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Sample Volume of a Capacitance Moisture Sensor in Function of its Geometry

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ABSTRACT
We have developed a permittivity-based sensor using a new and more accurate measurement technique. We now investigate the optimum geometry of sensor electrodes to apply models of conversion of medium permittivity into its water content (and salinity), while accounting for field constraints of compactness and simplicity. Models assume medium volume sampled by sensor to be large enough to consider a quasi-uniform permittivity. Using this hypothesis, and verifying ex-post its consistency, we carried out an exhaustive study of sample volume dependence on electrode geometry. In particular, we examine the role of electrode diameter, oversight by previous works. Besides, our approach permits a direct experimental validation with profiles of sensor sensitivity. For a two-rod design with electrode spacing $D$ and diameter $\Phi$, sample-volume size and localization is determined by the ratio $\alpha = \Phi / D$. For $\alpha$ lower than 0.25 or thin electrodes - a geometry extensively studied and commonly encountered - theory and measurements show that the volume is concentrated around electrode surface. As $\alpha$ is increased it becomes localized between electrodes with a more uniform sensitivity, like for a parallel-plate capacitor. Volume size at fixed $D$ is also assessed, with its highest value between $\alpha = 0.30$ and 0.50. We adopted this range for our sensors.

KEYWORDS
moisture sensor; sample volume; spatial distribution of sensor sensitivity; electrode geometry; electrostatic energy; dimensionless study

1. Introduction

We have developed a new type of sensor for determining the water content (and its salinity) in a porous medium, such as a soil, based on the original technique of a self-balanced impedance bridge (Chavanne and Frangi (2014, 2017)). Sensor principle relies on the sensitivity of medium relative dielectric permittivity $\varepsilon_r$ (medium permittivity normalized by vacuum permittivity $\varepsilon_0 = 8.854 \text{ pF} \cdot \text{m}^{-1}$) to its water content $\theta_v$. The quantity is deduced from the polarization of medium dipoles by the electric field generated by the sensor. The permittivity method is an indirect one by comparison with the standard determination by sample weightings, but permits to monitor continuously in situ medium moisture with cost-effective autonomous sensors owing to the continuous progress on the information and communication technologies (ICT). Various moisture sensors were also developed along the permittivity-based principle, but using different measurement techniques to determine $\varepsilon_r$, more or less indirectly (for a rapid review see Chavanne and Frangi (2014, 2017)). They can be broadly classified into two large families: sensors measuring the capacitance (and more rarely the conductance) of electrodes embedded in the medium under study, like in our case, and others detecting the phase velocity or travel time of an electromagnetic wave propagating along a guide in the medium, such as the well-known Time Domain Reflectometers (TDRs).

Whatever the technique, and assuming the instrument gives an accurate value of medium apparent permittivity $\varepsilon_r$ around its electrodes or waveguide, the next and final step of treatment is the conversion of $\varepsilon_r$ into the medium moisture. Various models exist to achieve it, from completely empirical relations - such as the correlation in Topp, Davis, and Annan (1980) - to approaches based on the application of the electromagnetic theory at the grain scale (Revil (2013)). They assume the medium to be quasi-homogenous, that is the medium volume sampled by sensors to generate a spatially-useful average of $\varepsilon_r$ is much larger than the characteristic dimensions of medium natural heterogeneity such as pore or grain (typically less than 1 mm; see for instance Musy and Soutter (1991); Revil, Koch, and Holliger (2012)). The volume of influence of the medium on the apparent permittivity measured by the sensor, or sample volume, depends on sensor electric field distribution, more precisely that of its...
energy, which, in turn, depends on electrode design and its dimensions. In the case of a sensor with parallel-plate electrodes, the volume corresponds to the space between the plates where the field is almost uniform. Consequently, the problem is a purely geometric and electrostatic one independent of the instrument technique, even for TDRs (see White, Knight, Zegelin, and Topp (1994) for a justification). From now and thereafter the article does not distinguish sensors according to their technique but their geometry. The larger is the probe the more important is the sample volume. On the other hand, field use and commercially-oriented sensors impose some constraints on the probe geometry, such as compactness for ease of insertion, which restrict the spacing between electrodes, and simplicity of fabrication. For these reasons our probe consists of two parallel cylindrical electrodes or rods, a geometry extensively used among other sensors along with the three-rod design (in particular for TDRs) and the configuration of annular electrodes on a same cylinder. The latter design may however result in a low volume of influence. The present study focuses on the two-rod configuration, with some remarks on the three-rod design. To meet both theoretical and practical demands an optimization of electrode geometry in function of the volume of influence is necessary. The volume is determined from the distribution of sensor sensitivity to the local permittivity at each part of the medium. The distribution is directly given by a fundamental relation between probe capacitance and its electrostatic energy expressed as an integral over the medium space. Accordingly, we base our study on this relation, as did previous authors on the same subject.

Among studies dedicated to derive theoretically and/or experimentally the extent and localization of the volume sampled by the sensor, we can mention the laboratory tests carried out by Baker and Lascano (1989) using a two-rod design. Their setup consisted in a row of water-filled tubes parallel to the probe axis and placed successively at different points in the sensor transverse plane to draw profiles of sensor sensitivity. Probes present a main axis, which coincides to electrode axis and usually to the largest dimension, and to which the electric field direction is mostly perpendicular. Variation of field amplitude is also much lower along the axis than in the transverse direction. As a result, the sample volume is approximately proportional to probe length. Studies thus largely focused on the section of the sample volume in the transverse plane, or sample area. Baker and Lascano (1989)’s setup was criticized by Knight (1991) as unable to derive the sample volume. By using a row of tubes they incorrectly assumed that the sample area depends separately on the two coordinates of the transverse plane. Knight (1992) laid a theoretical basis to obtain the sample volume of a sensor. From the expression of the electrostatic energy he introduced a spatial weighting function to measure the local sensor sensitivity. Using analytical expressions obtained by some simplifications he attempted to determine the best dimensions of a two-rod probe. However, his sample volume remained ill-defined and his conclusion was elusive suggesting without clear justification a distance between rods lower than 10 times their diameter. As a matter of fact most sensors using a multi-rod design present a geometry close to this ratio (such as TDRs, or GS3 by Decagon Devices Inc., USA). Interestingly, Knight (1992) demonstrated that small disturbances from a uniform permittivity have only a second-order effect on the electrostatic energy.

As a result of advances on ICT, numerical simulations were progressively used to solve the electrostatic problem associated with these sensors, even in three dimensions, specifically to study the influence on the apparent permittivity and sample volume of complex electrode geometries and particular distributions of medium permittivity (Bolvin, Chambarel, and Chanzy (2004); De Rosny et al. (2001); Knight, Ferré, Rudolph, and Kachanoski (1997); Robinson and Friedman (2000); Zhan, Mu,
and Chen (2016)). In particular, had been examined the impact of critical disturbances such as air-filled gaps or coating around electrodes, stones, roots... The simulations solved Laplace’s equation to deduce the electrical field in the medium and hence the energy density, or, in most cases, to derive directly the sensor apparent permittivity via the boundary conditions at electrode surfaces. The latter approach, at the difference of a work on the sample volume, permits a direct experimental test for which electrode geometry and permittivity distribution are reproduced in laboratory.

Among investigators looking at the sample volume and its dependence, Ferré, Knight, Rudolph, and Kachanoski (1998) assessed numerically Knight (1992)’s weighting function in the case of complex permittivity distributions around two or three-rod probes. The sample volume was then defined as regions of greatest sensitivity, or weight, and contributing to a fraction of sensor response such as 90% or 50%. In particular, the study showed that a coating or an air gap around rods reduce the sample area and enhance the contribution of the heterogeneity, all the more since their thickness becomes important relative to rod diameter and they present a permittivity smaller than the surrounding medium. One of their conclusions is that the bare two-rod configuration represents a better design to probe the medium than a three-rod configuration or with a coating. However, the dependence of the sample volume on probe dimensions were not fully investigated, being limited to a rod diameter at most one fifth of the rod spacing. Some experimental validations were tentatively provided by Nissen, Ferré, and Moldrup (2003).

Past studies focused on sensor sample volume appear to support our choice of a simple two-rod probe relative to more complex ones, but in cases of geometry limited to thin electrodes. We question this choice, very common in available sensors, as electrode diameter should also play a role in the extent and localization of the sample volume. Our main objective is the determination of the optimum probe dimensions for a better use of dielectric medium models to convert sensor apparent permittivity into medium water content (and salinity), while taking into account practical constraints. The models assume a quasi-homogeneous medium, or a uniform permittivity at the instrument scale. This hypothesis is supported by Knight (1992)’s work showing the second-order influence of small disturbances on the electrostatic energy and, therefore, sample volume. As a result, our approach assumes the condition of use of soil models, which brings some simplifications. We perform however an ex-post validation in which the result consistency with the assumption is verified. With regard to more important perturbations such as air gaps around electrodes, it is not possible during a field trial to know their existence and, if present, their shape and localization to assess their impact. We can only hope that careful sensor introduction reduces the risk of disturbances around the probe. Usually they diminish with time. In the present work, we verify that the deduced sample volume is not too localized in medium area where a disturbance have a much higher impact than elsewhere, such as around electrodes.

The simplifications introduced in the approach lead to analytical expressions, which permit to study the characteristics of the sample volume in function of probe dimensions over their complete range. For this type of comprehensive study, an analytical
expression is more convenient than its numerical simulation counterpart. Each change of electrode dimensions or medium permittivity requires to redefine the domain of simulation and perform another computation, in the same way as a laboratory experiment. Numerical simulations are not as well exempt of some defaults: they introduce their own artifacts (discretization of the fundamental equations, meshing close to electrode edges, conditions at the domain boundaries...), which necessitate the validation of the model against an analytical resolution in cases where it is available (see for instance Knight et al. (1997) for gaps around electrodes).

The next section provides the theoretical framework to determine the sample volume and its dependence. Our approach, although sharing a common premise with Knight (1992)’s work, will differ markedly to obtain a precise and comprehensive characterization. It also permits a direct comparison of the theoretical results with a laboratory validation. Experimental setup and results are presented in the third section. Outputs from the theoretical and laboratory works, as well as those from past studies, are discussed to draw a general conclusion about the sample volume of sensors with respect to rod geometry.

2. Theory

We first determine a quantity at point $r$ of the medium which measures the influence of this point to the apparent permittivity or, interchangeably, the sensor sensitivity to the local permittivity at this point. It is constructed from the density of sensor electrostatic energy in the medium, assuming the validity of the electrostatic theory. Sensors generate an alternating field at frequency $f$. However, the propagation effect in the case of capacitance sensors is negligible provided the associated wavelength is larger than probe size (Chavanne and Frangi (2017)). As a side effect, the sample volume does not depend on the field frequency $f$ (on the other hand, the apparent permittivity at same water content can vary with $f$ (Revil (2013)). But it is not the subject of the article).

Sensitivity derived from energy density decreases with the distance from electrode surfaces. The sample volume is thus defined as medium parts at which sensitivity presents a higher value than a certain cutoff.

With the hypothesis of uniform permittivity and in the case of a two-rod design we obtain analytical expressions of the field, hence of sensor sensitivity, in function of probes dimensions.

2.1. Spatial distribution of sensor sensitivity

2.1.1. Sensor Electrostatic Energy and Apparent permittivity

The electrostatic energy $W_{el}$ of a capacitor can be expressed with the two following formulas (see appendix A.1 for further details):

$$W_{el} = \frac{\varepsilon_0}{2} \int_{\text{medium}} \varepsilon_r E \cdot E \, d^3r,$$

$$= \frac{1}{2} CU^2. \quad (1)$$

where $E \cdot E$ is the amplitude squared of capacitor electric field $E$, $C$ the capacitance.
of the capacitor and $U$ the voltage applied between its electrodes. The integral in the first expression is applied over the whole medium.

The voltage $U$ is fixed by an external source and the capacitance $C$ is proportional to the apparent permittivity $\varepsilon_r$ with a constant geometric factor. Hence, we have a relation of proportionality between $\varepsilon_r$ and the local permittivity at point $r$:

$$\varepsilon_r \propto \int_{\text{medium}} \varepsilon_r(r) |\mathbf{E}|^2 d^3r.$$ (2)

As a result, the influence of the medium part in the apparent permittivity at point $r$ is fixed by the value of $|\mathbf{E}|^2$ at the point.

2.1.2. Assumption of uniform permittivity

In general the field $\mathbf{E}$ depends on the spatial distribution of medium permittivity through Laplace’s equation, $\nabla \cdot (\varepsilon_r(r) \mathbf{E}(r)) = 0$. Assuming a uniform permittivity leads to a simple resolution of the equation to obtain the spatial variation of $\mathbf{E}$ and the energy density (in the case of a two-rod design these quantities remain non-uniform). Justifications for the assumption were provided in introduction. At least for usual porous media with small perturbations the present analysis is sufficient.

On the other hand, it appears from previous studies that some disturbances do not have negligible effect on the apparent permittivity and the sample volume (such as those studied in Bolvin et al. (2004); Knight et al. (1997)). The effect can result from a large size, but not only. It also arises from low permittivity values - see the difference of impact between air-filled gaps and water-filled ones in Knight et al. (1997) - and a localization in area of high energy density. These effects can be understood with simple models derived from the electrostatic theory. A gap around an electrode is roughly equivalent to a capacitor in series with the capacitor associated with the surrounding medium. The resulting capacitance is dominated by the lowest of the two ones (The inverse is the sum of the inverses). At same gap thickness, because of its lower permittivity, air have a stronger influence than a water-filled gap. When applying the boundary conditions to the electric field $\mathbf{E}$ at the interface between the air gap (g) and the medium (m) with a higher permittivity $\varepsilon_{rm}$ - $n \cdot (\varepsilon_r \mathbf{E}_g - \varepsilon_{rm} \mathbf{E}_m) = 0$ with $n$ the unit vector normal to the interface, and assuming the interface parallel and close to electrode surface -, we have $\mathbf{E}_m = \mathbf{E}_g / \varepsilon_{rm}$. The electric field, and therefore the energy density, appear thus concentrated in the air-filled gap. The boundary conditions also explain that, in experiments with a moving liquid/air interface, contributions to permittivity when the interface is at the electrode level are concentrated in the low permittivity regions close to electrodes (Nissen et al. (2003)). The same air disturbance localized far from electrodes has a much lower effect.

By the choice of the sensitivity indicator, our approach, even with its simplifying assumption, can take into account these critical area with high energy density.

2.1.3. Sensor sensitivity from normalized energy density

The actual contribution of the local permittivity at point $r$ to apparent $\varepsilon_r$ is not fixed by the absolute value of $|\mathbf{E}|^2$ but by a relative one. Indeed, if probe voltage $U$ is modified, the density $|\mathbf{E}|^2$ varies accordingly, but $\varepsilon_r$ remains unchanged (in absence of any instrument bias). As a result, to assess the true contribution of a medium region, and to enhance as well the role of area of high density, the sensitivity is defined as $|\mathbf{E}|^2$ normalized by its maximum value encountered in the medium, $E_{max}^2$. The closer
to $E_{\text{max}}^2$ is the squared field at a medium part, the larger is the sensor sensitivity to that part and, consequently, the higher is its probability to be included in the sample volume.

The sample volume is then defined as all the medium points where $|E|^2$ is larger than a fraction $\eta$ of $E_{\text{max}}^2$ (with $\eta < 1$ by definition). The parameter $\eta$ represents an arbitrary sensitivity cut-off. The subsequent work consists in determining the contours $|E|^2/E_{\text{max}}^2 \equiv \eta$, or points of equi-sensitivity of the sensor, in function of electrode dimensions. They coincide to the outer boundary of the sample volume, the inner one being the surface of electrodes. The parameter $\eta$ is also varied to control its influence.

This normalization choice permits to enhance the maximal contribution to apparent permittivity, corresponding to the most critical area in terms of disturbances from a uniform permittivity.

### 2.2. The sample Volume in the case of a two-rod probe

#### 2.2.1. Dimensionless Analysis and other simplifications

The probe under study corresponds to a capacitor made of two parallel cylinders of height $h$, axis distance or spacing $D$ and diameter $\Phi$ (see Figure 1 in Chavanne and Frangi (2014)). Hence the sample volume depends on these three geometric variables. As noted in introduction we can separate the dependence on the dimension along the main axis, $h$, from that on the two others. This is equivalent to exclude fringing effects, or to assume that the capacitor electrical field is independent of $h$ and ideally parallel to the transverse plane. Fringing effects are either eliminated using guards or made negligible by using electrodes with a height larger than distance $D$ (from numerical simulations corroborated by some laboratory tests a corrective factor of 1.06 allows to account for the effect when $h = 4D$). Field attenuation occurs along probe axis and increases with $h$ and medium conductivity $\sigma$. However, it remains small for $h \leq 5$ cm and $\sigma$ below 1 S·m$^{-1}$.

The volume $\Gamma_\eta$ of a medium sampled by the sensor can be thus expressed as:

$$\Gamma_\eta = h \cdot S_\eta(D, \Phi),$$

where $S_\eta$ is the volume cross-section in the transverse plane.

From the definition of the sample volume in the previous section the surface $S_\eta$ corresponds to the points $(x,y)$ in the transverse plane where the quantity $|E|^2$ is larger than a fraction $\eta$ of $E_{\text{max}}^2$. Because physical phenomena do not depend on arbitrary units, the variation of $S_\eta$ with $D$ and $\Phi$ can be given by:

$$S_\eta(D, \Phi) = (D/2)^2 S^*_\eta(\alpha),$$

where $S^*_\eta$ is a dimensionless quantity only function of the geometric ratio $\alpha = \Phi/D$ ($\alpha < 1$).

From Equations 3 and 4 the sample volume $\Gamma_\eta$ becomes:

$$\Gamma_\eta = h (D/2)^2 S^*_\eta(\alpha).$$

The work to determine the dependence of $\Gamma_\eta$ on probe dimensions is thus reduced to
study the variation of a function, $S^*_\eta$, to its sole variable, $\alpha$. The choice of $D$ instead of another length such as $\Phi$ to make the surface $S_\eta$ dimensionless does not diminish the scope of the study. The use of $\Phi$ instead of $D$ would have lead to another function of the same ratio, $S^*_\eta(\alpha)$, such as $S_\eta(D, \Phi) = (\Phi/2)^2 S^*_\eta(\alpha)$. However, with the previous function we have the relation $S^*_\eta = (D/\Phi)^2 S^*_\eta$.

2.2.2. Expression of sensor sensitivity

The quantity $S^*_\eta$ requires the expression of field $\mathbf{E}$. In the case of the two-rod design it is established by the method of charge image (or principle of analogous problems in electrostatic theory as explained in chapter 6 of Feynman, Leighton, and Sands (1964)). Indeed, the problem of two cylindrical conductors centered respectively in $O$ and $O'$ with opposite charges per unit of height, $q$ and $-q$, is equivalent to the problem of two lines of same charges parallel to cylinder axes and located respectively in $A$ and $A'$ (see the cross section in Figure 1 for conductor and line positions), which is much easier to solve. The complete resolution is done in the appendix A.2. Only the results are given here.

In order to determine the normalized surface $S^*_\eta$, coordinates $x$ and $y$ are made dimensionless using the length $D/2$, $x^* = 2x/D$ and $y^* = 2y/D$. The expression of the squared amplitude of the electric field is:

$$\mathbf{E} \cdot \mathbf{E} = \left( \frac{q}{4 \pi \epsilon D/2} \right)^2 \left[ \frac{2(x^* + \sqrt{1 - \alpha^2})}{r_+^2} - \frac{2(x^* - \sqrt{1 - \alpha^2})}{r_-^2} \right]^2 + \left( \frac{2y^*}{r_+^2} \frac{2y^*}{r_-^2} \right)^2,$$

where $r_+^2$ and $r_-^2$ are:

$$\begin{cases} r_+^2 = \frac{r_+^2}{(D/2)^2} = y^* + (x^* + \sqrt{1 - \alpha^2})^2, \\ r_-^2 = \frac{r_-^2}{(D/2)^2} = y^* + (x^* - \sqrt{1 - \alpha^2})^2, \end{cases}$$

with $r_+$ and $r_-$ the distances between $A$ and $M$, and between $A'$ and $M$, respectively.

Figure 1. Cross-section of probe electrodes and symbols used for positions and distances. $A$ and $A'$ are the positions of the two lines with same charges and with same potential at point $M$ as for the electrodes.
The maximal value, $E_{\text{max}}^2$, is reached at $x^* = \pm (1 - \alpha)$ and $y^* = 0$, and equals:

$$E_{\text{max}}^2 = \left( \frac{q}{4 \pi \epsilon D/2} \right)^2 \frac{4(1 + \alpha)}{\alpha^2(1 - \alpha)}. \tag{8}$$

Hence the determination of the sample cross-section $S^\eta_*$ in Equation 4 is equivalent to deduce the surface of points $x^*$ and $y^*$ such as the ratio:

$$\frac{|E|^2}{E_{\text{max}}^2} = \left( \frac{\alpha^2(1 - \alpha)}{4(1 + \alpha)} \right) \left\{ \frac{2(x^* + \sqrt{1 - \alpha^2})}{r_+^2} - \frac{2(x^* - \sqrt{1 - \alpha^2})}{r_-^2} \right\}^2 + \left\{ \frac{2y^*}{r_+^2} - \frac{2y^*}{r_-^2} \right\}^2 \tag{9}$$

is larger than a sensitivity cut-off $\eta$ (paragraph 2.1).

At the probe center the ratio in Equation 9 becomes:

$$\frac{|E_0|^2}{E_{\text{max}}^2} = \frac{4\alpha^2}{(1 + \alpha)^2}. \tag{10}$$

The quantity decreases with the ratio $\alpha$. When $|E_0|^2/E_{\text{max}}^2$ is lower than the cut-off $\eta$, the probe center is outside the sample volume. For $\alpha = 0.50$ the ratio amounts to $|E_0|^2/E_{\text{max}}^2 = 0.44$, while at $\alpha = 0.20$ it is $|E_0|^2/E_{\text{max}}^2 = 0.11$.

2.2.3. Graphical resolution

2.2.3.1. Volume shape and localization. A programming tool such as Matlab is used to plot the contours of sensor sensitivity (Equation 9) in the $x^* y^*$ plane for three cutoffs, $\eta = 0.40, 0.25$ and 0.10. A contour coincide to the outer boundary of the sample surface $S^\eta_*$ for a fixed cutoff and ratio $\alpha$. The process is repeated for four values of $\alpha$ between its limits, 0 and 1, as shown in the graphs of Figures 2(a), 2(b), 2(c) and 2(d). The enclosed surfaces evolve from ring-like shapes around electrodes at low $\alpha$, to a strip between electrodes at large $\alpha$. From Figures 2(b) and 2(c) the transition takes place at a value of $\alpha$ slightly above 0.25, although it depends partly on the choice of the cutoff.

2.2.3.2. Volume size. Variation of $S^\eta_*$ with the ratio $\alpha$ are then assessed at a fixed cut-off $\eta$. The graphs in Figures 3(a) and 3(b) present the result for respectively $\eta = 0.25$ and $\eta = 0.10$. Table 1 provides some data on the function $S^\eta_*(\alpha)$.

Not surprisingly, the volume extent depends on the value of $\eta$. However, independently of $\eta$ value the maximum of $S^\eta_*$ is reached for $\alpha$ between 0.30 and 0.50. The surface decreases rapidly outside this range. Volume $\Gamma_\eta$ in Equation 5 will present the same behaviour at fixed dimensions $D$ and $h$, demonstrating the importance also of the electrode diameter.

For a complete validation, the result is compared with two extreme cases for each of which an analytical expression of the surface $S^\eta_*$ is derived.

2.2.3.3. Case of distant or thin electrodes. The case corresponds to small values of $\alpha$. The points O and A, as well as O' and A', in Figure 1 nearly coincide (see also Equation A11). This is equivalent to consider the two electrodes as independent from each other (it is approximately the situation in Figure 2(a)). Determination of the
In each figure they are assessed for three sensitivity cut-offs, \( \eta = 0.40 \) (filled with magenta color), \( \eta = 0.25 \) (purple) and \( \eta = 0.10 \) (blue).

**Figure 2.** Contours \( (|E|^2/E_{max}^2)_{\alpha} (x^*, y^*) \equiv \eta \) in the plane \( x^* = 2x/D, y^* = 2y/D \) for four ratios \( \alpha = \Phi/D \) from 0.12 to 0.85. \( D \) is the spacing of probe electrodes and \( \Phi \) their diameter. The function \( |E|^2/E_{max}^2 \) is expressed by Equation 9. Electrode surfaces and contours delimit the sensor sample area \( S_{\eta}(\alpha) \) (Equation 4).

### Table 1. Values of the function \( S_{\eta}(\alpha) \) when maximal, at \( \alpha = 0.12 \) and at \( \alpha = 0.85 \), according to the cut-off \( \eta \).

<table>
<thead>
<tr>
<th>( \eta )</th>
<th>0.4</th>
<th>0.25</th>
<th>0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha_{max} )</td>
<td>0.52</td>
<td>0.47</td>
<td>0.37</td>
</tr>
<tr>
<td>( S_{\eta}(0.12) )</td>
<td>0.10</td>
<td>0.22</td>
<td>0.64</td>
</tr>
<tr>
<td>( S_{\eta}(0.85) )</td>
<td>0.32</td>
<td>0.40</td>
<td>0.88</td>
</tr>
</tbody>
</table>

The expression of the field \( \mathbf{E} \) is carried out by only considering one electrode, for instance the one centered at \( O' \):

\[
\mathbf{E}(M) = \frac{-q}{2 \pi \epsilon r_-} \mathbf{U}_r
\]

where \( \mathbf{U}_r \) is the unity vector along the direction of \( r_- \).

The field amplitude is maximum at the surface electrode with \( E_{max} = 2q/(2 \pi \epsilon \Phi) \).
Hence the local sensitivity according to our definition is:

$$\frac{|E|^2}{E_{max}^2} = \frac{(\Phi/2)^2}{(r_-)^2}. \tag{12}$$

A contour of the function in the $xy$ plane for a cutoff $\eta$ is a circle centered in $O'$ with a radius $r_- = \Phi/2/\sqrt{\eta}$. Taking into account the second electrode, the sampled surface $S_\eta$ is $S_\eta = 2 \left[ \pi r_-^2 - \pi (\Phi/2)^2 \right] = 2 \pi (1/\eta - 1) (\Phi/2)^2$.

Hence the normalized surface $S_\eta^*$ in Equation 4 for this extreme probe geometry is:

$$S_\eta^* = 2 \pi (1/\eta - 1) \alpha^2. \tag{13}$$

The equation is reported in Figures 3(a) and 3(b) for low values of $\alpha$. It agrees with the general function $S_\eta^*$ obtained from the graphical resolution, especially at $\eta = 0.1$.

### 2.2.3.4. Case of close electrodes.

A simple adjustment of $S_\eta^*$ is also obtained for $\alpha$ close to 1. The surface converges to that of a parallel-plate capacitor (Figure 2(d)). Hence it is approximated by the product of the distance between electrodes, $x_{max}^* = 1 - \alpha$, with the distance between the two points of contour $|E|^2/E_{max}^2 \equiv \eta$ in the $y^*$ axis. The expression of their absolute coordinate $y_\eta^*$ is deduced from Equation 9:

$$\frac{|E|^2(x = 0)}{E_{max}^2} = \frac{\alpha^2(1 - \alpha)}{(1 + \alpha)} \frac{4(1 - \alpha^2)}{\left\{y_\eta^* + (1 - \alpha^2)\right\}^2} \equiv \eta,$$

which results in:

$$y_\eta^* = (1 - \alpha) \left(\frac{2\alpha}{\sqrt{\eta}} - \alpha - 1\right). \tag{14}$$
For $\alpha$ close enough to 1 this quantity is always positive, and the function $S^*_\eta$ is approximated by:

$$S^*_\eta \simeq 4 |y^*_\eta x^*_\eta_{\text{max}}| = 4 (1 - \alpha)^{3/2} \sqrt{\left( \frac{2}{\sqrt{\eta}} - 1 \right) \alpha - 1}.$$ (15)

The equation agrees with $S^*_\eta$ obtained from the calculus at values of $\alpha$ close to 1 (Figure 3(b)).

3. Laboratory Validation

3.1. Setup

![Cross-section drawing of the experimental setup. The sensitivity distribution of a sensor in air is measured owing to a water-filled tube successively placed at different points M of the transverse plane. Points are reported with the dimensionless coordinates $x^* = 2x/D$ and $y^* = 2y/D$, where $D$ is the electrode spacing.](image)

A vertical water-filled tube was used as a test probe of the spatial distribution of sensitivity of a two-rod probe in air (Figures 4 and 5). At the difference of a row of tubes (see the critic of Baker and Lascano (1989)’s setup by Knight (1991)), a sole tube at different points M in probe transverse plane permits to deduce the sensitivity dependence on the two plane coordinates $x$ and $y$. The tube is also considered a second-order disturbance of the energy density $|E|^2$, as demonstrated in Knight (1992), provided it is thinner than electrodes. As a result, our setup permitted to reconstruct the general quantity in Equation 6. The result is independent of the choice of the surrounding medium, air here, within the assumption of an homogeneous medium (see details further down). Two geometries of sensor electrodes were investigated: $\alpha = 0.20$, or an electrode spacing fivefold the electrode diameter (distance $D = 250$ mm), and $\alpha = 0.50$, or a spacing twice the diameter (distance $D = 100$ mm). Tube diameter was chosen small enough relative to electrode diameter to reduce the smoothing effect of its size on the sensitivity distribution. A big tube overestimates contribution of area with low sensitivity. A small diameter also permits to reduce the gap due to tube radius between electrode surface and the actual end point of profile measurements. In our setup the inner diameter of the tube was 14.4 mm, against 50 mm for electrode diameter. Contribution of glass wall was disregarded owing to its low thickness - 1 mm - and permittivity - $\varepsilon_r = 5$ - against water one. On the other hand the tube section was large enough for a good signal to noise ratio from sensor. Tube length was 180
mm, larger than the height of a sensor capacitor (Figure 5). Micrometers permitted to position the tube in the transverse plane with an accuracy better than 0.025 mm. At each position $r$ of the tube center a value of the capacitance $C(r)$ was measured by the sensor.

### 3.2. Sensitivity distribution from measurements

From Equation 1 the measured capacitance can be expressed as:

$$C = \varepsilon_0 \frac{U^2}{L^2} \int_{\text{medium}} \varepsilon_r \mathbf{E} \mathbf{E} d^3r = \varepsilon_0 \frac{U^2}{L^2} \left[ \int_{\text{air}} - \int_{\text{tube}} \mathbf{E} \mathbf{E} d^3r + \varepsilon_{rw} \int_{\text{tube}} \mathbf{E} \mathbf{E} d^3r \right]. \quad (16)$$

A measurement was also performed without the tube, $C_{\text{air}}$. The value was subtracted to all subsequent measurements with the tube. Besides, the maximum of the resulting series, $C_{\text{max}} - C_{\text{air}}$, was used to normalize it. Hence, from Equation 16 we derive a normalized capacitance, $(C)_{\text{nrmlzd}}$:

$$(C)_{\text{nrmlzd}} = \frac{\int_{\text{tube}} \mathbf{E} \mathbf{E} d^3r}{(\int_{\text{tube}} \mathbf{E} \mathbf{E} d^3r)_{\text{max}}} \approx \frac{|\mathbf{E}|^2}{E_{\text{max}}^2}. \quad (17)$$

The integral $\int_{\text{tube}} \mathbf{E} \mathbf{E} d^3r$ is approximated by $S_w |\mathbf{E}|^2(r)$ where $S_w$ is the tube inner section. Data thus normalized closely match the density ratio, or sensor local sensitivity, in Equation 9. Hence profiles of sensor sensitivity are obtained in sensor transverse plane for a probe geometry.

As the uncertainty on raw data $C$ is mostly absolute, $\delta C$, uncertainty on $(C)_{\text{nrmlzd}}$ is approximately the ratio of $\delta C$ to the maximal value $C_{\text{max}} - C_{\text{air}}$. With our sensor...
accuracy and experimental setup it amounts to $\pm 0.025$.

Measurements $(C)_{\text{nrmlzd}}$ along lines of the $x^*y^*$ plane are reported in Figures from 6 to 7 for $\alpha = 0.20$ and from 8 to 9 for $\alpha = 0.50$. If we exclude the data gap due to radius of the test tube, position of the maximum is close to the expected point of highest energy density (the point at electrode surface and along the line joining both electrodes). As the point is not exactly reached, normalized measurements $(C)_{\text{nrmlzd}}$ overestimate a little the sensitivity with respect to the theoretical results in paragraph 2.2.3. For the sake of convenience data in Figures from 6 to 9 have been replicated in the negative coordinates taking into account probe symmetry.

**Figure 6.** Profile along the line $x^* = 1$ of sensor sensitivity in the dimensionless $x^*y^*$ plane, obtained from laboratory measurements (see setup in Figures 4 and 5). Geometric ratio of electrodes is $\alpha = \Phi/D = 0.20$. Vertical lines represent electrode boundaries and axis. Data have been replicated in the negative coordinates by symmetry.

**Figure 7.** Profile of sensor sensitivity along $x^*$ axis ($y^* = 0$) for geometric ratio $\alpha = 0.20$. 

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4. Results and Discussion

4.1. Comparison between theoretical and laboratory results

Experimental profiles of sensor sensitivity in Figures from 6 to 9 are compared with the theoretical contours of sensitivity shown in Figures from 2(a) to 2(d). For the low ratio $\alpha = 0.20$, sensitivity profiles decrease from a maximum at electrode surface to 0 over a distance close to electrode diameter (Figures 6 and 7). The rate of decrease is nearly independent of the direction in the $x^*y^*$ plane, which implies a low influence of one electrode to the other. A ring-like distribution similar to the ones visible in Figure 2(a) or in Figure 2(b) is thus observed. Experimental results show even a more symmetrical distribution around electrodes than those seen in Figure 2(b). Besides, sensitivity value at probe center is close to zero, at least much lower than 10% of the maximum. These observations suggest that experiment is more in agreement with model outputs obtained for a cutoff $\eta$ around 0.25. This is also confirmed by results from the other geometry investigated.

![Figure 8](image)

**Figure 8.** Profile along $x^*$ axis ($y^* = 0$) of sensor normalized sensitivity (see Figure 6 for explanations). Geometric ratio of electrodes is $\alpha = 0.50$.

![Figure 9](image)

**Figure 9.** Same setup as in Figure 8 with sensitivity profile along line $x^* = 1$. Geometric ratio of electrodes is $\alpha = 0.50$.

For $\alpha = 0.50$ sensitivity distribution is not any more symmetrical (Figures 8 and 9).
Like the theoretical distributions presented in Figure 2(c), high sensitivity is located in the space between electrodes. The agreement between both studies departs only for points at the outer side of the probe; experimental data show a lower decline with distance.

According to these results, sensitivity distribution evolves with electrode geometry between two distinct configurations. For ratios $\alpha$ smaller than at least $\alpha = 0.25$ - corresponding to thin electrodes - it is concentrated around each electrode, whereas it tends to be confined between electrodes for larger ratios. It should be noted that if the spacing $D$ is increased under the assumption that larger $D$ produces larger sample volume, but retaining the same diameter, the gain is very small. Size of the sample volume becomes controlled by electrode diameter (and their height). Diameter must be thus much larger than medium intrinsic heterogeneities, pores or grains, in order to work with the dielectric models mentioned in introduction. Measurements using thin electrodes are more sensitive to perturbations resulting from probe introduction in the medium.

On the other end, above $\alpha = 0.50$, or thick electrodes, the size of the sample volume at fixed $D$ decreases markedly. Consequently, according to our methodology optimum probe geometry is obtained with $\alpha$ between 0.3 and 0.5.

4.2. Comparison with previous works

Our laboratory results can be compared to Baker and Lascano (1991)’s ones after the authors modified their setup, following Knight (1991)’s comments. With a two-rod probe of geometric ratio $\alpha = 0.065$, they obtained sensitivity contours qualitatively similar to ours in the case of a ratio $\alpha = 0.20$. They noted a decrease of sensitivity between rods steeper than the one observed in their previous and incorrect setup (Baker and Lascano (1989)). Besides, the diameter of their test tube was close to the rod one (3.2 mm), which had probably a smoothing effect overestimating low sensitivity areas.

Numerical works carried out by Ferré et al. (1998), and subsequent articles, to assess the sample area of a probe were based on a formula of sensitivity distribution different from Equation 9. They normalized the energy density with its integral over the whole domain instead of its maximum, as in our study. The sample volume was then introduced as the sum of regions of greatest sensitivity which contributes to a fraction $f$ of the overall energy, such as $f = 90\%$ or $f = 50\%$. This difference impedes a direct comparison with our results. Moreover, geometric ratios of the sensor investigated were restricted to $\alpha = 0.10$ and $\alpha = 0.20$. Sample areas defined with the fraction $f = 90\%$ appear larger than those in Figure 2(a) defined with a cut-off of sensitivity at 10%. Areas contributing to a fraction $f = 50\%$ of the response better agree with our results, and the ring like patterns around electrodes are visible. Areas to a fraction $f = 90\%$ correspond seemingly to a sensitivity cut-off lower than 10% with our definition. Extensive numerical works made by Zhan et al. (2016) with a three-rod probe confirm this conclusion. They investigated the effect of various sensor dimensions on the sample area using the definitions of Ferré et al. (1998). As a side remark, they did not take advantage of a dimensionless analysis to work with a unique variable such as $\alpha$, except to study the importance of confined area around electrodes (the ”skin effect”). As a consequence, some of their probes have the same ratio $\alpha$ using different dimensions and, as a result, present exactly the same shape of sensitivity distribution. Ratios of their simulations ranged from $\alpha = 0.075$ to $\alpha = 0.40$. Sample areas defined
by the fraction \( f = 50\% \) of sensor response clearly show the evolution from a ring-like pattern around each rod to a band between them.

Laboratory measurements to test indirectly the numerical results in Ferré et al. (1998) were carried out with a liquid/air interface at different distances from probe center (Nissen et al. (2003); Zhan et al. (2016)). They were performed with two-rod and three-rod probes of which geometric ratio varies from \( \alpha = 0.13 \) to \( \alpha = 0.20 \). Data on the apparent permittivity show a rapid change from air permittivity to liquid one when the interface crosses the rods up, and the reverse when going down. The change is not symmetric to probe plane due to the concentration of sensor sensitivity in the low permittivity region, air here. Nevertheless, interface displacement over which the change occurs is better related to rod diameter than their spacing. This result seems to confirm the confined nature of sensitivity around electrodes, which agrees with our results.

![Figure 10. Profiles of the normalized sensitivity of GS3 sensor in the dimensionless \( x^* y^* \) plane. Sensor geometry has a ratio \( \alpha = 0.13 \) with an electrode diameter of 3.15 mm (= 0.26 in \( x^* y^* \) plane).](image)

**4.3. Assessment of the sample volume of some two-rod and three-rod sensors**

Paragraph 2.2.3 provides graphics and formulas to assess the size of the sample volume according to the probe dimensions, in particular Equation 5, Figure 3(b) and table 1. With a sensitivity cut-off \( \eta \) of 10\%, the sensor presented in Figure 5, with a ratio \( \alpha = 0.50 \) a spacing \( D = 100 \) mm and a height per channel of \( h = 50 \) mm, develops a sample volume of about \( \Gamma_{0.1} = 0.40 \) l. Maintaining the same diameter but increasing the spacing would be beneficial: at \( \alpha = \Phi/D = 0.33 \) with \( D = 150 \) mm the sample volume becomes \( \Gamma_{0.1} = 0.90 \) l. On the other hand, sensitivity at probe center would be lower.

By comparison the capacitance sensor GS3 of Decagon Devices Inc. with the same two-rod design has a ratio \( \alpha = 0.13 \), an electrode spacing \( D = 25 \) mm and an height \( h = 50 \) mm. Its sample volume is about \( \Gamma_{0.1} = 0.5 \times (0.125)^2 \times 0.64 = 0.005 \) l. As a result of both low spacing and ratio \( \alpha \) the volume is 80 times as low as the volume for our
sensor. It is also confined around electrode surfaces, as shown in Figure 10 (data are obtained with the identical setup as the one described in Figure 5; the test tube had a diameter of 2 mm and the uncertainty on sensitivity was about ±0.075), which can impair medium representativity to assess its moisture and conductivity.

The three-rod TDR CS635 commercialized by Campbell Scientific presents a ratio \( \alpha = 0.20 \) with a distance between close electrodes of \( D = 15.5 \) mm and a height of \( h = 150 \) mm. To account for the three-rod design, we assume that the sample volume is proportional to the number of rods. This is probably correct for low \( \alpha \), due to field at each electrode acting independently, but uncertain for larger ratios. Assuming it is the case, the volume deduced from Equation 5 and the hypothesis amounts to \( \Gamma_{0.1} = \frac{3}{2} \left(0.155\right)^2 \times 2.1 = 0.11 \) l. A large height and sufficiently close rods compensate for their thin diameter and make the sample volume larger than the one for GS3 probe. However, like GS3, rod diameter is small, few mm.

5. Conclusions

The article has described a theoretical work, along with a laboratory validation, to establish the dependence of the medium volume sampled by a permittivity-based sensor on its dimensions over their whole range. The volume is determined from the spatial distribution of sensor sensitivity to the local permittivity around sensor electrodes. The local sensitivity is assessed with the density of the electrostatic energy generated by the sensor, or the squared amplitude of its electric field in the medium, \( |E|^2 \). In the present work the sensitivity is defined as the quantity \( |E|^2 \) normalized by its maximal value encountered in the medium, due to the critical importance of this contribution in the case of a large medium disturbance. The sample volume corresponds to medium parts where sensor sensitivity is higher than a cut-off of the maximum, such as 10% or 40%. The normalized quantity \( |E|^2 \) is determined analytically in the case of the common two-rod design for electrodes, with rod height \( h \), spacing \( D \) and diameter \( \Phi \). By neglecting fringing effects and performing a dimensionless analysis, the sample volume \( \Gamma \) is expressed as \( \Gamma = h (D/2)^2 S^* \), where \( S^* \) is only function of the geometric ratio \( \alpha = \Phi / D \). \( S^* \) corresponds to a dimensionless surface in the plane perpendicular to electrodes. It is delimited by electrode surface and medium points where sensitivity value reaches the cut-off. The surface \( S^* \) shape, localization and size have been thereafter assessed for different values of \( \alpha \) over its whole range.

Beside, using a water-filled tube placed at different positions around the electrodes of a sensor in air, sensitivity profiles have been experimentally drawn in the cases of \( \alpha = 0.20 \) and \( \alpha = 0.50 \).

Both works have shown the existence of two configurations of the sample volume depending on the ratio \( \alpha \). For \( \alpha \) lower than at least 0.25, that is for distant or thin electrodes, the volume is concentrated around electrodes with a thickness controlled by electrode diameter all the more as \( \alpha \) is small. On the other hand, at larger \( \alpha \) the volume tends to be confined between electrodes, like in the case of a parallel-plate capacitor. It is thus better defined and less subject to air gaps produced during electrode installation.

To derive the energy density we have considered a medium with quasi-uniform permittivity, as assumed by dielectric models of media to relate apparent permittivity with water content. This hypothesis requires at least a size of the sample volume larger than the dimension of medium intrinsic heterogeneities. Results of our study are consistent with the assumption provided rod diameter, which plays a determinant
role, larger than typical size of medium grains or pores.

Comparison with previous works carried out on the sample volume of sensors with a two-rod or three-rod design, either numerically or experimentally, is hampered by a quasi-systematic choice of geometric ratios \( \alpha \) lower than 0.2 (for the three-rod design \( \alpha \) is defined with two close rods). Therefore, they did not observe the change of configurations of the sample volume (it is however visible in the results from the simulation work by Zhan et al. (2016) in the case of a three-rod sensor at \( \alpha = 0.4 \)). Experimental profiles of sensitivity obtained by Baker and Lascano (1991) at \( \alpha = 0.064 \) with a setup close to ours, after correction, agrees well with our results for \( \alpha < 0.20 \). Comparison is also made difficult due to the difference of definitions for sensor sensitivity, and so the sample volume, between ours and those introduced in Ferré et al. (1998) and subsequent works. Latter ones define the volume as regions of highest energy density contributing to a fraction \( f \) of the overall energy, or sensor response. Both definitions can be conciliated if we consider a large fraction of sensor response to come from medium regions where sensor sensitivity, as defined in the present study, is very low. The sample volume associated with a fraction \( f = 50\% \) agrees reasonably with the volume defined by a sensitivity cut-off at 10\%. Laboratory experiments with a moving liquid/air interface show a rapid change of the apparent permittivity from air permittivity to the liquid one when the interface crosses the probe rods (Nissen et al. (2003); Zhan et al. (2016) with \( \alpha \) lower than 0.20). Interface displacement over which the change occurs is better related to rod diameter than their spacing. This result confirms the confined nature of sensitivity around each rod, which agrees with our results for \( \alpha \) lower than 0.25. Because of their limited geometric range these studies did not really question the widespread choice of a ratio \( \alpha \) lower than 0.20 among available two-rod or three-rod sensors. Concern about too small diameters is not addressed.

Regarding the size of the sample volume, our model shows that the largest volume \( \Gamma \) at fixed height \( h \) and spacing \( D \), is reached with \( \alpha \) between 0.30 and 0.50, i.e. a diameter from a third to an half of the rod spacing. Using a sensitivity cut-off of 10\% to define the sample volume, the maximal value is about \( \Gamma = 3.35 h (D/2)^2 \).

As a result of the present work about the best geometry of parallel-electrode sensors, we conclude to the importance of the electrode diameter for the medium volume sampled by sensor. It should be much larger than the typical size of medium intrinsic heterogeneities, more than few mm. It should also represent at least one fourth of electrode spacing in order to have a well defined and large sample volume between electrodes, and not only restricted to electrode-surface vicinity.

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Appendix A. Theoretical elements to derive sensor sensitivity to local permittivity

A.1. Electrostatic energy of a capacitor

The electrostatic energy $W_{el}$ of a distribution $\rho(r)$ of free charges corresponds to the work to build them from infinity (chapter 8 of Feynman et al. (1964) or section 1.11 of Jackson (1998)):

$$W_{el} = \frac{1}{2} \int \int_{\text{distr}} \rho(r_1) \frac{\rho(r_2)}{4 \pi \epsilon r_{12}} d^3r_1 d^3r_2$$

(A1)

Introducing the potential $V(r)$ of the electric field $E$ resulting from the charge action in the dielectric medium, the electrostatic energy $W_{el}$ becomes:

$$W_{el} = \frac{1}{2} \int \int_{\text{distr}} \rho(r) V(r) d^3r$$

(A2)

Taking into account the relation between the free charge density and the divergence of field $E$, and assuming a linear and isotropic dielectric medium of relative permittivity $\epsilon_r$, we have:

$$W_{el} = \frac{\epsilon_0}{2} \int_{\text{medium}} \nabla.(\epsilon_r E) V d^3r$$

(A3)

After integrating by parts with no field at infinity, $W_{el}$ results in:

$$W_{el} = \frac{\epsilon_0}{2} \int_{\text{medium}} \epsilon_r E.E d^3r$$

(A4)

where $E.E$ is the scalar product of $E$ with itself, or the square of its amplitude. In the last two equations the integration is applied to the whole medium.

We consider now a capacitor made of two conductors in electrostatic equilibrium, at potentials $V_+$ and $V_-$, respectively, and with charges of opposite signs. From Equation A2 the energy $W_{el}$ for this distribution of charges is:

$$W_{el} = 1/2 Q (V_+ - V_-) = 1/2 Q U = 1/2 C U^2$$

(A5)
with \( Q \) the total charge of each conductor in absolute value, \( U \) the voltage between electrodes, which is fixed by an external source, and \( C = Q/U \) the capacitance of the capacitor.

By combining Equations 1 and A5, the capacitance \( C \) is directly related to the integral of the squared amplitude of field \( E, |E|^2 \), over medium space. As a result, the weight of a medium part in \( C \), and consequently in the apparent permittivity, depends on the value of \( |E|^2 \) at this part.

**A.2. Determination of the energy density of a two-rod probe**

The potential of the two-rod probe at a point \( M \) of the medium (Figure 1) is:

\[
V(M) = V_{+q}(M) + V_{-q}(M) = \frac{-q}{2 \pi \epsilon} \ln \left( \frac{r_+}{r_-} \right) = \frac{-q}{4 \pi \epsilon} \ln \left( \frac{r_+}{r_-} \right)^2
\]  

(A6)

where \( r_+ \) and \( r_- \) are the distances between \( A \) and \( M \), and between \( A' \) and \( M \), respectively.

In order for the circle of radius \( \Phi/2 \) centered in \( O \) to be an equipotential for the two charged lines of in \( A \) and \( A' \), distances must obey the relation:

\[
OA'.OA = (\Phi/2)^2
\]

(A7)

The potential of the conductor is thus:

\[
V_+ = \frac{q}{2 \pi \epsilon} \ln \left( \frac{\Phi/2}{OA} \right)
\]

(A8)

Similarly, for the circle of radius \( \Phi/2 \) and centered in \( O' \) to be an equipotential of the two charged lines, the following relations are necessary:

\[
O'A'.O'A = (\Phi/2)^2
\]

(A9)

\[
V_- = \frac{-q}{2 \pi \epsilon} \ln \left( \frac{\Phi/2}{O'A'} \right)
\]

(A10)

Because in our case the circles have the same radius, we have \( O'A' = OA \). As a result, we obtain the relations between the dimensions \( OA, D \) and \( \Phi \), or the ratio \( \alpha = \Phi/D \), are:

\[
OA = 1/2 \left[ D - \sqrt{D^2 - \Phi^2} \right] = D/2 \left[ 1 - \sqrt{1 - \alpha^2} \right] = \Phi/2 \left[ \alpha^{-1} - \sqrt{\alpha^{-2} - 1} \right]
\]

(A11)

As expected the distance \( OA \) is lower than the electrode radius \( \Phi/2 \).

From Equations A8, A10 and A11 the voltage between the two conductors of the capacitor is:

\[
U = V_+ - V_- = \frac{q}{\pi \epsilon} \ln \left( \frac{\Phi/2}{OA} \right) = \frac{q}{\pi \epsilon} \ln \left( \frac{1}{\alpha^{-1} - \sqrt{\alpha^{-2} - 1}} \right) = \frac{q}{\pi \epsilon} \ln \left( \alpha^{-1} + \sqrt{\alpha^{-2} - 1} \right)
\]

(A12)
Hence the capacitance per unit of height is:

\[
C/h = \frac{\pi \epsilon}{\ln \left( \alpha^{-1} + \sqrt{\alpha^{-2} - 1} \right)} = \frac{\pi \epsilon}{\text{arcosh}(\alpha^{-1})} \tag{A13}
\]

The electric field \(\mathbf{E}\) at point M is determined from the potential \(V(M)\) in Equation A6:

\[
\mathbf{E}(M) = -\nabla V = -\frac{\partial V}{\partial x} \mathbf{U}_x - \frac{\partial V}{\partial y} \mathbf{U}_y = \frac{q}{4 \pi \epsilon} \left[ \left\{ \frac{2(x + d/2)}{r_+^2} - \frac{2(x - d/2)}{r_-^2} \right\} \mathbf{U}_x + \left\{ \frac{2y}{r_+^2} - \frac{2y}{r_-^2} \right\} \mathbf{U}_y \right] \tag{A14}
\]

The distances \(r_+\) and \(r_-\) are expressed using the coordinates \(x\) and \(y\) of point M:

\[
\begin{align*}
    r_+^2 &= y^2 + (x + d/2)^2, \\
    r_-^2 &= y^2 + (x - d/2)^2,
\end{align*} \tag{A15}
\]

where \(d\) is the length between A and A’. Its expression using \(D\) and \(\Phi\) is:

\[
d = D - 2 OA = D \sqrt{1 - \alpha^2} \tag{A16}
\]

Whatever the capacitor dimensions, the maximal amplitude of \(\mathbf{E}\) in the medium is reached at the two points where the distance \(r_+\), or \(r_-\), is minimal, i.e. on electrode surface and the line joining both electrode axes. Their coordinates are \(y = 0\) and \(x_{\text{max}} = \pm(D/2 - \Phi/2)\), and the value of \(\mathbf{E}\) is:

\[
E_{\text{max}} = \frac{q}{4 \pi \epsilon} \frac{2d}{(d/2)^2 - x_{\text{max}}^2} \tag{A17}
\]

At the probe center \((x = y = 0)\) the field is:

\[
E_0 = \frac{2q}{\pi \epsilon d} \tag{A18}
\]