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► **To cite this version:**

Cheikh Saliou Ndiaye. A NOTE ON BÉZOUT TYPE INEQUALITIES FOR MIXED VOLUMES AND MINKOWSKI SUMS. 2023. hal-04059439v1

HAL Id: hal-04059439

<https://cnrs.hal.science/hal-04059439v1>

Preprint submitted on 5 Apr 2023 (v1), last revised 13 Mar 2024 (v4)

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A NOTE ON BÉZOUT TYPE INEQUALITIES FOR MIXED VOLUMES AND MINKOWSKI SUMS.

CHEIKH SALIOU NDIAYE

ABSTRACT. In this note, we study Bézout type inequalities for mixed volume and Minkowski sum of convex bodies in \mathbb{R}^n . We first give a new proof and we extend inequalities of Jian Xiao on mixed discriminants. Then, we use mass transport method to deduce some Bézout type inequalities for mixed volumes. Finally, we apply these inequalities to obtain Bézout type inequalities for Minkowski sums.

Mathematics Subject Classification: 52A39, 52A40.

Keywords: Mixed volume, Minkowski sum, Convex body, optimal transport, Alexandrov-Fenchel, Bézout inequality.

1. INTRODUCTION

In 1779, Bézout gave the first version of his theorem about the number of intersection points of algebraic hypersurfaces having no component in common: if H_1, \dots, H_n are algebraic hypersurfaces in \mathbb{C}^n , then

$$\#(H_1, \dots, H_n) \leq \deg(H_1) \times \dots \times \deg(H_n). \quad (1)$$

From the Bernstein-Kushnirenko-Khovanskii theorem (see [5, 14, 15]), the quantities appearing in this inequality may be written in terms of mixed volumes of convex bodies. For any K_1, \dots, K_m convex bodies in \mathbb{R}^n , their mixed volume is defined as

$$V(K_1, \dots, K_n) = \frac{1}{n!} \sum_{k=1}^n (-1)^{n+k} \sum_{i_1 < \dots < i_k} |K_{i_1} + \dots + K_{i_k}|,$$

where $K_{i_1} + \dots + K_{i_k} := \{x_{i_1} + \dots + x_{i_k} \mid x_{i_j} \in K_{i_j}\}$ is the Minkowski sum and $|\cdot|$ is the Lebesgue measure. Minkowski showed that for $t_1, \dots, t_m \in \mathbb{R}_+$, $|t_1 K_1 + \dots + t_m K_m|$ can be extended as a polynomial function (see section 5.1 of [19]):

$$|t_1 K_1 + \dots + t_m K_m| = \sum_{|i|=n} \frac{n!}{i_1! i_2! \dots i_m!} V(K_1[i_1], \dots, K_m[i_m]) t_1^{i_1} \dots t_m^{i_m},$$

where $K[i_j]$ is K taken i_j times. Soprunov and Zvavitch showed in [20] that the inequality (1) can be rewritten in the following way: for $2 \leq r \leq n$, and for all P_1, \dots, P_r convex bodies,

$$|\Delta|^{r-1} V(P_1, \dots, P_r, \Delta[n-r]) \leq \prod_{i=1}^r V(P_i, \Delta[n-1]), \quad (2)$$

where Δ is a n -dimensional simplex. If H_1, \dots, H_r are generic hypersurfaces in $(\mathbb{C}^*)^n$, P_1, \dots, P_r their Newton polytope and H_{r+1}, \dots, H_n are generic linear forms, then the standard simplex $\text{conv}\{0, e_1, \dots, e_n\}$ is the Newton polytope of H_{r+1}, \dots, H_n and by the Bernstein-Kushnirenko-Khovanskii theorem, $\#(H_1 \cap \dots \cap H_n) = n! V(P_1, \dots, P_r, \Delta[n-r])$, for all $i \in \{1, \dots, r\}$, $\deg(H_i) = n! V(P_i, \Delta[n-1])$ and for all $i \in \{r+1, \dots, n\}$, $\deg(H_i) = 1$. Therefore, in this case, the inequality (1) becomes (2) by replacing $n!$ by $1/|\Delta|$. Soprunov and Zvavitch [20] conjectured that if A satisfies

$$|A|^{r-1} V(B_1, \dots, B_r, A[n-r]) \leq \prod_{i=1}^r V(B_i, A[n-1]),$$

This research is partly funded by the Bézout Labex, funded by ANR, reference ANR-10-LABX-58.

for all convex bodies B_1, \dots, B_r , then A is an n -dimensional simplex and some positive partial answers are given in [18, 20, 21]. In [20], an inequality in the same flavour where A can be an arbitrary convex body was also studied and the question of the best constant $b_{n,r}$ such that for all convex bodies A, B_1, \dots, B_r in \mathbb{R}^n ,

$$|A|^{r-1}V(B_1, \dots, B_r, A[n-r]) \leq b_{n,r} \prod_{i=1}^r V(B_i, A[n-1]) \quad (3)$$

was considered. Inspired by the works of Fradelizi, Giannopoulos, Hartzoulaki, Meyer, and Paouris in [9, 12], it was proved in [20] that $b_{n,r} \leq \frac{(nr)^r}{r!}$. Xiao used the inequality (7) below to show that $b_{n,r} \leq n^{r-1}$ in [23]. Brazitikos, Giannopoulos and Liakopoulos [4] showed that $b_{n,r} \leq 2^{2^{r-1}-1}$. Our first main result is the following improvement of the preceding bounds. We show that

$$b_{n,r} \leq \min_{k \in \{1, \dots, r\}} \left\{ 2^{\frac{k(k-1)}{2}} \frac{n^{r-k}}{(r-k)!} \right\}. \quad (4)$$

In particular, for $k = 1$, this gives $b_{n,r} \leq \frac{n^{r-1}}{(r-1)!}$ and for $k = r$, we get $b_{n,r} \leq 2^{\frac{r(r-1)}{2}}$. Our main tools are improved Xiao's argument and the following Fenchel's inequality ([8], see also inequality (7.76) in [19]):

$$V(A[2], K_1, \dots, K_{n-2})V(B, C, K_1, \dots, K_{n-2}) \leq 2V(A, B, K_1, \dots, K_{n-2})V(A, C, K_1, \dots, K_{n-2}), \quad (5)$$

for any $A, B, C, K_1, \dots, K_{n-2}$ convex bodies in \mathbb{R}^n . We establish (4) by using mixed discriminants which are defined in the following way: for M_1, \dots, M_m be semi-definite positive symmetric matrices in \mathbb{R}^n and $t_1, \dots, t_m \in \mathbb{R}_+$.

$$\det(t_1 M_1 + \dots + t_m M_m) = \sum_{|i|=n} \frac{n!}{i_1! i_2! \dots i_m!} D(M_1[i_1], \dots, M_m[i_m]) t_1^{i_1} \dots t_m^{i_m},$$

the coefficient $D(M_1[i_1], \dots, M_m[i_m])$ is the mixed discriminant. The mixed discriminants are non negative and for any linear invertible map T ,

$$D(TM_1, \dots, TM_n) = \det(T)D(M_1, \dots, M_n), \quad (6)$$

(see section 5.5 of [19]). In [23], Xiao proved the following inequality which we extend (7) in theorem 1 below: for any integers k, n such that $1 \leq k \leq n$ and any positive definite symmetric matrices $A, B, M_1, \dots, M_{n-k}$, we have

$$\det(A)D(B[k], M_1, \dots, M_{n-k}) \leq \binom{n}{k} D(A[n-k], B[k])D(A[k], M_1, \dots, M_{n-k}). \quad (7)$$

Our second main theorem is the application of these new Bézout type inequality for mixed volumes to establish a new Bézout type inequality for Minkowski sums. Let $c_{n,m}$ be the smallest constant such that for any convex bodies $A, B_1, \dots, B_m \subset \mathbb{R}^n$,

$$|A|^{m-1}|A + B_1 + \dots + B_m| \leq c_{n,m} \prod_{k=1}^m |A + B_k|.$$

Bobkov and Madiman established in [6] that $c_{n,m} \leq (m+1)^n$. Fradelizi, Madiman and Zvavitch used a consequence of (7) in [11] to show that $(\frac{4}{3} + o(1))^n \leq c_{n,2} \leq \varphi^n$ where φ is the golden ratio. They also gave more precise bounds for small dimensions: $c_{2,2} = 1$ $c_{3,2} = 4/3$ and they conjectured that $c_{4,2} = 3/2$. There are several similar results for Minkowski sums of zonoids in [10].

In Section 2, we give an extension of inequality (7) and then, we use optimal transport as introduced in [1] and developed in [16, 23] to deduce the same type of inequality for mixed volumes. In Section 3, we establish our Bézout inequality for mixed volumes (4) and we conclude with some Bézout type inequalities for Minkowski sums in Section 4.

Acknowledgement: We are grateful to Artem Zvavitch and Matthieu Fradelizi for introducing us to the question and their help. We also thank Pascal Dingoyan, Alfredo Hubbard and Maud Szusterman for interesting discussions about this work.

2. XIAO TYPE INEQUALITIES

Our first theorem is the following extension of inequality (7) due to Xiao [23].

Theorem 1. Let $m \geq 1, n \geq 2$ and $|i| := i_1 + \dots + i_m \leq n$ be integers. Then, for any semi-definite positive symmetric matrices $A, B_1, \dots, B_m, M_1, \dots, M_{n-|i|}$,

$$\frac{|i|!}{i_1! \dots i_m!} \det(A)^m D(B_1[i_1], \dots, B_m[i_m], M_1, \dots, M_{n-|i|}) \leq D(M_1, \dots, M_{n-|i|}, A[|i|]) \prod_{k=1}^m \binom{n}{i_k} D(B_k[i_k], A[n - i_k]). \quad (8)$$

Proof: We do this proof by induction on m . The first step is a consequence of (7) of which we give a new proof inspired in ideas from [3].

Case $m = 1$: (Xiao [23]) If $m = 1, B_1 = B$ and $i_1 = k \leq n$, (8) becomes (7). Since A and B_1 are semi-definite positive symmetric matrices, by simultaneous orthogonalization, there exists an invertible matrix P and a diagonal matrix Λ such that $A = PP^t$ and $B_1 = P\Lambda P^t$. Using (6) it is enough to assume in (7) that $A = I, B = \text{diag}(\lambda_1, \dots, \lambda_n)$ and to prove that

$$D(B[k], M_1, \dots, M_{n-k}) \leq \binom{n}{k} D(B[k], I[n-k]) D(I[k], M_1, \dots, M_{n-k}).$$

For any semi-definite positive symmetric matrices C_1, \dots, C_n , the polarization formula of the mixed discriminant is

$$D(C_1, \dots, C_n) = \frac{1}{n!} \sum_{\sigma \in S_n} \det(C_{\sigma(1)}^1, \dots, C_{\sigma(n)}^n),$$

where C_i^j denotes the j -th column of C_i (see section 5.5 of [19]). Taking $C_1 = \dots = C_k = B$ and for $i \in \{k+1, \dots, n\}, C_i = M_{i-k}$ and expanding the discriminants, one has

$$\begin{aligned} D(B[k], M_1, \dots, M_{n-k}) &= \frac{1}{n!} \sum_{|K|=k} k! \prod_{j \in K} \lambda_j \sum_{\substack{\sigma \in S_n \\ \{\sigma(1), \dots, \sigma(k)\} = K}} \det \left(\left(M_{\sigma(k+1)}^{K^c} \right)^1, \dots, \left(M_{\sigma(n)}^{K^c} \right)^{n-k} \right) \\ &= \frac{1}{\binom{n}{k}} \sum_{|K|=k} \prod_{j \in K} \lambda_j D(M_1^{K^c}, \dots, M_{n-k}^{K^c}), \end{aligned}$$

where for any K subset of $[n] := \{1, \dots, n\}$ of cardinality k and any n -dimensional matrix M , M^{K^c} is the $(n-k)$ dimensional matrix defined by $M^{K^c} = (M_{i,j})_{i,j \in [n] \setminus K}$ and $D(M_1^{K^c}, \dots, M_{n-k}^{K^c})$ their mixed discriminant in dimension $n-k$. In the same way, we have,

$$D(I[k], M_1, \dots, M_{n-k}) = \frac{1}{\binom{n}{k}} \sum_{|K|=k} D(M_1^{K^c}, \dots, M_{n-k}^{K^c})$$

$$\text{and} \quad D(I[k], B[n-k]) = \frac{1}{\binom{n}{k}} \sum_{|J|=k} \prod_{j \in J} \lambda_j.$$

It follows that

$$\begin{aligned} \binom{n}{k} D(I[n-k], B[k]) D(I[k], M_1, \dots, M_{n-k}) &= \frac{1}{\binom{n}{k}} \sum_{|J|=k} \prod_{j \in J} \lambda_j \sum_{|K|=k} D(M_1^{K^c}, \dots, M_{n-k}^{K^c}) \\ &\geq \frac{1}{\binom{n}{k}} \sum_{|K|=k} \prod_{j \in K} \lambda_j (M_1^{K^c}, \dots, M_{n-k}^{K^c}) \\ &= D(B[k], M_1, \dots, M_{n-k}). \end{aligned}$$

Hence, the case $m = 1$ is proved.

Induction step: Let us assume that inequality (8) is true for $m - 1$. We can assume that $A = I, B_m = \text{diag}(\delta_1, \dots, \delta_n)$ and we need to prove that

$$\begin{aligned} & \frac{(n - i_m)!}{i_1! \dots i_{m-1}!(n - |i|)!} \sum_{|J_m|=i_m} \prod_{j \in J_m} \delta_j D\left(B_1^{J_m^c}[i_1], \dots, B_{m-1}^{J_m^c}[i_{m-1}], M_1^{J_m^c}, \dots, M_{n-|i|}^{J_m^c}\right) \\ & \leq \sum_{|J|=|i|} D(M_1^{J^c}, \dots, M_{n-|i|}^{J^c}) \prod_{k=1}^m \sum_{|J_k|=i_k} \det(B_k^{J_k}) \\ & = \sum_{|J_m|=i_m} \prod_{j \in J_m} \delta_j \sum_{|J|=|i|} D(M_1^{J^c}, \dots, M_{n-|i|}^{J^c}) \prod_{k=1}^{m-1} \sum_{|J_k|=i_k} \det(B_k^{J_k}). \end{aligned}$$

Thus, by comparing term by term, it is enough to prove that for each J_m ,

$$\begin{aligned} & \frac{(n - i_m)!}{i_1! \dots i_{m-1}!(n - |i|)!} D(B_1^{J_m^c}[i_1], \dots, B_{m-1}^{J_m^c}[i_{m-1}], M_1^{J_m^c}, \dots, M_{n-|i|}^{J_m^c}) \\ & \leq \sum_{|J|=|i|} D(M_1^{J^c}, \dots, M_{n-|i|}^{J^c}) \prod_{k=1}^{m-1} \sum_{|J_k|=i_k} \det(B_k^{J_k}). \quad (9) \end{aligned}$$

But the induction allows to say that, for such a fixed J_m ,

$$\begin{aligned} & \frac{(n - i_m)!}{i_1! \dots i_{m-1}!(n - |i|)!} D(B_1^{J_m^c}[i_1], \dots, B_{m-1}^{J_m^c}[i_{m-1}], M_1^{J_m^c}, \dots, M_{n-|i|}^{J_m^c}) \\ & \leq \sum_{\substack{|J|=|i| \\ J^c \subset J_m^c}} D(M_1^{J^c}, \dots, M_{n-|i|}^{J^c}) \prod_{k=1}^{m-1} \sum_{\substack{|J_k|=i_k \\ J_k \subset J_m^c}} \det(B_k^{J_k}), \end{aligned}$$

which implies (9). \square

Remark 2. The method that we have used in the previous proof allows to answer positively to the Remark 3.6 in [16] where the author conjectured that for $|i| \leq n$ and for any semi-definite positive symmetric matrices A, B_1, \dots, B_m ,

$$\det(A)^{m-1} D(B_1[i_1], \dots, B_m[i_m], A[n - |i|]) \leq \frac{(n!)^{m-1} (n - |i|)!}{\prod_{j=1}^m (n - i_j)!} \prod_{k=1}^m D(B_k[i_k], A[n - i_k]).$$

Theorem 3. Let $m \geq 1, n \geq 2$ and $|i| := i_1 + \dots + i_m \leq n$ be integers. Then, for any convex bodies $A, B_1, \dots, B_m, K_1, \dots, K_{n-|i|}$ in \mathbb{R}^n ,

$$\begin{aligned} & \frac{|i|!}{i_1! \dots i_m!} |A|^m V(B_1[i_1], \dots, B_m[i_m], K_1, \dots, K_{n-|i|}) \\ & \leq V(K_1, \dots, K_{n-|i|}, A[|i|]) \prod_{k=1}^m \binom{n}{i_k} V(B_k[i_k], A[n - i_k]). \quad (10) \end{aligned}$$

Proof: We follow the method of Alesker, Dar and Milman [1], introduced in this context by Lehmann and Xiao [16]. Let γ_n be the gaussian measure on \mathbb{R}^n and for any convex body K , $\mathcal{U}(K)$ be the uniform distribution on K . By Brenier's theorem [7, 22], for $k = 1, \dots, m$, there exists a convex function f_k such that ∇f_k pushes forward γ_n onto $\mathcal{U}(B_k)$. Then $\nabla f_k(\mathbb{R}^n) = B$,

$$|B_k| = \int_{\mathbb{R}^n} \det(\nabla^2 f_k), \quad (11)$$

and for $j = 1, \dots, n - |i|$, there exists h_j such that ∇h_j pushes forward γ_n onto $\mathcal{U}(K_j)$. It was proved in [1] that if ∇g_1 and ∇g_2 are two Brenier's maps, then

$$\nabla g_1(\mathbb{R}^n) + \nabla g_2(\mathbb{R}^n) = (\nabla g_1 + \nabla g_2)(\mathbb{R}^n).$$

So, equality (11) also holds for Minkowski sums and therefore for mixed volume. Hence

$$V(B_1[i_1], \dots, B_m[i_m], K_1, \dots, K_{n-|i|}) = \int_{\mathbb{R}^n} \varphi(x) dx,$$

where $\varphi = D(\nabla^2 f_1[i_1], \dots, \nabla^2 f_m[i_m], \nabla^2 h_1, \dots, \nabla^2 h_{n-|i|})$. If $\int \varphi = 0$ the theorem is proved, so, we assume that $\int \varphi \neq 0$. Let μ be the probability measure having density $\frac{d\mu}{dx} = \frac{1}{\int \varphi} \varphi(x)$. There exists a convex function f_A such that ∇f_A pushes forward μ onto $\mathcal{U}(A)$. Since Brenier's map satisfies the Monge-Ampère equation (see Subsection 4.1.1 of [22]), one has

$$|A| \frac{\varphi(x)}{\int \varphi} = \det(\nabla^2 f_A(x)) \quad \mu - \text{almost everywhere in } \mathbb{R}^n. \quad (12)$$

Let $\psi = D(\nabla^2 f_A[n - |i|] \nabla^2 h_1, \dots, \nabla^2 h_{n-|i|})$, using Hölder inequality, we get that

$$\begin{aligned} & \binom{n}{|i|} V(K_1, \dots, K_{n-|i|}, A[|i|]) \prod_{k=1}^m \binom{n}{i_k} V(B_k[i_k], A[n - i_k]) \\ &= \binom{n}{|i|} \int_{\mathbb{R}^n} \psi(x) dx \prod_{k=1}^m \int_{\mathbb{R}^n} \binom{n}{i_k} D(\nabla^2 f_A(x)[n - i_m] \nabla^2 f_k(x)[i_m]) dx \\ &\geq \left[\int_{\mathbb{R}^n} \left(\binom{n}{|i|} \psi(x) \prod_{k=1}^m \binom{n}{i_k} D(\nabla^2 f_A(x)[n - i_m] \nabla^2 f_k(x)[i_m]) \right)^{\frac{1}{m+1}} dx \right]^{m+1}. \end{aligned}$$

The inequality (8) in Theorem 1 allows to get

$$\begin{aligned} & \left[\int_{\mathbb{R}^n} \left(\binom{n}{|i|} \psi(x) \prod_{k=1}^m \binom{n}{i_k} D(\nabla^2 f_A(x)[n - i_m] \nabla^2 f_k(x)[i_m]) \right)^{\frac{1}{m+1}} dx \right]^{m+1} \\ &\geq \left[\int_{\mathbb{R}^n} \left(\frac{n!}{i_1! \dots i_m! (n - |i|)!} \det(\nabla^2 f_A(x))^m \times \varphi(x) \right)^{\frac{1}{m+1}} dx \right]^{m+1} \\ &= \frac{n!}{i_1! \dots i_m! (n - |i|)!} V(B_1[i_1], \dots, B_m[i_m], K_1, \dots, K_{n-|i|}) |A|^m, \end{aligned}$$

by taking into account the equality (12). \square

Corollary 4. Let $1 \leq m \leq n$, $|i| := i_1 + \dots + i_m \leq n$ be integers and A, B_1, \dots, B_m be convex bodies in \mathbb{R}^n . Then, for any $j \in \{1, \dots, m\}$,

$$\frac{(|i| - i_j)! n!}{i_1! \dots i_m! (n - i_j)!} V(B_1[i_1], \dots, B_m[i_m], A[n - |i|]) |A|^{m-1} \leq \prod_{k=1}^m \binom{n}{i_k} V(B_k[i_k], A[n - i_k]). \quad (13)$$

Proof: Without loss of generality we can assume that $j = m$. Replacing m by $m - 1$ in (10) gives

$$\begin{aligned} & \frac{(|i| - i_m)!}{i_1! \dots i_{m-1}!} |A|^{m-1} V(B_1[i_1], \dots, B_{m-1}[i_{m-1}], K_1, \dots, K_{n-|i|+i_m}) \\ &\leq V(K_1, \dots, K_{n-|i|+i_m}, A[|i|]) \prod_{k=1}^{m-1} \binom{n}{i_k} V(B_k[i_k], A[n - i_k]). \quad (14) \end{aligned}$$

We obtain the desired result by taking $K_1 = \dots = K_{i_m} = B_m$ in and $K_{i_m+1} = \dots = K_{n-|i|+i_m} = A$ in (14) and by multiplying both sides by $\binom{n}{i_m}$. \square

Remark 5. Notice that Corollary 4 improves the following Xiao's inequality [23]: for any integers $2 \leq m \leq n$, $|i| := i_1 + \dots + i_m \leq n$ and convex bodies A, B_1, \dots, B_m in \mathbb{R}^n , for any $j \in \{1, \dots, m\}$,

$$\binom{n}{i_j} V(B_1[i_1], \dots, B_m[i_m], A[n - |i|]) |A|^{m-1} \leq \prod_{k=1}^m \binom{n}{i_k} V(B_k[i_k], A[n - i_k]).$$

(Similar results are given in [13] with A being the Euclidean ball.)

3. BÉZOUT INEQUALITY FOR MIXED VOLUMES.

In this section, we present upper bounds on the constant $b_{n,r}$ defined in (3). Here, our first argument is Lemma 7 below which is a generalization of Fenchel's inequality (5) and the second argument uses Corollary 4.

Theorem 6. Let $2 \leq r \leq n$ be two integers and $b_{n,r}$ be the best constant such that for all convex bodies A, B_1, \dots, B_r in \mathbb{R}^n

$$|A|^{r-1} V(B_1, \dots, B_r, A[n - r]) \leq b_{n,r} \prod_{i=1}^r V(B_i, A[n - 1]).$$

$$\text{Then, } b_{n,r} \leq \min_{k \in \{1, \dots, r\}} \left\{ 2^{\frac{k(k-1)}{2}} \frac{n^{r-k}}{(r-k)!} \right\}.$$

Before proving this theorem, we introduce the following generalization of Fenchel's inequality (5).

Lemma 7. Let $1 \leq m \leq n$ be two integers and A, B_1, \dots, B_m be convex bodies in \mathbb{R}^n . Then

$$|A| V(B_1, \dots, B_m, A[n - m]) \leq 2^{m-1} V(B_1, \dots, B_{m-1}, A[n - m + 1]) V(B_m, A[n - 1]) \quad (15)$$

and

$$|A|^{m-1} V(B_1, \dots, B_m, A[n - m]) \leq 2^{\frac{m(m-1)}{2}} \prod_{i=1}^m V(B_i, A[n - 1]). \quad (16)$$

Proof: We prove (15) by induction on m . The case $m = 1$ is trivial and the case $m = 2$ is Fenchel's inequality (5). Let us assume that (15) is verified for $m - 1$. From Fenchel's inequality,

$$\begin{aligned} & V(B_1, \dots, B_m, A[n - m]) V(B_1, \dots, B_{m-2}, A[n - m + 2]) \\ & \leq 2V(B_1, \dots, B_{m-1}, A[n - m + 1]) V(B_1, \dots, B_{m-2}, B_m, A[n - m + 1]). \end{aligned} \quad (17)$$

By the induction hypothesis, we have

$$\begin{aligned} & |A| V(B_1, \dots, B_{m-2}, B_m, A[n - m + 1]) \\ & \leq 2^{m-2} V(B_1, \dots, B_{m-2}, A[n - m + 2]) V(B_m, A[n - 1]). \end{aligned} \quad (18)$$

We end the proof by multiplying term by term (17) and (18).

The proof of (16) follows from (15) by induction on m . □

Proof of theorem 6: First we remark that, for any $k \in \{1, \dots, r\}$, by taking in (10) $i_1 = \dots = i_m = 1$, $m = r - k$ and for $1 \leq j \leq k$, $K_j = B_{r-k+j}$, for $k \leq j \leq n - |i|$, $K_j = A$, we get that

$$V(B_1, \dots, B_r, A[n - r]) |A|^{r-k} \leq \frac{n^{r-k}}{(r-k)!} V(B_{r-k+1}, \dots, B_r, A[n - k]) \prod_{i=1}^{r-k} V(B_i, A[n - 1]),$$

According to (16), we have

$$V(B_{r-k+1}, \dots, B_r, A[n - k]) |A|^{k-1} \leq 2^{\frac{k(k-1)}{2}} \prod_{i=r-k+1}^r V(B_i, A[n - 1]),$$

Hence the result follows. □

4. BÉZOUT INEQUALITY FOR MINKOWSKI SUMS.

In [17], Ruzsa showed that for any compact sets $A, B_1, \dots, B_m \subseteq \mathbb{R}^n$ and any $\epsilon > 0$, if $|A| \neq 0$, there exists a compact set $A' \subset A$ such that

$$|A|^m |A' + B_1 + \dots + B_m| \leq (1 + \epsilon) |A'| \prod_{k=1}^m |A + B_k|. \quad (19)$$

It was noticed in [11] that (19) gives

$$|A|^{m-1} |B_1 + \dots + B_m| \leq \prod_{k=1}^m |A + B_k|. \quad (20)$$

Notice that our methods allows to give a new simple proof of (20). Indeed, if $|i| = n$, inequality (10) becomes

$$\frac{n!}{i_1! \dots i_m!} |A|^{m-1} V(B_1[i_1], \dots, B_m[i_m]) \leq \prod_{k=1}^m \binom{n}{i_k} V(B_k[i_k], A[n - i_k]) \quad (21)$$

and since

$$|A|^{m-1} |B_1 + \dots + B_m| = \sum_{i_1 + \dots + i_m = n} \frac{n!}{i_1! \dots i_m!} |A|^{m-1} V(B_1[i_1], \dots, B_m[i_m]), \quad (22)$$

$$\prod_{k=1}^m |A + B_k| = \sum_{i_1=1}^n \dots \sum_{i_m=1}^n \prod_{k=1}^m \binom{n}{i_k} V(B_k[i_k], A[n - i_k]). \quad (23)$$

According to (21), each term of the right side of (22) is less than a corresponding term of the right side of (23). Hence, we get a new proof of (20). Another way of proving (20) is to first show that for all $m \geq 1$ and any semi-definite positive symmetric matrices A, B_1, \dots, B_m ,

$$\det(A)^{m-1} \det(B_1 + \dots + B_m) \leq \prod_{k=1}^m \det(A + B_k),$$

by extending each term into mixed discriminants like in (22) and (23) and applying Theorem 1. Then we use again the same optimal transport method.

Theorem 8. Let $m, n \geq 1$, two integers and $c_{n,m}$ the best positive constant such that

$$|A|^{m-1} |A + B_1 + \dots + B_m| \leq c_{n,m} \prod_{k=1}^m |A + B_k|, \quad (24)$$

for any convex bodies $A, B_1, \dots, B_m \subset \mathbb{R}^n$. Then

(i)

$$c_{n,m} \leq \left(\frac{1 - x_m}{1 - mx_m} \right) < 2^n,$$

where x_m is the unique real root of $P_r(x) = (1 - mx)^m - (m - 1)^{m-1} x^{m-1} (1 - x)$.

(ii)

$$c_{n,m} \geq \frac{e}{\sqrt{2\pi n(e-1)}} g(n, m)^{-(n+\frac{1}{2})},$$

where

$$g(n, m) = \left[1 - \frac{1}{m-1} \left(\frac{m-1}{m} \right)^m \right]^{m-1}.$$

Furthermore, $\frac{4}{3} \leq g(n, m) < e^{e-1}$

Note that Theorem 8 does not follow from (19) and the sets in (24) are convex bodies.

Remark 9. Here we give some simpler upper bounds of $c_{n,m}$

- (i) for fixed n , $(c_{n,m})_{m \geq 2}$ is an increasing sequence with respect to m , $1 \leq c_{n,2} \leq c_{n,m} \leq c_{n,m+1} \leq (c_{n,2})^m$ and for $q > 0$, we have $c_{n,m+q} \leq c_{n,m}c_{n,q+1}$. In fact, in (24), for $m \geq 3$, if $B_m = \{0\}$, we get

$$|A|^{m-2} |A + B_1 + \cdots + B_{m-1}| \leq c_{n,m} \prod_{k=1}^{m-1} |A + B_k|,$$

hence, $c_{n,m-1} \leq c_{n,m}$. If $B_k = \{0\}$ for all $1 \leq k \leq m$ then we obtain equality in (24) with $c_{n,m} = 1$, so $c_{n,m} \geq 1$. It is shown in [11] that $1 = c_{2,2} \leq c_{n,2}$ by using (5). In (24), if we replace B_m by $B_m + \cdots + B_{m+q}$, we get

$$|A|^{m-1} |A + B_1 + \cdots + B_{m+q}| \leq c_{n,m} |A + B_m + \cdots + B_{m+q}| \prod_{k=1}^{m-1} |A + B_k|. \quad (25)$$

On the other hand, we have

$$|A|^q |A + B_m + \cdots + B_{m+q}| \leq c_{n,q+1} \prod_{k=m}^{m+q} |A + B_k|. \quad (26)$$

Thus, (25) and (26) allow to say that $c_{n,m+q} \leq c_{n,m}c_{n,q+1}$, in particular, $c_{n,m} \leq (c_{n,2})^{m-1}$.

- (ii) We also remark that a weaker bound for $c_{n,m}$ can be obtained by applying (20) to $\frac{1}{m}A + B_i$ instead of B_i for $i \in \{1, \dots, m\}$ and $(1 - \frac{1}{m})A$ instead of A . This gives

$$|A|^{m-1} |A + B_1 + \cdots + B_m| \leq \left[\left(1 + \frac{1}{m-1} \right)^{m-1} \right]^n \prod_{k=1}^m |A + B_k|,$$

$$\text{so } c_{n,m} \leq \left[\left(1 + \frac{1}{m-1} \right)^{m-1} \right]^n \leq e^n.$$

- (iii) We extend again as sums both sides of (24) like (22) and (23) and compare each term of $|A|^{m-1} |A + B_1 + \cdots + B_m|$ by the term of the same index in $\prod_{k=1}^m |A + B_k|$. It follows that $c_{n,m} \leq d_{n,m}$ where $d_{n,m}$ satisfies for all $|i| := i_1 + \cdots + i_m \leq n$

$$\frac{n!}{i_1! \cdots i_m! (n - |i|)!} |A|^m V(A[n - |i|], B_1[i_1], \dots, B_m[i_m]) \leq d_{n,m} \prod_{k=1}^m \binom{n}{i_k} V(B_k[i_k], A[n - i_k]).$$

In (10), if $K_1 = \cdots = K_{n-|i|} = A$, it becomes

$$\frac{n!}{i_1! \cdots i_m! (n - |i|)!} |A|^{m-1} V(A[n - |i|], B_1[i_1], \dots, B_m[i_m]) \leq \binom{n}{|i|} \prod_{k=1}^m \binom{n}{i_k} V(B_k[i_k], A[n - i_k]),$$

so any $d_{n,m}$ such that $\binom{n}{|i|} \leq d_{n,m}$ for all $|i| \leq n$ gives an upper bound for $c_{n,m}$, hence

$$c_{n,m} \leq \max_{|i| \leq n} \binom{n}{|i|} = \binom{n}{\lfloor \frac{n}{2} \rfloor} < 2^n.$$

Proof of Theorem 8 (i) (upper bound): The following idea comes from [11] where the authors prove that $c_{n,2} \leq \varphi^n$ (where φ is the golden ratio). Similarly to computations (iii) from Remark 9, we recall that after expanding both sides of (24) and comparing term by term, one has $c_{n,m} \leq d_{n,m}$ where $d_{n,m}$ satisfies for all $|i| := i_1 + \cdots + i_m \leq n$

$$\frac{n!}{i_1! \cdots i_m! (n - |i|)!} |A|^m V(A[n - |i|], B_1[i_1], \dots, B_m[i_m]) \leq d_{n,m} \prod_{k=1}^m \binom{n}{i_k} V(B_k[i_k], A[n - i_k]),$$

thus

$$|A|^m V(A[n - |i|], B_1[i_1], \dots, B_m[i_m]) \leq \frac{i_1! \cdots i_m! (n - |i|)!}{n!} d_{n,m} \prod_{k=1}^m \binom{n}{i_k} V(B_k[i_k], A[n - i_k]).$$

For any $j \in \{1, \dots, m\}$, (13) can be rewritten as

$$V(B_1[i_1], \dots, B_m[i_m], A[n - |i|])|A|^{m-1} \leq \frac{i_1! \dots i_m!(n - i_j)!}{(|i| - i_j)!n!} \prod_{k=1}^m \binom{n}{i_k} V(B_k[i_k], A[n - i_k]).$$

Therefore, for a fixed $j \in \{1, \dots, m\}$, to obtain an upper bound on $c_{n,m}$, it is enough to find $d_{n,m}$ such that for any $|i| \leq n$

$$\frac{i_1! \dots i_m!(n - i_j)!}{(|i| - i_j)!n!} \leq \frac{i_1! \dots i_m!(n - |i|)!}{n!} d_{n,m},$$

then, $\max_{|i| \leq n} \binom{n - i_j}{|i| - i_j} \leq d_{n,m}$ and it follow that

$$c_{n,m} \leq \max_{|i| \leq n} \min_{j \in \{1, \dots, m\}} \binom{n - i_j}{|i| - i_j}.$$

Note that $\max_{|i| \leq n} \min_{j \in \{1, \dots, m\}} \binom{n - i_j}{|i| - i_j} \leq \max_{|i| \leq n} \binom{n}{|i|} < 2^n$ as claimed in Remark 9. For any $j \in \{1, \dots, m\}$, let $i_j = x_j n$ with $x_j \geq 0$ and $|x| := x_1 + \dots + x_m \leq 1$. According to Stirling formula, for any integers $k \leq n$, one has $\binom{n}{k} \leq \frac{n^n}{(n-k)^{n-k} k^k}$, therefore:

$$c_{n,m} \leq \max_{|x| \leq 1} \min_{j \in \{1, \dots, m\}} \left[\frac{(1 - x_j)^{1 - x_j}}{(1 - |x|)^{1 - |x|} (|x| - x_j)^{|x| - x_j}} \right]^n.$$

For $|x| \leq n$ and $j \in \{1, \dots, m\}$, let

$$F_j(x) = \frac{(1 - x_j)^{1 - x_j}}{(1 - |x|)^{1 - |x|} (|x| - x_j)^{|x| - x_j}}.$$

One has $\nabla^2 \log F_j(x) = \frac{-1}{1 - |x|} U U^t$ where $U_j = \sqrt{\frac{|x| - x_j}{1 - x_j}}$ and $U_k = 1/U_j$ for $k \neq j$, then, F_j is log-concave and $\min \log F_j$ also. Note that $\min_{1 \leq j \leq m} F_j$ is symmetric with respect to each hyperplane $\{x_i = x_k\}$, therefore $\max_{|x| \leq 1} \min_{j \in \{1, \dots, m\}} F_j(x)$ is reached only if $x_1 = \dots = x_m$. As a result, the problem becomes a simple study of a function having one variable:

$$c_{n,m} \leq \max_{x \leq 1/m} \left[\frac{(1 - x)^{1 - x}}{(1 - mx)^{1 - mx} ((m - 1)x)^{(m-1)x}} \right]^n$$

and this maximum is reached at the unique $x_m \in]0, 1/m[$ which is the real root of $P_r(x) = (1 - mx)^m - (m - 1)^{m-1} x^{m-1} (1 - x)$. After some simplifications, we get $c_{n,m} \leq \left(\frac{1 - x_m}{1 - mx_m} \right)^n < 2^n$.

For $m = 2$, $x_2 = \frac{5 - \sqrt{5}}{2}$ and $\frac{1 - x_2}{1 - 2x_2} = \varphi$ (golden ratio), for $m = 3$ we get that $c_{n,3} < (1.755)^n$. \square

For the lower bound of $c_{n,m}$, many results have been obtained in [2, 3, 9, 11, 10, 20]. For the case $m = 2$ we use methods from [2, 10, 11] and extend them to $m \geq 2$. The idea is to represent volumes in (24) as volumes of a projection of A to subspaces of lower dimension and optimize the lower bound with respect to those dimensions. We remark that (24) is equivalent to

$$\frac{|A|^{m-1} |A + B_1 + \dots + B_m|}{\prod_{k=1}^m |A + B_k|} \leq c_{n,m}.$$

So any choice of A, B_1, \dots, B_m gives a lower bound of $c_{n,m}$. Let Γ_2 be the set of vector subspaces $\{E_1, E_2\}$ such that $\dim(E_1) = i, \dim(E_2) = j, 0 < i, j \leq n, i + j > n$. It is obtained in [2] that for any A convex body in \mathbb{R}^n ,

$$c_{n,2} \geq \max_{A, \Gamma_2} \frac{|A|_n |P_{E_1 \cap E_2} A|_{i+j-n}}{|P_{E_1} A|_i |P_{E_2} A|_j} = \frac{\binom{i}{i+j-n} \binom{j}{i+j-n}}{\binom{n}{i+j-n}} \quad (27)$$

where A is a convex body in \mathbb{R}^n and $P_E A$ is the projection of A onto the subspace E . The equality case in (27) is reached for a sharp choice of E_1, E_2 and A . The authors of [11] conclude that

$$c_{n,2} \geq \max_{\substack{0 < i, j \leq n \\ i+j > n}} \frac{\binom{i}{i+j-n} \binom{j}{i+j-n}}{\binom{n}{i+j-n}} \approx \frac{2}{\sqrt{\pi n}} \left(\frac{4}{3}\right)^n.$$

By following the same idea, we get a lower bound for $c_{n,m}$.

Theorem 10. Let $\alpha_1, \dots, \alpha_m \in \{1, \dots, n\}$ be integers and $d = |\alpha| - (m-1)n > 0$. Let $A \subset \mathbb{R}^n$ be a convex body and Γ_m the set of collections of vector subspaces $\{E_1, \dots, E_m\}$ such that for $i = 1, \dots, m$, $\dim(E_i) = \alpha_i$ and $E_i^\perp \subset \bigcap_{j \in [m] \setminus \{i\}} E_j$. Then

$$c_{n,m} \geq \max_{\Gamma_m, A} \max_{\text{convex body}} \frac{|A|_n |P_{\bigcap_{i=1}^m E_i} A|_d}{\prod_{i=1}^m |P_{E_i} A|_{\alpha_i}}. \quad (28)$$

Proof: Let (e_1, \dots, e_n) be the canonical basis of \mathbb{R}^n . For $i = 1, \dots, m$, let $B_i = \sum_{k \in [n] \setminus K_i} [0, e_k]$, where $K_i = \{\alpha_1 + \dots + \alpha_{i-1} + k \mid 1 \leq k \leq \alpha_i\}$. Thus, for any $i = 1, \dots, m$ and any $t > 0$, one has

$$|A + tB_i|_n \underset{t \rightarrow \infty}{\sim} t^{n-\alpha_i} |P_{E_i} A|_{\alpha_i}$$

and

$$|A + tB_1 + \dots + tB_m|_n \underset{t \rightarrow \infty}{\sim} t^{nm-|\alpha|} \left| P_{\bigcap_{i=1}^m B_i} A \right|_d.$$

Let t to infinity, then we get

$$|A|_n^{m-1} |P_{\bigcap_{i=1}^m E_i} A|_d \leq c_{n,m} \prod_{i=1}^m |P_{E_i} A|_{\alpha_i}.$$

The desired result follows. \square

Corollary 11. Let $\alpha_1, \dots, \alpha_m \in [1, n]$ be integers and $d = |\alpha| - (m-1)n > 0$. Then

$$c_{n,m} \geq \max_{\alpha} \frac{\prod_{i=1}^m \binom{\alpha_i}{d}}{\binom{n}{d}}. \quad (29)$$

Proof: Let $B_\infty(d) = \sum_{k=n-d+1}^n [-e_k, e_k]$. For $i = 1, \dots, m$, let

$$J_i = [n - \alpha_1 + \dots + n - \alpha_{i-1} + 1 \quad ; \quad n - \alpha_1 + \dots + n - \alpha_i],$$

$B_1(i) = \text{conv}(\pm e_k)$, and $E_i = \text{vect}\{e_k, k \in [n] \setminus J_i\}$. If $A = \text{conv}\{\sum_{i=1}^m B_1(i), B_\infty(d)\}$, we get that

$$|A| = \prod_{i=1}^m |B_1(i)| \times |B_\infty(d)| / \binom{n}{d} \quad ; \quad |P_{\bigcap_{i=1}^m E_i} A| = |B_\infty(d)|$$

and for $i = 1, \dots, m$

$$|P_{E_i} A| = \prod_{k \in m \setminus \{i\}} |B_1(k)| \times |B_\infty(d)| / \binom{\alpha_i}{d}.$$

Thus simplifying the volumes we get,

$$\frac{|A|_n^{m-1} |P_{\bigcap_{i=1}^m E_i} A|_d}{\prod_{i=1}^m |P_{E_i} A|_{\alpha_i}} = \frac{\prod_{i=1}^m \binom{\alpha_i}{d}}{\binom{n}{d}^{m-1}},$$

for all α satisfying the given hypothesis. Hence the result follows. \square

Proof of Theorem 8 (ii) (lower bound): For the lower bound, Corollary 11 allows to say that

$$c_{n,m} \geq \max_{\substack{\alpha > 0 \\ d = |\alpha| - (m-1)n > 0}} \frac{\prod_{i=1}^m \binom{\alpha_i}{d}}{\binom{n}{d}^{m-1}}.$$

Now, using the Stirling formula, one has

$$\frac{\prod_{i=1}^m \binom{\alpha_i}{d}}{\binom{n}{d}^{m-1}} \geq \frac{1}{\sqrt{2\pi d}} \prod_{i=1}^m \sqrt{\frac{\alpha_i}{\alpha_i - d}} \times \frac{1}{d^d} \left[\sqrt{\frac{n-d}{n}} \frac{(n-d)^{n-d}}{n^n} \right]^{m-1} \prod_{i=1}^m \frac{\alpha_i^{\alpha_i}}{(\alpha_i - d)^{\alpha_i - d}} e^{o(1)}.$$

For $i = 1, \dots, m$, one assume that $\alpha_i = nx_i$ and $d = ny$, then $y = |x| - (m - 1)$ and

$$\max_{\alpha} \left(\frac{1}{d^d} \left[\frac{(n-d)^{n-d}}{n^n} \right]^{m-1} \prod_{i=1}^m \frac{\alpha_i^{\alpha_i}}{(\alpha_i - d)^{\alpha_i - d}} \right) = \max_x f(x)^n,$$

where

$$f(x_1, \dots, x_m) = \frac{(1-y)^{(1-y)(m-1)}}{y^y} \prod_{i=1}^m \frac{x_i^{x_i}}{(x_i - y)^{x_i - y}}.$$

By studying this function, we easily find that $\max f = \frac{x^m}{y} > 1$ where

$$x = (m - 1) \left[m - \left(\frac{m - 1}{m} \right)^{m-1} \right]^{-1}$$

and $y = m(x - 1) + 1$. It follows that

$$\begin{aligned} c_{n,m} &\geq \max \left(\frac{x^m}{y} \right)^n \frac{1}{\sqrt{2\pi ny}} (\sqrt{1-y})^{m-1} \left(\sqrt{\frac{x}{x-y}} \right)^m (1 + o(1)) \\ &= \frac{1}{\sqrt{2\pi n}} \left(\frac{x^m}{y} \right)^{n+1/2} \sqrt{\frac{(1-y)^{m-1}}{(x-y)^m}} (1 + o(1)). \end{aligned}$$

Besides, $\frac{x^m}{y} = \left[1 - \frac{1}{m-1} \left(\frac{m-1}{m} \right)^m \right]^{1-m}$ is a increasing sequence with respect to m while

$$\frac{(1-y)^{m-1}}{(x-y)^m} = \frac{\left(\frac{m-1}{m} \right)^{m-1} - \frac{1}{m}}{1 - \frac{1}{m} - \left(\frac{m-1}{m} \right)^m} \quad \text{is decreasing.}$$

Thus

$$\frac{e}{\sqrt{e-1}} \left(\frac{4}{3} \right)^{n+1/2} < \left(\frac{x^m}{y} \right)^{n+1/2} \sqrt{\frac{(1-y)^{m-1}}{(x-y)^m}} < \sqrt{6} (e^{e^{-1}})^{n+1/2},$$

it leads to

$$c_{n,m} \geq \frac{1}{\sqrt{2\pi n}} \frac{e}{\sqrt{e-1}} \left(\frac{4}{3} \right)^{n+1/2}.$$

For $m = 2$, $c_{n,2} \geq \frac{2}{\sqrt{\pi n}} \left(\frac{4}{3} \right)^n (1 + o(1))$ which was found in [11]. For $m = 3$, we find

$$c(n, 3) \geq \frac{81}{4\sqrt{115}\sqrt{\pi n}} \left(\frac{729}{529} \right)^n (1 + o(1)) \approx \frac{1.89}{\sqrt{\pi n}} (1.378)^n.$$

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