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A new class of L^2 -stable schemes for the isentropic Euler equations on staggered grids

Michaël Ndjinga and Katia Ait-Ameur

Abstract Staggered schemes for compressible flows are highly non linear and the stability analysis has historically been performed with a heuristic approach and the tuning of numerical parameters ([12]). We investigate the L^2 -stability of staggered schemes by analysing their numerical diffusion operator. The analysis of the numerical diffusion operator gives new insight into the scheme and is a step towards a proof of linear stability or stability for almost constant initial data. For most classical staggered schemes ([14, 9, 10, 11]), we are able to prove the positivity of the numerical diffusion only in specific cases (constant sign velocities).

We then propose a class of linearly L^2 -stable staggered schemes for the isentropic Euler equations based on a carefully chosen numerical diffusion operator. We give an example of such a scheme and present some first numerical results on a Riemann problem.

Key words: Euler equations, compressible flows, finite volumes, staggered grids, stability analysis, numerical diffusion

MSC (2010): 35L65, 35Q35, 65M08, 65M12

1 Introduction

As an introduction to the issue, we consider a 1D conservative non linear hyperbolic system

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$$\partial_t U(x, t) + \partial_x F(U)(x, t) = 0, \quad (1)$$

with unknown vector $U \in \mathbb{R}^m$ and Lipschitz flux $F : \mathbb{R}^m \rightarrow \mathbb{R}^m$ with real-diagonalisable Jacobian matrix $A(U) = \nabla_U F(U) \in \mathbb{R}^{m \times m}$.

When approximating smooth solutions U of (1) by a consistent numerical method on a regular mesh with space step Δx , the semi-discrete equations approximate to the first order in Δx the following perturbed version of equation (1) :

$$\partial_t U_{\Delta x} + \partial_x F(U_{\Delta x}) = \mathcal{D}(U_{\Delta x}, \Delta x) + o(\Delta x), \quad (2)$$

where $U_{\Delta x}$ is the numerical solution and \mathcal{D} is a second order differential operator.

When the flux function F is linear, linear numerical methods yield a linear diffusion operator $\mathcal{D}(U, \Delta x) = \Delta x \partial_x (D \partial_x U)$. The matrix D comes from the upwind (off-centered) contributions of the discrete equations and gives a first insight into the scheme precision and stability. In the non linear case (F Lipschitz), the numerical diffusion operator can often be approximated to the first order by a non linear diffusion operator :

$$\mathcal{D}(U, \Delta x) = \Delta x \partial_x (D(U) \partial_x U) + o(\Delta x). \quad (3)$$

This is the case for instance for collocated schemes based on characteristic upwinding such as Godunov [6], Roe [15], VFRoe [13] or VFFC [4] schemes where the non linear numerical diffusion tensor is $D(U) = |A(U)|$.

We recall that in the case of symmetric hyperbolic systems (${}^t A(U) = A(U)$), any entropy solution to (1) preserves the L^2 norm (see [5] Example 3.2 in the Introduction chapter) and we would like the discrete L^2 norm of any scheme to be bounded as well. In the case of non-symmetric systems, one first symmetrises the system using entropic variables $V(U) = \nabla_s(U)$ where s a strictly convex entropy of the system (1) is assumed to exist (see [5] Theorem 3.2 in the introduction chapter). The new symmetric system:

$$\partial_t V + \bar{A}(V) \partial_x V = 0, \text{ with } {}^t \bar{A} = \bar{A} \quad (4)$$

is linearly L^2 -stable. Any numerical scheme yields a numerical diffusion $\bar{D}(V)$ in the symmetrised basis and we require that the diffusion operator \bar{D} have positive symmetric part : ${}^t \bar{D} + \bar{D} \geq 0$.

In this first account of our research, we investigate the wave system which raises an issue that will remain with the more complex fluid model : the numerical treatment of the mass balance equation yields a non classical diffusion operator. In order to obtain a straightforwardly stable scheme we propose a new discretisation with positive numerical diffusion.

In Section 2, we determine the numerical diffusion of the staggered schemes and show it does not straightforwardly yield a linear stability. We then present a new class of staggered schemes and prove their linear stability in Section 3. Some first numerical results are given in Section 4.

2 The numerical diffusion of staggered schemes for the linear wave system

For compressible flows at low Mach numbers, it is customary to approximate the Euler system by the wave system (see for instance [2, 3]). The linear wave system is the first order reduction of the classical linear wave equation $\partial_{tt}\rho - c^2\partial_{xx}\rho = 0$ and can be written in the following conservative form :

$$\begin{cases} \partial_t \rho + \partial_x q = 0 \\ \partial_t q + c^2 \partial_x \rho = 0 \end{cases} \quad (5)$$

The analysis of the wave system can be extended to the Euler system but the calculations are lengthier and do not help the understanding of the main result.

The wave system (5) is a linear PDE system and the characteristic based upwind method is linear. Therefore the finite volume upwind scheme yields a classical linear diffusion operator (3) with constant diffusion tensor: $D = c\mathbb{1}_d$.

The upwind scheme can be proven to be stable. However the amount of numerical diffusion is proportional to the sound speed c and for low Mach number flows where $q \ll \rho c$ the schemes based on characteristics upwinding are not able to capture nearly incompressible solutions (see [2, 3] for more details).

On the contrary, staggered schemes are known to be more precise for low Mach number flows in practice and are very popular in the thermal hydraulics community ([14]). However, their stability analysis has historically been based on heuristics ([12]). Yet the conservative staggered schemes presented in [10, 9] is proven to be entropic and to satisfy a kinetic energy preservation [11]. Likewise in [1], the authors present a kinetic scheme on staggered grids for the barotropic Euler equations and derive stability conditions which preserve the positivity of the density and the decay of the discrete global entropy and satisfy a kinetic energy preservation. Unfortunately the boundedness of the entropy does not necessarily imply the boundedness of the solution. Indeed a strictly convex function is not necessarily bounded below. This is in particular the case for the full Euler system since the entropy involves the function $-\ln$ which is strictly convex but not bounded below (see [5] Example 3.3 in the introduction chapter). In the next subsection we show that the first order perturbed equation (2) associated to staggered schemes yields not the classical diffusion operator (3) but instead a strongly non linear numerical diffusion operator.

2.1 The staggered scheme of Herbin et al.

Using staggered schemes, the density and pressure are located on cells and the velocity on faces (nodes in 1D) [8, 7]. The momentum variable is usually split as a product between the density and the velocity : $\mathbf{q} = \rho \mathbf{u}$. The main difference be-

tween the various staggered schemes is the treatment of the convection term $\rho \mathbf{u} \otimes \mathbf{u}$ in the momentum equation. Since we work with the wave system, they differ only by the use of \mathbf{u} (non conservative approach, see [14] Section 11.2) or $\rho \mathbf{u}$ as main variable. We consider the staggered scheme of [11] which is conservative and entropic. For simplicity, we present the semi-discrete equations of the fully implicit variant ([10] Section 3, [9] Section 2.1) for the 1D wave system :

$$\begin{cases} \partial_t \rho_i + \frac{1}{\Delta x} (\rho_{i+\frac{1}{2}}^{up} u_{i+\frac{1}{2}} - \rho_{i-\frac{1}{2}}^{up} u_{i-\frac{1}{2}}) = 0 \\ \partial_t \left(\bar{\rho}_{i+\frac{1}{2}} u_{i+\frac{1}{2}} \right) + c^2 \frac{1}{\Delta x} (\rho_{i+1} - \rho_i) = 0. \end{cases} \quad (6)$$

The mass flux ρu at the cell interfaces is defined using an upwind density $\rho_{i+\frac{1}{2}}^{up}$ defined as :

$$\begin{aligned} \rho_{i+\frac{1}{2}}^{up} &= \begin{cases} \rho_i & \text{if } u_{i+\frac{1}{2}} > 0 \\ \rho_{i+1} & \text{if } u_{i+\frac{1}{2}} \leq 0 \end{cases} \\ &= \frac{\rho_i + \rho_{i+1}}{2} + \text{sign}(u_{i+\frac{1}{2}}) \frac{\rho_i - \rho_{i+1}}{2}, \end{aligned} \quad (7)$$

which is the sum of a centered and an upwind term.

The expression of $\bar{\rho}_{i+\frac{1}{2}}$ in the discrete momentum equation amounts to an average of the neighbouring densities

$$\bar{\rho}_{i+\frac{1}{2}} = \frac{1}{2} (\rho_i + \rho_{i+1}). \quad (8)$$

2.2 The numerical diffusion

In this section, we assume that the exact solution is smooth and we determine the numerical diffusion of the scheme (6). The first order momentum perturbed equation is straightforward :

$$\partial_t (\rho u)(x_{i+\frac{1}{2}}, t) + c^2 \partial_x \rho(x_{i+\frac{1}{2}}, t) = \frac{c^2}{2} (\Delta x) \partial_{xx} \rho(x_{i+\frac{1}{2}}, t) + \mathcal{O}(\Delta x^2).$$

After some calculations, we find that the mass flux is given to the first order :

$$\frac{\rho_{i+\frac{1}{2}}^{up} u_{i+\frac{1}{2}} - \rho_{i-\frac{1}{2}}^{up} u_{i-\frac{1}{2}}}{\Delta x} = \partial_x (\rho u)(x_i, t) - \frac{\Delta x}{2} \text{sign}(u(x_i, t)) \partial_x (u \partial_x \rho)(x_i, t) + \mathcal{O}(\Delta x^2).$$

Hence the following result on the numerical diffusion operator of the staggered scheme (6).

Theorem 1 (Numerical diffusion of staggered schemes) *The second order perturbation operator associated to the staggered scheme (6) on a 1D regular mesh with space step Δx is the strongly non linear operator :*

$$\mathcal{D}(U, \Delta x) = \Delta x \begin{pmatrix} \text{sign}(u) & 0 \\ 0 & 1 \end{pmatrix} \partial_x \left[\begin{pmatrix} u & 0 \\ c^2 & 0 \end{pmatrix} \partial_x \begin{pmatrix} \rho \\ q \end{pmatrix} \right] + o(\Delta x). \quad (9)$$

The numerical diffusion associated to the mass conservation law is the term $\text{sign}(u)\partial_x(u\partial_x\rho)$ and is decoupled from the momentum diffusion. The linear stability analysis of such a strongly non linear diffusion is not classical and we are not aware of any reference in the litterature.

If we assume that u does not change sign then the diffusion term simplifies to the weakly non linear diffusion term $\partial_x(|u|\partial_x\rho)$ which involves a positive diffusion coefficient $|u|$. The weakly non linear diffusion term $\partial_x(|u|\partial_x\rho)$ can be linearised around a constant state $(\rho_0, u_0 \neq 0)$ as $\partial_x(|u_0|\partial_x\rho) + \partial_x(|u|\partial_x\rho_0) = |u_0|\partial_{xx}\rho$. Hence if $u > 0$ or $u < 0$ the mass equation has a positive contribution on the diagonal of the numerical diffusion tensor D and thus has a stabilising effect.

If we allow u to change sign then the multiplication with $\text{sign}(u)$ makes things more complicated and we can not rule out potential instabilities. The linearisation is not trivial, even taking u smooth enough, since the function $\text{sign}(u)$ is not continuous. The consistency analysis is only a first step that requires smooth solutions but the final goal of capturing discontinuous weak solutions with velocity that change sign will raise even more issues.

3 A new class of schemes for the isentropic Euler equations

We now propose a class of staggered schemes for conservation laws (1) which admits a classical diffusion operator (3) and such that the diffusion tensor satisfies: $\bar{D} + {}^t\bar{D} \geq 0$. We will prove that these schemes are linearly L^2 -stable.

We specify this new class in the particular case of the following 1D isothermal Euler equations in conservative form:

$$\begin{cases} \partial_t \rho + \partial_x q = 0 \\ \partial_t q + \partial_x \frac{q^2}{\rho} + \partial_x p = 0. \end{cases} \quad (10)$$

which takes the form (1) with

$$U = \begin{pmatrix} \rho \\ q \end{pmatrix}, \quad F(U) = \begin{pmatrix} q \\ \frac{q^2}{\rho} + p \end{pmatrix}.$$

The results can be extended to the multidimensional Euler system but the calculations are lengthier and do not help the understanding of the main properties.

We consider the class *Stag* of discrete staggered conservative schemes of the form:

$$U_i'(t) + \frac{F_{i,i+1} - F_{i-1,i}}{\Delta x} = 0, \quad \text{with: } U_i = \begin{pmatrix} \rho_i \\ q_{i+\frac{1}{2}} \end{pmatrix}, \quad \text{and:} \quad (11)$$

$$F_{i,i+1} = \frac{F(U_i) + F(U_{i+1})}{2} + D_{Stag}(U_i, U_{i+1}) \frac{U_i - U_{i+1}}{2}, \quad (12)$$

135 where D_{Stag} is a 2×2 matrix-valued function. An example of a scheme in the class *Stag* is the staggered centered scheme, which correspond to the case $D_{Stag} = 0$. Schemes of the class *Stag* admit a classical diffusion operator (3).

Theorem 2 (Classical diffusion of *Stag* schemes) Let $D_{Stag} : \mathbb{R}^2 \rightarrow \mathbb{R}^{2 \times 2}$ be a matrix valued Lipschitz function. A staggered conservative scheme (11) with a numerical flux $F_{i,i+1}$ of the form (12) admits the following classical diffusion operator

$$\mathcal{D}(U, \Delta x) = \Delta x \partial_x (D_{Stag}(U, U) \partial_x U) + o(\Delta x).$$

on a regular mesh with space step Δx .

Schemes of the class *Stag* are L^2 -stable.

Theorem 3 (L^2 -stability of *Stag* schemes)

145 A staggered conservative scheme (11) with a numerical flux $F_{i,i+1}$ of the form (12) such that the diffusion operator D_{Stag} satisfies: $\bar{D}_{Stag} + {}^t \bar{D}_{Stag} \geq 0$, is linearly L^2 -stable.

Proof After the symmetrisation (4) and linearisation, the isentropic Euler system takes the form:

$$\partial_t \bar{V} + \bar{A}(V_0) \partial_x \bar{V} = 0, \quad \bar{V} = \begin{pmatrix} c\rho \\ q - \rho u_0 \end{pmatrix}$$

We consider \bar{V} a solution of the conservative scheme:

$$\bar{V}_i'(t) + \frac{\bar{F}_{i,i+1} - \bar{F}_{i-1,i}}{\Delta x} = 0, \quad \text{where: } \bar{F}_{i,i+1} = \bar{A}(V_0) \frac{\bar{V}_i + \bar{V}_{i+1}}{2} + \bar{D}_{Stag}(V_0) \frac{\bar{V}_i - \bar{V}_{i+1}}{2}$$

We compute the evolution in time of $\|\bar{V}\|_2^2$ using the symmetry of $\bar{A}(V_0)$ and the positiveness of $\bar{D}_{Stag}(V_0)$.

$$\begin{aligned} \frac{1}{2} \frac{d\|\bar{V}\|_2^2}{dt} &= \bar{V} \cdot \frac{d\bar{V}}{dt} = \sum_i \Delta x \bar{V}_i \cdot \frac{d\bar{V}_i}{dt} = - \sum_i \bar{V}_i \cdot (\bar{F}_{i,i+1} - \bar{F}_{i-1,i}) \\ &= - \frac{1}{2} \sum_i \bar{V}_i \cdot \bar{A}(V_0) (\bar{V}_{i+1} - \bar{V}_{i-1}) - \frac{1}{2} \sum_i \bar{V}_i \cdot \bar{D}_{Stag}(V_0) (\bar{V}_i - \bar{V}_{i+1}) \\ &\quad + \frac{1}{2} \sum_i \bar{V}_i \cdot \bar{D}_{Stag}(V_0) (\bar{V}_{i-1} - \bar{V}_i) \end{aligned}$$

150 Since: $\bar{A}(V_0) = {}^t \bar{A}(V_0)$, we have: $\sum_i \bar{V}_i \cdot \bar{A}(V_0) (\bar{V}_{i+1} - \bar{V}_{i-1}) = 0$, and:

$$\frac{1}{2} \frac{d\|\bar{V}\|_2^2}{dt} = -\frac{1}{2} \sum_i (\bar{V}_i - \bar{V}_{i+1}) \cdot \bar{D}_{Stag}(V_0)(\bar{V}_i - \bar{V}_{i+1})$$

Since: $\bar{D}_{Stag}(V_0) + {}^t\bar{D}_{Stag}(V_0) \geq 0$, we obtain: $\frac{1}{2} \frac{d\|\bar{V}\|_2^2}{dt} \leq 0$. □

Corollary 1 *The numerical scheme (11)-(12) based on the following numerical flux:*

$$F_{i,i+1} = \begin{pmatrix} \bar{q}_{i+\frac{1}{2}} \\ \frac{\bar{q}_{i+1}^2}{\rho_{i+1}} + p_{i+1} \end{pmatrix}, \text{ with:}$$

$$\begin{aligned} \bar{q}_{i+\frac{1}{2}} &= q_{i+\frac{1}{2}} + (|u| - u) \frac{\rho_i - \rho_{i+1}}{2} \\ \frac{\bar{q}_{i+1}^2}{\rho_{i+1}} &= \frac{q_{i+\frac{1}{2}}^2}{\rho_i} + (|u| - u) \frac{q_{i+\frac{1}{2}} - q_{i+\frac{3}{2}}}{2} \end{aligned}$$

where u is the Roe average velocity ([15]): $u = \frac{q_{i+\frac{1}{2}} + q_{i+\frac{3}{2}}}{\frac{\sqrt{\rho_i}}{\sqrt{\rho_i + \sqrt{\rho_{i+1}}}} + \frac{\sqrt{\rho_{i+1}}}{\sqrt{\rho_i + \sqrt{\rho_{i+1}}}}}$, is linearly L^2 -stable.

Proof The diffusion operator of the proposed scheme is :

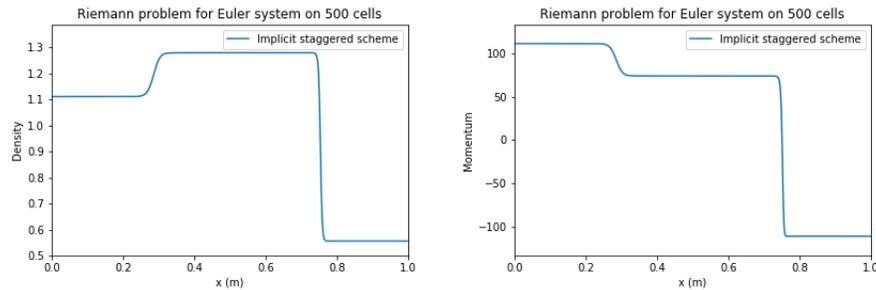
$$D_{Stag} = \begin{pmatrix} |u| - u & 1 \\ -c^2 - u^2 & |u| + u \end{pmatrix}, \quad \bar{D}_{Stag} = \begin{pmatrix} |u| & c \\ -c & |u| \end{pmatrix}. \quad (13)$$

Since D_{Stag} verifies the property of Theorem 3, this numerical scheme is linearly L^2 -stable. □

In the next section, we implement an implicit version of this new numerical scheme.

4 Numerical results and conclusion

We assess the robustness of our new staggered scheme on a compressible fluid with isothermal equation of state $p = \rho c^2$ where the sound speed is $c = 300m/s$. We consider a Riemann problem for the isentropic Euler system (10) with left state $(\rho_L = \frac{10}{9}, q_L = 100\rho_L)$ and right state $(\rho_R = \frac{\rho_L}{2}, q_R = -100\rho_L)$. The solution displays a rarefaction (smooth) wave followed by a (discontinuous) shock wave. Our new method is able to capture both waves in a distinct and stable way. In a forthcoming longer paper we will present the details of the 2D version of the scheme and study its entropic character.



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