\partial x_i}(W(h_1), \dots, W(h_n)) h_i.$$

The operator  $D$  is continuous from  $\mathbb{D}^{\alpha,p}$  into  $\mathbb{D}^{\alpha-1,p}(H)$ .

We will intensively use the product formula for multiple integrals. It is well-known that for  $f \in H^{\odot n}$  and  $g \in H^{\odot m}$

$$I_n(f)I_m(g) = \sum_{r=0}^{n \wedge m} r! \binom{n}{r} \binom{m}{r} I_{m+n-2r}(f \otimes_r g) \quad (47)$$

where  $f \otimes_r g$  means the  $r$ -contraction of  $f$  and  $g$  (see e.g. Section 1.1.2 in [8]).

The Fourth Moment Theorem states as follows.

**Theorem 3.** ([9] and [5]) *Fix  $n \in \mathbb{N}$ . Consider a sequence  $(F_k = I_n(f_k))_{k \in \mathbb{N}}$  of square integrable random variables in the  $n$ -th Wiener chaos. Assume that*

$$\lim_{k \rightarrow \infty} \mathbf{E}[F_k^2] = \lim_{k \rightarrow \infty} \|f_k\|_{H^{\odot n}}^2 = 1. \quad (48)$$

*Then, the following statements are equivalent.*

1. *The sequence of random variables  $(F_k = I_n(f_k))_{k \geq 1}$  converges to the standard normal law in distribution as  $k \rightarrow \infty$ .*
2.  $\lim_{k \rightarrow \infty} \mathbf{E}[F_k^4] = 3$ .
3.  $\lim_{k \rightarrow \infty} \|f_k \otimes_l f_k\|_{H^{\otimes 2(n-l)}} = 0$  for  $l = 1, 2, \dots, n-1$ .
4.  $\|DF_k\|_H^2$  converges to  $n$  in  $L^2(\Omega)$  as  $k \rightarrow \infty$ .

We also need to introduce the Skorohod integral (or the divergence operator), denoted by  $\delta$ , which is the adjoint operator of  $D$ . Its domain is

$$\text{Dom}(\delta) = \{u \in L^2(\Omega; H), \mathbf{E}|\langle DF, u \rangle_H| \leq C\|F\|_2\}$$

and we have the duality relationship

$$\mathbf{E}F\delta(u) = \mathbf{E}\langle DF, u \rangle_H, \quad F \in \mathcal{S}, u \in \text{Dom}(\delta). \quad (49)$$

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