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Multidimensional Stein method and quantitative asymptotic independence

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Abstract

If \mathbb{Y} is a random vector in \mathbb{R}^d , we denote by $P_{\mathbb{Y}}$ its probability distribution. Consider a random variable X and a d -dimensional random vector \mathbb{Y} . Inspired by [15], we develop a multidimensional Stein-Malliavin calculus which allows to measure the Wasserstein distance between the law $P_{(X, \mathbb{Y})}$ and the probability distribution $P_Z \otimes P_{\mathbb{Y}}$, where Z is a Gaussian random variable. That is, we give estimates, in terms of the Malliavin operators, for the distance between the law of the random vector (X, \mathbb{Y}) and the law of the vector (Z, \mathbb{Y}) , where Z is Gaussian and independent of \mathbb{Y} . Then we focus on the particular case of random vectors in Wiener chaos and we give an asymptotic version of this result. In this situation, this variant of the Stein-Malliavin calculus has strong and unexpected consequences. Let $(X_k, k \geq 1)$ be a sequence of random variables in the p th Wiener chaos ($p \geq 2$), which converges in law, as $k \rightarrow \infty$, to the Gaussian distribution $N(0, \sigma^2)$. Also consider $(\mathbb{Y}_k, k \geq 1)$ a d -dimensional random sequence converging in $L^2(\Omega)$, as $k \rightarrow \infty$, to an arbitrary random vector \mathbb{U} in \mathbb{R}^d and assume that the two sequences are asymptotically uncorrelated. We prove that, under very light assumptions on \mathbb{Y}_k , we have the joint convergence of $((X_k, \mathbb{Y}_k), k \geq 1)$ to (Z, \mathbb{U}) where $Z \sim N(0, \sigma^2)$ is independent of \mathbb{U} . These assumptions are automatically satisfied when the components of the vector \mathbb{Y}_k belong to a finite sum of Wiener chaoses or when $\mathbb{Y}_k = Y$ for every $k \geq 1$, where \mathbb{Y} belongs to the Sobolev-Malliavin space $\mathbb{D}^{1,2}$.

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Key words: Stein's method, Malliavin calculus, multiple stochastic integrals, asymptotic independence.

1 Introduction

The Stein's method constitutes a collection of mathematical techniques that allow to give quantitative bounds for the distance between the probability distributions of random variables. It has been initially introduced in the paper [18] and then developed by many authors. We refer, among many others to the monographs and surveys [3], [16], [17], [19] for a detailed description of this method. Of particular interest is the situation when one random variable is Gaussian, but the cases of other target distributions have been analyzed in the literature.

A more recent theory is the so-called Stein-Malliavin calculus which combines the Stein's method with the techniques of the Malliavin calculus. The first work in this direction is [7] (see [8] for a more detailed exposition) and since, numerous authors extended, refined or applied this theory. In this theory, the bounds obtained for the distance between the law of an arbitrary random variable and the target distribution are given in terms of the Malliavin operators.

The starting point of the Stein's method for normal approximation is the following observation: $Z \sim N(0, \sigma^2)$ with $\sigma > 0$ if and only if

$$\sigma^2 \mathbf{E}f'(Z) - \mathbf{E}Zf(Z) = 0$$

for every absolutely continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $\mathbf{E}|f'(Z)| < \infty$. Then, one can think that if a random variable X has the property that $\sigma^2 \mathbf{E}f'(X) - \mathbf{E}Xf(X)$ is close to zero for a large class of functions f , then the probability distribution of X should be close to $N(0, \sigma^2)$. From this observation, the whole Stein's theory has been constructed, leading to various bounds for the distance between the probability law of the random variable X and the normal distribution $N(0, \sigma^2)$.

In this work, we deal with a variant of this method recently developed in the reference [15] that allows to measure the distance between the components of a random vector (X_1, X_2) , where $X_1 \sim N(0, \sigma^2)$ and X_2 has an arbitrary distribution. The nice observation made in [15] is that $X_1 \sim N(0, \sigma^2)$ and X_1 is independent of X_2 if and only if

$$\sigma^2 \mathbf{E}\partial_{x_1}f(X_1, X_2) - \mathbf{E}X_1f(X_1, X_2) = 0$$

for a large class of differentiable functions $f : \mathbb{R}^2 \rightarrow \mathbb{R}$. We denoted by $\partial_{x_1}f$ the partial derivative of f with respect to its first variable. As in the standard Stein's method, one follows the intuition that if some random vector (X_1, X_2) satisfies that $\sigma^2 \mathbf{E}\partial_{x_1}f(X_1, X_2) - \mathbf{E}X_1f(X_1, X_2)$ is close to zero, then X_1 should be close in law to $Z \sim N(0, \sigma^2)$ and $P_{(X_1, X_2)}$ should be close to $P_Z \otimes P_{X_2}$. By combining this idea with Malliavin calculus, in [15] one gives bounds for the Wasserstein distance between $P_{(X_1, X_2)}$ and $P_{X_1} \otimes P_{X_2}$ in terms of the Malliavin operators.

Our purpose is, in a first step, to generalize the above idea by considering random vectors of arbitrary dimension. This extension of the Stein's method com-

bined with Malliavin calculus allows to obtain the following estimate: if $X \in \mathbb{D}^{1,2}$ and $\mathbb{Y} = (Y_1, \dots, Y_d)$ is such that $Y_j \in \mathbb{D}^{1,2}$ for all $j = 1, \dots, d$, then (we denote by d_W the Wasserstein distance and $Z \sim N(0, \sigma^2)$)

$$d_W(P_{(X, \mathbb{Y})}, P_Z \otimes P_{\mathbb{Y}}) \leq C \left[\mathbf{E} |\sigma^2 - \langle D(-L)^{-1}X, DX \rangle_H| + \mathbf{E} \sum_{j=1}^d |\langle D(-L)^{-1}X, DY_j \rangle_H| \right], \quad (1)$$

with $C > 0$. We denoted by D, L the Malliavin derivative and the Ornstein-Uhlenbeck operator with respect to an isonormal process $(W(h), h \in H)$, where $(H, \langle \cdot, \cdot \rangle_H)$ is a real and separable Hilbert space.

Then, we focus on the particular case of sequences of random variables belonging to a Wiener chaos and we give asymptotic-type results. We will here show that the convergence of a sequence of multiple stochastic integrals to the Gaussian law has other strong and unexpected consequences. Let H be an Hilbert space and let I_p denote the multiple integral of order $p \geq 1$ with respect to an isonormal process $(W(h), h \in H)$. Assume that $p \geq 2$ is an integer number and for every $k \geq 1$, $X_k = I_p(f_k)$ where $f_k \in H^{\otimes p}$ are symmetric functions. Suppose that

$$X_k \xrightarrow[k \rightarrow \infty]{(d)} Z \sim N(0, \sigma^2),$$

where $\sigma > 0$ and " $\xrightarrow{(d)}$ " stands for the convergence in distribution. Then the following facts hold true:

- If $\mathbb{Y} = (Y_1, \dots, Y_d)$ is a d -dimensional random vector with components in the Malliavin-Sobolev space $\mathbb{D}^{1,2}$ and X_k, \mathbb{Y} are asymptotically uncorrelated (i.e. $\mathbf{E}X_k Y_j \xrightarrow[k \rightarrow \infty]{} 0$ for every $j = 1, \dots, d$), then

$$(X_k, \mathbb{Y}) \xrightarrow[k \rightarrow \infty]{(d)} (Z', \mathbb{Y}),$$

with $Z' \sim N(0, \sigma^2)$ independent of \mathbb{Y} .

- Let $(\mathbb{Y}_k = (Y_{1,k}, \dots, Y_{d,k}), k \geq 1)$ be a sequence of random vectors such that each component belongs to the sum of the first q th Wiener chaoses with $q \leq p$ and $\mathbb{Y}_k \xrightarrow{(d)} \mathbb{U}$ (\mathbb{U} is an arbitrary random vector). Then, if X_k, \mathbb{Y}_k are asymptotically uncorrelated (i.e. for every $j = 1, \dots, d$, $\mathbf{E}X_k Y_{j,k} \xrightarrow[k \rightarrow \infty]{} 0$), then

$$(X_k, \mathbb{Y}_k) \xrightarrow[k \rightarrow \infty]{(d)} (Z', \mathbb{U}),$$

where $Z' \sim N(0, \sigma^2)$ and Z', \mathbb{U} are independent.

- Let $(\mathbb{Y}_k = (Y_{1,k}, \dots, Y_{d,k}), k \geq 1)$ be a sequence of random vectors such that each component belongs to $\mathbb{D}^{1,2}$ and satisfies an additional (pretty natural) condition

(assumption (19) in Theorem 3). Suppose that $\mathbb{Y}_k \rightarrow \mathbb{U}$ in $L^2(\Omega)$, with \mathbb{U} is an arbitrary d -dimensional random vector, and X_k, \mathbb{Y}_k are asymptotically uncorrelated. Then

$$(X_k, \mathbb{Y}_k) \xrightarrow[k \rightarrow \infty]{(d)} (Z', \mathbb{U}),$$

where $Z' \sim N(0, \sigma^2)$ and Z', \mathbb{U} are independent.

- If $(Y_k, k \geq 1)$ is random sequence in the q th Wiener chaos with $q > p$ which converges only in law to U , then the joint convergence of $((X_k, Y_k), k \geq 1)$ to (Z, U) with Z, U independent does not hold. See the counter-example in Section 4.5.

These findings may have direct consequences to statistics and limit theorems since many estimators can be expressed as multiple stochastic integrals (see e.g. [22]). The main idea of the proof consists in combining the Fourth Moment Theorem with the multi-dimensional Stein-Malliavin bound (1), and it also involves some interesting technical lemmas (Lemmas 6 and 6), which may have their own interest. Let us emphasize that the assumption $p \geq 2$ is crucial. When $p = 1$, we cannot expect to have results as those listed above. Indeed, take $X = I_1(h)$ with $h \in H, \|h\| = 1$, so $X \sim N(0, 1)$. Then $Y = I_1(h)^2 - 1 = I_2(h^{\otimes 2})$ is an element of the second Wiener chaos, but X and Y are not independent (see e.g. the independence criterion in [23]).

We organized the paper as follows. In Section 2, we develop in a multidimensional context the variant of the Stein-Malliavin calculus introduced in [15]. Section 3 contains the statement of our main result concerning the asymptotic independence on Wiener chaos and a short discussion around it and its consequences. Section 4 contains the proof of the main result, which is detailed into several steps. In Section 5 we included several applications of our theory, while Section 6 is the the appendix where we present the basic tools needed throughout our work.

2 Multidimensional Stein method

In this paragraph, we generalize the variant of the Stein's method introduced in Section 5 of [15] to any dimension $d \geq 1$. Then, we combine it with the techniques of the Malliavin calculus in order to obtain the estimate (1).

2.1 The method

The basis of the Stein's method consists in the definition of the Stein's operator and of the Stein's equation. For the normal approximation, the standard operator is

$$\mathcal{L}f(x) = \sigma^2 f'(x) - xf(x), \quad x \in \mathbb{R},$$

which acts on suitable differentiable functions $f : \mathbb{R} \rightarrow \mathbb{R}$. This operator satisfies $\mathbf{E}\mathcal{L}f(Z) = 0$ for every $f : \mathbb{R} \rightarrow \mathbb{R}$ differentiable with $\mathbf{E}|f'(Z)| < \infty$ if and only if $Z \sim N(0, \sigma^2)$. The corresponding Stein's equation is

$$\mathcal{L}f(x) = \mathbf{E}h(x) - \mathbf{E}h(Z), \quad x \in \mathbb{R},$$

where $h : \mathbb{R} \rightarrow \mathbb{R}$ is a given function such that $\mathbf{E}|h(Z)| < \infty$. The idea of the Stein's method is to find a solution f_h to the Stein's equation with nice properties and to use it in order to obtain estimates for $\mathbf{E}h(X) - \mathbf{E}h(Z)$ for an arbitrary random variable X .

We follow the same line in a multidimensional context. Now, the purpose is not the normal approximation but to quantify the distance between the probability distribution of a random vector (X, \mathbb{Y}) and the random vector (Z, \mathbb{Y}) where Z is a centered Gaussian random variable with variance σ^2 and it is independent of \mathbb{Y} .

Let us consider the operator \mathcal{N} given by

$$\mathcal{N}f(x, \mathbf{y}) = \sigma^2 \partial_x f(x, \mathbf{y}) - xf(x, \mathbf{y}), \quad x \in \mathbb{R}, \mathbf{y} \in \mathbb{R}^d, \quad (2)$$

where $\partial_{x_1} f$ denotes the partial derivative of f with respect to its first variable. The operator \mathcal{N} acts on the set of differentiable functions $f : \mathbb{R}^{d+1} \rightarrow \mathbb{R}$.

Recall that if \mathbb{Y} is a random vector, we denote by $P_{\mathbb{Y}}$ its probability distribution. The following two lemmas show that the operator (2) characterizes the law of X and the independence of X and \mathbb{Y} . The material from this section is inspired from Section 5 in [15].

Lemma 1 *Assume $X \sim N(0, \sigma^2)$ and X is independent of the random vector \mathbb{Y} . Then $\mathbf{E}\mathcal{N}f(X, \mathbb{Y}) = 0$ for all $f : \mathbb{R}^{d+1} \rightarrow \mathbb{R}$ differentiable with $\mathbf{E}|\partial_x f(X, \mathbb{Y})| < \infty$.*

Proof: By the standard Stein method, for all $\mathbf{y} \in \mathbb{R}^d$,

$$\sigma^2 \mathbf{E}\partial_x f(X, \mathbf{y}) = \mathbf{E}Xf(X, \mathbf{y})$$

or

$$\sigma^2 \int_{\mathbb{R}} \partial_x f(x, \mathbf{y}) dP_X(x) = \int_{\mathbb{R}} xf(x, \mathbf{y}) dP_X(x).$$

Let us integrate with respect to the probability measure $P_{\mathbb{Y}}$. We have (the use of Fubini's theorem is based on Lemma 2.1 in [17])

$$\begin{aligned} & \sigma^2 \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}} \partial_x f(x, \mathbf{y}) dP_X(x) \right) dP_{\mathbb{Y}}(\mathbf{y}) \\ &= \sigma^2 \int_{\mathbb{R}^{d+1}} \partial_x f(x, \mathbf{y}) dP_X(x) \otimes dP_{\mathbb{Y}}(\mathbf{y}) \\ &= \sigma^2 \int_{\mathbb{R}^{d+1}} \partial_x f(x, \mathbf{y}) dP_{(X, \mathbb{Y})}(x, \mathbf{y}) = \sigma^2 \mathbf{E}\partial_x f(X, \mathbb{Y}), \end{aligned}$$

where we used the independence of X and \mathbb{Y} for the first equality on the above line. Similarly,

$$\begin{aligned} & \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}} xf(x, \mathbf{y}) dP_X(x) \right) dP_{\mathbb{Y}}(\mathbf{y}) \\ &= \int_{\mathbb{R}^{d+1}} xf(x, \mathbf{y}) dP_X(x) \otimes P_{\mathbb{Y}}(\mathbf{y}) = \int_{\mathbb{R}^{d+1}} xf(x, \mathbf{y}) dP_{(X, \mathbb{Y})}(x, \mathbf{y}) \\ &= \mathbf{E}Xf(X, \mathbb{Y}). \end{aligned}$$

■

We also have a lemma in the converse direction. By $\|\cdot\|_{\infty}$ we denote the infinity norm on \mathbb{R}^{d+1} .

Lemma 2 Consider a random vector (X, \mathbb{Y}) with $\mathbf{E}|X| < \infty$. Assume that

$$\mathbf{E}\mathcal{N}f(X, \mathbb{Y}) = 0 \tag{3}$$

for all differentiable functions $f : \mathbb{R}^{d+1} \rightarrow \mathbb{R}$ with $\|\partial_x f\|_{\infty} < \infty$. Then $X \sim N(0, \sigma^2)$ and X is independent of \mathbb{Y} .

Proof: Let φ be the characteristic function of the vector (X, \mathbb{Y}) , i.e.

$$\varphi(\lambda_1, \boldsymbol{\lambda}) = \mathbf{E} \left(e^{i(\lambda_1 X + \boldsymbol{\lambda} \mathbb{Y})} \right),$$

for $\lambda_1 \in \mathbb{R}$ and $\boldsymbol{\lambda} \in \mathbb{R}^d$. By applying (3) for the real and imaginary parts of φ , we get

$$\begin{aligned} \partial_{\lambda_1} \varphi(\lambda_1, \boldsymbol{\lambda}) &= i \mathbf{E} \left(X e^{i(\lambda_1 X + \boldsymbol{\lambda} \mathbb{Y})} \right) \\ &= i \sigma^2 \mathbf{E} \left(\partial_x e^{i(\lambda_1 X + \boldsymbol{\lambda} \mathbb{Y})} \right) = -\lambda_1 \sigma^2 \varphi(\lambda_1, \boldsymbol{\lambda}). \end{aligned}$$

By noticing that for every $\boldsymbol{\lambda} \in \mathbb{R}^d$, $\varphi(0, \boldsymbol{\lambda}) = \varphi_{\mathbb{Y}}(\boldsymbol{\lambda})$ (the characteristic function of the vector \mathbb{Y}), we obtain

$$\varphi(\lambda_1, \boldsymbol{\lambda}) = \varphi_{\mathbb{Y}}(\boldsymbol{\lambda}) e^{-\frac{\sigma^2 \lambda_1^2}{2}},$$

and this implies $X \sim N(0, \sigma^2)$ and X independent of \mathbb{Y} . ■

Let us now introduce the multidimensional Stein's equation

$$\mathcal{N}f(x, \mathbf{y}) = h(x, \mathbf{y}) - \mathbf{E}h(Z, \mathbf{y}), \quad x \in \mathbb{R}, \mathbf{y} \in \mathbb{R}^d \tag{4}$$

where $Z \sim N(0, \sigma^2)$. In (4), $h : \mathbb{R}^{d+1} \rightarrow \mathbb{R}$ is given and we assume that h is continuously differentiable with bounded partial derivatives. Let us show that (4) admits a solution with suitable properties.

Proposition 1 Let $h : \mathbb{R}^{d+1} \rightarrow \mathbb{R}$ be continuously differentiable with bounded partial derivatives. Then (4) admits a unique bounded solution which is given by

$$f_h(x, \mathbf{y}) = -\frac{1}{\sigma^2} \int_0^1 \frac{1}{2\sqrt{t(1-t)}} \mathbf{E} \left[Zh \left(\sqrt{t}x + \sqrt{1-t}Z, \mathbf{y} \right) \right] dt. \quad (5)$$

Moreover, we have the following bounds:

1.

$$\|f_h\|_\infty \leq \|\partial_{x_1} h\|_\infty. \quad (6)$$

2.

$$\|\partial_x f_h\|_\infty \leq \frac{1}{\sigma} \sqrt{\frac{2}{\pi}} \|\partial_x h\|_\infty. \quad (7)$$

3. For $j = 1, \dots, d$, if $\mathbf{y} = (y_1, \dots, y_d)$,

$$\|\partial_{y_j} f_h\|_\infty \leq \frac{1}{\sigma} \sqrt{\frac{\pi}{2}} \|\partial_{x_j} h\|_\infty, \quad (8)$$

Proof: By using the dominated convergence theorem, we get, by taking the derivative with respect to x in (5),

$$\partial_x f_h(x, \mathbf{y}) = -\frac{1}{\sigma^2} \int_0^1 \frac{1}{2\sqrt{1-t}} \mathbf{E} \left[Z \partial_x h \left(\sqrt{t}x + \sqrt{1-t}Z, \mathbf{y} \right) \right]. \quad (9)$$

Now, we apply the standard Stein identity to the function $g(z) = h(\sqrt{t}x + \sqrt{1-t}z, \mathbf{y})$ and we obtain

$$\begin{aligned} & \mathbf{E} \left[Z \partial_x h \left(\sqrt{t}x + \sqrt{1-t}Z, \mathbf{y} \right) \right] \\ &= \mathbf{E} g'(Z) = \sigma^2 \sqrt{1-t} \mathbf{E} \left[\partial_x h \left(\sqrt{t}x + \sqrt{1-t}Z, \mathbf{y} \right) \right]. \end{aligned} \quad (10)$$

By plugging (10) into (5), the function f_h can be written as

$$f_h(x, \mathbf{y}) = -\int_0^1 \frac{1}{2\sqrt{t}} \mathbf{E} \left[\partial_x h \left(\sqrt{t}x + \sqrt{1-t}Z, \mathbf{y} \right) \right] dt. \quad (11)$$

By (9) and (11), we can write

$$\begin{aligned} & \partial_x f_h(x, \mathbf{y}) - x f_h(x, \mathbf{y}) \\ &= \int_0^1 \mathbf{E} \left[\left(-\frac{Z}{2\sqrt{1-t}} + \frac{x}{2\sqrt{t}} \right) \partial_x h \left(\sqrt{t}x + \sqrt{1-t}Z, \mathbf{y} \right) \right] \\ &= \mathbf{E} \int_0^1 \frac{d}{dt} h \left(\sqrt{t}x + \sqrt{1-t}Z, \mathbf{y} \right) dt = h(x, \mathbf{y}) - \mathbf{E} h(Z, \mathbf{y}). \end{aligned}$$

Consequently, f_h given by (5) is a solution to (4). To prove (6), we use (11) to get

$$\|f_h\|_\infty \leq \int_0^1 \frac{1}{2\sqrt{t}} \|\partial_{x_1} h\|_\infty dt \leq \|\partial_x h\|_\infty$$

The bound (7) follows from (9) since

$$\|\partial_x f_h\|_\infty \leq \frac{\mathbf{E}|Z|}{\sigma^2} \|\partial_x h\|_\infty \leq \sigma^{-1} \sqrt{\frac{2}{\pi}} \|\partial_x h\|_\infty.$$

To prove (8), we differentiate with respect to $y_j, j = 1, \dots, d$ in (5),

$$\partial_{y_j} f_h(x, \mathbf{y}) = -\frac{1}{\sigma^2} \int_0^1 \frac{1}{2\sqrt{t(1-t)}} \mathbf{E} \left[Z \partial_{y_j} h \left(\sqrt{t}x + \sqrt{1-t}Z, \mathbf{y} \right) \right] dt$$

and

$$\|\partial_{y_j} f_h\|_\infty \leq \frac{\mathbf{E}|Z|}{\sigma^2} \|\partial_{y_j} h\|_\infty \int_0^1 \frac{1}{2\sqrt{t(1-t)}} dt = \frac{1}{\sigma} \sqrt{\frac{\pi}{2}} \|\partial_{y_j} h\|_\infty.$$

To finish the proof, we notice that for any other solution g_h to (4), one has

$$\partial_x \left(e^{-\frac{x^2}{2\sigma^2}} (f_h(x, \mathbf{y}) - g_h(x, \mathbf{y})) \right) = 0$$

so $g_h(x, \mathbf{y}) = f_h(x, \mathbf{y}) + e^{\frac{x^2}{2\sigma^2}} c(\mathbf{y})$ so g_h is bounded if and only if $c(\mathbf{y}) = 0$. ■

By Proposition 1, if f_h is the solution (5) to the Stein's equation (4), we have

$$\sigma^2 \partial_x f_h(x, \mathbf{y}) - x f_h(x, \mathbf{y}) = h(x, \mathbf{y}) - \mathbf{E}h(Z, \mathbf{y})$$

for any h differentiable with bounded partial derivatives. Let X, \mathbb{Y} be random vectors with $\mathbf{E}|X| < \infty$. Let us integrate with respect to $\theta := P_{(X, \mathbb{Y})}$ in the above identity. We have

$$\int_{\mathbb{R}^{d+1}} h(x, \mathbf{y}) d\theta(x, \mathbf{y}) = \mathbf{E}h(X, \mathbb{Y})$$

and

$$\begin{aligned} & \int_{\mathbb{R}^{d+1}} \mathbf{E}h(Z, \mathbf{y}) d\theta(x, \mathbf{y}) \\ &= \int_{\mathbb{R}^{d+1}} \left(\int_{\mathbb{R}} h(z, \mathbf{y}) dP_Z(z) \right) d\theta(x, \mathbf{y}) \\ &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}} h(z, \mathbf{y}) dP_Z(z) \right) dP_{\mathbb{Y}}(\mathbf{y}) \\ &= \int_{\mathbb{R}^{d+1}} h(x, \mathbf{y}) dP_Z \otimes P_{\mathbb{Y}}(\mathbf{y}) = \int_{\mathbb{R}^{d+1}} h(x, \mathbf{y}) d\eta(x, \mathbf{y}), \end{aligned}$$

with

$$\eta = P_Z \otimes P_{\mathbb{Y}}.$$

Therefore

$$\begin{aligned} & \sigma^2 \mathbf{E} \partial_x f_h(X, \mathbb{Y}) - \mathbf{E} X f_h(X, \mathbb{Y}) = \mathbf{E} h(X, \mathbb{Y}) - \mathbf{E} h(Z', \mathbb{Y}) \\ &= \int_{\mathbb{R}^{d+1}} h(x, \mathbf{y}) d\theta(x, \mathbf{y}) - \int_{\mathbb{R}^{d+1}} h(x, \mathbf{y}) d\eta(x, \mathbf{y}) \end{aligned} \quad (12)$$

where Z' has the same law as $Z \sim N(0, \sigma^2)$ and Z' is independent of \mathbb{Y} .

2.2 Stein method and Malliavin calculus

Let

$$\mathcal{A} = \{h : \mathbb{R}^n \rightarrow \mathbb{R}, h \text{ is Lipschitz continuous with } \|h\|_{Lip} \leq 1\}$$

and let F, G be two n -dimensional random vectors such that $h(F), h(G) \in L^1(\Omega)$ for every $h \in \mathcal{A}$. Then the Wasserstein distance between the probability distributions of F and G is defined by

$$d_W(P_F, P_G) = \sup_{h \in \mathcal{A}} |\mathbf{E} h(F) - \mathbf{E} h(G)|. \quad (13)$$

We denote by $\|h\|_{Lip}$ the Lipschitz norm of h given by

$$\|h\|_{Lip} = \sup_{x, y \in \mathbb{R}^n, x \neq y} \frac{|h(x) - h(y)|}{\|x - y\|_{\mathbb{R}^n}},$$

with $\|\cdot\|_{\mathbb{R}^n}$ the Euclidean norm in \mathbb{R}^n . The operators D, L, δ below are defined with respect to an isonormal process $(W(h), h \in H)$, see the Appendix. By $\langle \cdot, \cdot \rangle$ we denote the scalar product in the Hilbert space H .

We use the ideas of the Stein method for normal approximation (see [8]) to prove the following result.

Theorem 1 *Let X be a centered random variable in $\mathbb{D}^{1,2}$ and let $\mathbb{Y} = (Y_1, \dots, Y_d)$ be such that $Y_j \in \mathbb{D}^{1,2}$ for all $j = 1, \dots, d$. Let $\theta = P_{(X, \mathbb{Y})}$ and $\eta = P_Z \otimes P_{\mathbb{Y}}$, where $Z \sim N(0, \sigma^2)$. Then*

$$d_W(\theta, \eta) \leq C \left(\mathbf{E} |\sigma^2 - \langle D(-L)^{-1} X, DX \rangle| + \sum_{j=1}^d \mathbf{E} |\langle D(-L)^{-1} X, DY_j \rangle| \right). \quad (14)$$

Proof: Let $h : \mathbb{R}^{d+1} \rightarrow \mathbb{R}$ be continuously differentiable with bounded derivatives and let f_h be the corresponding solution to the Stein's equation (4). By using the well-known

formula $X = \delta D(-L)^{-1}X$ in (12), we obtain, by integrating by parts

$$\begin{aligned}
& \int_{\mathbb{R}^{d+1}} h(x, \mathbf{y}) d\theta(x, \mathbf{y}) - \int_{\mathbb{R}^{d+1}} h(x, \mathbf{y}) d\eta(x, \mathbf{y}) \\
&= \sigma^2 \mathbf{E} \partial_x f_h(X, \mathbb{Y}) - \mathbf{E} \delta D(-L)^{-1} X f_h(X, \mathbb{Y}) \\
&= \sigma^2 \mathbf{E} \partial_x f_h(X, \mathbb{Y}) - \mathbf{E} \langle D(-L)^{-1} X, Df_h(X, \mathbb{Y}) \rangle \\
&= \mathbf{E} \partial_x f_h(X, \mathbb{Y}) (\sigma^2 - \langle D(-L)^{-1} X, DX \rangle) \\
&\quad - \mathbf{E} \sum_{j=1}^d \partial_{x_j} f_h(X, \mathbb{Y}) \langle D(-L)^{-1} X, DY_j \rangle.
\end{aligned}$$

Hence, by using inequalities (7) and (8) in Proposition 1,

$$\begin{aligned}
& \left| \int_{\mathbb{R}^{d+1}} h(x, \mathbf{y}) d\theta(x, \mathbf{y}) - \int_{\mathbb{R}^{d+1}} h(x, \mathbf{y}) d\eta(x, \mathbf{y}) \right| \\
&\leq C \left(\mathbf{E} |\sigma^2 - \langle D(-L)^{-1} X, DX \rangle| + \sum_{j=1}^d \mathbf{E} |\langle D(-L)^{-1} X, DY_j \rangle| \right). \quad (15)
\end{aligned}$$

To finish the proof, we borrow again an argument from [15] (proof of Lemma 9 in this reference) to approximate a Lipschitz function by continuously differentiable functions with bounded derivatives. Indeed, if $h \in \mathcal{A}$ and $\varepsilon > 0$, then consider

$$h_\varepsilon(x, y_1, \dots, y_d) = \mathbf{E} h(x + \sqrt{\varepsilon} N, y_1 + \sqrt{\varepsilon} N_1, \dots, y_d + \sqrt{\varepsilon} N_d),$$

where N, N_1, \dots, N_d are independent standard normal random variables. Then h_ε is differentiable and it satisfies

$$\|h_\varepsilon - h\|_\infty \rightarrow_{\varepsilon \rightarrow 0} 0, \quad \|\partial_x h_\varepsilon\|_\infty \leq \|h_\varepsilon\|_{Lip} \leq \|h\|_{Lip} \leq 1$$

and

$$\max_{j=1, \dots, d} \|\partial_{y_j} h_\varepsilon\|_\infty \leq \|h_\varepsilon\|_{Lip} \leq \|h\|_{Lip} \leq 1.$$

Therefore, by (15),

$$\begin{aligned}
& \left| \int_{\mathbb{R}^{d+1}} h(x, \mathbf{y}) d\theta(x, \mathbf{y}) - \int_{\mathbb{R}^{d+1}} h(x, \mathbf{y}) d\eta(x, \mathbf{y}) \right| \\
&\leq 2\|h_\varepsilon - h\|_\infty + \left| \int_{\mathbb{R}^{d+1}} h_\varepsilon(x, \mathbf{y}) d\theta(x, \mathbf{y}) - \int_{\mathbb{R}^{d+1}} h_\varepsilon(x, \mathbf{y}) d\eta(x, \mathbf{y}) \right| \\
&\leq 2\|h_\varepsilon - h\|_\infty + C \left(\mathbf{E} |\sigma^2 - \langle D(-L)^{-1} X, DX \rangle| + \sum_{j=1}^d \mathbf{E} |\langle D(-L)^{-1} X, DY_j \rangle| \right)
\end{aligned}$$

and we conclude by letting $\varepsilon \rightarrow 0$. ■

The corollary below is used to deal with random vectors with components in Wiener chaos.

Corollary 1 *With the notation from Theorem 1, if $X, Y_1, \dots, Y_d \in \mathbb{D}^{1,4}$, then*

$$d_W(\theta, \eta) \leq C \left[\left(\mathbf{E} |\sigma^2 - \langle D(-L)^{-1}X, DX \rangle|^2 \right)^{\frac{1}{2}} + \sum_{j=1}^d \left(\mathbf{E} |\langle D(-L)^{-1}X, DY_j \rangle|^2 \right)^{\frac{1}{2}} \right]. \quad (16)$$

Proof: The proof follows from Theorem 1, by using Cauchy-Schwarz's inequality in the right-hand side of (14) and by noticing that $\langle D(-L)^{-1}X, DY_j \rangle$ belongs to $L^2(\Omega)$ when $X, Y_j \in \mathbb{D}^{1,4}$, for $j = 1, 2, \dots, d$. ■

Remark 1 *As a particular case of relation (14) in Theorem 1, it follows that if $X_1 \sim N(0, \sigma^2)$ and $\langle DX_1, DX_2 \rangle = 0$ almost surely, then X_1 is independent of X_2 . In particular, this means that, if $X_1 = I_1(h)$ and $X_2 = \sum_{n \geq 0} I_n(g_n)$ (with $h \in H, g_n \in H^{\odot n}$ for every $n \geq 1$), then $h \otimes_1 g_n = 0$ almost everywhere on $H^{\otimes n-1}$ implies the independence of X_1 and X_2 . This is related to the independence criterion for multiple stochastic integrals in [23], which states that two random variables $I_p(f)$ and $I_q(g)$ (with $f \in H^{\odot p}, g \in H^{\odot q}$) are independent if and only if $f \otimes_1 g$ vanishes almost everywhere on $H^{\otimes p+q-2}$.*

3 Asymptotic independence on Wiener chaos

The variant of the Stein's method presented in Section 2 lead to some strong consequences when it is applied to sequences of multiple stochastic integrals. Here we describe and discuss our main findings in the case of the Wiener chaos. The proofs will be detailed in the next section.

3.1 Preliminary tools

Let us start with some auxiliary results that will be used several times in the sequel. Recall that H is a real and separable Hilbert space and $W = (W(h), h \in H)$ is an isonormal process on the probability space (Ω, \mathcal{G}, P) , where \mathcal{G} is the sigma-algebra generated by W . The operators D, L and the multiple stochastic integral $I_p, p \geq 1$ are all with respect to W .

This our first auxiliary result. The contraction of two kernels has been defined in the appendix (see (92)).

Lemma 3 Let $f_1, f_3 \in H^{\odot p}$ and $f_2, f_4 \in H^{\odot q}$ with $p, q \geq 1$. Then, for every $r = 0, \dots, p \wedge q$,

$$\langle f_1 \otimes_r f_2, f_3 \otimes_r f_4 \rangle_{H^{\otimes p+q-2r}} = \langle f_1 \otimes_{p-r} f_3, f_2 \otimes_{q-r} f_4 \rangle_{H^{\otimes 2r}}.$$

Proof: This is e.g. Lemma 4.4 in [21]. ■

The following well-known result allows to express the L^2 -norm of $\langle D(-L)^{-1}X, DY \rangle$ when X and Y are multiple stochastic integrals.

Lemma 4 Let $X = I_p(f)$ and $Y = I_q(g)$ with $p, q \geq 1$ and $f \in H^{\odot p}, g \in H^{\odot q}$. Then

$$\mathbf{E} \langle D(-L)^{-1}X, DY \rangle_H^2 = (\mathbf{E}(XY))^2 1_{p=q} + \sum_{r=1}^{p \wedge q} c(r, p, q) \|f \tilde{\otimes}_r g\|_{H^{\otimes p+q-2r}}^2,$$

where $c(r, p, q)$ are strictly positive combinatorial constants for $r = 1, \dots, (p \wedge q) - 1$ and

$$c(p \wedge q, p, q) = \begin{cases} 0, & \text{if } p = q \\ > 0, & \text{if } p \neq q. \end{cases}$$

Proof: See e.g. [8], Lemma 6.2.1. ■

We will also need the celebrated Fourth Moment Theorem proven in [13]. See also [12] for point 4. below.

Theorem 2 ([13] and [8]) Fix an integer $n \geq 1$. Consider a sequence $(F_k = I_n(f_k), k \geq 1)$ of square integrable random variables in the n th Wiener chaos. Assume that

$$\lim_{k \rightarrow \infty} \mathbf{E}[F_k^2] = \lim_{k \rightarrow \infty} n! \|f_k\|_{H^{\odot n}}^2 = 1. \quad (17)$$

Then, the following statements are equivalent.

1. The sequence of random variables $(F_k = I_n(f_k), k \geq 1)$ converges to the standard normal law in distribution as $k \rightarrow \infty$.
2. $\lim_{k \rightarrow \infty} \mathbf{E}[F_k^4] = 3$.
3. $\lim_{k \rightarrow \infty} \|f_k \otimes_l f_k\|_{H^{\otimes 2(n-l)}} = 0$ for $l = 1, 2, \dots, n-1$.
4. $\|DF_k\|_H^2$ converges to n in $L^2(\Omega)$ as $k \rightarrow \infty$.

3.2 Main result

In this paragraph, we state our main findings and we discuss some consequences. The main result of this work states as follows. The notation d_W below stands for the Wasserstein distance, see (13).

Theorem 3 *Let us consider the integer numbers $p \geq 2$, $d \geq 1$. Let $(X_k, k \geq 1)$ be a sequence of random variables such that for every $k \geq 1$, $X_k = I_p(f_k)$ with $f_k \in H^{\odot p}$. Assume that*

$$X_k \xrightarrow[k \rightarrow \infty]{(d)} Z \sim N(0, \sigma^2). \quad (18)$$

Let $(\mathbb{Y}_k, k \geq 1) = ((Y_{1,k}, \dots, Y_{d,k}), k \geq 1)$ be a sequence of random vectors such that, for every $j = 1, \dots, d$, the random variable $Y_{j,k}$ belongs to $\mathbb{D}^{1,2}$, and it admits the chaos expansion

$$Y_{j,k} = \sum_{n=0}^{\infty} I_n(g_{n,k}^{(j)}) \text{ with } g_{n,k}^{(j)} \in H^{\odot n}$$

and

$$\sup_{k \geq 1} \sum_{n=M+1}^{\infty} n! \|g_{n,k}\|_{H^{\otimes n}}^2 \xrightarrow{M \rightarrow \infty} 0. \quad (19)$$

Suppose that there exists a random vector \mathbb{U} in \mathbb{R}^d such that

$$\mathbb{Y}_k \xrightarrow[k \rightarrow \infty]{} \mathbb{U} \text{ in } L^2(\Omega). \quad (20)$$

Then, if

$$\mathbf{E} X_k Y_{j,k} \xrightarrow[k \rightarrow \infty]{} 0 \text{ for every } j = 1, \dots, d \quad (21)$$

we have

$$(X_k, \mathbb{Y}_k) \xrightarrow[k \rightarrow \infty]{(d)} (Z', \mathbb{U}), \quad (22)$$

where $Z' \sim N(0, \sigma^2)$ and Z' is independent by the random vector \mathbb{U} . Moreover, for every $k \geq 1$,

$$\begin{aligned} & d_W(P_{(X_k, Y_k)}, P_{Z'} \otimes P_{\mathbb{U}}) \\ & \leq C \left[\mathbf{E} |\sigma^2 - \langle D(-L)^{-1} X_k, D X_k \rangle| + \sum_{j=1}^d \mathbf{E} |\langle D(-L)^{-1} X_k, D Y_{j,k} \rangle_H| \right] + d_W(\mathbb{Y}_k, \mathbb{U}). \end{aligned} \quad (23)$$

Let us make some comment around Theorem 3.

- Condition (19) is automatically verified when $X_{j,k}$ belongs to a finite sum of Wiener chaoses or when $Y_{j,k} = Y_j$ for every $k \geq 1$ (this is stated in Corollary 2). On the other hand, this case (when the components of \mathbb{Y}_k are in a finite sum of Wiener chaoses) will be proven before the main result, as a step of the proof of Theorem 3.

- In Proposition 4, we show that if the components of \mathbb{Y}_k belong to the sum of the first q Wiener chaoses ($q \leq p$), then it is enough to assume, instead of (20), only the convergence in law of $(\mathbb{Y}_k, k \geq 1)$ in order to obtain (22).
- The assumption (19) also appears in the paper [5], in the context of the normal approximation of Wiener space (see also Theorem 6.3.1 in [8]).
- The quantitative bound (23) is a direct consequence of the results in Section 2. It will be actually used inside the proof of the main result (Theorem 3).
- The uncorrelation condition (20) is obviously crucial for the joint convergence of (X_k, \mathbb{Y}_k) in Theorem 3. Another interesting question is what happens if we assume, instead of (21), that

$$\mathbf{E}X_k Y_{j,k} \rightarrow_{k \rightarrow \infty} c_j,$$

with $c_j \neq 0$ for $j = 1, \dots, d$. Can we deduce the joint convergence of (X_k, \mathbb{Y}_k) to a random vector with marginals Z and \mathbb{U} ? In the case when \mathbb{U} follows a Gaussian distribution, the answer is given by the main result in [14]. In order to give a complete answer, we need to know how to characterize the law of the vector (Z, \mathbb{U}) when $Z \sim N(0, \sigma^2)$ is not independent of \mathbb{U} and the law of \mathbb{U} is not Gaussian.

Let us state the following corollary of the above theorem.

Corollary 2 *Consider the sequence $(X_k, k \geq 1)$ as in Theorem 3 and $\mathbb{Y} = (Y_1, \dots, Y_d)$ be a random vector in \mathbb{R}^d . Assume that for every $j = 1, \dots, d$, $Y_j \in \mathbb{D}^{1,2}$. Also assume*

$$\mathbf{E}X_k Y_j \rightarrow_{k \rightarrow \infty} 0. \quad (24)$$

Then

$$(X_k, \mathbb{Y}) \rightarrow^{(d)} (Z', \mathbb{Y}) \quad (25)$$

with $Z' \sim N(0, \sigma^2)$ independent of \mathbb{Y} and for $k \geq 1$,

$$\begin{aligned} & d_W(P_{(X_k, \mathbb{Y})}, P_{Z'} \otimes P_{\mathbb{Y}}) \\ & \leq C \left[\mathbf{E} |\sigma^2 - \langle D(-L)^{-1} X_k, DX_k \rangle| + \sum_{j=1}^d \mathbf{E} |\langle D(-L)^{-1} X_k, DY_j \rangle_H| \right]. \end{aligned} \quad (26)$$

Proof: It is an immediate consequence of Theorem 3, since (19) is obviously satisfied.

Remark 2 *Corollary 2 actually says that any sequence in the p th Wiener chaos with $p \geq 2$ is asymptotically independent of any (regular enough) d -dimensional random vector in $L^2(\Omega, \mathcal{G}, P)$ (with components in $\mathbb{D}^{1,2}$) if the uncorrelation assumption (24) is satisfied.*

Let us give a possible explanation of this phenomenon. Since $(X_k, k \geq 1)$ satisfies (18), it follows from Theorem 2 that, for $r = 1, \dots, p-1$,

$$\|f_k \otimes_r f_k\|_{H^{\otimes 2p-2r}} \rightarrow_{k \rightarrow \infty} 0.$$

Let $h \in H$. Then, by Lemma 3 and Cauchy-Schwarz' inequality,

$$\begin{aligned} \|f_k \otimes_1 h\|_{H^{\otimes p-1}} &= \langle f_k \otimes_1 h, f_k \otimes_1 h \rangle_{H^{\otimes p-1}} \\ &= \langle f_k \otimes_{p-1} f_k, h \otimes h \rangle_{H^{\otimes 2}} \leq \|f_k \otimes_{p-1} f_k\|_{H^{\otimes 2}} \|h\|_H^2 \rightarrow_{k \rightarrow \infty} 0. \end{aligned}$$

This intuitively means, taking into account the independence criterion of two multiple integrals proven in [23], that $X_k = I_p(f_k)$ and $W(h) = I_1(h)$ are asymptotically independent for any $h \in H$. Then X_k is asymptotically independent by any functional of W and by density by any random variable in $L^2(\Omega, \mathcal{G}, P)$ (recall that \mathcal{G} is the sigma-algebra generated by W).

4 Proof of the main result

The proof of the main result will be done into several steps. We start with an (intriguing) technical lemma (Lemma 5 below) which plays a crucial role in our proofs. Then we prove the result in the case when the components of \mathbb{Y}_k belong each of them to a Wiener chaos of fixed order, we continue with the case when these components are in a finite sum of Wiener chaos and finally we conclude the proof of Theorem 3. Our arguments use intensively the auxiliary tools recalled in Section 3.1, the Lemma 5 and the Stein-Malliavin bounds (14), (16) obtained in Section 2.

4.1 A key lemma

As mentioned, the below lemma is a central point in our approach.

Lemma 5 *Let $p \geq 2$ and $q \geq 1$ be two integer numbers. Let $(X_k, k \geq 1)$ be that such for every $k \geq 1$, $X_k = I_p(f_k)$ with $f_k \in H^{\odot p}$. Assume*

$$X_k \xrightarrow[k \rightarrow \infty]{(d)} Z \sim N(0, \sigma^2). \quad (27)$$

Then, for every $g \in H^{\odot q}$,

$$\|f_k \otimes_r g\|_{H^{p+q-2r}} \rightarrow_{k \rightarrow \infty} 0 \text{ for every } \begin{cases} r = 1, \dots, p \wedge q \text{ if } p \neq q \\ r = 1, \dots, (p \wedge q) - 1 \text{ if } p = q. \end{cases}$$

Proof: Without loss of generality, we can assume that $H = L^2(T, \mathcal{B}, \nu)$, where ν is a sigma-finite measure without atoms.

Let $p > q$. Then the conclusion follows easily from Lemma 3 and point 3. in the Fourth Moment Theorem (Theorem 2). Indeed, for every $1 \leq r \leq q < p$,

$$\begin{aligned} \|f_k \otimes_r g\|_{H^{\otimes p+q-2r}}^2 &= \langle f_k \otimes_r g, f_k \otimes_r g \rangle_{H^{\otimes p+q-2r}} = \langle f_k \otimes_{p-r} f_k, g \otimes_{q-r} g \rangle_{H^{\otimes 2r}} \\ &\leq \|f_k \otimes_{p-r} f_k\|_{H^{\otimes 2r}} \|g \otimes_{q-r} g\|_{H^{\otimes 2r}} \end{aligned} \quad (28)$$

and $\|f_k \otimes_{p-r} f_k\|_{H^{\otimes 2r}} \rightarrow_{k \rightarrow \infty} 0$ by Theorem 2 since $1 \leq p-r \leq p-1$. We employ the same argument holds when $p = q$ and $1 \leq r \leq p-1$.

Assume now $p < q$. If $1 \leq r \leq p-1$, then the above argument still holds, due to the inequality

$$\|f_k \otimes_r g\|_{H^{\otimes p+q-2r}}^2 \leq \|f_k \otimes_{p-r} f_k\|_{H^{\otimes 2r}} \|g \otimes_{q-r} g\|_{H^{\otimes 2r}}$$

and of the fact that $1 \leq p-r \leq p-1$.

It remains to prove that, for $2 \leq p < q$,

$$\|f_k \otimes_p g\|_{L^2(T^{q-p})} \rightarrow_{k \rightarrow \infty} 0. \quad (29)$$

To prove (29), we will proceed into two steps.

Step 1. We show that for every $h_1, \dots, h_q \in H = L^2(T)$, we have

$$\|f_k \otimes_p (h_1 \tilde{\otimes} \dots \tilde{\otimes} h_q)\|_{L^2(T^{q-p})} \rightarrow_{k \rightarrow \infty} 0.$$

We have

$$h_1 \tilde{\otimes} \dots \tilde{\otimes} h_q = \frac{1}{q!} \sum_{\sigma \in S_q} h_{\sigma(1)} \otimes \dots \otimes h_{\sigma(q)},$$

where S_q is the set of permutations of $\{1, \dots, q\}$. Then, via the definition of the contraction

(92),

$$\begin{aligned}
& (f_k \otimes_p (h_1 \tilde{\otimes} \dots \tilde{\otimes} h_q)) (t_1, \dots, t_{q-p}) \\
&= \frac{1}{q!} \sum_{\sigma \in S_q} \int_{T^p} f_k(u_1, \dots, u_p) (h_{\sigma(1)} \otimes \dots \otimes h_{\sigma(q)}) (u_1, \dots, u_p, t_1, \dots, t_{q-p}) du_1 \dots du_p \\
&= \frac{1}{q!} \sum_{\sigma \in S_q} (h_{\sigma(p+1)} \otimes \dots \otimes h_{\sigma(q)}) (t_1, \dots, t_{q-p}) \\
&\quad \times \int_{T^p} f_k(u_1, \dots, u_p) (h_{\sigma(1)} \otimes \dots \otimes h_{\sigma(p)}) (u_1, \dots, u_p) du_1 \dots du_p \\
&= \frac{1}{q!} \sum_{\sigma \in S_q} (h_{\sigma(p+1)} \otimes \dots \otimes h_{\sigma(q)}) (t_1, \dots, t_{q-p}) \\
&\quad \times \int_{T^{p-1}} \left(\int_T f_k(u_1, \dots, u_p) h_{\sigma(1)}(u_1) du_1 \right) (h_{\sigma(2)} \otimes \dots \otimes h_{\sigma(p)}) (u_2, \dots, u_p) du_2 \dots du_p \\
&= \frac{1}{q!} \sum_{\sigma \in S_q} (h_{\sigma(p+1)} \otimes \dots \otimes h_{\sigma(q)}) (t_1, \dots, t_{q-p}) \\
&\quad \times \int_{T^{p-1}} (f_k \otimes_1 h_{\sigma(1)})(u_2, \dots, u_p) (h_{\sigma(2)} \otimes \dots \otimes h_{\sigma(p)}) (u_2, \dots, u_p) du_2 \dots du_p \\
&= \frac{1}{q!} \sum_{\sigma \in S_q} (h_{\sigma(p+1)} \otimes \dots \otimes h_{\sigma(q)}) (t_1, \dots, t_{q-p}) \langle f_k \otimes_1 h_{\sigma(1)}, h_{\sigma(2)} \otimes \dots \otimes h_{\sigma(p)} \rangle_{L^2(T^{p-1})}
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \|f_k \otimes_p (h_1 \tilde{\otimes} \dots \tilde{\otimes} h_q)\|_{L^2(T^{q-p})} \\
&\leq \frac{1}{q!} \sum_{\sigma \in S_q} \|h_{\sigma(p+1)}\|_{L^2(T)} \dots \|h_{\sigma(q)}\|_{L^2(T)} |\langle f_k \otimes_1 h_{\sigma(1)}, h_{\sigma(2)} \otimes \dots \otimes h_{\sigma(p)} \rangle_{L^2(T^{p-1})}| \\
&\leq \frac{1}{q!} \sum_{\sigma \in S_q} \|h_{\sigma(p+1)}\|_{L^2(T)} \dots \|h_{\sigma(q)}\|_{L^2(T)} \|f_k \otimes_1 h_{\sigma(1)}\|_{L^2(T^{p-1})} \|h_{\sigma(2)} \otimes \dots \otimes h_{\sigma(p)}\|_{L^2(T^{p-1})} \\
&\leq \frac{1}{q!} \sum_{\sigma \in S_q} \|h_{\sigma(1)}\|_{L^2(T)} \dots \|h_{\sigma(q)}\|_{L^2(T)} \sqrt{\|f_k \otimes_{p-1} \otimes f_k\|_{L^2(T^2)}},
\end{aligned}$$

where we used Lemma 3 and Cauchy-Schwarz's inequality. We obtained

$$\|f_k \otimes_p (h_1 \tilde{\otimes} \dots \tilde{\otimes} h_q)\|_{L^2(T^{q-p})} \leq \left(\prod_{j=1}^q \|h_j\|_{L^2(T)} \right) \sqrt{\|f_k \otimes_{p-1} \otimes f_k\|_{L^2(T^2)}},$$

and this goes to zero as $k \rightarrow \infty$ by point 3. in Theorem 2.

Step 2. We prove the claim (29) for $g \in L^2_{\mathcal{S}}(T^q)$ (the set of symmetric functions in $L^2(T^q)$). Consider the sequence $(g^M, M \geq 1)$ given by

$$g^M = \sum_{j_1, \dots, j_q=1}^M \langle g, h_{j_1} \otimes \dots \otimes h_{j_q} \rangle_{L^2(T^q)} h_{j_1} \otimes \dots \otimes h_{j_q} = \sum_{j_1, \dots, j_q=1}^M \langle g, h_{j_1} \otimes \dots \otimes h_{j_q} \rangle_{L^2(T^q)} h_{j_1} \tilde{\otimes} \dots \tilde{\otimes} h_{j_q}$$

where $(h_i, i \geq 1)$ is an orthonormal basis of $H = L^2(T)$. Then g^M are symmetric functions and $\|g^M - g\|_{L^2(T^q)} \rightarrow_{M \rightarrow \infty} 0$. We write

$$f_k \otimes_p g = f_k \otimes_p g^M + f_k \otimes_p g - f_k \otimes_p g^M$$

and

$$\|f_k \otimes_p g\|_{L^2(T^{q-p})} \leq \|f_k \otimes_p g^M\|_{L^2(T^{q-p})} + \|f_k \otimes_p g - f_k \otimes_p g^M\|_{L^2(T^{q-p})}. \quad (30)$$

Now, for every $M \geq 1$,

$$\begin{aligned} \|f_k \otimes_p g - f_k \otimes_p g^M\|_{L^2(T^{q-p})} &= \|f_k \otimes_p (g - g^M)\|_{L^2(T^{q-p})} \\ &\leq \|f_k\|_{L^2(T^p)} \|g - g^M\|_{L^2(T^q)} \leq C \|g - g^M\|_{L^2(T^q)}. \end{aligned} \quad (31)$$

We used the fact that, by (27), $q! \|f_k\|_{L^2(T^p)}^2 \rightarrow_{k \rightarrow \infty} \sigma^2$ so the sequence $(f_k, k \geq 1)$ is bounded in $L^2(T^p)$.

Let $\varepsilon > 0$. By (31), there exists $M_0 \geq 1$ such that for any $M \geq M_0$

$$\|f_k \otimes_p g - f_k \otimes_p g^M\|_{L^2(T^{q-p})} \leq \frac{\varepsilon}{2}. \quad (32)$$

Take $M \geq M_0$. Then

$$f_k \otimes_p g^M = \sum_{j_1, \dots, j_q=1}^M \langle g, h_{j_1} \otimes \dots \otimes h_{j_q} \rangle_{L^2(T^q)} (f_k \otimes_p (h_{j_1} \tilde{\otimes} \dots \tilde{\otimes} h_{j_q})).$$

By Step 1,

$$\|f_k \otimes_p g^M\|_{L^2(T^{q-p})} \rightarrow_{k \rightarrow \infty} 0,$$

so for k large enough,

$$\|f_k \otimes_p g^M\|_{L^2(T^{q-p})} \leq \frac{\varepsilon}{2}. \quad (33)$$

By plugging (32) and (33) into (30), we get the claim (29).

We state an immediate consequence of Lemma 5

Lemma 6 *Let $p \geq 2$ and $q \geq 1$ be two integer numbers. Let $(X_k, k \geq 1)$ be that such for every $k \geq 1$, $X_k = I_p(f_k)$ with $f_k \in H^{\odot p}$. Assume (18).*

1. *Let $(g_k, k \geq 1)$ (with $g_k \in H^{\odot q}$ for every $k \geq 1$) be a sequence that converges in $H^{\odot q}$. Then*

$$\|f_k \otimes_r g_k\|_{H^{p+q-2r}} \xrightarrow{k \rightarrow \infty} 0 \text{ for every } \begin{cases} r = 1, \dots, p \wedge q \text{ if } p \neq q \\ r = 1, \dots, (p \wedge q) - 1 \text{ if } p = q. \end{cases} \quad (34)$$

2. *Let $(g_k, k \geq 1)$ (with $g_k \in H^{\odot q}$ for every $k \geq 1$) be a sequence bounded in $H^{\odot q}$. Assume $q \leq p$. Then (34) holds true.*

Proof: Denote by g the limit in $H^{\odot q}$ of the sequence $(g_k, k \geq 1)$. Then, for $r = 1, \dots, p \wedge q$ (if $p \neq q$) or $r = 1, \dots, p \wedge q - 1$ (when $p = q$), we have

$$\begin{aligned} \|f_k \otimes_r g_k\|_{H^{\otimes p+q-2r}} &\leq \|f_k \otimes_r g\|_{H^{\otimes p+q-2r}} + \|f_k \otimes_r (g_k - g)\|_{H^{\otimes p+q-2r}} \\ &\leq \|f_k \otimes_r g\|_{H^{\otimes p+q-2r}} + \|f_k\|_{H^{\otimes p}} \|g_k - g\|_{H^{\otimes q}} \\ &\leq \|f_k \otimes_r g\|_{H^{\otimes p+q-2r}} + C \|g_k - g\|_{H^{\otimes q}}, \end{aligned}$$

since $(f_k, k \geq 1)$ is bounded in $H^{\otimes p}$. It suffices to apply Lemma 5 to conclude point 1. Point 2. follows immediately from the bound (28) since

$$\begin{aligned} \|f_k \otimes_r g_k\|_{H^{\otimes p+q-2r}}^2 &\leq \|f_k \otimes_{p-r} f_k\|_{H^{\otimes 2r}} \|g_k \otimes_{q-r} g_k\|_{H^{\otimes 2r}} \\ &\leq \|f_k \otimes_{p-r} f_k\|_{H^{\otimes 2r}} \|g_k\|_{H^{\otimes q}}^2 \leq C \|f_k \otimes_{p-r} f_k\|_{H^{\otimes 2r}}. \end{aligned}$$

4.2 The proof of the main result when the components of \mathbb{Y}_k belongs to a Wiener chaos

Let us make a first step to prove the main result, by dealing with the case when the random vector \mathbb{Y}_k from the statement of Theorem 3 has components that belong each of them in a Wiener chaos of fixed (but possibly different) order.

Proposition 2 *Let $p \geq 2$ and let $q_1, \dots, q_d \geq 1$ be integer numbers. Assume that $(X_k, k \geq 1)$ is such that $X_k = I_p(f_k)$, $f_k \in H^{\odot p}$ and (18) holds true. Let $(\mathbb{Y}_k, k \geq 1) = ((Y_{1,k}, \dots, Y_{d,k}), k \geq 1)$ be a sequence of random vectors such that for every $k \geq 1, j = 1, \dots, d$,*

$$Y_{j,k} = I_{q_j}(g_{j,k}) \text{ with } g_{j,k} \in H^{\odot q_j}.$$

Suppose (20) and (21). Then

$$(X_k, \mathbb{Y}_k) \xrightarrow[k \rightarrow \infty]{(d)} (Z', \mathbb{U}), \quad (35)$$

where $Z' \sim N(0, \sigma^2)$ and Z' is independent by \mathbb{U} . Moreover, we have the estimate (23).

Proof: We first notice that (23) is a direct consequence of (16) of the triangle's inequality. Indeed, for every $k \geq 1$,

$$\begin{aligned} d_W(P_{(X_k, \mathbb{Y}_k)}, P_{Z'} \otimes P_{\mathbb{U}}) &\leq d_W(P_{(X_k, \mathbb{Y}_k)}, P_{Z'} \otimes P_{\mathbb{Y}_k}) + d_W(P_{Z'} \otimes P_{\mathbb{Y}_k}, P_{Z'} \otimes P_{\mathbb{U}}) \\ &\leq d_W(P_{(X_k, \mathbb{Y}_k)}, P_{Z'} \otimes P_{\mathbb{Y}_k}) + d_W(P_{\mathbb{Y}_k}, P_{\mathbb{U}}) \\ &\leq d_W(P_{(X_k, \mathbb{Y}_k)}, P_{Z'} \otimes P_{\mathbb{Y}_k}) + \mathbf{E}\|\mathbb{Y}_k - \mathbb{U}\|_{\mathbb{R}^d}. \end{aligned} \quad (36)$$

and then we use (16). For the rest of the proof, we will again proceed into several steps.

Step 1. We prove that for every $j = 1, \dots, d$,

$$\mathbf{E}\langle D(-L)^{-1}X_k, DY_{j,k} \rangle^2 \rightarrow_{k \rightarrow \infty} 0. \quad (37)$$

By Lemma 4, we have, for every $k \geq 1$ and $j = 1, \dots, d$,

$$\mathbf{E}\langle D(-L)^{-1}X_k, DY_{j,k} \rangle^2 = (\mathbf{E}X_k Y_{j,k})^2 1_{p=q_j} + \sum_{r=1}^{p \wedge q_j} c(r, p, q_j) \|f_k \tilde{\otimes}_r g_{j,k}\|_{H^{\otimes p+q_j-2r}}^2, \quad (38)$$

where $c(r, p, q_j)$ are as in Lemma 4. In particular, recall that $c(p \wedge q_j, p, q_j) = 0$ if $p \neq q_j$. By Lemma 6,

$$\|f_k \tilde{\otimes}_r g_{j,k}\|_{H^{\otimes p+q_j-2r}}^2 \leq \|f_k \otimes_r g_{j,k}\|_{H^{\otimes p+q_j-2r}}^2 \rightarrow_{k \rightarrow \infty} 0 \quad (39)$$

for every $r = 1, \dots, p \wedge q_j$ (if $p \neq q_j$) and $r = 1, \dots, (p \wedge q_j) - 1$ (if $p = q_j$). The relation (39) and the assumption (21) imply the conclusion (37) of this step.

Step 2. Let us use the notation

$$\theta_k = P_{(X_k, \mathbb{Y}_k)}, \quad \eta_k = P_Z \otimes P_{\mathbb{Y}_k}, \quad \eta = P_Z \otimes P_{\mathbb{U}}. \quad (40)$$

In this step, we prove that

$$d_W(\theta_k, \eta_k) \rightarrow_{k \rightarrow \infty} 0. \quad (41)$$

We know from (16) that

$$d_W(\theta_k, \eta_k) \leq C \left[\left(\mathbf{E}(\langle D(-L)^{-1}X_k, DX_k \rangle - \sigma^2)^2 \right)^{\frac{1}{2}} + \sum_{j=1}^d \left(\mathbf{E}\langle D(-L)^{-1}X_k, DY_{j,k} \rangle^2 \right)^{\frac{1}{2}} \right] \quad (42)$$

The assumption (18) and the Fourth Moment Theorem implies that (see Section 5 in [8]),

$$\mathbf{E}(\langle D(-L)^{-1}X_k, DX_k \rangle - \sigma^2)^2 \rightarrow_{k \rightarrow \infty} 0.$$

This fact, together with Step 1, implies (41).

The conclusion is obtained by Step 2 and the bound (36).

Remark 3 It is possible to write a quantitative bound for $d_W(\theta_k, \eta)$ in terms of the norms of the contractions of the kernels f_k and $g_{j,k}$ (with the notation from Proposition 2). Indeed, assume $d = 1$ and $q_1 = q$. Then, by using (23), Lemma 4, the inequality (28) and the fact that a sequence of random variables that converges in distribution is bounded in $L^r(\Omega)$ for every $r > 1$ (see [4] or [8]), we can write

$$d_W(\theta_k, \eta) \leq C \left[(\mathbf{E}X_k Y_k)^2 1_{p=q} + \sum_{r=1}^{p-1} \|f_k \otimes_r f_k\|_{H^{\otimes 2p-2r}}^2 + \sum_{r=1}^{(p \wedge q)-1} \|f_k \otimes_{p-r} f_k\|_{H^{\otimes 2r}} \right. \\ \left. + \|f_k \otimes_q f_k\|_{H^{\otimes p-q}} 1_{p>q} + \|f_k \otimes_p g_k\|_{H^{q-p}}^2 1_{p<q} \right]^{\frac{1}{2}}.$$

Taking into account point 3. in Theorem 2, we can also write, for k large enough,

$$d_W(\theta_k, \eta) \leq C \left[\langle f_k, g_k \rangle_{H^{\otimes p}}^2 1_{p=q} + \sum_{r=1}^{p-1} \|f_k \otimes_r f_k\|_{H^{\otimes 2p-2r}} + \|f_k \otimes_p g_k\|_{H^{q-p}}^2 1_{p<q} \right]^{\frac{1}{2}}. \quad (43)$$

The above bound may be not optimal in some cases (see Remark 5 in Section 5.3).

4.3 The components of \mathbb{Y}_k belong to a finite sum of Wiener chaoses

Let us first notice that if a sequence of random variables $(Y_k, k \geq 1)$ converges in $L^2(\Omega)$ as $k \rightarrow \infty$ and

$$Y_k = \sum_{n=0}^{\infty} I_n(g_{n,k}), \quad g_{n,k} \in H^{\odot n},$$

then for every $n \geq 1$, the sequence $(g_{n,k}, k \geq 1)$ converges in $H^{\otimes n}$.

We make a further step to get the main result by extending the result in Proposition 2.

Proposition 3 Assume that the sequence $(X_k, k \geq 1)$ is as in Proposition 2 and let $\mathbb{Y}_k = (Y_{1,k}, \dots, Y_{d,k})$ be such that for every $j = 1, \dots, d$ and for every $k \geq 1$,

$$Y_{j,k} = \sum_{n=0}^{N_0} I_n(g_{n,k}^{(j)}),$$

with $N_0 \geq 1$, $g_{n,k}^{(j)} \in H^{\odot n}$ for $n \geq 0, k \geq 1$ and $j = 1, \dots, d$. Assume (20) and (21). Then

$$(X_k, \mathbb{Y}_k) \xrightarrow[k \rightarrow \infty]{(d)} (Z', \mathbb{U}) \quad (44)$$

where $Z \sim N(0, \sigma^2)$ and Z', \mathbb{U} are independent. Moreover, the estimate (23) holds true.

Proof: Recall the notation (40). Again, the Stein-Malliavin bound (23) follows directly from (16). By using this estimate (16),

$$d_W(\theta_k, \eta_k) \leq C \left(\mathbf{E} |\sigma^2 - \langle D(-L)^{-1} X_k, DX_k \rangle_H| + \sum_{j=1}^d \mathbf{E} |\langle D(-L)^{-1} X_k, DY_{j,k} \rangle_H| \right).$$

We also have, for every $j = 1, \dots, d$ and $k \geq 1$,

$$\begin{aligned} \mathbf{E} |\langle D(-L)^{-1}, X_k, DY_{j,k} \rangle_H| &= \mathbf{E} \left| \langle D(-L)^{-1} X_k, \sum_{n=0}^{N_0} DI_n(g_{n,k}^{(j)}) \rangle_H \right| \\ &\leq \sum_{n=0}^{N_0} \mathbf{E} \left| \langle D(-L)^{-1} X_k, DI_n(g_{n,k}^{(j)}) \rangle_H \right| \leq \sum_{n=0}^{N_0} \left(\mathbf{E} \left| \langle D(-L)^{-1} X_k, DI_n(g_{n,k}^{(j)}) \rangle_H \right|^2 \right)^{\frac{1}{2}} \end{aligned}$$

We notice that (21) and the isometry of multiple stochastic integrals (89) implies that

$$\mathbf{E} X_k I_n(g_{n,k}^{(j)}) \rightarrow_{k \rightarrow \infty} 0, \quad (45)$$

for every $j = 1, \dots, d$ and for every $n = 0, \dots, N_0$. We use Lemma 4 to express the quantity $\mathbf{E} \left| \langle D(-L)^{-1} X_k, DI_n(g_{n,k}^{(j)}) \rangle_H \right|^2$, and then by using (45) and Lemma 6, we deduce that

$$\mathbf{E} \left| \langle D(-L)^{-1} X_k, DI_n(g_{n,k}^{(j)}) \rangle_H \right|^2 \rightarrow_{k \rightarrow \infty} 0,$$

for every $j = 1, \dots, d$ and $n = 0, 1, \dots, N_0$. Thus

$$\mathbf{E} |\langle D(-L)^{-1}, X_k, DY_{j,k} \rangle_H| \rightarrow_{k \rightarrow \infty} 0,$$

for every $j = 1, \dots, d$ and this implies

$$d_W(\theta_k, \eta_k) \rightarrow_{k \rightarrow \infty} 0.$$

To deduce (44), it suffices to apply (36) in the proof of Proposition 2 and to use the hypothesis (20). \blacksquare

4.4 Proof of the main result (Theorem 3)

Let $\varepsilon > 0$. For $M \geq 1$, let us define,

$$Y_{j,k}^M = \sum_{n=0}^M I_n(g_{n,k}^{(j)}), \quad j = 1, \dots, d,$$

and consider the random vector in \mathbb{R}^d

$$\mathbb{Y}_k^M = (Y_{1,k}^M, \dots, Y_{d,k}^M), \quad k \geq 1. \quad (46)$$

Clearly, for every $k \geq 1$,

$$\mathbf{E} \|\mathbb{Y}_k^M - \mathbb{Y}_k\|_{\mathbb{R}^d}^2 \rightarrow_{M \rightarrow \infty} 0.$$

Recall that by $\|\cdot\|_{\mathbb{R}^d}$ and $\langle \cdot, \cdot \rangle_{\mathbb{R}^d}$ we denote the Euclidean norm and the Euclidean scalar product in \mathbb{R}^d . By (21) and the orthogonality of multiple stochastic integrals of different orders (89), for every $j = 1, \dots, d$ and for every $M \geq 1$,

$$\mathbf{E} X_k Y_{j,k}^M \rightarrow_{k \rightarrow \infty} 0. \quad (47)$$

Now, for any $\lambda_1 \in \mathbb{R}$ and $\boldsymbol{\lambda} \in \mathbb{R}^d$,

$$\begin{aligned} & \left| \mathbf{E} e^{i\lambda_1 X_k + i\langle \boldsymbol{\lambda}, \mathbb{Y}_k \rangle_{\mathbb{R}^d}} - \mathbf{E} e^{i\lambda_1 Z'} \mathbf{E} e^{i\langle \boldsymbol{\lambda}, \mathbb{U} \rangle_{\mathbb{R}^d}} \right| \\ & \leq \left| \mathbf{E} e^{i\lambda_1 X_k + i\langle \boldsymbol{\lambda}, \mathbb{Y}_k \rangle_{\mathbb{R}^d}} - \mathbf{E} e^{i\lambda_1 X_k + i\langle \boldsymbol{\lambda}, \mathbb{Y}_k^M \rangle_{\mathbb{R}^d}} \right| + \left| \mathbf{E} e^{i\lambda_1 X_k + i\langle \boldsymbol{\lambda}, \mathbb{Y}_k^M \rangle_{\mathbb{R}^d}} - \mathbf{E} e^{i\lambda_1 Z'} \mathbf{E} e^{i\langle \boldsymbol{\lambda}, \mathbb{Y}_k^M \rangle_{\mathbb{R}^d}} \right| \\ & \quad + \left| \mathbf{E} e^{i\lambda_1 Z'} \mathbf{E} e^{i\langle \boldsymbol{\lambda}, \mathbb{Y}_k^M \rangle_{\mathbb{R}^d}} - \mathbf{E} e^{i\lambda_1 Z'} \mathbf{E} e^{i\langle \boldsymbol{\lambda}, \mathbb{U} \rangle_{\mathbb{R}^d}} \right| \\ & = a_{M,k} + b_{M,k} + c_{M,k}. \end{aligned} \quad (48)$$

Let us estimate separately the three summands from above.

Estimation of $a_{M,k}$. By the mean value theorem,

$$\begin{aligned} a_{M,k} & \leq \mathbf{E} \left| e^{i\langle \boldsymbol{\lambda}, \mathbb{Y}_k \rangle_{\mathbb{R}^d}} - e^{i\langle \boldsymbol{\lambda}, \mathbb{Y}_k^M \rangle_{\mathbb{R}^d}} \right| \leq \mathbf{E} \|\mathbb{Y}_k^M - \mathbb{Y}_k\|_{\mathbb{R}^d} \\ & \leq \sqrt{\sum_{j=1}^d \mathbf{E} (Y_{j,k}^M - Y_{j,k})^2} = \sqrt{\sum_{j=1}^d \sum_{n=M+1}^{\infty} n! \|g_{n,k}^{(j)}\|_{H^{\otimes n}}^2} \\ & \leq \sqrt{\sum_{j=1}^d \sup_{k \geq 1} \sum_{n=M+1}^{\infty} n! \|g_{n,k}^{(j)}\|_{H^{\otimes n}}^2} \end{aligned} \quad (49)$$

and the last quantity goes to zero as $M \rightarrow \infty$ due to (19). So, for $M \geq M_1$ large, $a_{M,k} \leq \varepsilon$.

Estimation of $b_{M,k}$. Basically, the convergence of this term follows from Proposition 3, since the components of \mathbb{Y}_k^M belong to a finite sum of Wiener chaoses. For $M \geq M_1$, we have

$$\begin{aligned} & d_W \left(P_{(X_k, \mathbb{Y}_k^M)}, P_{Z'} \otimes P_{\mathbb{Y}_k^M} \right) \\ & \leq C \left[\mathbf{E} |\sigma^2 - \langle D(-L)^{-1} X_k, DX_k \rangle_H| + \sum_{j=1}^d \mathbf{E} \left| \sigma^2 - \langle D(-L)^{-1} X_k, DY_{j,k}^M \rangle_H \right| \right] \end{aligned}$$

Using (47), as in Proposition 3, the both summands in the right-hand above converge to zero as $k \rightarrow \infty$. So, for k large, $b_{M,k} \leq \varepsilon$.

Estimation of $c_{M,k}$. First notice that

$$c_{M,k} \leq \left| \mathbf{E}e^{i\langle \lambda, \mathbb{Y}_k^M \rangle_{\mathbb{R}^d}} - \mathbf{E}e^{i\langle \lambda, \mathbb{U} \rangle_{\mathbb{R}^d}} \right|$$

Let $\varepsilon > 0$. We show that for M, k large enough,

$$\left| \mathbf{E}e^{i\langle \lambda, \mathbb{Y}_k^M \rangle_{\mathbb{R}^d}} - \mathbf{E}e^{i\langle \lambda, \mathbb{U} \rangle_{\mathbb{R}^d}} \right| \leq \varepsilon. \quad (50)$$

We have

$$\begin{aligned} \left| \mathbf{E}e^{i\langle \lambda, \mathbb{Y}_k^M \rangle_{\mathbb{R}^d}} - \mathbf{E}e^{i\langle \lambda, \mathbb{U} \rangle_{\mathbb{R}^d}} \right| &\leq \left| \mathbf{E}e^{i\langle \lambda, \mathbb{Y}_k^M \rangle_{\mathbb{R}^d}} - \mathbf{E}e^{i\langle \lambda, \mathbb{Y}_k \rangle_{\mathbb{R}^d}} \right| + \left| \mathbf{E}e^{i\langle \lambda, \mathbb{Y}_k \rangle_{\mathbb{R}^d}} - \mathbf{E}e^{i\langle \lambda, \mathbb{U} \rangle_{\mathbb{R}^d}} \right| \\ &\leq C\mathbf{E}\|\mathbb{Y}_k^M - \mathbb{Y}_k\|_{\mathbb{R}^d} + \left| \mathbf{E}e^{i\langle \lambda, \mathbb{Y}_k \rangle_{\mathbb{R}^d}} - \mathbf{E}e^{i\langle \lambda, \mathbb{U} \rangle_{\mathbb{R}^d}} \right|. \end{aligned} \quad (51)$$

We use the estimate (49)

$$\mathbf{E}\|\mathbb{Y}_k^M - \mathbb{Y}_k\|_{\mathbb{R}^d} \leq \sqrt{\sum_{j=1}^d \sup_{k \geq 1} \sum_{n=M+1}^{\infty} n! \|g_{n,k}^{(j)}\|_{H^{\otimes n}}^2}$$

and the last quantity goes to zero as $M \rightarrow \infty$ due to (19). By using this inequality and (20) in (51), we get (50). Therefore, for k, M large, $c_{M,k} \leq \varepsilon$.

Consequently, the left-hand side of (48) goes to zero as $k \rightarrow \infty$. \blacksquare

It is possible to assume only the convergence in law of the sequence $(\mathbb{Y}_k, k \geq 1)$ instead of (20) if the components of \mathbb{Y}_k belongs to the sum of the first q Wiener chaos with $q \leq p$.

Proposition 4 *Let us consider the integer numbers $p \geq 2$, $d \geq 1$. Let $(X_k, k \geq 1)$ be a sequence of random variables such that for every $k \geq 1$, $X_k = I_p(f_k)$ with $f_k \in H^{\odot p}$ that satisfies (18).*

Let $(\mathbb{Y}_k, k \geq 1) = ((Y_{1,k}, \dots, Y_{d,k}), k \geq 1)$ be a sequence of random vectors such that, for every $j = 1, \dots, d$, the random variable $Y_{j,k}$ belongs to $\mathbb{D}^{1,2}$, and it admits the chaos expansion

$$Y_{j,k} = \sum_{n=0}^q I_n(g_{n,k}^{(j)}) \text{ with } g_{n,k}^{(j)} \in H^{\odot n}$$

with $q \leq p$. Suppose that there exists a random vector \mathbb{U} in \mathbb{R}^d such that

$$\mathbb{Y}_k \xrightarrow[k \rightarrow \infty]{(d)} \mathbb{U}. \quad (52)$$

Then, if (21) holds true, we have

$$(X_k, \mathbb{Y}_k) \xrightarrow[k \rightarrow \infty]{(d)} (Z', \mathbb{U}),$$

where $Z' \sim N(0, \sigma^2)$ and Z' is independent by the random vector \mathbb{U} . Moreover, (23) holds true.

Proof: The proof can be done by following the lines of the proof of Proposition 3, by using point 2. in Lemma 6. We use the notation (40). Via the bound (42) and point 2. in Lemma 6, we obtain that

$$d_W(\theta_k, \eta_k) \xrightarrow[k \rightarrow \infty]{} 0. \quad (53)$$

Let $f : \mathbb{R}^{d+1} \rightarrow \mathbb{R}$ be a continuous and bounded function. By using the triangle's inequality, we have

$$\begin{aligned} & \left| \int_{\mathbb{R}^{d+1}} f(x) d\theta_k(x) - \int_{\mathbb{R}^{d+1}} f(x) d\eta(x) \right| \\ & \leq \left| \int_{\mathbb{R}^{d+1}} f(x) d\theta_k(x) - \int_{\mathbb{R}^{d+1}} f(x) d\eta_k(x) \right| + \left| \int_{\mathbb{R}^{d+1}} f(x) d\eta_k(x) - \int_{\mathbb{R}^{d+1}} f(x) d\eta(x) \right|. \end{aligned}$$

The first summand in the right-hand side converges to zero as $k \rightarrow \infty$ by (53). The second summand in the right-hand side also goes to zero as k tends to infinity due to the assumption (52). Then, the conclusion is obtained.

4.5 A counter-example

Assume $(X_k = I_p(f_k), k \geq 1)$ with $f_k \in H^{\odot p}$ be such that $X_k \xrightarrow[k \rightarrow \infty]{} Z \sim N(0, \sigma^2)$. Let $(Y_k, k \geq 1)$ be a sequence in the q th Wiener chaos, $Y_k = I_q(g_k), g_k \in H^{\odot q}$. Assume that $q > p$

$$Y_k \xrightarrow[k \rightarrow \infty]{} U.$$

Can we deduce the joint convergence of (X_k, Y_k) to (Z', U) where $Z' \sim N(0, \sigma^2)$ and Z', U are independent? By Theorem 3 and Proposition 4, the conclusion is true if the convergence of $(Y_k, k \geq 1)$ holds in $L^2(\Omega)$ or if $p \geq q$ (and (21) holds). For $q > p$, the answer is negative as illustrated by the following example. Let

$$g_k = f_k \tilde{\otimes} f_k, \quad k \geq 1,$$

and $Y_k = I_{2p}(g_k), k \geq 1$. Then, by the product formula (91),

$$X_k^2 - \mathbf{E}X_k^2 = Y_k + R_k,$$

where $R_k \xrightarrow[k \rightarrow \infty]{} 0$ in $L^2(\Omega)$ (this comes from point 3. in Theorem 2). Consequently,

$$(X_k, Y_k) \xrightarrow[k \rightarrow \infty]{(d)} (Z, Z^2 - \sigma^2),$$

and obviously the components of the limit vector are not independent.

5 Applications

We illustrate our results by four examples. In the first example, we deduce from Proposition 2 the joint convergence of the Hermite variations of $d + 1$ correlated fractional Brownian motions. The second example constitutes an application of Theorem 3, by considering a random variable with infinite chaos expansion. In the third example, we treat a two-dimensional sequence in Wiener chaos, one component being asymptotically Gaussian and the second component satisfying a non-central limit theorem. Such estimates are new in the literature and they cannot be obtained via the standard Stein method. Finally, in the last example, to evaluate the dependence structure between the solution a stochastic differential equation and the random noise.

5.1 Hermite variations of correlated fractional Brownian motions

Let $(W_t, t \geq 0)$ be a Wiener process and for $H \in (0, 1), t \geq 0$ consider the kernel

$$f_{t,H}(s) = d(H) \left((t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}} \right), \quad s \in \mathbb{R}.$$

where $d(H)$ is a normalizing constant that ensures that $\int_{\mathbb{R}} f_{t,H}(s)^2 ds = t^{2H}$. Let $H_0, H_1, \dots, H_d \in (0, 1)$ and define, for $i = 0, 1, \dots, d$,

$$B_t^{H_i} = \int_{\mathbb{R}} f_{t,H_i}(s) dW_s, \quad t \geq 0. \quad (54)$$

Then, for $i = 0, 1, \dots, d$, $(B^{H_i}, t \geq 0)$ are $d + 1$ (correlated) fractional Brownian motions with Hurst parameters H_i . We write, for any integer number $k \geq 0$,

$$B_{k+1}^{H_i} - B_k^{H_i} = I_1(L_{k,H_i}), \quad i = 0, 1, \dots, d,$$

where I_q stands for the multiple stochastic integral of order $q \geq 1$ with respect to the Wiener process W and for $k \geq 0$,

$$L_{k,H_i} = f_{k+1,H_i} - f_{k,H_i}. \quad (55)$$

For $N \geq 1$ integer, we set

$$X_N = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} I_p \left(L_{k,H_0}^{\otimes p} \right) = I_p(f_N) \quad (56)$$

and for $j = 1, \dots, d$,

$$Y_{N,j} = N^{q_j(1-H_j)-1} \sum_{k=0}^{N-1} I_{q_j} \left(L_{k,H_j}^{\otimes q_j} \right) = I_{q_j}(g_{N,j}). \quad (57)$$

We used the notation

$$f_N = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} L_{k,H_0}^{\otimes p} \text{ and } g_{N,j} = N^{q_j(1-H_j)-1} \sum_{k=0}^{N-1} L_{k,H_j}^{\otimes q_j} \quad (58)$$

From the classical Breuer -Major theorem (see [1]) we know the limit behavior in distribution of the sequence (56) while the Non-Central limit theorem (see e.g. [20]) gives the limit behavior of (57). More precisely, we have the following.

Theorem 4 *Consider the sequences $(X_N, N \geq 1)$ and $(Y_{N,j}, N \geq 1)$ given by (56), (57), respectively. Then*

1. *If $H_0 \in (0, 1 - \frac{1}{p})$,*

$$X_N \xrightarrow{N \rightarrow \infty} N(0, \sigma_{p,H_0}^2).$$

2. *If $H_j \in (1 - \frac{1}{2q_j}, 1)$ for $j = 1, \dots, d$,*

$$Y_{N,j} \xrightarrow{N \rightarrow \infty} c_{q_j, H_j} R_1^{\gamma_j},$$

where $R_1^{\gamma_j}$ is a Hermite random variable with Hurst parameter $\gamma_j = 1 + q(H - 1)$. The explicit expression of the constants $\sigma_{p,H_0}, c_{q_j, H_j} > 0$ can be found in e.g. [1], [20].

Recall that the Hermite random variable has a non-Gaussian law (it actually lives in q th Wiener chaos) and it represents the value at time $t = 1$ of a Hermite process. For more details on Hermite processes, see e.g. [22].

Let

$$\mathbb{Y}_N = (Y_{N,1}, \dots, Y_{N,d}), \quad N \geq 1.$$

The purpose is to show the joint convergence of the two-dimensional random sequence $((X_N, \mathbb{Y}_N), N \geq 1)$. Let us recall some facts. For every integers $k, l \geq 1$ and for $i, j = 0, 1, \dots, d$ (see [6]),

$$\mathbf{E}(B_{k+1}^{H_i} - B_k^{H_i})(B_{l+1}^{H_j} - B_l^{H_j}) = \langle L_{k,H_i}, L_{l,H_j} \rangle_{L^2(\mathbb{R})} = D(H_i, H_j) \rho_{\frac{H_i+H_j}{2}}(k-l),$$

where $D(H_i, H_j)$ is a constant depending on H_i, H_j and for $v \in \mathbb{Z}$,

$$\rho_H(v) = \frac{1}{2} (|v+1|^{2H} + |v-1| - 2|v|^{2H}). \quad (59)$$

For v sufficiently large, one has

$$|\rho_H(v)| \leq C_H v^{2H-2} \quad (60)$$

We have the following result.

Proposition 5 Let $p \geq 1, q_1, \dots, q_d \geq 2$ be integer numbers such that $p \geq \max(q_1, \dots, q_d)$ and assume that for $j = 1, \dots, d$,

$$0 < H_1 < 1 - \frac{1}{2p} \text{ and } 1 - \frac{1}{2q_j} < H_j < 1. \quad (61)$$

Consider the sequences $(X_N, N \geq 1)$ and $(\mathbb{Y}_N, N \geq 1)$ given by (56) and (57), respectively. Then

$$(X_N, \mathbb{Y}_N) \rightarrow_{N \rightarrow \infty}^{(d)} (Z, c_{q_j, H_j} R_1^{\gamma_j}, j = 1, \dots, d),$$

where $Z \sim N(0, \sigma_{p, H_0}^2)$ and $R_1^{\gamma_j}$ stands for a Hermite random variable (with Hurst index γ_j) independent of Z . The constants σ_{p, H_0} and c_{q_j, H_j} are those from Theorem 4.

Proof: First, we notice that, as $N \rightarrow \infty$,

$$\mathbb{Y}_N \rightarrow^{(d)} (c_{q_1, H_1} R_1^{\gamma_1}, \dots, c_{q_d, H_d} R_1^{\gamma_d}). \quad (62)$$

The above claim can be argued in the following way: for every $c > 0$, we have the scaling property

$$\left(B_{ct}^{H_1}, \dots, B_{ct}^{H_d}, t \geq 0 \right) \equiv^{(d)} \left(c^{H_1} B_t^{H_1}, \dots, c^{H_d} B_t^{H_d}, t \geq 0 \right),$$

where " $\equiv^{(d)}$ " means the equivalence of finite dimensional distributions. This is a consequence of (54) and of the scaling property of the Wiener process W . Then, for all $N \geq 1$, we have the equality in law

$$(Y_{N,1}, \dots, Y_{N,d}) \equiv^{(d)} (Y'_{N,1}, \dots, Y'_{N,d})$$

where, for every $j = 1, \dots, d$,

$$Y'_{N,j} = q_j! N^{q_j-1} \sum_{k=0}^{N-1} H_{q_j} \left(B_{\frac{k+1}{N}}^{H_j} - B_{\frac{k}{N}}^{H_j} \right)$$

with H_q the Hermite polynomial of degree q . On the other hand, for every $j = 1, \dots, d$, the sequence $(Y'_{N,j}, N \geq 1)$ converges in $L^2(\Omega)$, as $N \rightarrow \infty$, to $c_{q_j, H_j} R_1^{\gamma_j}$ (see e.g. [8]). This implies (62).

In order to apply Proposition 2, we just need to check (21). Obviously, this holds for $p \neq q_j$, since in this situation $\mathbf{E}X_N Y_{N,j} = 0$ for all $N \geq 1$ and for all $j = 1, \dots, d$. We calculate $\mathbf{E}X_N Y_{N,j}$ for $p = q_j$. We have, by the isometry formula (89),

$$\begin{aligned} \mathbf{E}X_N Y_{N,j} &= p! N^{p(1-H_j)-\frac{3}{2}} \sum_{k,l=0}^{N-1} \langle L_{k,H_0}, L_{l,H_j} \rangle_{L^2(\mathbb{R})}^p \\ &= p! D(H_0, H_j)^p N^{p(1-H_j)-\frac{3}{2}} \sum_{k,l=0}^{N-1} \rho_{\frac{H_0+H_j}{2}}(k-l)^p, \end{aligned}$$

and for N large enough, by (60),

$$\begin{aligned} |\mathbf{E}X_N Y_{N,j}| &\leq c(H_0, H_j, p) N^{p(1-H_j)-\frac{3}{2}} \left(1 + \sum_{k=1}^N (N-k) k^{(H_0+H_j-2)p} \right) \\ &\leq c(H_0, H_j, p) N^{p(1-H_j)-\frac{3}{2}} \left(1 + N \sum_{k=1}^N k^{(H_0+H_j-2)p} \right). \end{aligned}$$

Assume $(H_0 + H_j - 2)p < -1$. In this case, the series $\sum_{k \geq 1} k^{(H_0+H_j-2)p}$ converges and we get

$$|\mathbf{E}X_N Y_{N,j}| \leq c(H_0, H_j, p) N^{p(1-H_j)-\frac{1}{2}} \rightarrow_{N \rightarrow \infty} 0$$

since $H_j > 1 - \frac{1}{2p}$.

Assume $(H_0 + H_j - 2)p > -1$. Then the sequence $\sum_{k=1}^N k^{(H_0+H_j-2)p}$ behaves as $N^{(H_0+H_j-2)p+1}$ for N large and thus

$$\begin{aligned} |\mathbf{E}X_N Y_{N,j}| &\leq c(H_0, H_j, p) N^{p(1-H_j)-\frac{3}{2}} \left(1 + N^{(H_0+H_j-2)p+1} \right) \\ &= c(H_0, H_j, p) \left(N^{p(1-H_j)-\frac{3}{2}} + N^{-p(1-H_0)+\frac{1}{2}} \right) \rightarrow_{N \rightarrow \infty} 0, \end{aligned}$$

since $H_0 < 1 - \frac{1}{2p}$ and $H_j > 1 - \frac{1}{2p}$.

If $(H_0 + H_j - 2)p = -1$, then $\sum_{k=1}^N k^{(H_0+H_j-2)p}$ behaves as $\log(N)$ and

$$|\mathbf{E}X_N Y_{N,j}| \leq c(H_0, H_j, p) N^{p(1-H_j)-\frac{1}{2}} \log(N) \rightarrow_{N \rightarrow \infty} 0.$$

We obtained

$$|\mathbf{E}X_N Y_{N,j}| \leq c(H_0, H_j, p) \begin{cases} N^{p(1-H_j)-\frac{1}{2}} & \text{if } (H_0 + H_j - 2)p < -1 \\ N^{p(1-H_j)-\frac{1}{2}} \log(N), & \text{if } (H_0 + H_j - 2)p = -1 \\ N^{p(1-H_j)-\frac{3}{2}} + N^{-p(1-H_0)+\frac{1}{2}} & \text{if } (H_0 + H_j - 2)p > -1. \end{cases}$$

In particular $\mathbf{E}X_N Y_{N,j} \rightarrow_{N \rightarrow \infty} 0$ and (21) holds. The conclusion follows by Proposition 2. \blacksquare

Remark 4 1. A quantitative bound in Proposition 5 can be obtained via (23) or (43).

2. Let the above notation prevail. It is also possible to apply Proposition 5 to the estimation of the Hurst parameter (H_0, H_1, \dots, H_d) from the discrete observations $\left(B_{\frac{i}{N}}^{H_j}, i = 0, 1, \dots, N, j = 0, 1, \dots, d \right)$. Denote, for $j = 0, 1, \dots, d$,

$$S_{N,j} = \frac{1}{N} \sum_{i=0}^{N-1} \left(B_{\frac{i+1}{N}}^{H_j} - B_{\frac{i}{N}}^{H_j} \right)^2.$$

Then

$$\widehat{H}_{N,j} = -\frac{\log(S_{N,j})}{2\log(N)}, \quad j = 0, 1, \dots, d$$

are consistent estimators for the Hurst index H_j and (see e.g. Section 5.5 in [22])

$$2\sqrt{N}(\widehat{H}_{N,0} - H_0) = X_N + R_{N,0}$$

and for $j = 1, \dots, d$,

$$2N^{2-2H_j}(\widehat{H}_{N,j} - H_j) = Y_{N,j} + R_{N,j}$$

where $R_{N,j}, j = 0, 1, \dots, d$ converge almost surely to zero as $N \rightarrow \infty$. From Proposition 5, we get the joint convergence in law, as $N \rightarrow \infty$, of

$$\left(2\sqrt{N}(\widehat{H}_{N,0} - H_0), 2N^{2-2H_j}(\widehat{H}_{N,j} - H_j)\right)$$

to

$$\left(Z, c_{2,H_j} R_1^{2H_j-1}, j = 1, \dots, d\right),$$

$Z \sim N(0, \sigma_{p,H_0}^2)$ and Z is independent of $R_1^{2H_j-1}, j = 1, \dots, d$.

5.2 Infinite chaos expansion

Let $(W(h), h \in H)$ be an isonormal process and let $(h_i, i \geq 1)$ be a family of elements of H such that for every $i, j \geq 1$

$$\langle h_i, h_j \rangle_H = \rho_H(i - j),$$

where ρ_H is the auto-correlation function of the fractional noise given by (59). Consider the sequence $(V_N, N \geq 1)$ given by

$$V_N = \frac{1}{\sqrt{N}} \sum_{k=1}^N I_p(h_k^{\otimes p}). \quad (63)$$

and let

$$Y = e^{W(h_1)} = \sqrt{e} \sum_{n \geq 0} \frac{1}{n!} I_n(h_1^{\otimes n}). \quad (64)$$

Obviously $(V_N, N \geq 1)$ has the same finite-dimensional distribution as (56) (when $H = H_0$). Assume

$$0 < H < 1 - \frac{1}{2p}. \quad (65)$$

By Theorem 4, if (65) holds true, then $(V_N, N \geq 1)$ converges in law, as $N \rightarrow \infty$, to $Z \sim N(0, \sigma_{p,H}^2)$. Moreover, we have the following estimate for the Wasserstein distance (see [7]): if N is large,

$$d_W(V_N, Z) \leq C \begin{cases} n^{-\frac{1}{2}}, & \text{if } H \in (0, \frac{1}{2}] \\ n^{H-1}, & \text{if } H \in [\frac{1}{2}, \frac{2p-3}{2p-2}) \\ n^{pH-p+\frac{1}{2}}, & \text{if } H \in [\frac{2p-3}{2p-2}, \frac{2p-1}{2p}). \end{cases} \quad (66)$$

We check the joint convergence in law of the couple (X_N, Y) when $N \rightarrow \infty$ and we evaluate the Wasserstein distance associated to it.

Proposition 6 *Let V_N, Y be given by (63), (64), respectively. Then*

$$(V_N, Y) \xrightarrow{(d)} (Z, Y)$$

where $Z \sim N(0, \sigma_{p,H}^2)$ is independent of Y . Moreover, for N large

$$d_W(P_{(V_N, Y)}, P_Z \otimes P_Y) \leq C \begin{cases} n^{-\frac{1}{2}}, & \text{if } H \in (0, \frac{1}{2}] \\ n^{H-1}, & \text{if } H \in [\frac{1}{2}, \frac{3}{4}) \\ n^{H-1} + n^{pH-p+\frac{1}{2}}, & \text{if } H \in [\frac{3}{4}, \frac{2p-1}{2p}). \end{cases} \quad (67)$$

Proof: In order to get the joint convergence of $((V_N, Y), N \geq 1)$, we need to check (24). We have

$$\begin{aligned} \mathbf{E}(V_N Y) &= \sqrt{e} \frac{1}{\sqrt{N}} \sum_{k=1}^N \mathbf{E} I_p(h_k^{\otimes p}) Y = \sqrt{e} \frac{1}{\sqrt{N}} \sum_{k=1}^N \mathbf{E} I_p(h_k^{\otimes p}) \frac{1}{p!} I_p(h_1^{\otimes p}) \\ &= \sqrt{e} \frac{1}{\sqrt{N}} \sum_{k=1}^N \langle h_k, h_1 \rangle_p = \sqrt{e} \frac{1}{\sqrt{N}} \sum_{k=1}^N \rho_H (k-1)^p. \end{aligned}$$

By isolating the term with $k = 1$, we have

$$\mathbf{E}(V_N Y) = \sqrt{e} \frac{1}{\sqrt{N}} \left(1 + \sum_{k \geq 2} (k-1)^{(2H-2)p} \right) \leq C \frac{1}{\sqrt{N}},$$

since the series $\sum_{k \geq 1} k^{(2H-2)p}$ is convergent due to (65). Then, by Theorem 3,

$$(V_N, Y) \xrightarrow[N \rightarrow \infty]{(d)} (Z, Y), \quad (68)$$

where $Z \sim N(0, \sigma_{p,H}^2)$ and Z, Y are independent random variables.

Let us evaluate the rate of convergence under the Wasserstein distance for (68). We compute the quantity $\mathbf{E}\langle D(-L)^{-1}V_N, DY \rangle_H^2$. We have

$$D(-L)^{-1}V_N = \frac{1}{\sqrt{N}} \sum_{k=1}^N I_{p-1}(h_k^{\otimes p-1})h_k, \quad DY = Yh_1$$

and

$$\langle D(-L)^{-1}V_N, DY \rangle_H = \frac{1}{\sqrt{N}} \sum_{k=1}^N I_{p-1}(h_k^{\otimes p-1})Y \langle h_k, h_1 \rangle_H.$$

Hence,

$$\begin{aligned} \mathbf{E}\langle D(-L)^{-1}V_N, DY \rangle_H^2 &= \frac{1}{N} \sum_{k,l=1}^N I_{p-1}(h_k^{\otimes p-1})I_{p-1}(h_l^{\otimes p-1})Y^2 \langle h_k, h_1 \rangle_H \langle h_l, h_1 \rangle_H \\ &= \frac{1}{N} \sum_{k,l=1}^N \sum_{r=0}^{p-1} r!(C_{p-1}^r)^2 \mathbf{E}I_{2p-2r-2} \left(h_k^{\otimes p-1} \otimes_r h_l^{\otimes p-1} \right) Y^2 \langle h_k, h_1 \rangle_H \langle h_l, h_1 \rangle_H, \end{aligned}$$

where we applied the product formula (91). Since

$$Y^2 = e^{2W(h_1)} = e \sum_{n \geq 0} \frac{2^n}{n!} I_n(h_1^{\otimes n}),$$

we have, for $r = 0, \dots, p-1$,

$$\begin{aligned} &\mathbf{E}I_{2p-2r-2} \left(h_k^{\otimes p-1} \otimes_r h_l^{\otimes p-1} \right) Y^2 \\ &= e \frac{2^{2p-2r-2}}{(2p-2r-2)!} \mathbf{E}I_{2p-2r-2} \left(h_k^{\otimes p-1} \otimes_r h_l^{\otimes p-1} \right) I_{2p-2r-2}(h_1^{\otimes 2p-2r-2}) \\ &= e 2^{2p-2r-2} \langle (h_k^{\otimes p-1} \tilde{\otimes}_r h_l^{\otimes p-1}, h_1^{\otimes 2p-2r-2})_{H^{\otimes 2p-2r-2}} \rangle \\ &= e 2^{2p-2r-2} \langle h_k, h_l \rangle_H^r \langle h_k, h_1 \rangle_H^{p-r-1} \langle h_l, h_1 \rangle_H^{p-r-1}. \end{aligned}$$

Consequently,

$$\mathbf{E}\langle D(-L)^{-1}V_N, DY \rangle_H^2 = e \sum_{r=0}^{p-1} r!(C_{p-1}^r)^2 2^{2p-2r-2} T(r, p, N)$$

with

$$\begin{aligned} T(r, p, N) &= \frac{1}{N} \sum_{k,l=1}^N \langle h_k, h_1 \rangle_H^{p-r} \langle h_k, h_1 \rangle_H^{p-r} \langle h_l, h_1 \rangle_H^{p-r} \\ &= \frac{1}{N} \sum_{k,l=1}^N \rho_H(k-l)^r \rho_H(k-1)^{p-r} \rho_H(l-1)^{p-r}. \end{aligned} \quad (69)$$

We now evaluate $T(r, p, N)$ for $r = 0, 1, \dots, p-1$. We write

$$\begin{aligned} T(r, p, N) &= \frac{1}{N} \sum_{k=1}^N \rho_H(k-1)^{2(p-r)} + \frac{1}{N} \sum_{k,l=1; k \neq l}^N \rho_H(k-l)^r \rho_H(k-1)^{p-r} \rho_H(l-1)^{p-r} \\ &:= T_1(r, p, N) + T_2(r, p, N). \end{aligned}$$

Let us first treat the term $T_1(r, p, N)$ with $r = 0, 1, \dots, p-1$. One has

$$\begin{aligned} T_1(r, p, N) &= \frac{1}{N} \left(1 + \sum_{k \geq 2} \rho_H(k-1)^{2(p-r)} \right) \leq C \frac{1}{N} \left(1 + \sum_{k \geq 2} (k-1)^{(2H-2)(2p-2r)} \right) \\ &\leq C \frac{1}{N} \left(1 + \sum_{k \geq 1} k^{2H-2} \right) \leq C \frac{1}{N} (1 + N^{2H-1}) \\ &\leq C (N^{-1} + N^{2H-2}). \end{aligned}$$

For $T_2(r, p, N)$, we can write

$$\begin{aligned} T_2(r, p, N) &= 2 \frac{1}{N} \sum_{k,l=1; k > l}^N \rho_H(k-l)^r \rho_H(k-1)^{p-r} \rho_H(l-1)^{p-r} \\ &\leq C \frac{1}{N} \left(\sum_{k=2}^N \rho_H(k-l)^p + \sum_{k > l \geq 2} (k-l)^{(2H-2)r} (k-1)^{(2H-2)(p-r)} (l-1)^{(2H-2)(p-r)} \right) \end{aligned}$$

By (65), $\sum_{k=2}^N \rho_H(k-l)^p < \infty$ and so

$$T_2(0, p, N) \leq C \frac{1}{N} \left(1 + \left(\sum_{k \geq 2} (k-1)^{(2H-2)p} \right)^2 \right) \leq C \frac{1}{N}$$

and for $r = 1, \dots, p-1$, since $(k-1)^{(2H-2)(p-r)} \leq (k-l)^{(2H-2)(p-r)}$,

$$\begin{aligned} T_2(r, p, N) &\leq C \frac{1}{N} \left(1 + \sum_{k > l \geq 2} (k-l)^{(2H-2)p} (l-1)^{(2H-2)(p-r)} \right) \\ &\leq C \frac{1}{N} \left(1 + \sum_{l=2}^N (l-1)^{2H-2} \sum_{k \geq 1} k^{(2H-2)p} \right) \\ &\leq C \frac{1}{N} (1 + N^{2H-1}) \leq C (N^{-1} + N^{2H-2}). \end{aligned}$$

From the above computations, we deduce that for N sufficiently large,

$$\mathbf{E}\langle D(-L)^{-1}V_N, DY \rangle_H^2 \leq C(N^{-1} + N^{2H-2}). \quad (70)$$

By combining (66) and (70), we get (67).

5.3 Quantitative bounds in a central-noncentral limit theorem

Our approach allows to give qualitative bounds for the multidimensional sequences of multiple stochastic integral when only one of these sequences converges to a normal distribution. Here we illustrate the method by treating a two-dimensional sequence in Wiener chaos, one component

being asymptotically Gaussian and the second component satisfying a non-central limit theorem. Such estimates are new in the literature and they cannot be obtained via the standard Stein method. Let $(B_t^H, t \geq 0)$ be a fractional Brownian motion with Hurst index $H \in (0, 1)$. For $N \geq 1$, define

$$V_N = q! \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} H_q (B_{k+1}^H - B_k^H), \quad (71)$$

where H_q is the Hermite polynomial of degree q . Then, the Breuer-Major theorem (see [1] or Theorem 4) states that, if $H \in \left(0, 1 - \frac{1}{2q}\right)$ the sequence $(V_N, N \geq 1)$ converges to a Gaussian random variable $Z \sim N(0, \sigma_{q,H}^2)$, where the variance $\sigma_{q,H}^2$ is explicitly known.

On the other hand, the sequence $(U_N, N \geq 1)$ given by

$$U_N = 2N^{1-2H} \sum_{k=0}^{N-1} H_2 (B_{k+1}^H - B_k^H), \quad N \geq 1, \quad (72)$$

converges in distribution, for $H \in \left(\frac{3}{4}, 1\right)$, to $c_{2,H}R^{(2H-1)}$ where $R^{(2H-1)}$ is a Rosenblatt random variable with Hurst parameter $2H-1$ and again the constant $c_{2,H} > 0$ is known.

Moreover, the random sequence (V_N, U_N) converges in law, as $N \rightarrow \infty$, to $(Z, c_{2,H}R^{(2H-1)})$, with Z independent of $R^{(2H-1)}$. This can be obtained from the main findings in [9] or [10] but it also follows from our Theorem 3. The purpose is to find the rate of convergence, under the Wasserstein distance, for this two-dimensional limit theorem.

We have the following result.

Proposition 7 *Let V_N, U_N be given by (71), (72), respectively. Assume*

$$H \in \left(\frac{3}{4}, 1 - \frac{1}{2q}\right) \Rightarrow q \geq 3. \quad (73)$$

Then

$$(V_N, U_N) \xrightarrow{N \rightarrow \infty (d)} (Z, c_{2,H} R^{(2H-1)})$$

where $Z \sim N(0, \sigma_{q,H}^2)$ and Z is independent from the Rosenblatt random variable $R^{(2H-1)}$.

Moreover

$$d_W \left((V_N, U_N), (Z, c_{2,H} R^{(2H-1)}) \right) \leq c_{q,H} \begin{cases} N^{H-1} + N^{\frac{3}{2}-2H} & \text{for } H \in \left(\frac{3}{4}, 1 - \frac{1}{2(q-1)} \right) \\ N^{(H-1)q+\frac{1}{2}} + N^{\frac{3}{2}-2H} & \text{for } \left(1 - \frac{1}{2(q-1)}, 1 - \frac{1}{2q} \right). \end{cases} \quad (74)$$

Proof: The joint convergence of $((V_N, U_N), N \geq 1)$ is obtained via Proposition 4. By Theorem 3, we have

$$\begin{aligned} & d_W \left(P_{(V_N, U_N)}, P_Z \otimes P_{c_{2,H} R^{(2H-1)}} \right) \\ & \leq C \left[\left(\mathbf{E} \left(\sigma^2 - \langle DV_N, D(-L)^{-1} V_N \rangle \right)^2 \right)^{\frac{1}{2}} + d_W(P_{U_N}, P_{c_{2,H} R^{(2H-1)}}) + \sqrt{\mathbf{E} \left(\langle DV_N, DU_N \rangle \right)^2} \right]. \end{aligned}$$

We know the rate of convergence to their limits for each of the sequences $(V_N, N \geq 1)$ and $(U_N, N \geq 1)$. If one assumes (73), then (see Theorem 4.1 in [7])

$$\left(\mathbf{E} \left(\sigma^2 - \langle DV_N, D(-L)^{-1} V_N \rangle \right)^2 \right)^{\frac{1}{2}} \leq C_{H,q} \begin{cases} N^{H-1} & \text{if } H \in \left(\frac{3}{4}, \frac{2q-3}{2q-2} \right] \\ N^{qH-q+\frac{1}{2}} & \text{if } H \in \left[\frac{2q-3}{2q-2}, \frac{2q-1}{2q} \right). \end{cases} \quad (75)$$

Moreover, for any H satisfying (73) (see [2] or [8], relation (7.4.13))

$$d_W(U_N, c_{2,H} R^{(2H-1)}) \leq C_H N^{\frac{3}{2}-2H}. \quad (76)$$

In particular, if $q = 3$, it follows from (75) and (76) that

$$\begin{aligned} & d_W(V_N, Z) + d_W(U_N, c_{2,H} R^{(2H-1)}) \leq C_H \left(N^{\frac{3}{2}-2H} + N^{3H-\frac{5}{2}} \right) \\ & \leq C_H \begin{cases} N^{\frac{3}{2}-2H} & \text{if } H \in \left(\frac{3}{4}, \frac{4}{5} \right) \\ N^{3H-\frac{5}{2}} & \text{if } H \in \left[\frac{4}{5}, \frac{5}{6} \right) \end{cases} \end{aligned} \quad (77)$$

Let us estimate the quantity $\sqrt{\mathbf{E} \left(\langle DV_N, DU_N \rangle \right)^2}$. Denote by \mathcal{H} the canonical Hilbert space associated to the fractional Brownian motion, defined as the closure of the set of step functions on the positive real line with respect to the scalar product

$$\langle 1_{[0,t]}, 1_{[0,s]} \rangle_{\mathcal{H}} = \mathbf{E} B_t^H B_s^H = \frac{1}{2} (t^{2H} + s^{2H} - |t-s|^{2H}).$$

We can write, if I_q is the multiple stochastic integral with respect to the isonormal process generated by B^H ,

$$V_N = I_q(f_N) \text{ with } f_N = \frac{1}{\sqrt{N}} \sum_{k=1}^N h_k^{\otimes q}$$

and

$$U_N = I_2(g_N) \text{ with } g_N = N^{1-2H} \sum_{l=1}^N h_l^{\otimes 2},$$

where $h_k = 1_{[k-1, k]}$ for $k = 1, \dots, N$. In particular $\|h_k\|_{\mathcal{H}} = 1$ and

$$\langle h_k, h_l \rangle_{\mathcal{H}} = \rho_H(k-l) \quad (78)$$

with ρ_H from (59). Thus

$$\begin{aligned} \langle DV_N, DU_N \rangle &= 2qN^{\frac{1}{2}-2H} \sum_{k,l=1}^N I_{q-1}(h_k^{\otimes(q-1)}) I_1(h_l) \langle h_k, h_l \rangle \\ &= 2qN^{\frac{1}{2}-2H} \sum_{k,l=1}^N \left[I_q(h_k^{\otimes(q-1)} \otimes h_l) + (q-1) I_{q-2}(h_k^{\otimes(q-1)} \otimes_1 h_l) \right] \langle h_k, h_l \rangle \\ &= 2qN^{\frac{1}{2}-2H} \sum_{k,l=1}^N \left[I_q(h_k^{\otimes(q-1)} \otimes h_l) + (q-1) I_{q-2}(h_k^{\otimes(q-2)}) \langle h_k, h_l \rangle \right] \langle h_k, h_l \rangle, \end{aligned}$$

where we applied the product formula (91). Consequently,

$$\begin{aligned} &\mathbf{E} \langle DV_N, DU_N \rangle^2 \\ &\leq c_q N^{1-4H} \left[\sum_{i,j,k,l=1}^N \langle h_i^{\otimes(q-1)} \tilde{\otimes} h_j, h_k^{\otimes(q-1)} \tilde{\otimes} h_l \rangle \langle h_i, h_j \rangle \langle h_k, h_l \rangle + \langle h_i, h_k \rangle^{q-2} \langle h_i, h_j \rangle^2 \langle h_k, h_l \rangle^2 \right] \\ &\leq c_q N^{1-4H} \left[\sum_{i,j,k,l=1}^N \langle h_i, h_k \rangle^{q-1} \langle h_i, h_j \rangle \langle h_k, h_l \rangle \langle h_j, h_l \rangle + \sum_{i,j,k,l=1}^N \langle h_i, h_k \rangle^{q-2} \langle h_i, h_j \rangle \langle h_k, h_l \rangle \langle h_i, h_l \rangle \langle h_j, h_k \rangle \right. \\ &\quad \left. + \sum_{i,j,k,l=1}^N \langle h_i, h_k \rangle^{q-2} \langle h_i, h_j \rangle^2 \langle h_k, h_l \rangle^2 \right] =: a_{1,N} + a_{2,N} + a_{3,N}. \end{aligned}$$

We used Lemma 4.5 in [22] in order to express the scalar product $\langle h_i^{\otimes(q-1)} \tilde{\otimes} h_j, h_k^{\otimes(q-1)} \tilde{\otimes} h_l \rangle$. Using the inequality

$$\langle h_i, h_j \rangle \langle h_k, h_l \rangle \langle h_i, h_l \rangle \langle h_j, h_k \rangle \leq \frac{1}{2} (\langle h_i, h_j \rangle^2 \langle h_k, h_l \rangle^2 + \langle h_i, h_l \rangle^2 \langle h_k, h_j \rangle^2),$$

we get $a_{2,N} \leq a_{3,N}$ so we have to estimate $a_{1,N}$ and $a_{3,N}$. Now, by (78),

$$\begin{aligned} a_{3,N} &= c_q N^{1-4H} \sum_{i,j,k,l=1}^N \rho_H(i-k)^{q-2} \rho_H(i-j)^2 \rho_H(k-l)^2 \\ &\leq c_q N^{1-4H} \sum_{i,k=1}^N \rho_H(i-k)^{q-2} \left(\sum_{a=-N}^N \rho_H(a)^2 \right)^2. \end{aligned}$$

By using the bound $\sum_{a=-N}^N \rho_H(a)^2 \leq c_H N^{4H-3}$ we obtain

$$\begin{aligned} a_{3,N} &\leq c_{q,H} N^{4H-5} \sum_{i,k=1}^N \rho_H(i-k)^{q-2} \leq c_{q,H} N^{4H-4} \sum_{k \geq 1} k^{(2H-2)(q-2)} \\ &\leq c_{q,H} N^{4H-4} \begin{cases} 1, & \text{if } H < 1 - \frac{1}{2(q-2)} \\ \log(N) & \text{if } H = 1 - \frac{1}{2(q-2)} \\ N^{(2H-2)(q-2)+1} & \text{if } H \in \left(1 - \frac{1}{2(q-2)}, 1 - \frac{1}{2q}\right). \end{cases} \end{aligned}$$

For $q = 3$, we have for $H \in \left(\frac{3}{4}, \frac{5}{6}\right)$,

$$a_{3,N} \leq c_H N^{6H-5} \quad (79)$$

Let us deal with

$$a_{1,N} = c_{q,H} N^{1-4H} \sum_{i,j,k,l=1}^N \rho_H(i-k)^{q-1} \rho_H(i-j) \rho_H(k-l) \rho_H(j-l).$$

This summand is the most complicated. Similar quantities (but not exactly the same!) have been treated in e.g. [7], proof of Theorem 4.1. We decompose the sum over $(i, j, k, l) \in \{1, \dots, N\}^4$ upon the following cases:

1. $(i = j = k = l)$,
2. $((i = j = k, l \neq i), (i = j = l, k \neq i), (i = k = l, j \neq i), (j = k = l, i \neq j))$,
3. $((i = j, k = l, k \neq i), (i = k, j = l, j \neq i), (i = l, j = k, j \neq i))$,
- 4.

$$\begin{aligned} &((i = j, k \neq i, k \neq l, l \neq i), (i = k, j \neq i, j \neq l, k \neq l), (i = l, k \neq i, k \neq j, j \neq i), \\ &(j = k, k \neq i, k \neq l, l \neq i), (j = l, k \neq i, k \neq l, j \neq i), (k = l, k \neq i, k \neq j, j \neq i)). \end{aligned}$$

5. i, j, k, l are all different.

We denote by $a_{1,N}^{(j)}, j = 1, 2, 3, 4, 5$ the sum of all the terms from the groups 1.-5. defined above. The first of these terms can be easily estimated since

$$a_{1,N}^{(1)} = c_{q,H} N^{1-4H} \sum_{i=1}^N \rho_H(0)^{q+2} = c_{q,H} N^{2-4H}. \quad (80)$$

For, the first sum from point 2.

$$c_{q,H} N^{1-4H} \sum_{i,l=1}^N \rho_H(i-l)^2 \leq c_{q,H} N^{2-4H} \sum_{i=1}^N i^{4H-4} \leq c_{q,H} N^{2-4H} N^{4H-3} = c_{q,H} N^{-1}$$

while the second from point 2.

$$c_{q,H} N^{1-4H} \sum_{i,k=1}^N \rho_H(i-k)^q \leq c_{q,H} N^{2-4H} \sum_{k \in \mathbb{Z}} \rho_H(k)^q \leq c_{q,H} N^{2-4H}.$$

So, by symmetry,

$$a_{2,N}^{(2)} \leq c_{q,H} (N^{-1} + N^{2-4H}) \leq c_{q,H} N^{-1}. \quad (81)$$

The sums from group 3. are similar to the those from group 2. and we get

$$a_{1,N}^{(3)} \leq c_{q,H} N^{-1}. \quad (82)$$

Let us with the summands corresponding to point 4. The first one in this set reads

$$\begin{aligned} & c_{q,H} N^{1-4H} \sum_{i \neq k \neq l \neq i} \rho_H(i-k)^{q-1} \rho_H(k-l) \rho_H(i-l) \\ & \leq c_q N^{2-4H} \sum_{a,b=-N}^N |\rho_H|(a-b)^{q-1} |\rho_H|(a) |\rho_H|(b) \leq c_q N^{2-4H} \sum_{a,b=-N}^N |\rho_H|(a-b)^{q-1} |\rho_H|(a)^2 \\ & \leq c_{q,H} N^{2-4H} \sum_{a=-N}^N |a|^{4H-4} \sum_{b=-2N}^{2N} |b|^{(2H-2)(q-1)}. \end{aligned}$$

It follows that this term is less than

$$c_{q,H} \begin{cases} N^{-1} & \text{if } H < 1 - \frac{1}{2(q-1)} \\ N^{-1} \log N & \text{if } H = 1 - \frac{1}{2(q-1)} \\ N^{(2H-2)(q-1)+2} & \text{if } H \in \left(1 - \frac{1}{2(q-1)}, 1 - \frac{1}{2q}\right). \end{cases}$$

Regarding the second summand in 4., we can bound as follows

$$\begin{aligned}
& c_{q,H} N^{1-4H} \sum_{i \neq j \neq l \neq i} \rho_H(i-j) \rho_H(i-l) \rho_H(j-l) \\
& \leq c_{q,H} N^{1-4H} N^3 N^{6H-6} \frac{1}{N^3} \sum_{i \neq j \neq l \neq i} \left(\frac{|i-j|}{N} \right)^{2H-2} \left(\frac{|i-l|}{N} \right)^{2H-2} \left(\frac{|j-l|}{N} \right)^{2H-2} \\
& = c_{q,H} N^{2H-2} \frac{1}{N^3} \sum_{i \neq j \neq l \neq i} \left(\frac{|i-j|}{N} \right)^{2H-2} \left(\frac{|i-l|}{N} \right)^{2H-2} \left(\frac{|j-l|}{N} \right)^{2H-2} \leq c_{q,H} N^{2H-2},
\end{aligned}$$

since the quantity $\frac{1}{N^3} \sum_{i \neq j \neq l \neq i} \left(\frac{|i-j|}{N} \right)^{2H-2} \left(\frac{|i-l|}{N} \right)^{2H-2} \left(\frac{|j-l|}{N} \right)^{2H-2}$ is a Riemann sum that converges to $\int_{[0,1]^3} |x-y|^{2H-2} |y-z|^{2H-2} |z-x|^{2H-2} dx dy dz < \infty$. We have similar bounds for the other terms and we get

$$a_{1,N}^{(4)} \leq c_{q,H} N^{2H-2}. \quad (83)$$

Notice that the estimation of the dominant term, the second in this group is sharp.

For the only summand in group 5., we separate its analysis upon all the possible orders: $i > j > k > l, i > j > l > k, \dots$. The first summand is treated as follows

$$\begin{aligned}
& c_q N^{1-4H} \sum_{i > j > k > l} \rho_H(i-k)^{q-1} \rho_H(i-j) \rho_H(k-l) \rho_H(j-l) \\
& \leq c_{q,H} N^{1-4H} \sum_{i > j > k > l} |i-k|^{(2H-2)(q-1)} |i-j|^{2H-2} |k-l|^{2H-2} |j-l|^{2H-2} \\
& \leq c_{q,H} N^{1-4H} \sum_{i > j > k > l} |i-k|^{(2H-2)(q-1)} |i-j|^{2H-2} |k-l|^{4H-4} \\
& \leq c_{q,H} N^{1-4H} \sum_{i > j > k} |i-k|^{(2H-2)(q-1)} |i-j|^{2H-2} \sum_{l=-N}^N |l|^{4H-4} \\
& \leq c_{q,H} N^{-2} \sum_{i > j > k} |i-k|^{(2H-2)(q-1)} |i-j|^{2H-2} \\
& \leq c_{q,H} N^{-2} \sum_{i > k} |i-k|^{(2H-2)(q-1)} \sum_{j=-N}^N |j|^{2H-2} \leq c_{q,H} N^{2H-3} \sum_{i > k} |i-k|^{(2H-2)(q-1)} \\
& \leq c_{q,H} N^{2H-2} \sum_{k=1}^N k^{(2H-2)(q-1)}.
\end{aligned}$$

With analogous estimates for the other cases of point 5., we obtain

$$a_{1,N}^{(5)} \leq c_{q,H} \begin{cases} N^{2H-2} & \text{if } H < 1 - \frac{1}{2(q-1)} \\ N^{2H-2} \log N & \text{if } H = 1 - \frac{1}{2(q-1)} \\ N^{(2H-2)q+1} & \text{if } H \in \left(1 - \frac{1}{2(q-1)}, 1 - \frac{1}{2q}\right). \end{cases} \quad (84)$$

So, by (80), (81), (82), (83) and (84)

$$a_{1,N} \leq c_{q,H} \begin{cases} N^{2H-2} & \text{if } H \in \left(\frac{3}{4}, 1 - \frac{1}{2(q-1)}\right) \\ N^{(2H-2)q+1} & \text{if } H \in \left(1 - \frac{1}{2(q-1)}, 1 - \frac{1}{2q}\right). \end{cases}$$

Thus

$$\mathbf{E}\langle DV_N, DU_N \rangle^2 \leq c_{q,H} \begin{cases} N^{2H-2} & \text{if } H \in \left(\frac{3}{4}, 1 - \frac{1}{2(q-1)}\right) \\ N^{(2H-2)q+1} & \text{if } H \in \left(1 - \frac{1}{2(q-1)}, 1 - \frac{1}{2q}\right), \end{cases} \quad (85)$$

the bound on the first branch being immaterial for $q = 3, 4$. If $q = 3$, then

$$\mathbf{E}\langle DV_N, DU_N \rangle^2 \leq c_H N^{6H-5}. \quad (86)$$

We then obtain (74). ■

Remark 5 1. For $q = 3$, we have from (77), (79) and (86),

$$d_W \left((V_N, U_N), (Z, c_{2,H} R^{(2H-1)}) \right) \leq C_H \begin{cases} N^{\frac{3}{2}-2H} & \text{if } H \in \left(\frac{3}{4}, \frac{4}{5}\right) \\ N^{3H-\frac{5}{2}} & \text{if } H \in \left[\frac{4}{5}, \frac{5}{6}\right). \end{cases} \quad (87)$$

2. It follows from the above calculation that the quantity $(\mathbf{E}\langle DV_N, DU_N \rangle^2)^{\frac{1}{2}}$, which somehow measures the correlation between V_N and U_N has the same size, for N large, as $d_W(V_N, Z)$ (compare (75) and (85)).
3. A quantitative bound for the above limit theorem can be also obtained by using the estimate (43) in Remark 3. Notice that (43) gives

$$\mathbf{E}\langle DV_N, DU_N \rangle^2 \leq C_H \mathbf{E}(\|f_N \otimes_1 f_N\| + \|f_N \otimes_2 f_N\|).$$

By using the calculations in the proof of Theorem 4.1 in [7] and since $\mathbf{E}G_N \leq C_H$ (with $C_H > 0$ not depending on N), we get

$$\mathbf{E}\langle DV_N, DU_N \rangle^2 \leq C_H \left(N^{-\frac{1}{2}} + N^{H-1} + N^{1-q(1-H)} \right),$$

which is in general less good than (74). For instance, if $q = 3$, we have

$$\mathbf{E}\langle DV_N, DU_N \rangle^2 \leq C_H \left(N^{-\frac{1}{2}} + N^{H-1} + N^{3H-2} \right),$$

and leads, for $H \in (\frac{3}{4}, \frac{5}{6})$, to

$$d_W \left((V_N, U_N), (Z, c_{2,H} R^{(2H-1)}) \right) \leq C_H N^{\frac{3H}{2}-1},$$

which clearly is less optimal than (87).

5.4 The evolution of the solution to a semilinear stochastic equation

The theory developed in Section 2 can also be applied to quantify the evolution of a stochastic system defined by a stochastic differential equation. We present here a very simple example (a more complex situation, in the KPZ context, has been treated in [15]). Let $\lambda \in \mathbb{R}$ and consider the stochastic equation

$$X_t^\lambda = X_0 + \lambda \int_0^t b(X_s^\lambda) ds + W_t, \quad t \geq 0 \quad (88)$$

where $(W_t, t \geq 0)$ is a Wiener process. We assume that the drift $b : \mathbb{R} \rightarrow \mathbb{R}$ is differentiable and satisfies $|b'(x)| \leq M$ for every $x \in \mathbb{R}$. Then (88) admits a unique solution which is Malliavin differentiable and (see e.g. Exercice 2.2.1 in [11]) for $a < t$,

$$D_a X_t^\lambda = e^{\int_a^t b'(X_s^\lambda) ds}.$$

The solution to (88) is a Gaussian process for $\lambda = 0$ and for $\lambda \neq 0$, its law is non-Gaussian if b is nonlinear. Theorem 1 allows to quantify the dependence structure between the components of the vector (X_t^λ, X_t^0) at each time $t > 0$. Indeed, by Theorem 1,

$$\begin{aligned} d_W \left(P_{(X_t^\lambda, X_t^0)}, P_{X_t^\lambda} \otimes P_{X_t^0} \right) &\leq C \int_0^t D_a X_t^\lambda da \\ &\leq C \int_0^t e^{\int_a^t b'(X_s^\lambda) ds} da \leq C \int_0^t e^{\lambda M(t-\lambda)} = \frac{C}{M\lambda} (e^{M\lambda t} - 1) := g(\lambda). \end{aligned}$$

The function g provides a quantitative estimate for the dependence between X^λ and X^0 for any λ , at any time. This function converges to a constant when $\lambda \rightarrow 0$ and to infinity as $\lambda \rightarrow \infty$. When λ tends to $-\infty$, $g(\lambda)$ converges to zero, i.e. the drift forces the solution to (88) to be independent of the noise at each time.

6 Appendix: Wiener-Chaos and Malliavin derivatives

Here we describe the elements from stochastic analysis that we will need in the paper. Consider H a real separable Hilbert space and $(W(h), h \in H)$ an isonormal Gaussian process on a probability space (Ω, \mathcal{A}, P) , which is a centered Gaussian family of random variables such that $\mathbf{E}[W(\varphi)W(\psi)] = \langle \varphi, \psi \rangle_H$. Denote by I_n the multiple stochastic integral with respect to B (see [11]). This mapping I_n is actually an isometry between the Hilbert space $H^{\odot n}$ (symmetric tensor product) equipped with the scaled norm $\frac{1}{\sqrt{n!}} \|\cdot\|_{H^{\otimes n}}$ and the Wiener chaos of order n which is defined as the closed linear span of the random variables $H_n(W(h))$ where $h \in H, \|h\|_H = 1$ and H_n is the Hermite polynomial of degree $n \in \mathbb{N}$

$$H_n(x) = \frac{(-1)^n}{n!} \exp\left(\frac{x^2}{2}\right) \frac{d^n}{dx^n} \left(\exp\left(-\frac{x^2}{2}\right) \right), \quad x \in \mathbb{R}.$$

The isometry of multiple integrals can be written as follows: for m, n positive integers,

$$\begin{aligned} \mathbf{E}(I_n(f)I_m(g)) &= n! \langle \tilde{f}, \tilde{g} \rangle_{H^{\otimes n}} \quad \text{if } m = n, \\ \mathbf{E}(I_n(f)I_m(g)) &= 0 \quad \text{if } m \neq n. \end{aligned} \tag{89}$$

It also holds that

$$I_n(f) = I_n(\tilde{f})$$

where \tilde{f} denotes the symmetrization of f defined by the formula

$$\tilde{f}(x_1, \dots, x_n) = \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} f(x_{\sigma(1)}, \dots, x_{\sigma(n)}).$$

We recall that any square integrable random variable which is measurable with respect to the σ -algebra generated by W can be expanded into an orthogonal sum of multiple stochastic integrals

$$F = \sum_{n=0}^{\infty} I_n(f_n) \tag{90}$$

where $f_n \in H^{\odot n}$ are (uniquely determined) symmetric functions and $I_0(f_0) = \mathbf{E}[F]$.

Let L be the Ornstein-Uhlenbeck operator

$$LF = - \sum_{n \geq 0} n I_n(f_n)$$

if F is given by (90) and it is such that $\sum_{n=1}^{\infty} n^2 n! \|f_n\|_{\mathcal{H}^{\otimes n}}^2 < \infty$.

For $p > 1$ and $\alpha \in \mathbb{R}$ we introduce the Sobolev-Watanabe space $\mathbb{D}^{\alpha,p}$ as the closure of the set of polynomial random variables with respect to the norm

$$\|F\|_{\alpha,p} = \|(I - L)^{\frac{\alpha}{2}} F\|_{L^p(\Omega)}$$

where I represents the identity. We denote by D the Malliavin derivative operator that acts on smooth functions of the form $F = g(W(h_1), \dots, W(h_n))$ (g is a smooth function with compact support and $h_i \in H$)

$$DF = \sum_{i=1}^n \frac{\partial g}{\partial x_i}(W(h_1), \dots, W(h_n)) h_i.$$

The operator D is continuous from $\mathbb{D}^{\alpha,p}$ into $\mathbb{D}^{\alpha-1,p}(H)$. The adjoint of D is the divergence integral, denoted by δ . It acts from $\mathbb{D}^{\alpha-1,p}(H)$ onto $\mathbb{D}^{\alpha,p}$.

We will intensively use the product formula for multiple integrals. It is well-known that for $f \in H^{\odot n}$ and $g \in H^{\odot m}$

$$I_n(f)I_m(g) = \sum_{r=0}^{n \wedge m} r! \binom{n}{r} \binom{m}{r} I_{m+n-2r}(f \otimes_r g) \quad (91)$$

where $f \otimes_r g$ means the r -contraction of f and g (see e.g. Section 1.1.2 in [11]). This contraction is defined, when $H = L^2(T, \mathbb{B}, \nu)$ (where ν is a sigma-finite measure without atoms)

$$\begin{aligned} & (f \otimes_r g)(t_1, \dots, t_{n+m-2r}) \\ &= \int_{T^r} f(u_1, \dots, u_r, t_1, \dots, t_{n-r}) g(u_1, \dots, u_r, t_{n-r+1}, \dots, t_{n+m-2r}) du_1 \dots du_r, \end{aligned} \quad (92)$$

for $r = 1, \dots, n \wedge m$ and $f \otimes_0 g = f \otimes g$, the tensor product. It holds that $f \otimes_r g \in H^{\otimes n+m-2r} = L^2(T^{n+m-2r})$. In general, the contraction $f \otimes_r g$ is not symmetric and we denote by $f \tilde{\otimes}_r g$ its symmetrization.

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