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# A NOTE ON BÉZOUT TYPE INEQUALITIES FOR MIXED VOLUMES AND MINKOWSKI SUMS.

CHEIKH SALIOU NDIAYE

ABSTRACT. In this note, we study Bézout type inequalities for mixed volumes and Minkowski sums of convex bodies in  $\mathbb{R}^n$ . We first give a new proof and we extend inequalities of Jian Xiao on mixed discriminants. Then, we use the mass transport method to deduce some Bézout type inequalities for mixed volumes. Finally, we apply these inequalities to obtain Bézout type inequalities for Minkowski sums.

**Mathematics Subject Classification:** 52A39, 52A40.

**Keywords:** Mixed volume, Minkowski sum, Convex body, optimal transport, Alexandrov-Fenchel, Bézout inequality.

## 1. INTRODUCTION

In 1779, Bézout gave the first version of his theorem about the number of intersection points of algebraic hypersurfaces: let  $f_1, \dots, f_n$  be polynomials with coefficients in  $\mathbb{C}^n$  such that the sets  $X_i := \{x \in \mathbb{C}^n \mid f_i(x) = 0\}$  for  $i \in \{1, \dots, n\}$  have no component in common, then

$$\#(X_1 \cap \dots \cap X_n) \leq \deg(f_1) \dots \deg(f_n). \quad (1)$$

From the Bernstein-Kushnirenko-Khovanskii theorem (see [5, 14, 15]), the quantities appearing in this inequality may be written in terms of mixed volumes of convex bodies. For any convex bodies  $K_1, \dots, K_n$  in  $\mathbb{R}^n$ , their mixed volume is defined as

$$V(K_1, \dots, K_n) = \frac{1}{n!} \sum_{k=1}^n (-1)^{n+k} \sum_{i_1 < \dots < i_k \leq n} |K_{i_1} + \dots + K_{i_k}|,$$

where  $K_{i_1} + \dots + K_{i_k} := \{x_{i_1} + \dots + x_{i_k} \mid x_{i_j} \in K_{i_j}\}$  is the Minkowski sum and  $|\cdot|$  is the Lebesgue measure. Minkowski showed that for any convex bodies  $K_1, \dots, K_m$  and non-negative  $t_1, \dots, t_m \in \mathbb{R}_+$ , the function  $|t_1 K_1 + \dots + t_m K_m|$  can be extended as a polynomial function (see section 5.1 of [19]):

$$|t_1 K_1 + \dots + t_m K_m| = \sum_{|i|=n} \frac{n!}{i_1! i_2! \dots i_m!} V(K_1[i_1], \dots, K_m[i_m]) t_1^{i_1} \dots t_m^{i_m},$$

where  $K[i_j]$  denotes  $K$  repeated  $i_j$  times. Soprunov and Zvavitch showed in [20] that the inequality (1) can be rewritten in the following way: for  $2 \leq r \leq n$ , and for all choices of convex bodies  $P_1, \dots, P_r$ , it holds

$$|\Delta|^{r-1} V(P_1, \dots, P_r, \Delta[n-r]) \leq \prod_{i=1}^r V(P_i, \Delta[n-1]), \quad (2)$$

where  $\Delta$  is a  $n$ -dimensional simplex. Indeed, let  $P_1, \dots, P_r$  be Newton polytopes and  $f_1, \dots, f_r$  be polynomials having respectively  $P_1, \dots, P_r$  as Newton polytopes and for  $i = 1, \dots, r$ ,  $X_i := \{x \in \mathbb{C}^n \mid f_i(x) = 0\}$ . Let  $H_{r+1}, \dots, H_n$  be linear forms, then for all  $i \in \{r+1, \dots, n\}$ , the standard simplex  $\text{conv}\{0, e_1, \dots, e_n\}$  is the Newton polytope of  $H_i$  and  $\deg(H_i) = 1$ . The Bernstein-Kushnirenko-Khovanskii theorem states that

$$\#(X_1 \cap \dots \cap X_r \cap H_{r+1} \cap \dots \cap H_n) = n! V(P_1, \dots, P_r, \Delta[n-r])$$

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and for all  $i \in \{1, \dots, r\}$ ,  $\deg(H_i) = n!V(P_i, \Delta[n-1])$  (see [23, 20] for more details). Therefore, in this case, the inequality (1) becomes (2) by replacing  $n!$  by  $1/|\Delta|$ . Soprunov and Zvavitch [20] conjectured that if a convex body  $A$  satisfies

$$|A|^{r-1}V(B_1, \dots, B_r, A[n-r]) \leq \prod_{i=1}^r V(B_i, A[n-1]),$$

for all convex bodies  $B_1, \dots, B_r$ , then  $A$  is an  $n$ -dimensional simplex. Some positive partial answers about this conjecture are given in [18, 20, 21]. In [20], an inequality of the same flavor where  $A$  can be an arbitrary convex body was also studied and the question of the best constant  $b_{n,r}$  such that for all convex bodies  $A, B_1, \dots, B_r$  in  $\mathbb{R}^n$ , we have

$$|A|^{r-1}V(B_1, \dots, B_r, A[n-r]) \leq b_{n,r} \prod_{i=1}^r V(B_i, A[n-1]) \quad (3)$$

was considered. Inspired by the works of Fradelizi, Giannopoulos, Hartzoulaki, Meyer, and Paouris in [9, 12], it was proved in [20] that  $b_{n,r} \leq \frac{(nr)^r}{r!}$ . Xiao used inequality (7) below to show that  $b_{n,r} \leq n^{r-1}$  in [23]. Brazitikos, Giannopoulos and Liakopoulos [4] showed that  $b_{n,r} \leq 2^{2^{r-1}-1}$ . Our first main result is the following improvement of the previously known bounds. We show that

$$b_{n,r} \leq \min_{k \in \{1, \dots, r\}} \left\{ 2^{\frac{k(k-1)}{2}} \frac{n^{r-k}}{(r-k)!} \right\}. \quad (4)$$

In particular, for  $k=1$ , this gives  $b_{n,r} \leq \frac{n^{r-1}}{(r-1)!}$  and for  $k=r$ , we get  $b_{n,r} \leq 2^{\frac{r(r-1)}{2}}$ . Our main tools improve Xiao's argument and the following Fenchel's inequality ([8], see also inequality (7.76) in [19]):

$$V(A[2], K_1, \dots, K_{n-2})V(B, C, K_1, \dots, K_{n-2}) \leq 2V(A, B, K_1, \dots, K_{n-2})V(A, C, K_1, \dots, K_{n-2}), \quad (5)$$

for any convex bodies  $A, B, C, K_1, \dots, K_{n-2}$  convex bodies in  $\mathbb{R}^n$ . We establish (4) by using mixed discriminants. Let  $M_1, \dots, M_m$  be positive semi-definite symmetric matrices in  $\mathbb{R}^n$  and  $t_1, \dots, t_m \in \mathbb{R}_+$  then

$$\det(t_1 M_1 + \dots + t_m M_m) = \sum_{|i|=n} \frac{n!}{i_1! i_2! \dots i_m!} D(M_1[i_1], \dots, M_m[i_m]) t_1^{i_1} \dots t_m^{i_m},$$

where the coefficient  $D(M_1[i_1], \dots, M_m[i_m])$  is called the mixed discriminant. Mixed discriminants are non negative and for any invertible map  $T$ ,  $D(TM_1, \dots, TM_n)$  is defined as

$$D(TM_1, \dots, TM_n) := |\det(T)| D(M_1, \dots, M_n), \quad (6)$$

(see section 5.5 of [19]). In [23], Xiao proved inequality (7) which we extend in theorem 1 below. Xiao's inequality states that for any integers  $k, n$  such that  $1 \leq k \leq n$  and any positive definite symmetric matrices  $A, B, M_1, \dots, M_{n-k}$ , we have

$$\det(A)D(B[k], M_1, \dots, M_{n-k}) \leq \binom{n}{k} D(A[n-k], B[k])D(A[k], M_1, \dots, M_{n-k}). \quad (7)$$

Our second main theorem is the application of this new Bézout type inequality for mixed volumes to establish a new Bézout type inequality for Minkowski sums. Let  $c_{n,m}$  be the smallest constant such that for any convex bodies  $A, B_1, \dots, B_m \subset \mathbb{R}^n$ ,

$$|A|^{m-1}|A + B_1 + \dots + B_m| \leq c_{n,m} \prod_{k=1}^m |A + B_k|.$$

Bobkov and Madiman established in [6] that  $c_{n,m} \leq (m+1)^n$ . Fradelizi, Madiman and Zvavitch used a consequence of (7) in [11] to show that  $(\frac{4}{3} + o(1))^n \leq c_{n,2} \leq \varphi^n$  where  $\varphi$  is the golden ratio. They also gave more precise bounds for small dimensions:  $c_{2,2} = 1$   $c_{3,2} = 4/3$  and they conjectured that  $c_{4,2} = 3/2$ . Several similar results for Minkowski sums of zonoids appear in [10].

In Section 2, we give an extension of inequality (7) and then we use optimal transport as introduced in [1] and developed in [16, 23] to deduce the same type of inequality for mixed volumes. In Section 3, we establish our Bézout inequality for mixed volumes (4) and we conclude with some Bézout type inequalities for Minkowski sums in Section 4.

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## 2. XIAO TYPE INEQUALITIES

Our first theorem is the following extension of inequality (7) due to Xiao [23].

**Theorem 1.** Let  $m \geq 1, n \geq 2$  and  $i_1, \dots, i_m \geq 0$  be integers such that  $|i| := i_1 + \dots + i_m \leq n$ . Then, for any positive semi-definite symmetric matrices  $A, B_1, \dots, B_m, M_1, \dots, M_{n-|i|}$ ,

$$\begin{aligned} \frac{n!}{i_1! \dots i_m! (n - |i|)!} \det(A)^m D(B_1[i_1], \dots, B_m[i_m], M_1, \dots, M_{n-|i|}) \\ \leq \binom{n}{|i|} D(M_1, \dots, M_{n-|i|}, A[|i|]) \prod_{k=1}^m \binom{n}{i_k} D(B_k[i_k], A[n - i_k]). \end{aligned} \quad (8)$$

*Proof:* The proof is by induction on  $m$ . The case  $m = 1$  is the inequality (7) of which we give a new proof inspired by ideas from [3].

Case  $m = 1$ : If  $m = 1, B_1 = B$  and  $i_1 = k \leq n$ , (8) becomes (7). Since  $A$  and  $B$  are positive semi-definite symmetric matrices, by simultaneous orthogonalization, there exists an invertible matrix  $P$  and a diagonal matrix  $\Lambda$  such that  $A = PP^t$  and  $B_1 = P\Lambda P^t$ . Using (6) it is enough to assume in (7) that  $A = I, B = \Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$  and to prove that

$$D(B[k], M_1, \dots, M_{n-k}) \leq \binom{n}{k} D(B[k], I[n - k]) D(I[k], M_1, \dots, M_{n-k}).$$

For any positive semi-definite symmetric matrices  $C_1, \dots, C_n$ , the polarization formula of the mixed discriminant is

$$D(C_1, \dots, C_n) = \frac{1}{n!} \sum_{\sigma \in S_n} \det \left( C_{\sigma(1)}^1, \dots, C_{\sigma(n)}^n \right),$$

where  $C_i^j$  denotes the  $j$ -th column of  $C_i$  (see section 5.5 of [19]). Taking  $C_1 = \dots = C_k = B$  and for  $i \in \{k + 1, \dots, n\}, C_i = M_{i-k}$  and expanding the discriminants, one has

$$\begin{aligned} D(B[k], M_1, \dots, M_{n-k}) &= \frac{1}{n!} \sum_{|K|=k} k! \lambda_K \sum_{\substack{\sigma \in S_n \\ \{\sigma(1), \dots, \sigma(k)\} = K}} \det \left( \left( M_{\sigma(k+1)}^{K^c} \right)^1, \dots, \left( M_{\sigma(n)}^{K^c} \right)^{n-k} \right) \\ &= \frac{1}{\binom{n}{k}} \sum_{|K|=k} \lambda_K D(M_1^{K^c}, \dots, M_{n-k}^{K^c}), \end{aligned}$$

where for any  $K \subset [n] : \{1, \dots, n\}$  such that  $|K| = k$ ,  $\lambda_K$  denotes  $\prod_{j \in K} \lambda_j$ ,  $K^c$  is the complementary of  $K$  in  $[n]$  and  $M^{K^c}$  is the  $(n - k)$  dimensional matrix defined by  $M^{K^c} = (M_{i,j})_{i,j \in K^c}$  and then  $D(M_1^{K^c}, \dots, M_{n-k}^{K^c})$  is a mixed discriminant in dimension  $n - k$ . In particular, we have,

$$D(I[k], M_1, \dots, M_{n-k}) = \frac{1}{\binom{n}{k}} \sum_{|K|=k} D(M_1^{K^c}, \dots, M_{n-k}^{K^c})$$

$$\text{and} \quad D(I[n - k], B[k]) = \frac{1}{\binom{n}{k}} \sum_{|J|=k} \det(B^J) = \frac{1}{\binom{n}{k}} \sum_{|J|=k} \lambda_J.$$

It follows that

$$\begin{aligned} \binom{n}{k} D(I[n-k], B[k]) D(I[k], M_1, \dots, M_{n-k}) &= \frac{1}{\binom{n}{k}} \sum_{|J|=k} \lambda_J \sum_{|K|=k} D(M_1^{K^c}, \dots, M_{n-k}^{K^c}) \\ &\geq \frac{1}{\binom{n}{k}} \sum_{|K|=k} \lambda_K D(M_1^{K^c}, \dots, M_{n-k}^{K^c}) \\ &= D(B[k], M_1, \dots, M_{n-k}). \end{aligned}$$

Hence, the case  $m = 1$  is proved.

Induction step: Let us assume that inequality (8) is true for  $m - 1$ . We can assume that  $A = I, B_m = \text{diag}(\delta_1, \dots, \delta_n)$ , then we have

$$\begin{aligned} &\frac{n!}{i_1! \dots i_m! (n - |i|)!} D(B_1[i_1], \dots, B_m[i_m], M_1, \dots, M_{n-|i|}) \\ &= \frac{n!}{i_1! \dots i_m! (n - |i|)!} \sum_{|J_m|=i_m} \frac{1}{\binom{n}{i_m}} \delta_{J_m} D(B_1^{J_m^c}[i_1], \dots, B_{m-1}^{J_m^c}[i_{m-1}], M_1^{J_m^c}, \dots, M_{n-|i|}^{J_m^c}) \\ &= \frac{(n - i_m)!}{i_1! \dots i_{m-1}! (n - |i|)!} \sum_{|J_m|=i_m} \delta_{J_m} D(B_1^{J_m^c}[i_1], \dots, B_{m-1}^{J_m^c}[i_{m-1}], M_1^{J_m^c}, \dots, M_{n-|i|}^{J_m^c}). \end{aligned}$$

On the other hand,

$$\begin{aligned} &\binom{n}{|i|} D(I[|i|], M_1, \dots, M_{n-|i|}) \prod_{k=1}^m \binom{n}{i_k} D(I[n - i_k], B_k[i_k]) \\ &= \sum_{|J|=|i|} D(M_1^{J^c}, \dots, M_{n-|i|}^{J^c}) \prod_{k=1}^m \sum_{|J_k|=i_k} \det(B_k^{J_k}) \\ &= \sum_{|J_m|=i_m} \delta_{J_m} \sum_{|J|=|i|} D(M_1^{J^c}, \dots, M_{n-|i|}^{J^c}) \prod_{k=1}^{m-1} \sum_{|J_k|=i_k} \det(B_k^{J_k}). \end{aligned}$$

Thus, we need to prove that

$$\begin{aligned} &\frac{(n - i_m)!}{i_1! \dots i_{m-1}! (n - |i|)!} \sum_{|J_m|=i_m} \delta_{J_m} D(B_1^{J_m^c}[i_1], \dots, B_{m-1}^{J_m^c}[i_{m-1}], M_1^{J_m^c}, \dots, M_{n-|i|}^{J_m^c}) \\ &\leq \sum_{|J_m|=i_m} \delta_{J_m} \sum_{|J|=|i|} D(M_1^{J^c}, \dots, M_{n-|i|}^{J^c}) \prod_{k=1}^{m-1} \sum_{|J_k|=i_k} \det(B_k^{J_k}). \end{aligned}$$

By comparing term by term, it is enough to prove that for each  $J_m$ ,

$$\begin{aligned} &\frac{(n - i_m)!}{i_1! \dots i_{m-1}! (n - |i|)!} D(B_1^{J_m^c}[i_1], \dots, B_{m-1}^{J_m^c}[i_{m-1}], M_1^{J_m^c}, \dots, M_{n-|i|}^{J_m^c}) \\ &\leq \sum_{|J|=|i|} D(M_1^{J^c}, \dots, M_{n-|i|}^{J^c}) \prod_{k=1}^{m-1} \sum_{|J_k|=i_k} \det(B_k^{J_k}). \quad (9) \end{aligned}$$

The induction hypothesis allows to say that, for such a fixed  $J_m$ ,

$$\begin{aligned} &\frac{(n - i_m)!}{i_1! \dots i_{m-1}! (n - |i|)!} D(B_1^{J_m^c}[i_1], \dots, B_{m-1}^{J_m^c}[i_{m-1}], M_1^{J_m^c}, \dots, M_{n-|i|}^{J_m^c}) \\ &\leq \sum_{\substack{|J|=|i| \\ J^c \subset J_m^c}} D(M_1^{J^c}, \dots, M_{n-|i|}^{J^c}) \prod_{k=1}^{m-1} \sum_{\substack{|J_k|=i_k \\ J_k \subset J_m^c}} \det(B_k^{J_k}), \end{aligned}$$

which implies (9).  $\square$

**Remark 2.** In [16] (Remark 3.6), the author affirmed that for any  $|i| \leq n$ ,

$$\det(A)^{m-1} D(B_1[i_1], \dots, B_m[i_m], A[n - |i|]) \leq \frac{(n!)^{m-1} (n - |i|)!}{\prod_{j=1}^m (n - i_j)!} \prod_{k=1}^m D(B_k[i_k], A[n - i_k]).$$

is true if  $A, B_1, \dots, B_m$  are diagonal matrices and asked that whether it holds for any positive semi-definite symmetric matrices. The induction method that we have used in the previous proof allows to answer positively to this question.

**Theorem 3.** Let  $m \geq 1, n \geq 2$  and  $|i| := i_1 + \dots + i_m \leq n$  be integers. Then, for any convex bodies  $A, B_1, \dots, B_m, K_1, \dots, K_{n-|i|}$  in  $\mathbb{R}^n$ ,

$$\begin{aligned} \frac{n!}{i_1! \dots i_m! (n - |i|)!} |A|^m V(B_1[i_1], \dots, B_m[i_m], K_1, \dots, K_{n-|i|}) \\ \leq \binom{n}{|i|} V(K_1, \dots, K_{n-|i|}, A[|i|]) \prod_{k=1}^m \binom{n}{i_k} V(B_k[i_k], A[n - i_k]). \end{aligned} \quad (10)$$

*Proof:* We follow the method of Alesker, Dar and Milman [1], introduced in this context by Lehmann and Xiao [16]. Let  $\gamma_n$  be the gaussian measure on  $\mathbb{R}^n$  and for any convex body  $K$ ,  $\mathcal{U}(K)$  be the uniform distribution on  $K$ . By Brenier's theorem [7, 22], for  $k = 1, \dots, m$ , there exists a convex function  $f_k$  such that  $\nabla f_k$  pushes forward  $\gamma_n$  onto  $\mathcal{U}(B_k)$ . Then  $\nabla f_k(\mathbb{R}^n) = B$ ,

$$|B_k| = \int_{\mathbb{R}^n} \det(\nabla^2 f_k), \quad (11)$$

and for  $j = 1, \dots, n - |i|$ , there exists a convex function  $h_j$  such that  $\nabla h_j$  pushes forward  $\gamma_n$  onto  $\mathcal{U}(K_j)$ . It was proved [1] that if  $\nabla g_1$  and  $\nabla g_2$  are two Brenier's maps, then

$$\nabla g_1(\mathbb{R}^n) + \nabla g_2(\mathbb{R}^n) = (\nabla g_1 + \nabla g_2)(\mathbb{R}^n).$$

So, equality (11) also holds for Minkowski sums and therefore for mixed volumes. Hence

$$V(B_1[i_1], \dots, B_m[i_m], K_1, \dots, K_{n-|i|}) = \int_{\mathbb{R}^n} \varphi(x) dx,$$

where  $\varphi = D(\nabla^2 f_1[i_1], \dots, \nabla^2 f_m[i_m], \nabla^2 h_1, \dots, \nabla^2 h_{n-|i|})$ . If  $\int \varphi = 0$  the theorem is proved, so, we assume that  $\int \varphi \neq 0$ . Let  $\mu$  be the probability measure having density  $\frac{d\mu}{dx} = \frac{1}{\int \varphi} \varphi(x)$ . There exists a convex function  $f_A$  such that  $\nabla f_A$  pushes forward  $\mu$  onto  $\mathcal{U}(A)$ . Since Brenier's map satisfies the Monge-Ampère equation (see Subsection 4.1.1 of [22]), one has

$$|A| \frac{\varphi(x)}{\int \varphi} = \det(\nabla^2 f_A(x)) \quad \mu - \text{almost everywhere in } \mathbb{R}^n. \quad (12)$$

Let  $\psi = D(\nabla^2 f_A[|i|], \nabla^2 h_1, \dots, \nabla^2 h_{n-|i|})$ , using Hölder inequality, we get that

$$\begin{aligned} \binom{n}{|i|} V(K_1, \dots, K_{n-|i|}, A[|i|]) \prod_{k=1}^m \binom{n}{i_k} V(B_k[i_k], A[n - i_k]) \\ = \binom{n}{|i|} \int_{\mathbb{R}^n} \psi(x) dx \prod_{k=1}^m \int_{\mathbb{R}^n} \binom{n}{i_k} D(\nabla^2 f_k(x)[i_k], \nabla^2 f_A(x)[n - i_k]) dx \\ \geq \left[ \int_{\mathbb{R}^n} \left( \binom{n}{|i|} \psi(x) \prod_{k=1}^m \binom{n}{i_k} D(\nabla^2 f_k(x)[i_k], \nabla^2 f_A(x)[n - i_k]) \right)^{\frac{1}{m+1}} dx \right]^{m+1}. \end{aligned}$$

The inequality (8) in Theorem 1 allows to get

$$\begin{aligned} & \left[ \int_{\mathbb{R}^n} \left( \binom{n}{|i|} \psi(x) \prod_{k=1}^m \binom{n}{i_k} D(\nabla^2 f_k(x)[i_k], \nabla^2 f_A(x)[n - i_k]) \right)^{\frac{1}{m+1}} dx \right]^{m+1} \\ & \geq \left[ \int_{\mathbb{R}^n} \left( \frac{n!}{i_1! \dots i_m! (n - |i|)!} \det(\nabla^2 f_A(x))^m \varphi(x) \right)^{\frac{1}{m+1}} dx \right]^{m+1} \\ & = \frac{n!}{i_1! \dots i_m! (n - |i|)!} V(B_1[i_1], \dots, B_m[i_m], K_1, \dots, K_{n-|i|}) |A|^m, \end{aligned}$$

by taking into account the equality (12).  $\square$

**Corollary 4.** Let  $1 \leq m \leq n$ ,  $|i| := i_1 + \dots + i_m \leq n$  be integers and  $A, B_1, \dots, B_m$  be convex bodies in  $\mathbb{R}^n$ . Then, for any  $j \in \{1, \dots, m\}$ ,

$$\frac{n!(|i| - i_j)!}{i_1! \dots i_m! (n - i_j)!} |A|^{m-1} V(B_1[i_1], \dots, B_m[i_m], A[n - |i|]) \leq \prod_{k=1}^m \binom{n}{i_k} V(B_k[i_k], A[n - i_k]). \quad (13)$$

*Proof:* Without loss of generality we can assume that  $j = m$ . Firstly we replace  $m$  by  $m - 1$  in (10) (and maintain  $|i| := i_1 + \dots + i_m$ ), then one has

$$\begin{aligned} & \frac{n!}{i_1! \dots i_{m-1}! (n - |i| + i_m)!} |A|^{m-1} V(B_1[i_1], \dots, B_{m-1}[i_{m-1}], K_1, \dots, K_{n-|i|+i_m}) \\ & \leq \binom{n}{|i| - i_m} V(K_1, \dots, K_{n-|i|+i_m}, A[|i| - i_m]) \prod_{k=1}^{m-1} \binom{n}{i_k} V(B_k[i_k], A[n - i_k]). \quad (14) \end{aligned}$$

Taking  $K_1 = \dots = K_{i_m} = B_m$  in and  $K_{i_m+1} = \dots = K_{n-|i|+i_m} = A$  in (14) gives

$$\begin{aligned} & \frac{(|i| - i_m)!}{i_1! \dots i_{m-1}!} |A|^{m-1} V(B_1[i_1], \dots, B_m[i_m], A[n - |i|]) \\ & \leq V(B_m[i_m], A[n - i_m]) \prod_{k=1}^{m-1} \binom{n}{i_k} V(B_k[i_k], A[n - i_k]). \quad (15) \end{aligned}$$

We obtain the desired result by multiplying both sides of (15) by  $\binom{n}{i_m}$ .  $\square$

**Remark 5.** Notice that Corollary 4 improves the following Xiao's inequality [23]: for any integers  $2 \leq m \leq n$ ,  $|i| := i_1 + \dots + i_m \leq n$  and convex bodies  $A, B_1, \dots, B_m$  in  $\mathbb{R}^n$ , for any  $j \in \{1, \dots, m\}$ ,

$$\binom{n}{i_j} V(B_1[i_1], \dots, B_m[i_m], A[n - |i|]) |A|^{m-1} \leq \prod_{k=1}^m \binom{n}{i_k} V(B_k[i_k], A[n - i_k]).$$

(Similar results are given in [13] with  $A$  being the Euclidean ball.)

### 3. BÉZOUT INEQUALITY FOR MIXED VOLUMES.

In this section, we present upper bounds on the constant  $b_{n,r}$  defined in (3). Here, our first argument is Lemma 7 below which is a generalization of Fenchel's inequality (5) and the second argument uses Corollary 4.

**Theorem 6.** Let  $2 \leq r \leq n$  be two integers and  $b_{n,r}$  be the best constant such that for all convex bodies  $A, B_1, \dots, B_r$  in  $\mathbb{R}^n$ , it holds

$$|A|^{r-1} V(B_1, \dots, B_r, A[n - r]) \leq b_{n,r} \prod_{i=1}^r V(B_i, A[n - 1]).$$

Then,  $b_{n,r} \leq \min_{k \in \{1, \dots, r\}} \left\{ 2^{\frac{k(k-1)}{2}} \frac{n^{r-k}}{(r-k)!} \right\}$ .

Before proving this theorem, we introduce the following generalization of Fenchel's inequality (5).

**Lemma 7.** Let  $1 \leq m \leq n$  be two integers and  $A, B_1, \dots, B_m$  be convex bodies in  $\mathbb{R}^n$ . Then

$$|A|V(B_1, \dots, B_m, A[n-m]) \leq 2^{m-1}V(B_1, \dots, B_{m-1}, A[n-m+1])V(B_m, A[n-1]) \quad (16)$$

and

$$|A|^{m-1}V(B_1, \dots, B_m, A[n-m]) \leq 2^{\frac{m(m-1)}{2}} \prod_{i=1}^m V(B_i, A[n-1]). \quad (17)$$

*Proof:* We prove (16) by induction on  $m$ . The case  $m = 1$  is trivial and the case  $m = 2$  is Fenchel's inequality (5). Let us assume that (16) is verified for  $m - 1$ . From Fenchel's inequality,

$$\begin{aligned} & V(B_1, \dots, B_m, A[n-m])V(B_1, \dots, B_{m-2}, A[n-m+2]) \\ & \leq 2V(B_1, \dots, B_{m-1}, A[n-m+1])V(B_1, \dots, B_{m-2}, B_m, A[n-m+1]). \end{aligned} \quad (18)$$

By the induction hypothesis, we have

$$\begin{aligned} & |A|V(B_1, \dots, B_{m-2}, B_m, A[n-m+1]) \\ & \leq 2^{m-2}V(B_1, \dots, B_{m-2}, A[n-m+2])V(B_m, A[n-1]). \end{aligned} \quad (19)$$

We end the proof by multiplying term by term (18) and (19).

The proof of (17) follows from (16) by induction on  $m$ .  $\square$

*Proof of theorem 6:* First we remark that, for any  $k \in \{1, \dots, r\}$ , by taking in (10)  $i_1 = \dots = i_m = 1, m = r - k$  and for  $1 \leq j \leq k, K_j = B_{r-k+j}$ , for  $k \leq j \leq n - |i|, K_j = A$ , we get that

$$V(B_1, \dots, B_r, A[n-r])|A|^{r-k} \leq \frac{n^{r-k}}{(r-k)!}V(B_{r-k+1}, \dots, B_r, A[n-k]) \prod_{i=1}^{r-k} V(B_i, A[n-1]),$$

According to (17), we have

$$V(B_{r-k+1}, \dots, B_r, A[n-k])|A|^{k-1} \leq 2^{\frac{k(k-1)}{2}} \prod_{i=r-k+1}^r V(B_i, A[n-1]),$$

Hence the result follows.  $\square$

#### 4. BÉZOUT INEQUALITY FOR MINKOWSKI SUMS.

In [17], Ruzsa showed that for any compact sets  $A, B_1, \dots, B_m \subseteq \mathbb{R}^n$  and any  $\epsilon > 0$ , if  $|A| \neq 0$ , there exists a compact set  $A' \subset A$  such that

$$|A|^m|A' + B_1 + \dots + B_m| \leq (1 + \epsilon)|A'| \prod_{k=1}^m |A + B_k|. \quad (20)$$

It was noticed in [11] that (20) gives

$$|A|^{m-1}|B_1 + \dots + B_m| \leq \prod_{k=1}^m |A + B_k|. \quad (21)$$

Notice that our methods allow to give a new simple proof of (21). Indeed, if  $|i| = n$ , inequality (10) becomes

$$\frac{n!}{i_1! \dots i_m!} |A|^{m-1}V(B_1[i_1], \dots, B_m[i_m]) \leq \prod_{k=1}^m \binom{n}{i_k} V(B_k[i_k], A[n-i_k]) \quad (22)$$

and since

$$|A|^{m-1}|B_1 + \dots + B_m| = \sum_{i_1 + \dots + i_m = n} \frac{n!}{i_1! \dots i_m!} |A|^{m-1}V(B_1[i_1], \dots, B_m[i_m]), \quad (23)$$



$$\prod_{k=1}^m |A + B_k| = \sum_{i_1=1}^n \cdots \sum_{i_m=1}^n \prod_{k=1}^m \binom{n}{i_k} V(B_k[i_k], A[n - i_k]). \quad (24)$$

According to (22), each term of the right side of (23) is less than a corresponding term of the right side of (24). Hence, we get a new proof of (21). Another way of proving (21) is to first show that for all  $m \geq 1$  and any positive semi-definite symmetric matrices  $A, B_1, \dots, B_m$ ,

$$\det(A)^{m-1} \det(B_1 + \cdots + B_m) \leq \prod_{k=1}^m \det(A + B_k),$$

by extending each term into mixed discriminants like in (23) and (24) and applying Theorem 1. Then we use again the same optimal transport method.

**Theorem 8.** Let  $m, n \geq 1$ , two integers and  $c_{n,m}$  the best positive constant such that

$$|A|^{m-1} |A + B_1 + \cdots + B_m| \leq c_{n,m} \prod_{k=1}^m |A + B_k|, \quad (25)$$

for any convex bodies  $A, B_1, \dots, B_m \subset \mathbb{R}^n$ . Then

(i)

$$c_{n,m} \leq \left( \frac{1 - x_m}{1 - mx_m} \right)^n < 2^n,$$

where  $x_m$  is the unique real root of  $P_r(x) = (1 - mx)^m - (m - 1)^{m-1} x^{m-1} (1 - x)$ .

(ii)

$$c_{n,m} \geq \frac{e}{\sqrt{2\pi n}(e-1)} g(n, m)^{-(n+\frac{1}{2})},$$

where

$$g(n, m) = \left[ 1 - \frac{1}{m-1} \left( \frac{m-1}{m} \right)^m \right]^{m-1}.$$

Furthermore,  $\frac{4}{3} \leq g(n, m) < e^{e^{-1}}$

Note that Theorem 8 does not follow from (20) and the sets in (25) are convex bodies.

**Remark 9.** Here we give some upper bounds of  $c_{n,m}$  simpler but less optimal than that given in theorem 8

(i) for fixed  $n$ ,  $(c_{n,m})_{m \geq 2}$  is a increasing sequence with respect to  $m$ ,  $1 \leq c_{n,2} \leq c_{n,m} \leq c_{n,m+1} \leq (c_{n,2})^m$  and for  $q > 0$ , we have  $c_{n,m+q} \leq c_{n,m} c_{n,q+1}$ . In fact, in (25), for  $m \geq 3$ , if  $B_m = \{0\}$ , we get

$$|A|^{m-2} |A + B_1 + \cdots + B_{m-1}| \leq c_{n,m} \prod_{k=1}^{m-1} |A + B_k|,$$

hence,  $c_{n,m-1} \leq c_{n,m}$ . If  $B_k = \{0\}$  for all  $1 \leq k \leq m$  then we obtain equality in (25) with  $c_{n,m} = 1$ , so  $c_{n,m} \geq 1$ . It is shown in [11] that  $1 = c_{2,2} \leq c_{n,2}$  by using (5). In (25), if we replace  $B_m$  by  $B_m + \cdots + B_{m+q}$ , we get

$$|A|^{m-1} |A + B_1 + \cdots + B_{m+q}| \leq c_{n,m} |A + B_m + \cdots + B_{m+q}| \prod_{k=1}^{m-1} |A + B_k|. \quad (26)$$

On the other hand, we have

$$|A|^q |A + B_m + \cdots + B_{m+q}| \leq c_{n,q+1} \prod_{k=m}^{m+q} |A + B_k|. \quad (27)$$

Thus, (26) and (27) allow to say that  $c_{n,m+q} \leq c_{n,m} c_{n,q+1}$ , in particular,  $c_{n,m} \leq (c_{n,2})^{m-1}$ .

(ii) We also remark that a weaker bound for  $c_{n,m}$  can be obtained by applying (21) to  $\frac{1}{m}A + B_i$  instead of  $B_i$  for  $i \in \{1, \dots, m\}$  and  $(1 - \frac{1}{m})A$  instead of  $A$ . This gives

$$|A|^{m-1}|A + B_1 + \dots + B_m| \leq \left[ \left(1 + \frac{1}{m-1}\right)^{m-1} \right]^n \prod_{k=1}^m |A + B_k|,$$

$$\text{so } c_{n,m} \leq \left[ \left(1 + \frac{1}{m-1}\right)^{m-1} \right]^n \leq e^n.$$

(iii) We extend again as sums both sides of (25) like (23) and (24) and compare each term of  $|A|^{m-1}|A + B_1 + \dots + B_m|$  by the term of the same index in  $\prod_{k=1}^m |A + B_k|$ . It follows that  $c_{n,m} \leq d_{n,m}$  where  $d_{n,m}$  satisfies for all  $|i| := i_1 + \dots + i_m \leq n$

$$\frac{n!}{i_1! \dots i_m!(n - |i|)!} |A|^m V(A[n - |i|], B_1[i_1], \dots, B_m[i_m]) \leq d_{n,m} \prod_{k=1}^m \binom{n}{i_k} V(B_k[i_k], A[n - i_k]).$$

In (10), if  $K_1 = \dots = K_{n-|i|} = A$ , it becomes

$$\frac{n!}{i_1! \dots i_m!(n - |i|)!} |A|^{m-1} V(A[n - |i|], B_1[i_1], \dots, B_m[i_m]) \leq \binom{n}{|i|} \prod_{k=1}^m \binom{n}{i_k} V(B_k[i_k], A[n - i_k]),$$

so any  $d_{n,m}$  such that  $\binom{n}{|i|} \leq d_{n,m}$  for all  $|i| \leq n$  gives an upper bound for  $c_{n,m}$ , hence

$$c_{n,m} \leq \max_{|i| \leq n} \binom{n}{|i|} = \binom{n}{\lfloor \frac{n}{2} \rfloor} < 2^n.$$

*Proof of Theorem 8 (i) (upper bound):* The following idea comes from [11] where the authors prove that  $c_{n,2} \leq \varphi^n$  (where  $\varphi$  is the golden ratio). Similarly to computations (iii) from Remark 9, we recall that after expanding both sides of (25) and comparing term by term, one has  $c_{n,m} \leq d_{n,m}$  where  $d_{n,m}$  satisfies for all  $|i| := i_1 + \dots + i_m \leq n$

$$\frac{n!}{i_1! \dots i_m!(n - |i|)!} |A|^m V(A[n - |i|], B_1[i_1], \dots, B_m[i_m]) \leq d_{n,m} \prod_{k=1}^m \binom{n}{i_k} V(B_k[i_k], A[n - i_k]),$$

thus

$$|A|^m V(A[n - |i|], B_1[i_1], \dots, B_m[i_m]) \leq \frac{i_1! \dots i_m!(n - |i|)!}{n!} d_{n,m} \prod_{k=1}^m \binom{n}{i_k} V(B_k[i_k], A[n - i_k]).$$

For any  $j \in \{1, \dots, m\}$ , (13) can be rewritten as

$$V(B_1[i_1], \dots, B_m[i_m], A[n - |i|]) |A|^{m-1} \leq \frac{i_1! \dots i_m!(n - i_j)!}{(|i| - i_j)! n!} \prod_{k=1}^m \binom{n}{i_k} V(B_k[i_k], A[n - i_k]).$$

Therefore, for a fixed  $j \in \{1, \dots, m\}$ , to obtain an upper bound on  $c_{n,m}$ , it is enough to find  $d_{n,m}$  such that for any  $|i| \leq n$

$$\frac{i_1! \dots i_m!(n - i_j)!}{(|i| - i_j)! n!} \leq \frac{i_1! \dots i_m!(n - |i|)!}{n!} d_{n,m},$$

then,  $\max_{|i| \leq n} \binom{n-i_j}{|i|-i_j} \leq d_{n,m}$  and it follow that

$$c_{n,m} \leq \max_{|i| \leq n} \min_{j \in \{1, \dots, m\}} \binom{n - i_j}{|i| - i_j}.$$

Note that  $\max_{|i| \leq n} \min_{j \in \{1, \dots, m\}} \binom{n - i_j}{|i| - i_j} \leq \max_{|i| \leq n} \binom{n}{|i|} < 2^n$  as claimed in Remark 9. For any  $j \in \{1, \dots, m\}$ , let  $i_j = x_j n$  with  $x_j \geq 0$  and  $|x| := x_1 + \dots + x_m \leq 1$ . According to Stirling

formula, for any integers  $k \leq n$ , one has  $\binom{n}{k} \leq \frac{n^n}{(n-k)^{n-k}k^k}$ , therefore:

$$c_{n,m} \leq \max_{|x| \leq 1} \min_{j \in \{1, \dots, m\}} \left[ \frac{(1-x_j)^{1-x_j}}{(1-|x|)^{1-|x|}(|x-x_j|^{|x-x_j|})} \right]^n.$$

For  $|x| \leq n$  and  $j \in \{1, \dots, m\}$ , let

$$F_j(x) = \frac{(1-x_j)^{1-x_j}}{(1-|x|)^{1-|x|}(|x-x_j|^{|x-x_j|})}.$$

One has  $\nabla^2 \log F_j(x) = \frac{-1}{1-|x|} U U^t$  where  $U_j = \sqrt{\frac{|x-x_j|}{1-x_j}}$  and  $U_k = 1/U_j$  for  $k \neq j$ , then,  $F_j$  is log-concave and  $\min \log F_j$  also. Note that  $\min_{1 \leq j \leq m} F_j$  is symmetric with respect to each hyperplane  $\{x_i = x_k\}$ , therefore  $\max_{|x| \leq 1} \min_{j \in \{1, \dots, m\}} F_j(x)$  is reached only if  $x_1 = \dots = x_m$ . As a result, the problem becomes a simple study of a function having one variable:

$$c_{n,m} \leq \max_{x \leq 1/m} \left[ \frac{(1-x)^{1-x}}{(1-mx)^{1-mx}((m-1)x)^{(m-1)x}} \right]^n$$

and this maximum is reached at the unique  $x_m \in ]0, 1/m[$  which is the real root of  $P_r(x) = (1-mx)^m - (m-1)^{m-1}x^{m-1}(1-x)$ . After some simplifications, we get  $c_{n,m} \leq \left(\frac{1-x_m}{1-mx_m}\right)^n < 2^n$ .

For  $m = 2$ ,  $x_2 = \frac{5-\sqrt{5}}{2}$  and  $\frac{1-x_2}{1-2x_2} = \varphi$  (golden ratio), for  $m = 3$  we get that  $c_{n,3} < (1.755)^n$ .  $\square$

For the lower bound of  $c_{n,m}$ , many results have been obtained in [2, 3, 9, 11, 10, 20]. For the case  $m = 2$  we use methods from [2, 10, 11] and extend them to  $m \geq 2$ . The idea is to represent volumes in (25) as volumes of a projection of  $A$  to subspaces of lower dimension and optimize the lower bound with respect to those dimensions. We remark that (25) is equivalent to

$$\frac{|A|^{m-1} |A + B_1 + \dots + B_m|}{\prod_{k=1}^m |A + B_k|} \leq c_{n,m}.$$

So any choice of  $A, B_1, \dots, B_m$  gives a lower bound of  $c_{n,m}$ . Let  $\Gamma_2$  be the set of vector subspaces  $\{E_1, E_2\}$  such that  $\dim(E_1) = i, \dim(E_2) = j, 0 < i, j \leq n, i + j > n$ . It is obtained in [2] that for any  $A$  convex body in  $\mathbb{R}^n$ ,

$$c_{n,2} \geq \max_{A, \Gamma_2} \frac{|A|_n |P_{E_1 \cap E_2} A|_{i+j-n}}{|P_{E_1} A|_i |P_{E_2} A|_j} = \frac{\binom{i}{i+j-n} \binom{j}{i+j-n}}{\binom{n}{i+j-n}} \quad (28)$$

where  $A$  is a convex body in  $\mathbb{R}^n$  and  $P_E A$  is the projection of  $A$  onto the subspace  $E$ . The equality case in (28) is reached for a sharp choice of  $E_1, E_2$  and  $A$ . The authors of [11] conclude that

$$c_{n,2} \geq \max_{\substack{0 < i, j \leq n \\ i+j > n}} \frac{\binom{i}{i+j-n} \binom{j}{i+j-n}}{\binom{n}{i+j-n}} \approx \frac{2}{\sqrt{\pi n}} \left(\frac{4}{3}\right)^n.$$

By following the same idea, we get a lower bound for  $c_{n,m}$ .

**Theorem 10.** Let  $\alpha_1, \dots, \alpha_m \in \{1, \dots, n\}$  be integers and  $d = |\alpha| - (m-1)n > 0$ . Let  $A \subset \mathbb{R}^n$  be a convex body and  $\Gamma_m$  the set of collections of vector subspaces  $\{E_1, \dots, E_m\}$  such that for  $i = 1, \dots, m$ ,  $\dim(E_i) = \alpha_i$  and  $E_i^\perp \subset \bigcap_{j \in [m] \setminus \{i\}} E_j$ . Then

$$c_{n,m} \geq \max_{\Gamma_m, A} \max_{\text{convex body}} \frac{|A|_n |P_{\bigcap_{i=1}^m E_i} A|_d}{\prod_{i=1}^m |P_{E_i} A|_{\alpha_i}}. \quad (29)$$

*Proof:* Let  $(e_1, \dots, e_n)$  be the canonical basis of  $\mathbb{R}^n$ . For  $i = 1, \dots, m$ , let  $B_i = \sum_{k \in [n] \setminus K_i} [0, e_k]$ , where  $K_i = \{\alpha_1 + \dots + \alpha_{i-1} + k \mid 1 \leq k \leq \alpha_i\}$ . Thus, for any  $i = 1, \dots, m$  and any  $t > 0$ , one has

$$|A + tB_i|_n \underset{t \rightarrow \infty}{\sim} t^{n-\alpha_i} |P_{E_i} A|_{\alpha_i}$$

and

$$|A + tB_1 + \cdots + tB_m|_n \underset{t \rightarrow \infty}{\sim} t^{nm-|\alpha|} \left| P_{\bigcap_{i=1}^m B_i} A \right|_d.$$

Let  $t$  to infinity, then we get

$$|A|_n^{m-1} |P_{\bigcap_{i=1}^m E_i} A|_d \leq c_{n,m} \prod_{i=1}^m |P_{E_i} A|_{\alpha_i}.$$

The desired result follows.  $\square$

**Corollary 11.** Let  $\alpha_1, \dots, \alpha_m \in [1, n]$  be integers and  $d = |\alpha| - (m-1)n > 0$ . Then

$$c_{n,m} \geq \max_{\alpha} \frac{\prod_{i=1}^m \binom{\alpha_i}{d}}{\binom{n}{d}}. \quad (30)$$

*Proof:* Let  $B_{\infty}(d) = \sum_{k=n-d+1}^n [-e_k, e_k]$ . For  $i = 1, \dots, m$ , let

$$J_i = [n - \alpha_1 + \dots + n - \alpha_{i-1} + 1 \quad ; \quad n - \alpha_1 + \dots + n - \alpha_i],$$

$B_1(i) = \text{conv}_{J_i}(\pm e_k)$ , and  $E_i = \text{vect}\{e_k, k \in [n] \setminus J_i\}$ . If  $A = \text{conv}\{\sum_{i=1}^m B_1(i), B_{\infty}(d)\}$ , we get that

$$|A| = \prod_{i=1}^m |B_1(i)| \times |B_{\infty}(d)| / \binom{n}{d} \quad ; \quad |P_{\bigcap_{i=1}^m E_i} A| = |B_{\infty}(d)|$$

and for  $i = 1, \dots, m$

$$|P_{E_i} A| = \prod_{k \in m \setminus \{i\}} |B_1(k)| \times |B_{\infty}(d)| / \binom{\alpha_i}{d}.$$

Thus simplifying the volumes we get,

$$\frac{|A|_n^{m-1} |P_{\bigcap_{i=1}^m E_i} A|_d}{\prod_{i=1}^m |P_{E_i} A|_{\alpha_i}} = \frac{\prod_{i=1}^m \binom{\alpha_i}{d}}{\binom{n}{d}^{m-1}},$$

for all  $\alpha$  satisfying the given hypothesis. Hence the result follows.  $\square$

*Proof of Theorem 8 (ii) (lower bound):* For the lower bound, Corollary 11 allows to say that

$$c_{n,m} \geq \max_{\substack{\alpha > 0 \\ d = |\alpha| - (m-1)n > 0}} \frac{\prod_{i=1}^m \binom{\alpha_i}{d}}{\binom{n}{d}^{m-1}}.$$

Now, using the Stirling formula, one has

$$\frac{\prod_{i=1}^m \binom{\alpha_i}{d}}{\binom{n}{d}^{m-1}} \geq \frac{1}{\sqrt{2\pi d}} \prod_{i=1}^m \sqrt{\frac{\alpha_i}{\alpha_i - d}} \times \frac{1}{d^d} \left[ \sqrt{\frac{n-d}{n}} \frac{(n-d)^{n-d}}{n^n} \right]^{m-1} \prod_{i=1}^m \frac{\alpha_i^{\alpha_i}}{(\alpha_i - d)^{\alpha_i - d}} e^{o(1)}.$$

For  $i = 1, \dots, m$ , one assume that  $\alpha_i = nx_i$  and  $d = ny$ , then  $y = |x| - (m-1)$  and

$$\max_{\alpha} \left( \frac{1}{d^d} \left[ \frac{(n-d)^{n-d}}{n^n} \right]^{m-1} \prod_{i=1}^m \frac{\alpha_i^{\alpha_i}}{(\alpha_i - d)^{\alpha_i - d}} \right) = \max_x f(x)^n,$$

where

$$f(x_1, \dots, x_m) = \frac{(1-y)^{(1-y)(m-1)}}{y^y} \prod_{i=1}^m \frac{x_i^{x_i}}{(x_i - y)^{x_i - y}}.$$

By studying this function, we easily find that  $\max f = \frac{x^m}{y} > 1$  where

$$x = (m-1) \left[ m - \left( \frac{m-1}{m} \right)^{m-1} \right]^{-1}$$

and  $y = m(x - 1) + 1$ . It follows that

$$\begin{aligned} c_{n,m} &\geq \max \left( \frac{x^m}{y} \right)^n \frac{1}{\sqrt{2\pi n y}} (\sqrt{1-y})^{m-1} \left( \sqrt{\frac{x}{x-y}} \right)^m (1 + o(1)) \\ &= \frac{1}{\sqrt{2\pi n}} \left( \frac{x^m}{y} \right)^{n+1/2} \sqrt{\frac{(1-y)^{m-1}}{(x-y)^m}} (1 + o(1)). \end{aligned}$$

Besides,  $\frac{x^m}{y} = \left[ 1 - \frac{1}{m-1} \left( \frac{m-1}{m} \right)^m \right]^{1-m}$  is a increasing sequence with respect to  $m$  while

$$\frac{(1-y)^{m-1}}{(x-y)^m} = \frac{\left( \frac{m-1}{m} \right)^{m-1} - \frac{1}{m}}{1 - \frac{1}{m} - \left( \frac{m-1}{m} \right)^m} \quad \text{is decreasing.}$$

Thus

$$\frac{e}{\sqrt{e-1}} \left( \frac{4}{3} \right)^{n+1/2} < \left( \frac{x^m}{y} \right)^{n+1/2} \sqrt{\frac{(1-y)^{m-1}}{(x-y)^m}} < \sqrt{6} (e^{e^{-1}})^{n+1/2},$$

it leads to

$$c_{n,m} \geq \frac{1}{\sqrt{2\pi n}} \frac{e}{\sqrt{e-1}} \left( \frac{4}{3} \right)^{n+1/2}.$$

For  $m = 2$ ,  $c_{n,2} \geq \frac{2}{\sqrt{\pi n}} \left( \frac{4}{3} \right)^n (1 + o(1))$  which was found in [11]. For  $m = 3$ , we find

$$c(n, 3) \geq \frac{81}{4\sqrt{115}\sqrt{\pi n}} \left( \frac{729}{529} \right)^n (1 + o(1)) \approx \frac{1.89}{\sqrt{\pi n}} (1.378)^n.$$

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