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MATRICES OF LINEAR FORMS OF CONSTANT RANK FROM VECTOR BUNDLES ON PROJECTIVE SPACES

LAURENT MANIVEL AND ROSA M. MIRÓ-ROIG

ABSTRACT. We consider the problem of constructing matrices of linear forms of constant rank by focusing on the associated vector bundles on projective spaces. Important examples are given by the classical Steiner bundles, as well as some special (duals of) syzygy bundles that we call Drézet bundles. Using the classification of globally generated vector bundles with small first Chern class on projective spaces, we are able to describe completely the indecomposable matrices of constant rank up to six; some of them come from rigid homogeneous vector bundles, some other from Drézet bundles related either to plane quartics or to instanton bundles on \mathbb{P}^3 .

It is a classical problem in both algebraic geometry and linear algebra to construct linear spaces of matrices of constant rank (constant outside the origin). The seminal modern reference for this problem is [13], as well as for the closely related problem of constructing linear spaces of matrices of bounded rank (in the sense that the rank is never maximal). It was followed by an important literature, to which this paper is a contribution.

Two dimensional spaces of matrices (including those of constant or bounded rank, of course) were classified by Kronecker and Weierstrass. Atkinson [3] classified spaces of bounded rank at most three. He showed that there is a unique primitive three dimensional space of constant rank two, $\Lambda^2\mathbb{C}^3 \subset \text{End}(\mathbb{C}^3)$, and a unique primitive four dimensional space of constant rank three corresponding to the natural inclusion $\mathbb{C}^4 \hookrightarrow \text{Hom}(\mathbb{C}^4, \Lambda^2\mathbb{C}^4)$. All other spaces are obtained by restricting these two subspaces. More generally, $\mathbb{C}^n \hookrightarrow \text{Hom}(\Lambda^k\mathbb{C}^n, \Lambda^{k+1}\mathbb{C}^n)$ has always constant rank. This example, dating at least back to Westwick [35], was vastly generalized in [25].

Beyond that, only isolated examples were known. Westwick in [36] produced a 4-dimensional space of skew-symmetric 10×10 matrices of constant rank 8 that remained mysterious until 2012 when Boralevi, Faenzi, and Mezzetti [4] gave an explanation for it in terms of instanton bundles and generalized it to a family of such. They also found four dimensional families of 14×14 skew-symmetric matrices of constant rank 12.

Eisenbud and Harris [13] also observed that spaces of syzygies furnish spaces of bounded rank, at least in the case where the syzygies in some degree are only linear. We can even get matrices of constant rank by restricting to suitable linear subspaces. Many instances of such linear syzygies are known,

suggesting that the classification of matrices of constant rank must be a wild problem.

There is a strong relationship with the study of vector bundles on projective spaces, another classical topic that attracts considerable attention. Indeed, a vector space of dimension $n + 1$ of matrices of size $a \times b$ can be seen as a matrix with linear entries, or equivalently as a morphism of sheaves $\psi : \mathcal{O}_{\mathbb{P}^n}^{\oplus a} \rightarrow \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus b}$. If the rank is constant, equal to e , the image of this morphism is a vector bundle \mathcal{E} of rank e . Letting \mathcal{K} and \mathcal{C} denote the kernel and cokernel bundles, respectively, we get a diagram

$$\begin{array}{ccccc}
 & & \mathcal{K} & & \\
 & & \searrow & & \\
 & & \mathcal{O}_{\mathbb{P}^n}^{\oplus a} & \xrightarrow{\psi} & \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus b} & \nearrow & \mathcal{C} \\
 & & \searrow & & \nearrow & & \\
 & & \mathcal{E} & & & &
 \end{array}$$

whose diagonals are short exact sequences. The vector bundle \mathcal{E} has very special properties, in particular:

- \mathcal{E} and $\mathcal{E}^\vee(1)$ are generated by global sections;
- as a consequence, \mathcal{E} is uniform, in the sense that its restriction to every line $L \subset \mathbb{P}^n$ splits in the same way, as

$$\mathcal{E}|_L \simeq \mathcal{O}_L(1)^{\oplus c_1(E)} \oplus \mathcal{O}_L^{\oplus (e - c_1(E))}.$$

Uniform vector bundles have been classified up to rank $e \leq n + 1$. In this range, for $n \geq 3$ they are sums of line bundles and the tautological quotient bundle Q or its dual (see [19] and references therein. In that paper the same result is conjectured to hold for $n \geq 5$ and $e < 2n$). The general philosophy is that there should exist very few uniform vector bundles on \mathbb{P}^n of small rank, but they are much easier to construct when the rank is large. Conversely, if \mathcal{E} is a rank e vector bundle on \mathbb{P}^n , such that \mathcal{E} and $\mathcal{E}^\vee(1)$ are generated by global sections (so that in particular \mathcal{E} is uniform), the natural morphism

$$\psi_{\mathcal{E}} : H^0(\mathbb{P}^n, \mathcal{E}) \otimes \mathcal{O}_{\mathbb{P}^n} \rightarrow H^0(\mathbb{P}^n, \mathcal{E}^\vee(1))^\vee \otimes \mathcal{O}_{\mathbb{P}^n}(1)$$

has constant rank e . This yields a nice recipe to construct matrices of constant rank, that we have applied concretely on a series of significant examples.

The point of view chosen in this paper is to understand families of matrices of constant rank in terms of the corresponding vector bundles and their moduli spaces. Indeed, if we have family of matrices of constant rank, parametrized by a certain scheme \mathcal{X} , there is an induced morphism $\mathcal{X} \rightarrow \text{Quot}(\mathbb{P}^n)$, mapping $x \in \mathcal{X}$ to a rank e vector bundle \mathcal{E}_x on \mathbb{P}^n . If moreover \mathcal{E}_x is semistable for each $x \in \mathcal{X}$, we even get a morphism

$\mathcal{X} \rightarrow \mathcal{M}(e, c)$ to the coarse moduli space of rank e semistable vector bundles on \mathbb{P}^n with total Chern class c . Conversely, we deduce that there is an open subset of the space of matrices of constant rank e that is essentially parametrized by some locally closed subset $\mathcal{M}(e, c)_{gen}$ of $\mathcal{M}(e, c)$. Indeed, after restricting if necessary to an open subset of the family, we may suppose that $h = h^0(\mathcal{E}_x)$ and $k = h^0(\mathcal{E}_x^\vee(1))$ are constant. Then locally, $\psi_{\mathcal{E}_x}$ only deforms to morphisms of the same type among spaces of matrices of size $h \times k$ and of constant rank e . A similar conclusion holds for matrices of size $a \times b$ with $a \leq h$ and $b \leq k$ giving rise to vector bundles from the same family; we get a moduli space which maps to $\mathcal{M}(e, c)_{gen}$ with fibers that are open subsets of the product of Grassmannians $Gr(a, h) \times Gr(b, k)$.

Our main results are classification theorems for matrices of linear forms of constant rank $e \leq 6$. To obtain these results we build on the classifications of globally generated vector bundles on projective spaces with small first Chern class that were obtained some years ago [2, 27, 1, 31, 32]. Given such a vector bundle \mathcal{E} , the main problem becomes: can $\mathcal{E}^\vee(1)$ also be globally generated? Important instances are given by the so-called Steiner bundles, defined by short exact sequences of type

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus c} \rightarrow \mathcal{O}_{\mathbb{P}^n}^{\oplus e+c} \rightarrow \mathcal{E} \rightarrow 0,$$

a family of vector bundles that has already been well-studied [11, 28]. A closely related family is that of vector bundles defined by an exact sequence of type

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(-c) \rightarrow \mathcal{O}_{\mathbb{P}^n}^{\oplus e+1} \rightarrow \mathcal{E} \rightarrow 0,$$

which are special classes of (duals of) syzygy bundles; we call them Drézet bundles because they were used by Drézet to construct examples of uniform, non-homogeneous vector bundles [12]. After a section of preliminaries, we study Steiner and Drézet bundles under our perspective. Then we focus on vector bundles with first Chern class $c = 2$ and we deduce a classification of matrices of linear forms of constant rank four or five.

Theorem 1. *Given an indecomposable matrix of linear forms of constant rank 4, the associated vector bundle \mathcal{E} must be, up to switching with $\mathcal{E}^\vee(1)$, either*

- (1) ($c_1 = 1$) the quotient bundle Q on \mathbb{P}^4 , or
- (2) ($c_1 = 2$) the unique (up to isomorphism) Drézet bundle of rank four on \mathbb{P}^2 defined by a non-degenerate conic.

For an indecomposable matrix of linear forms of constant rank 5, the associated vector bundle \mathcal{E} must be, up to switching with $\mathcal{E}^\vee(1)$,

- (1) ($c_1 = 1$) the quotient bundle Q on \mathbb{P}^5 , or
- (2) ($c_1 = 2$) the unique homogeneous rank five Drézet bundle on \mathbb{P}^2 .

The corresponding matrices are described in subsection 4.2 and 4.3. Note the surprising fact that in rank five, the possible vector bundles are unique. In rank four they are unique up to isomorphism. Also the homogeneous

bundles considered in [25] play an important rôle in the classification. But when the rank increases there are certainly many more possibilities. Already in rank six some vector bundles with non-trivial moduli spaces appear in the classification:

Theorem 2. *Given an indecomposable matrix of linear forms of constant rank 6, the associated vector bundle \mathcal{E} must be, up to switching with $\mathcal{E}^\vee(1)$, of one the following types:*

- (1) ($c_1 = 1$) the quotient bundle Q on \mathbb{P}^6 , or
- (2) ($c_1 = 2$) a Drézet bundle on \mathbb{P}^3 defined by a net of quadrics, or
- (3) ($c_1 = 3$) the second exterior power $\wedge^2 Q$ on \mathbb{P}^4 ,
- (4) ($c_1 = 3$) a Drézet bundle on \mathbb{P}^2 defined by the cubics apolar to a plane quartic, or
- (5) ($c_1 = 3$) a non-trivial extension of the quotient bundle Q on \mathbb{P}^2 by a rank 4 non-generic Drézet bundle.

Cases (1) and (3) are homogeneous and rigid, while cases (2) and (5) have interesting moduli. Indeed they are both closely related to plane quartics, and (4) has also nice unexpected connections with instanton bundles of charge 3 on \mathbb{P}^2 . Case (5) is the most surprising one, being built from a rank 4 Drézet bundle that gives rise to a matrix of linear forms of bounded but not constant rank: a defect that the non-triviality of the extension turns out to correct.

For all these cases we provide explicit matrices of linear forms of constant rank. In particular this will illustrate the nice fact that although we focus on the abstract study of the associated vector bundles, the method is actually quite concrete.

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1. PRELIMINARIES

1.1. Generalities. Giving a matrix of linear forms in $n+1$ variables, of size $a \times b$ and constant rank e , up to linear combinations of rows and columns, is equivalent to providing the following data:

- a vector bundle \mathcal{E} of rank e on \mathbb{P}^n , such that \mathcal{E} and $\mathcal{E}^\vee(1)$ are both generated by global sections;
- a subspace A of $H^0(\mathbb{P}^n, \mathcal{E})$, of dimension a , generating \mathcal{E} at every point;
- a subspace B of $H^0(\mathbb{P}^n, \mathcal{E}^\vee(1))$, of dimension b , generating $\mathcal{E}^\vee(1)$ at every point.

The most important datum is obviously the vector bundle \mathcal{E} . Once it is given, one can wonder how many sections are necessary to generate it at

every point. By general principles, $e + n$ general sections will always suffice. A more precise answer is provided by Segre classes. Indeed, given a subspace A of $H^0(\mathbb{P}^n, \mathcal{E})$ of dimension a , the evaluation morphism for sections of \mathcal{E} restricts to a morphism

$$ev_A : A \otimes \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{E}.$$

Suppose $a \geq e$. If ev_A is surjective, the kernel \mathcal{K} is a vector bundle of rank $a - e$, hence $c_{a-e+1}(\mathcal{K}) = 0$ and therefore $s_{a-e+1}(\mathcal{E}) = 0$. In general, the locus D_A where ev_A fails to be surjective has expected codimension $a - e + 1$, and this expected codimension is the correct one for A generic. In particular, ev_A is everywhere surjective if A is generic of dimension $a \geq e + n$. When $a < e + n$ and D_A has the expected codimension, the Thom-Porteous formula asserts that the degree of D_A is the degree of the Segre class $s_{a-e+1}(\mathcal{E})$. This applies in particular when D_A is empty, and we get a contradiction if $s_{a-e+1}(\mathcal{E}) \neq 0$. We conclude that:

- the minimal number of sections needed to generate \mathcal{E} at all points is

$$a_{min}(\mathcal{E}) := e - 1 + \min\{k \geq 0 \text{ such that } s_k(\mathcal{E}) = 0\};$$

- any general subspace A of $H^0(\mathbb{P}^n, \mathcal{E})$, of dimension $a \geq a_{min}(\mathcal{E})$, generates \mathcal{E} at every point.

Similar considerations apply to $\mathcal{E}^\vee(1)$. This means that we can, in a sense, focus on the case where $A = H^0(\mathbb{P}^n, \mathcal{E})$ and $B = H^0(\mathbb{P}^n, \mathcal{E}^\vee(1))$. The corresponding matrix of linear forms can be described invariantly as given by the natural pairing defined by the composition

$$m_{\mathcal{E}} : H^0(\mathbb{P}^n, \mathcal{E}) \otimes H^0(\mathbb{P}^n, \mathcal{E}^\vee(1)) \rightarrow H^0(\mathbb{P}^n, \mathcal{E} \otimes \mathcal{E}^\vee(1)) \rightarrow H^0(\mathbb{P}^n, \mathcal{O}(1)).$$

If we choose basis s_1, \dots, s_N of $H^0(\mathbb{P}^n, \mathcal{E})$ and $\sigma_1, \dots, \sigma_M$ of $H^0(\mathbb{P}^n, \mathcal{E}^\vee(1))$, we get the matrix $M_{\mathcal{E}}$ whose entry (i, j) is the linear form $m_{\mathcal{E}}(s_i, \sigma_j)$. This matrix has constant rank e . Moreover, it remains of constant rank e when we take any $a \times b$ submatrix given by a general linear combinations of rows and b general linear combinations of columns, as long as $a \geq a_{min}(\mathcal{E})$ and $b \geq b_{min}(\mathcal{E}) := a_{min}(\mathcal{E}^\vee(1))$. For special linear combinations of rows and columns, or for $a < a_{min}(\mathcal{E})$ or $b < b_{min}(\mathcal{E})$, the rank will no longer be constant but only bounded by e .

Notice the complete symmetry between $\mathcal{F} = \mathcal{E}^\vee(1)$ and $\mathcal{E} = \mathcal{F}^\vee(1)$. Exchanging these two bundles, once basis are chosen for their spaces of global sections, simply amounts to taking the transpose of the associated matrix of linear forms. Since $c_1(\mathcal{E}^\vee(1)) = e - c_1(\mathcal{E})$, we can in particular always suppose that $2c_1(\mathcal{E}) \leq e$, which will simplify the classifications.

Remark. Eisenbud and Harris proposed in [13] a specific terminology for linear spaces of matrices of bounded (i.e. everywhere non-maximal) rank, by distinguishing *decomposable and strongly indecomposable, primitive, unextendable, unliftable* and finally *basic* spaces. For the constant rank case that we discuss in this paper, we focus on the indecomposability of the associated bundle. It would certainly be interesting to understand how our

results should be interpreted in the wider setting of spaces of matrices of bounded rank.

1.2. Uniformity. We have already mentioned the following important fact. We include a proof for the reader's convenience. Recall that a vector bundle \mathcal{E} is *uniform* if for every line $L \subset \mathbb{P}^n$, the restriction

$$\mathcal{E}_L \simeq \mathcal{O}_L(m_1) \oplus \cdots \oplus \mathcal{O}_L(m_e)$$

for some integers m_1, \dots, m_e that do not depend on L . We say \mathcal{E} is 1-uniform if $0 \leq m_1, \dots, m_e \leq 1$. In this case the number of integers k such that $m_k = 1$ is equal to the first Chern class $c_1(\mathcal{E})$.

Proposition 3. *Suppose \mathcal{E} is a rank e vector bundle on \mathbb{P}^n , such that \mathcal{E} and $\mathcal{E}^\vee(1)$ are both generated by global sections. Then*

- (1) \mathcal{E} is 1-uniform;
- (2) if $e \leq n + 1$ and \mathcal{E} is indecomposable, then $\mathcal{E} = \mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(1), Q$ or $Q^\vee(1)$.

Proof. (1) Being globally generated is preserved by restriction to a subvariety. So if L is a line and

$$\mathcal{E}_L \simeq \mathcal{O}_L(m_1(L)) \oplus \cdots \oplus \mathcal{O}_L(m_e(L)),$$

then $m_k(L) \geq 0$ for any k . Similarly, if $\mathcal{E}^\vee(1)$ is also generated by global sections, we get that $m_k(L) \leq 1$ for any k . Since the sum $m_1(L) + \cdots + m_e(L) = c_1(\mathcal{E})$ is fixed, the first assertion follows.

(2) Uniform bundles of small rank have been classified [33, 15, 16]. A uniform vector bundle \mathcal{E} of rank $e \leq n + 1$ is a direct sum of line bundles and twists of the quotient bundle Q and its dual. If it is moreover 1-uniform, the only possible simple factors are $\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(1), Q$ and $Q^\vee(1)$. \square

Example 4. When $\mathcal{E} = Q$, let $V = H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))^\vee$. Then

$$H^0(\mathcal{E}) = V, \quad H^0(\mathcal{E}^\vee(1)) = \wedge^2 V^\vee,$$

and $m_{\mathcal{E}}$ is the natural pairing

$$V \otimes \wedge^2 V^\vee \rightarrow V^\vee, \quad v \otimes \omega \mapsto \omega(v, \bullet).$$

This classical example was vastly generalized in [25].

1.3. Factoring out sections. When \mathcal{E} is globally generated of rank $e > n$, a general section $s \in H^0(\mathbb{P}^n, \mathcal{E})$ does not vanish anywhere and therefore defines a vector bundle \mathcal{F} of rank $f = e - 1$, with an exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n} \xrightarrow{s} \mathcal{E} \rightarrow \mathcal{F} \rightarrow 0.$$

In particular, \mathcal{F} is globally generated and $H^0(\mathbb{P}^n, \mathcal{F}) = H^0(\mathbb{P}^n, \mathcal{E})/\mathbb{C}s$. When $\mathcal{E}^\vee(1)$ is generated by global sections, can it also be the case of $\mathcal{F}^\vee(1)$? We have a short exact sequence

$$0 \rightarrow \mathcal{F}^\vee(1) \rightarrow \mathcal{E}^\vee(1) \rightarrow \mathcal{O}_{\mathbb{P}^n}(1) \rightarrow 0,$$

showing that $H^0(\mathbb{P}^n, \mathcal{F}^\vee(1)) \subset H^0(\mathbb{P}^n, \mathcal{E}^\vee(1))$ is the kernel of the morphism

$$m_{\mathcal{E}}(s, \bullet) : H^0(\mathbb{P}^n, \mathcal{E}^\vee(1)) \longrightarrow H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) = V^\vee.$$

Suppose that the morphism $m_{\mathcal{E}}(s, \bullet)$ is surjective. In order to check whether $\mathcal{F}^\vee(1)$ is generated by global sections, we can use the following diagram, where the horizontal and vertical short sequences are exact:

$$\begin{array}{ccccc} & & \mathcal{K}_{\mathcal{E}^\vee(1)} & \longrightarrow & Q^\vee \\ & & \downarrow & & \downarrow \\ H^0(\mathcal{F}^\vee(1)) \otimes \mathcal{O}_{\mathbb{P}^n} & \longrightarrow & H^0(\mathcal{E}^\vee(1)) \otimes \mathcal{O}_{\mathbb{P}^n} & \longrightarrow & V^\vee \otimes \mathcal{O}_{\mathbb{P}^n} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{F}^\vee(1) & \longrightarrow & \mathcal{E}^\vee(1) & \longrightarrow & \mathcal{O}_{\mathbb{P}^n}(1) \end{array}$$

We denoted by $\mathcal{K}_{\mathcal{E}^\vee(1)}$ the kernel of the evaluation morphism for $\mathcal{E}^\vee(1)$. By the snake lemma, we deduce:

Proposition 5. *The evaluation morphism*

$$H^0(\mathcal{F}^\vee(1)) \otimes \mathcal{O}_{\mathbb{P}^n} \rightarrow \mathcal{F}^\vee(1)$$

for $\mathcal{F}^\vee(1)$ is surjective at $[v]$ if and only if the natural map

$$V/\mathbb{C}v \longrightarrow \mathcal{K}_{\mathcal{E}^\vee(1),v}^\vee, \quad \bar{w} \mapsto (\sigma \mapsto m_{\mathcal{E}}(s, \sigma)(w))$$

is injective.

In concrete terms, this means that we can choose a basis s_1, \dots, s_N of $H^0(\mathcal{E})$ with $s_1 = s$, and a basis $\sigma_1, \dots, \sigma_M$ of $H^0(\mathcal{E}^\vee(1))$ such that $m_{\mathcal{E}}(s, \sigma_1), \dots, m_{\mathcal{E}}(s, \sigma_{n+1})$ form a basis of $H^0(\mathcal{O}_{\mathbb{P}^n}(1))$, say x_0, \dots, x_n , while $m_{\mathcal{E}}(s, \sigma_k) = 0$ for $k > n+1$, meaning that $\sigma_{n+2}, \dots, \sigma_M$ form a basis of $H^0(\mathcal{F}^\vee(1))$. The corresponding matrix $M_{\mathcal{E}}$ is then of the form

$$M_{\mathcal{E}} = \begin{pmatrix} x_0 & * & * & * & * & * \\ x_1 & * & * & * & * & * \\ \dots & * & * & * & * & * \\ \dots & * & * & * & * & * \\ x_n & * & * & * & * & * \\ 0 & & & & & \\ \dots & & & & & \\ \dots & & & & M_{\mathcal{F}} & \\ \dots & & & & & \\ 0 & & & & & \end{pmatrix}$$

where the matrix $M_{\mathcal{F}}$ has constant rank $f = e - 1$. We can apply this process several times if we want to factor out several sections of \mathcal{E} , and the innocent looking criterion given by the previous proposition is easy to apply concretely.

Conversely, the short exact sequence $0 \rightarrow \mathcal{F}^\vee(1) \rightarrow \mathcal{E}^\vee(1) \rightarrow \mathcal{O}_{\mathbb{P}^n}(1) \rightarrow 0$ indicates that we can expect that $\mathcal{E}^\vee(1)$ is globally generated when $\mathcal{F}^\vee(1)$ is. In other words, if $\mathcal{E}^\vee(1)$ fails to be generated, we cannot expect to correct this failure by extracting sections. It is not true in general that extensions of globally generated vector bundles must be globally generated; but this can be checked to be true in some concrete situations, and the previous principle will apply. A typical case will be that of ordinary Drézet bundles.

In the next two sections we discuss two different types of vector bundles, which have been studied a lot in the literature, but that we consider under our specific point of view: Steiner bundles first, and then those we call Drézet bundles, which are special instances of (duals of) the more general syzygy bundles.

2. STEINER BUNDLES

Steiner bundles were defined in [11] as vector bundles \mathcal{E} on \mathbb{P}^n arising from exact sequences of the form

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus c} \rightarrow \mathcal{O}_{\mathbb{P}^n}^{\oplus e+c} \rightarrow \mathcal{E} \rightarrow 0.$$

2.1. First properties. It may be more natural to rewrite the previous exact sequence as

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^V}(-1) \otimes B_c \xrightarrow{M} \mathcal{O}_{\mathbb{P}^V} \otimes W_{e+c} \rightarrow \mathcal{E} \rightarrow 0,$$

for a matrix M defined by a tensor in $V^\vee \otimes B_c^\vee \otimes W_{e+c}$.

Using the Beilinson spectral sequence, the property of being Steiner was characterized in [11] by certain cohomological conditions; as a consequence, this is an open property. It was proved by Bohnhorst and Spindler (cited erroneously in [11]) that any rank n Steiner bundle on \mathbb{P}^n is stable (note that n is the smallest possible rank of a Steiner bundle, since all its Chern classes are nonzero). This was extended in [8, Theorem 1.7] to Steiner bundles such that $n \leq e < (n-1)c$ (actually under a slightly less restrictive condition).

Remark 6. From the defining exact sequence of a Steiner bundle, we immediately get that its Chern classes are $c_k = \binom{c+k-1}{k}$, independent of the rank. From these relations it is in general easy to check that a given vector bundle is not Steiner.

Those Steiner bundles we are interested in are necessarily 1-uniform, and as such have been studied in [28], where the following statement is proved:

Proposition 7. *If a Steiner bundle \mathcal{E} is 1-uniform and has no rank one summand, then $c + 2n - 2 \leq e \leq cn$. All these values are possible, but for $e = cn$ the only possibility is $\mathcal{E} = Q^{\oplus c}$.*

It is therefore quite natural to construct Steiner bundles of rank smaller than cn by factoring out sections from $Q^{\oplus c}$. The commutative diagram (where we omit zeroes)

$$\begin{array}{ccccc}
& & \mathcal{O}_{\mathbb{P}^n}(-1) \otimes B_c & \xlongequal{\quad} & \mathcal{O}_{\mathbb{P}^n}(-1) \otimes B_c \\
& & \downarrow & & \downarrow \\
\mathcal{O}_{\mathbb{P}^n} \otimes A_k & \longrightarrow & \mathcal{O}_{\mathbb{P}^n} \otimes V \otimes B_c & \longrightarrow & \mathcal{O}_{\mathbb{P}^n} \otimes W_{e+c} \\
\parallel & & \downarrow & & \downarrow \\
\mathcal{O}_{\mathbb{P}^n} \otimes A_k & \longrightarrow & Q \otimes B_c & \longrightarrow & \mathcal{E}
\end{array}$$

exchanges the subspace A_k of $V \otimes B_c$ with subspace B_c of $V^\vee \otimes W_{e+c}$, where $e = cn - k$. This is an instance of the classical *castling transform*.

2.2. Global generation. From our perspective, the main problem is to understand when $\mathcal{E}^\vee(1)$ is globally generated. For the general Steiner bundle, an essentially complete answer to this question was obtained in [17].

Proposition 8. *Suppose that $e \geq \frac{n+1}{2}c + 1$. If $c \leq 3$ suppose moreover that $e \geq n$ for $c = 1$, $e \geq 2n$ for $c = 2$, $e \geq 2n + 1$ for $c = 3$ (and even $e \geq 6$ for $c = 3$ and $n = 2$).*

Then for the generic Steiner bundle \mathcal{E} of rank e on \mathbb{P}^n with $c = c_1(\mathcal{E})$, the twisted dual $\mathcal{E}^\vee(1)$ is generated by global sections.

Moreover $\mathcal{E}^\vee(1)$ is acyclic and its sections are given by either one of the exact sequences

$$\begin{aligned}
0 &\rightarrow H^0(\mathcal{E}^\vee(1)) \rightarrow W_{e+c}^\vee \otimes V^\vee \rightarrow B_c^\vee \otimes S^2V^\vee \rightarrow 0, \\
0 &\rightarrow H^0(\mathcal{E}^\vee(1)) \rightarrow B_c^\vee \otimes \wedge^2V^\vee \rightarrow A_k^\vee \otimes V^\vee \rightarrow 0.
\end{aligned}$$

Remark 9. A very similar situation, but with vector bundles $\mathcal{E}^\vee(1)$ which do not have enough sections to be generated, was discussed in [18] and provides examples of spaces of matrices of bounded rank.

Example 10. Suppose that $n = 2$, $e = 5$, $c = 3$. By castling transform, \mathcal{E} is defined by an exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^2} \rightarrow Q \otimes B_3 \rightarrow \mathcal{E} \rightarrow 0$$

where the first morphism is given by a tensor in $V \otimes B_3$. In general this tensor has maximal rank and identifies B_3 with V^\vee , and then $H^0(\mathcal{E})$ with $\mathfrak{sl}(V)$. It follows that \mathcal{E} is in fact homogeneous, and that $H^0(\mathcal{E}^\vee(1))$ can be identified with $S^2V \otimes \det(V)^\vee$. Moreover $\mathcal{E}^\vee(1)$ is globally generated although it has only 6 sections, and one can check that $s_2(\mathcal{E}^\vee(1)) = 0$ (in fact $\mathcal{E}^\vee(1)$ is a Steiner bundle). So we get a matrix of constant rank, which can be interpreted as the equivariant morphism $V \rightarrow \text{Hom}(S_2V, S_{21}V)$. This is one of the toy examples generalized in [25].

For $c = 1$, the condition $e \geq n$ is already necessary for the mere existence of a Steiner bundles, as we already mentioned. For $c = 2$, the condition that $e \geq 2n$ is necessary, in view of the following (see also Proposition 7).

Example 11. Consider the direct sum of two copies of the quotient bundle Q and let us try to factor out a general section $s \in H^0(Q \oplus Q) = V \oplus V$. So $s = (s_1, s_2)$ for two independent vectors $s_1, s_2 \in V$. Applying Proposition 5, we see that the corresponding bundle $\mathcal{E}^\vee(1)$ is generated by global sections at $[v]$ if and only if the natural morphism

$$V/\mathbb{C}v \rightarrow \wedge^2(V/\mathbb{C}v) \oplus \wedge^2(V/\mathbb{C}v), \quad \bar{w} \mapsto (\bar{w} \wedge \bar{s}_1, \bar{w} \wedge \bar{s}_2)$$

is injective. But this fails! Indeed it clearly fails when $v = s_1$ since \bar{s}_2 is sent to zero. In fact it fails exactly on the projective line joining $[s_1]$ to $[s_2]$, and $\mathcal{E}^\vee(1)$ fails to be globally generated exactly on this line.

For $c = 3$ this problem does not happen.

Example 12. Consider the direct sum of three copies of the quotient bundle Q , and let us try to factor out a general section $s \in H^0(Q \oplus Q \oplus Q) = V \oplus V \oplus V$. So $s = (s_1, s_2, s_3)$ for three independent vectors $s_1, s_2, s_3 \in V$. Then the corresponding vector bundle $\mathcal{E}^\vee(1)$ is generated by global sections at $[v]$ if and only if the natural morphism

$$V/\mathbb{C}v \rightarrow \wedge^2(V/\mathbb{C}v)^{\oplus 3}, \quad \bar{w} \mapsto (\bar{w} \wedge \bar{s}_1, \bar{w} \wedge \bar{s}_2, \bar{w} \wedge \bar{s}_3)$$

is injective. Observe that $\bar{w} \wedge \bar{s}_1$ is zero exactly when $w \in \langle v, s_1 \rangle$. So if also $\bar{w} \wedge \bar{s}_2 = 0$ and $\bar{w} \wedge \bar{s}_3 = 0$, then

$$w \in \langle v, s_1 \rangle \cap \langle v, s_2 \rangle \cap \langle v, s_3 \rangle = \mathbb{C}v$$

for any v since s_1, s_2, s_3 are independent. So in this case $\mathcal{E}^\vee(1)$ is generated by global sections, with

$$H^0(\mathcal{E}^\vee(1)) = \left\{ (\omega_1, \omega_2, \omega_3) \in \wedge^2 V^\vee, \quad \omega_1(s_1, \bullet) + \omega_2(s_2, \bullet) + \omega_3(s_3, \bullet) = 0 \right\}.$$

A natural question is whether extracting sections from a decomposable vector bundle, we can get an indecomposable vector bundle. Consider once again an exact sequence

$$0 \rightarrow A_k \otimes \mathcal{O}_{\mathbb{P}^n} \rightarrow B_c \otimes Q \rightarrow \mathcal{E} \rightarrow 0$$

defined by a tensor $M \in A_k^\vee \otimes B_c \otimes V$. Note that \mathcal{E} is a vector bundle if and only if the image of the induced map $A_k \rightarrow B_c \otimes V$ does not contain any rank one tensor. Dualizing, we get $H^q(\mathcal{E}^\vee) = \delta_{q,1} A_k^\vee$, and after tensoring by Q and taking cohomology, we obtain the exact sequence

$$0 \rightarrow H^0(\mathcal{E}^\vee \otimes Q) \rightarrow B_c^\vee \rightarrow A_k^\vee \otimes V \rightarrow H^1(\mathcal{E}^\vee \otimes Q) \rightarrow 0.$$

If we suppose the morphism $B_c^\vee \rightarrow A_k^\vee \otimes V$ to be injective, we deduce the exact sequence

$$0 \rightarrow H^0(\text{End}(\mathcal{E})) \rightarrow \text{End}(A_k) \rightarrow B_c \otimes (A_k^\vee \otimes V)/B_c^\vee \rightarrow H^1(\text{End}(\mathcal{E})) \rightarrow 0.$$

Note that $B_c \otimes (A_k^\vee \otimes V)/B_c^\vee$ is the tangent space at B_c^\vee to the Grassmannian $Gr(c, A_k^\vee \otimes V)$. We can see this Grassmannian as the quotient of $A_k^\vee \otimes B_c \otimes V$ by $GL(B_c)$. Hence the following statement:

Proposition 13. *Let G denote the image of the group $GL(A_k) \times GL(B_c)$ inside $GL(A_k^\vee \otimes B_c)$. Suppose that $M \in A_k^\vee \otimes B_c \otimes V$ has finite stabilizer in G , and that the induced maps $A_k \rightarrow B_c \otimes V$ and $B_c^\vee \rightarrow A_k^\vee \otimes V$ are both injective.*

Then \mathcal{E} is simple, and in particular indecomposable.

A more precise simplicity criterion is given in [5]. Under our conditions we can also conclude that the moduli space or stack parametrizing deformations of \mathcal{E} is made of vector bundles of the same type, and is essentially the quotient of $A_k^\vee \otimes B_c \otimes V$ by $G \simeq GL(A_k) \times GL(B_c)/\mathbb{C}^*$. Up to the action of $GL(V)$ we get the quotient of $A_k^\vee \otimes B_c \otimes V$ by $GL(A_k) \times GL(B_c) \times GL(V)$, which has been much studied.

Remark 14. Finiteness of the generic stabilizer is to be expected in general, and in fact it holds as soon as it is compatible with dimensions. A general classification of irreducible representations of semisimple complex Lie groups with generic stabilizer of positive dimension was obtained in [14].

Example 15. A well-known case is $k = 2, c = n = 3$, for which we get the variety of twisted cubics in \mathbb{P}^3 , as a parameter space for a family of rank seven vector bundles. Moreover the action of PGL_4 on this quotient is quasi-homogeneous, with an open subset corresponding to the smooth twisted cubics; hence an essentially unique matrix of constant rank 7 that we are going to compute. The starting point is the presentation of the ideal sheaf of the twisted cubic, as being generated by the 2×2 minors of

$$\begin{pmatrix} x_1 & x_2 & x_3 \\ x_2 & x_3 & x_4 \end{pmatrix}.$$

The corresponding tensor in $A_2^\vee \otimes B_3 \otimes V$ is

$$M = \alpha_1 \otimes (b_1 \otimes c_1 + b_2 \otimes c_2 + b_3 \otimes c_3) + \alpha_2 \otimes (b_1 \otimes c_2 + b_2 \otimes c_3 + b_3 \otimes c_4).$$

This means that $H^0(\mathcal{E})$ can be identified with the quotient of $B_3 \otimes V$ by the pencil generated by $b_1 \otimes c_1 + b_2 \otimes c_2 + b_3 \otimes c_3$ and $b_1 \otimes c_2 + b_2 \otimes c_3 + b_3 \otimes c_4$. We choose for basis of this 10-dimensional space the classes of $b_1 \otimes c_3, b_1 \otimes c_4, b_2 \otimes c_1, \dots, b_3 \otimes c_4$.

On the other hand, $H^0(\mathcal{E}^\vee(1))$ is the kernel of the contraction by M from $B_3^\vee \otimes \wedge^2 V^\vee$ to $A_2^\vee \otimes V^\vee$. We check that this kernel has the expected dimension 10, with a basis given by the tensors

$$\beta_1 \otimes x_{34}, \beta_2 \otimes x_{14}, \beta_3 \otimes x_{12}, \beta_2 \otimes x_{12} - \beta_3 \otimes x_{13}, \beta_1 \otimes x_{24} - \beta_2 \otimes x_{34},$$

$$\beta_1 \otimes x_{13} - \beta_2 \otimes x_{23} + \beta_3 \otimes x_{24}, \beta_2 \otimes x_{13} - \beta_3 \otimes x_{14}, \beta_1 \otimes x_{14} - \beta_2 \otimes x_{24},$$

$$\beta_1 \otimes x_{12} - \beta_3 \otimes (x_{14} - x_{23}), \beta_1 \otimes (x_{14} - x_{23}) - \beta_3 \otimes x_{34}.$$

The resulting 10×10 matrix of constant rank 7 is the following:

$$\begin{pmatrix} x_4 & -x_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_4 & 0 & 0 & -x_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x_2 & -x_1 & 0 & 0 \\ 0 & 0 & x_2 & -x_1 & 0 & 0 & -x_3 & 0 & x_1 & 0 \\ 0 & -x_2 & 0 & 0 & x_4 & -x_3 & 0 & 0 & 0 & 0 \\ -x_1 & 0 & 0 & x_3 & -x_2 & 0 & 0 & x_4 & 0 & -x_2 \\ 0 & 0 & x_3 & 0 & -x_1 & 0 & -x_4 & 0 & 0 & x_1 \\ 0 & -x_1 & 0 & -x_4 & 0 & x_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -x_4 & x_3 & -x_2 & x_1 \\ x_2 & -x_1 & 0 & 0 & 0 & 0 & 0 & 0 & x_4 & -x_3 \end{pmatrix}.$$

3. DRÉZET BUNDLES

In this section we discuss a class of vector bundles that were used by Drézet to construct examples of uniform bundles of small rank which are not homogeneous; namely, rank $2n$ non-homogeneous uniform vector bundles on \mathbb{P}^n for any $n \geq 2$ [12].

3.1. Definition and first examples. Our terminology is the following.

Definition. A *Drézet bundle* on \mathbb{P}^n is a vector bundle \mathcal{E} fitting into an exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(-c) \rightarrow \mathcal{O}_{\mathbb{P}^n}^{\oplus e+1} \rightarrow \mathcal{E} \rightarrow 0.$$

Of course $c = c_1(\mathcal{E})$ and $s(\mathcal{E}) = 1 - ch$.

The dual vector bundle \mathcal{E}^\vee is a special case of what is called a *syzygy bundle*. The stability of syzygy bundles was discussed in [26]. It follows from [26, Theorem 4.6] that a general Drézet bundle is stable when $n \leq e \leq \binom{n+c}{c}$.

Example 16. Consider the space \mathcal{A}_m of skew-symmetric matrices of size m . If $m = 2p + 1$ is odd, the generic matrix in \mathcal{A}_m has rank $2p$, hence a one dimensional kernel given by its p -th power as a skew-symmetric two-form. Moreover the closed subset of matrices of rank smaller than $2p$ has codimension three. Taking a general plane $P \simeq \mathbb{P}^2$ inside $\mathbb{P}(\mathcal{A}_m)$, we thus get a Drézet bundle

$$0 \rightarrow \mathcal{O}_P(-p) \rightarrow \mathcal{O}_P^{\oplus 2p+1} \rightarrow \mathcal{E}_P \rightarrow 0.$$

Hence a family of vector bundles on \mathbb{P}^2 parametrized by $Gr(3, \mathcal{A}_m)$. In this case $\mathcal{E}_P^\vee(1) \cong \mathcal{E}_P$, in particular it is generated by sections [7]. This self-duality property implies that \mathcal{E}_P is uniform of splitting type $(0^p, 1^p)$. In the sequel we will call these bundles *Pfaffian*, since P is generated by the Pfaffians of the $2p \times 2p$ skew-symmetric minors of the skew-symmetric matrix \mathcal{A}_{2p+1} . Note that Pfaffian bundles only exist on \mathbb{P}^2 .

Example 17. The universal Drézet bundle on \mathbb{P}^n with first Chern class c is the homogeneous bundle defined by the sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(-c) \rightarrow S^c V \otimes \mathcal{O}_{\mathbb{P}^n} \rightarrow \mathcal{E}_c \rightarrow 0,$$

where $V = \mathbb{C}^{n+1}$ and $\mathbb{P}^n = \mathbb{P}(V)$. This homogeneous bundle is discussed in [25]. Any Drézet bundle of higher rank with the same first Chern class is the direct sum of \mathcal{E}_c with a trivial factor. Any Drézet bundle of smaller rank is obtained by factoring out sections of \mathcal{E}_c .

3.2. Deformations. A Drézet bundle \mathcal{E} of rank e with first Chern class c is defined by a collection P_0, \dots, P_e of degree c polynomials, or rather by the subspace $P = \langle P_0, \dots, P_e \rangle$ they generate in $S^c V^\vee$. We will always suppose P to be of dimension $e + 1$, since otherwise \mathcal{E} has a trivial factor. We thus consider the family of Drézet bundles as parametrized by the open subset of $Gr(e + 1, S^c V^\vee)$ parametrizing spaces of polynomials with no common zero, and we write their defining sequence as

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(-c) \rightarrow P^\vee \otimes \mathcal{O}_{\mathbb{P}^n} \rightarrow \mathcal{E}_P \rightarrow 0.$$

After dualizing, we get $H^1(\mathcal{E}_P^\vee) = S^c V^\vee / P$, while $H^q(\mathcal{E}_P^\vee) = 0$ for $q \neq 1$. Twisting by $\mathcal{O}_{\mathbb{P}^n}(-c)$ and supposing $c \leq n$, we get $H^1(\mathcal{E}_P^\vee(-c)) = \mathbb{C}$ and $H^q(\mathcal{E}_P^\vee(-c)) = 0$ for $q \neq 1$. The short sequence

$$0 \rightarrow \mathcal{E}_P^\vee(-c) \rightarrow P^\vee \otimes \mathcal{E}_P^\vee \rightarrow \mathcal{E}_P^\vee \otimes \mathcal{E}_P \rightarrow 0$$

then yields the following consequences:

Proposition 18. *Any Drézet bundle \mathcal{E}_P on \mathbb{P}^n with fist Chern class $c_1(\mathcal{E}_P) = c \leq n$ is simple. Moreover*

$$H^1(\text{End}(\mathcal{E}_P)) = \text{Hom}(P, S^c V^\vee / P) = T_{[P]} Gr(e + 1, S^c V^\vee),$$

and $H^q(\text{End}(\mathcal{E}_P)) = 0$ for any $q > 1$.

In particular, for $c \leq n$ local deformations of Drézet bundles are again Drézet bundles, and we get a smooth family of vector bundles parametrized by an open subset of the Grassmannian $Gr(e + 1, S^c V^\vee)$. For $n = 2$ and $c > 2$ it is no longer true that local deformations of Drézet bundles are again Drézet bundles, see [9, Theorem 4.4].

3.3. Sections and global generation of the twisted dual. From the short exact sequence that defines \mathcal{E}_P , after dualizing, twisting and taking cohomology we get the exact sequence

$$0 \rightarrow H^0(\mathcal{E}_P^\vee(1)) \rightarrow P \otimes V^\vee \rightarrow S^{c+1} V^\vee \rightarrow H^1(\mathcal{E}_P^\vee(1)) \rightarrow 0.$$

An obvious consequence is:

Proposition 19. $H^0(\mathcal{E}_P^\vee(1)) \simeq \text{Syz}_1(P)$, the space of linear syzygies between polynomials in P .

Definition. The Drézet bundle \mathcal{E}_P on \mathbb{P}^n is *ordinary* if $H^1(\mathcal{E}_P^\vee(1)) = 0$. This is equivalent to the condition that

$$\forall C \in S^{c+1} V^\vee, \exists \ell_0, \dots, \ell_e \in V^\vee \quad C = \ell_0 P_0 + \dots + \ell_e P_e.$$

Of course this is only possible when e is large enough. More precisely, it immediately follows from [23] that when

$$(e+1)(n+1) \geq \binom{n+c+1}{c+1},$$

a general Drézet bundle is ordinary. In addition, If e is too small for this condition to be fulfilled, then a general Drézet bundle verifies $H^0(\mathcal{E}_P^\vee(1)) = 0$.

For ordinary Drézet bundles, we can use the snake lemma to reformulate the condition that $\mathcal{E}_P^\vee(1)$ is globally generated at $[v]$ into the following condition:

$$\begin{aligned} \forall C \in S^{c+1}V^\vee, C(v) = 0, \quad \exists \ell_0, \dots, \ell_e \in V^\vee, \\ \ell_0(v) = \dots = \ell_e(v) = 0, \quad C = \ell_0 P_0 + \dots + \ell_e P_e. \end{aligned}$$

Dualizing, we get the following equivalent condition:

Lemma 20. *Suppose the Drézet bundle \mathcal{E}_P is ordinary. Then $\mathcal{E}_P^\vee(1)$ is globally generated at $[v]$ if and only if the map*

$$S^{c+1}V/\mathbb{C}v^{c+1} \rightarrow (V/\mathbb{C}v)^{e+1}, \quad T \mapsto (P_0]T, \dots, P_e]T)$$

is injective.

Here $Q]T$ is just the vector obtained by contracting the tensor T with the polynomial Q . Note the obvious consequence:

Corollary 21. *Suppose that $P' \subset P \subset S^cV^\vee$. If $\mathcal{E}_{P'}$ is an ordinary Drézet bundle and $\mathcal{E}_{P'}^\vee(1)$ is generated by global sections, then \mathcal{E}_P is also an ordinary Drézet bundle and $\mathcal{E}_P^\vee(1)$ is also generated by global sections.*

Applying Lemma 20 we can prove that a generic Drézet bundle of high enough rank always gives rise to a matrix of linear forms of constant rank.

Theorem 22. *Suppose that the rank e is large enough, more precisely that*

$$\dim P > \left(\frac{1}{c} + \frac{1}{n+1} \right) \dim S^cV^\vee.$$

Then the general Drézet bundle \mathcal{E}_P is ordinary, and $\mathcal{E}_P^\vee(1)$ is generated by global sections.

Proof. First observe that the map θ_T from S^cV^\vee to V that sends Q to $Q]T$ is in general surjective. Indeed, if it is not, suppose its image is contained in a hyperplane $H = \langle e_1, \dots, e_n \rangle$ of $V = \langle e_0, \dots, e_n \rangle$; then the tensor T , when written in this basis, can have only zero coefficient on any monomial $e_0 M$ involving e_0 ; otherwise, contracting with the monomial M^\vee in the dual coordinates we would get a vector with a nonzero coefficient on e_0 , a contradiction. In other words T has to belong to $S^{c+1}H$. More generally, if $W_T \subset V$ is the image of θ_T , then T has to belong to $S^{c+1}W_T$, and W_T is minimal for this property. Let us call the dimension of W_T the *global rank* of T .

Then consider the set I where global generation fails, that is the set

$$I := \left\{ (P, [v], [T]) \in Gr(e+1, S^c V^\vee) \times \mathbb{P}(V) \times \mathbb{P}(S^{c+1}V), \right. \\ \left. T \notin \mathbb{C}v^{c+1}, P_0 \rfloor T, \dots, P_e \rfloor T \in \mathbb{C}v \right\}.$$

Let us show that the dimension of I is strictly smaller than the dimension of $Gr(e+1, S^c V^\vee)$. For this we project to $\mathbb{P}(V) \times \mathbb{P}(S^{c+1}V)$. If $([v], [T])$ is in the image, then necessarily $v \in W_T$. Denote the dimension of W_T by d . Then P must be a subspace of $\theta_T^{-1}(\mathbb{C}v)$, which has codimension $d-1$. Note that if $d=1$, T has to be a multiple of some t^{c+1} with t independent of v , and then $P_0 \rfloor T, \dots, P_e \rfloor T$ are multiples of t ; since they are also multiples of v , they must in fact be zero, which means that $P_0(t) = \dots = P_e(t)$. In other words P has a common zero and cannot be generic if $e \geq n$. So we can exclude this case and bound the dimension of I by the maximum, for $2 \leq d \leq n+1$, of

$$\dim Gr(d, n+1) + \dim S^{c+1}\mathbb{C}^d + \dim Gr(e+1, \dim S^c \mathbb{C}^{n+1} - (d-1)).$$

Let us rewrite this quantity as $\dim Gr(e+1, \dim S^c \mathbb{C}^{n+1}) - N_d$ where

$$N_d = (d-1)(e+1) - d(n+1-d) - \binom{c+d}{c+1}.$$

We will therefore be able to conclude that for a general $P \in Gr(e+1, \dim S^c \mathbb{C}^{n+1})$, there is no $([v], [T])$ contradicting the global generation of $\mathcal{E}_P^\vee(1)$ as soon as $N_d > 0$ for all d . Note that

$$N_d - N_{d+1} = n - e + \binom{c+d}{c}$$

implies that N_d is convex as a function of d . So we only need to check that N_2 and N_{n+1} are strictly positive, which amounts to

$$e \geq 2n + c \quad \text{and} \quad e + 1 > \frac{1}{n} \binom{c+n+1}{c+1}.$$

Since $\dim S^c V^\vee = \binom{c+n}{c} = \frac{c+1}{c+n+n} \binom{c+n+1}{c+1}$, our claim follows. \square

Remark. In case the Drézet bundle is not ordinary, a substitute to Lemma 20 is the following criterion for global generation:

Lemma 23. *The Drézet bundle \mathcal{E}_P is generated by global sections at $[v]$ if and only if, for any $q_0 \in P$ such that $q_0(v) = 0$, one has $q_0 V^\vee \subset P v^\perp$ in $S^c V^\vee$.*

Example 24. On $\mathbb{P}^2 = \mathbb{P}(V_3)$, the universal Drézet bundle \mathcal{E}_2 with first Chern class $c=2$ has rank 5. It is an extension of $S^2 Q$ by $Q(-1)$ and

$H^0(\mathcal{E}_2^\vee(1)) \simeq \mathfrak{sl}_3$. This vector bundle gives rise to the following 6×8 matrix of constant rank five [25, Proposition 11]:

$$\begin{pmatrix} -x_1 & -x_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_0 & 0 & -x_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_0 & 0 & x_1 & 0 & 0 \\ x_0 & 0 & -x_1 & 0 & 0 & 0 & -x_2 & x_2 \\ 0 & x_0 & 0 & -x_2 & 0 & 0 & 0 & -x_1 \\ 0 & 0 & 0 & 0 & x_1 & -x_2 & x_0 & 0 \end{pmatrix}.$$

This is in fact the twisted dual of the Steiner bundle discussed in Example 10.

We can factor out a generic section by restricting to the hyperplane of quadrics orthogonal to the tensor $t = e_0e_1 + e_2^2$, which is

$$H = \langle q_1 = x_0x_1 - x_2^2, q_2 = x_0x_2, q_3 = x_1x_2, q_4 = x_0^2, q_5 = x_1^2 \rangle.$$

Note that t is invariant under the action of a copy of the orthogonal group in three variables, which has only two orbits in \mathbb{P}^2 . Therefore, to prove that $\mathcal{E}_H^\vee(1)$ is globally generated, it is enough to check it at one point of the closed orbit, say $[e_0]$. So consider a cubic symmetric tensor $T = \sum_{ijk} T_{ijk}e_i e_j e_k$, and suppose that the contractions with all quadrics in H give multiples of e_0 . Applying this successively to q_2, q_3, q_4, q_5 yields $T_{012} = T_{022} = 0$, $T_{112} = T_{122} = 0$, $T_{001} = T_{002} = 0$ and $T_{111} = T_{112} = 0$. There only remains $T = T_{000}e_0^3 + 3T_{011}e_0e_1^2 + T_{222}e_2^3$, whose contraction with the remaining quadric q_1 is $3T_{011}e_1 - T_{222}e_2$. This is a multiple of e_0 if and only if $= T_{000}e_0^3$, and we conclude that $\mathcal{E}_H^\vee(1)$ is generated by global sections. It has five independent sections given by the five syzygies

$$x_0q_1 - x_1q_4 + x_2q_2, x_1q_1 - x_0q_5 + x_2q_3, x_0q_2 - x_2q_4, x_1q_1 - x_0q_3, x_1q_3 - x_2q_5,$$

hence the following 5×5 matrix of constant rank four:

$$\begin{pmatrix} x_0 & x_2 & 0 & -x_1 & 0 \\ x_1 & 0 & x_2 & 0 & -x_0 \\ 0 & x_0 & 0 & -x_2 & 0 \\ 0 & x_1 & -x_0 & 0 & 0 \\ 0 & 0 & x_1 & 0 & -x_2 \end{pmatrix}.$$

Note that if we replace t by some rank two tensor $t' = e_0e_1$, any quadric in its orthogonal hyperplane H' contracts the tensor $T = e_0e_1^2$ to a multiple of e_0 , showing that $\mathcal{E}_{H'}^\vee(1)$ is not globally generated at $[e_0]$.

If we degenerate further to a rank one tensor $t'' = e_0^2$, then the orthogonal hyperplane $H'' \subset S^2V^\vee$ is such that $H''V^\vee$ is only a hyperplane in S^3V^\vee , so that $\mathcal{E}_{H''}$ is not ordinary: it has $h^1(\mathcal{E}_{H''}^\vee(1)) = 1$ and $h^0(\mathcal{E}_{H''}^\vee(1)) = 6$. We

get in this case a 6×5 matrix of rank at most four:

$$\begin{pmatrix} x_0 & 0 & -x_1 & 0 & 0 \\ x_2 & 0 & 0 & -x_1 & 0 \\ 0 & x_0 & 0 & 0 & -x_2 \\ 0 & x_1 & 0 & -x_2 & 0 \\ 0 & 0 & x_2 & -x_0 & 0 \\ 0 & 0 & x_2 & 0 & -x_1 \end{pmatrix}.$$

The minor corresponding to rows 2345 (resp. 1246) and columns 1345 (resp. 2345) is x_2^4 (resp. x_1^4), showing that the rank can drop only at $[e_0]$, where it is indeed equal to three. In particular, although $\mathcal{E}_{H''}(1)$ has extra global sections, it is not globally generated.

4. CLASSIFICATION FOR $c_1 = 2$ AND MATRICES OF CONSTANT RANK FOUR AND FIVE

Globally generated vector bundles on projective spaces were classified for $c_1 \leq 5$ [31, 32, 2, 1]. We need to determine inside this classification which vector bundles \mathcal{E} are such that $\mathcal{E}^\vee(1)$ is also generated by global sections. Recall that this implies that \mathcal{E} is uniform, so we can suppose that the rank of \mathcal{E} is $e \geq n + 2$; otherwise \mathcal{E} is known to be a direct sum of irreducible homogeneous bundles (line bundles, the quotient bundle and its twisted dual).

We can also suppose that \mathcal{E} is indecomposable. In particular, since $\mathcal{E}^\vee(1)$ is globally generated and has no trivial factor, $H^0(\mathcal{E}(-1))$ must vanish. Together with the rank condition $e \geq n + 2$, this drastically simplifies the classification in case the first Chern class $c = c_1(\mathcal{E}) \leq 3$, and only a handful of cases have to be discussed. Note that there is also a partial classification for $c = 4$ and even $c = 5$, but only under the hypothesis that $h^1(\mathcal{E}^\vee) = 0$, which for our matters is unrealistic.

For $c = 1$, the fact that \mathcal{E} can only be $\mathcal{O}_{\mathbb{P}^n}(1)$ or the quotient bundle Q already appears in [13, Theorem 2.4]. In section 4 of the same paper, Eisenbud and Harris suggest to focus (for constant and also bounded rank) on the case $c_1 = 2$, which we shall discuss now.

4.1. Vector bundles with $c_1 = 2$. According to [31, Theorem 1.1], there are only two possibilities.

Proposition 25. *An indecomposable, globally generated vector bundle \mathcal{E} on \mathbb{P}^n of rank $e \geq n + 2$, with $c_1(\mathcal{E}) = 2$, must be either a Steiner bundle or a Drézet bundle.*

4.1.1. The Steiner case. Steiner bundles with $c_1 = 2$ are defined by an exact sequence of the form

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(-1) \oplus \mathcal{O}_{\mathbb{P}^n}(-1) \rightarrow \mathcal{O}_{\mathbb{P}^n}^{\oplus e+2} \rightarrow \mathcal{E} \rightarrow 0.$$

Observe by dualizing this sequence that $H^0(\mathcal{E}^\vee)$ has to be nonzero as soon as $e > 2n$, and then \mathcal{E} admits a trivial factor. If $e = 2n$ and $H^0(\mathcal{E}^\vee) = 0$ then necessarily $\mathcal{E} = Q \oplus Q$ is again decomposable.

Proposition 26. *There is no indecomposable Steiner bundle \mathcal{E} such that $c_1(\mathcal{E}) = 2$ and $\mathcal{E}^\vee(1)$ is generated by global sections.*

Proof. This is an immediate consequence of [28, Theorem 13], where it is proved that if \mathcal{E} is a 1-uniform Steiner bundle on \mathbb{P}^n bundle of rank e with no trivial factor, then $c + 2n - 2 \leq e \leq cn$ where $c = c_1(\mathcal{E})$. This applies in our setting and for $c = 2$ yields $e = 2n$, in which case we know that $\mathcal{E} = Q \oplus Q$ is decomposable. \square

4.1.2. *The Drézet case.* As a consequence of Theorem 22 and its proof, a generic Drézet bundle \mathcal{E} of rank $e > \frac{n^2+6n+11}{6}$ is ordinary, and $\mathcal{E}^\vee(1)$ is generated by global sections, which are given by linear syzygies between the quadrics defining \mathcal{E} . The maximal case is that of the homogeneous bundle \mathcal{E}_2 , of rank $e = \frac{n(n+3)}{2}$.

Now consider \mathcal{E}_P for P a hyperplane in S^2V^\vee . Its orthogonal in S^2V is the line generated by some tensor t of rank k . Note that \mathcal{E}_P is a vector bundle only if $k > 1$. (In general, \mathcal{E}_P is a vector bundle exactly when P^\perp contains no rank one tensor.)

Proposition 27. *For any $k \geq 2$, \mathcal{E}_P is an ordinary Drézet bundle. Moreover $\mathcal{E}_P^\vee(1)$ is generated by global sections if and only if $k > 2$.*

Proof. Let us choose a basis of V such that $t = e_0e_1 + e_2^2 + \cdots + e_{k-1}^2$. Its orthogonal P is generated by the quadratic forms $x_0^2, x_1^2, x_0x_1 - \frac{1}{2}x_2^2, x_2^2 - x_3^2, \dots, x_{k-2}^2 - x_{k-1}^2, x_k^2, \dots, x_n^2$, and the x_ix_j for $i < j$ and $(i, j) \neq (0, 1)$.

In order to check whether $\mathcal{E}_P^\vee(1)$ is generated by global sections we can use the fact that t has a big stabilizer, whose orbits in $\mathbb{P}(V)$ are: the support $\mathbb{P}(S_t) = \mathbb{P}(\langle e_0, \dots, e_{k-1} \rangle)$ of t , the quadric $Q_t \subset \mathbb{P}(S_t)$ defined by t , and the complement of $\mathbb{P}(S_t)$. Since the locus where $\mathcal{E}_P^\vee(1)$ is generated by global sections is closed and preserved by the stabilizer of t , we just need to check whether $\mathcal{E}_P^\vee(1)$ is generated or not at one given point of Q_t , say $[e_0]$.

So we consider a cubic tensor $C \in S^3V$, such that $Q]C$ is a multiple of e_0 for any $Q \in P$, and we must decide whether C must be a multiple of e_0^3 . So we may suppose that $C = e_0^2\lambda + e_0\kappa + \gamma$, where λ, κ, γ do not depend on e_0 . That $x_0^2]C$ is a multiple of e_0 implies $\lambda = 0$. That $x_ix_j]C$ is a multiple of e_0 for any $0 < i < j$ implies that $\gamma_{ijk} = 0$ for any triple ijk with a least two distinct indices; so we can write $\gamma = \sum_{i>0} \gamma_i e_i^3$, and then the fact that $Q]\gamma$ must be zero for $Q = x_1^2, x_2^2 - x_3^2, \dots, x_{k-2}^2 - x_{k-1}^2, x_k^2, \dots, x_n^2$ implies that $\gamma = 0$. So we remain with $C = e_0\kappa$, and the condition that $x_0x_i]C$ is a multiple of e_0 for any $i > 1$ implies that $x_i]\kappa = 0$. Since this is true for any $i > 1$ we only remain with $\kappa = \kappa_1 e_1^2$, hence $C = \kappa_1 e_0 e_1^2$.

If $k > 2$, we can use the extra condition that $(x_0x_1 - \frac{1}{2}x_2^2)]C$ is a multiple of e_0 to conclude that $\kappa_1 = 0$, and therefore that $\mathcal{E}_P^\vee(1)$ is generated at

$[e_0]$. But if $k = 2$, this extra condition is not available and we arrive at the opposite conclusion: the cubic $C = e_0e_1^2$ shows that $\mathcal{E}_P^\vee(1)$ is not generated at $[e_0]$. \square

We could use the classification of pencils and webs of conics to get inequivalent families of matrices of constant rank $e = \frac{n(n+3)}{2} - 2$ or 3 .

For a generic pencil $\langle t_1, t_2 \rangle$ in S^2V the two tensors t_1, t_2 are simultaneously diagonalizable: there exists a basis e_0, \dots, e_n and coefficients λ_0, \dots, μ_n such that

$$t_1 = \lambda_0e_0^2 + \dots + \lambda_n e_n^2, \quad t_2 = \mu_0e_0^2 + \dots + \mu_n e_n^2.$$

Some explicit computations yield the following conclusion:

Proposition 28. *For $P = \langle t_1, t_2 \rangle^\perp \subset S^2V^\vee$, the vector bundle \mathcal{E}_P is an ordinary Drézet bundle and $\mathcal{E}_P^\vee(1)$ is generated by global sections if and only if the pencil $\langle t_1, t_2 \rangle$ contains no tensor of rank two.*

Example 29. Suppose $n = 3$ and consider the pencil generated by

$$t_1 = e_0^2 + Te_2^2 - e_3^2, \quad t_2 = e_1^2 - e_2^2 + e_3^2.$$

The other two rank three tensors in the pencil are $t_1 + t_2$ and $t_1 + Tt_2$, which with t_1 and t_2 gives four points on the pencil with cross-ratio T . In particular, we will get inequivalent matrices of linear forms by varying T .

In order to get these matrices, as always we have to choose basis of $P = \langle t_1, t_2 \rangle^\perp$ and $Syz_1(P)$, whose dimensions are 8 and 12, respectively. For a basis of P we choose the six quadrics $Q_{ij} = x_i x_j$, plus $Q_0 = x_1^2 + Tx_3^2 - x_4^2$ and $Q_1 = x_2^2 - x_3^2 + x_4^2$. The syzygies are:

$$x_1Q_{23} = x_2Q_{13} = x_3Q_{12}, \quad x_1Q_{24} = x_2Q_{14} = x_4Q_{12},$$

$$x_1Q_{34} = x_3Q_{14} = x_4Q_{13}, \quad x_2Q_{34} = x_3Q_{24} = x_4Q_{23},$$

$$x_1Q_1 - x_2Q_{12} + x_3Q_{13} - x_4Q_{14} = 0,$$

$$x_2Q_0 - x_1Q_{12} - Tx_3Q_{23} + x_4Q_{24} = 0,$$

$$x_3(Q_0 + TQ_1) - x_1Q_{13} - Tx_2Q_{23} + (1 - T)x_4Q_{34} = 0,$$

$$x_4(Q_0 + Q_1) - x_1Q_{14} - x_2Q_{24} + (1 - T)x_3Q_{34} = 0.$$

We interpret each of the twelve $=$ signs in these formulas as one of our twelve basis of syzygies. Then we immediately read the associated 8×12 matrix of linear forms of constant rank 7, depending on the parameter $T \neq 0, 1$:

$$\begin{pmatrix} x_3 & -x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_2 & -x_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_4 & 0 & -x_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_2 & 0 & -x_1 & 0 & 0 & 0 & 0 \\ 0 & x_4 & -x_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_3 & 0 & 0 & -x_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_4 & -x_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_3 & -x_2 & 0 & 0 & 0 \\ -x_2 & x_3 & -x_4 & 0 & 0 & 0 & 0 & 0 & x_1 \\ -x_1 & 0 & 0 & -Tx_3 & x_4 & 0 & x_2 & 0 & 0 \\ 0 & -x_1 & 0 & -Tx_2 & 0 & (1-T)x_4 & x_3 & Tx_3 & 0 \\ 0 & 0 & -x_1 & 0 & -x_2 & (1-T)x_3 & x_4 & x_4 & 0 \end{pmatrix}$$

4.2. Matrices of constant rank four. We have now enough information to classify matrices of linear forms of constant rank four. (The question of classifying matrices of linear forms of rank everywhere bounded by four is raised in [13]). Let \mathcal{E} be the rank four vector bundle on \mathbb{P}^n defined by such a matrix. We may suppose that \mathcal{E} is indecomposable. Moreover since $c_1(\mathcal{E}) + c_1(\mathcal{E}^\vee(1)) = 4$, up to exchanging the two vector bundles, that is, up to transposing the matrix, we may suppose that $c_1(\mathcal{E}) \leq 2$. If $c_1(\mathcal{E}) = 1$, then \mathcal{E} must be the quotient bundle Q on \mathbb{P}^4 .

So suppose $c_1(\mathcal{E}) = 2$. If the rank $e = 4 \leq n + 1$, we know that \mathcal{E} is a direct sum of homogeneous bundles, in particular it must be decomposable. So we must have $n = 2$. According to Propositions 25 and 26, \mathcal{E} must then be a Drézet bundle \mathcal{E}_P , for P a hyperplane in S^2V^\vee (where $\mathbb{P}^2 = \mathbb{P}(V)$). By Proposition 27, the orthogonal of P in S^2V must be spanned by a non-degenerate tensor. We deduce:

Proposition 30. *Any indecomposable matrix of linear forms of constant rank four can be obtained from one of the following two matrices:*

$$M_{Quot} = \begin{pmatrix} x_2 & x_3 & x_4 & x_5 & 0 & 0 & 0 & 0 & 0 & 0 \\ -x_1 & 0 & 0 & 0 & x_3 & x_4 & x_5 & 0 & 0 & 0 \\ 0 & -x_1 & 0 & 0 & -x_2 & 0 & 0 & x_4 & x_5 & 0 \\ 0 & 0 & -x_1 & 0 & 0 & -x_2 & 0 & -x_3 & 0 & x_5 \\ 0 & 0 & 0 & -x_1 & 0 & 0 & -x_2 & 0 & -x_3 & -x_4 \end{pmatrix},$$

$$M_{Drez} = \begin{pmatrix} 0 & x_1 & 0 & 0 & x_3 \\ -x_1 & 0 & x_2 & x_3 & 0 \\ 0 & -x_2 & 0 & x_1 & 0 \\ 0 & -x_3 & -x_1 & 0 & x_2 \\ -x_3 & 0 & 0 & -x_2 & 0 \end{pmatrix}.$$

Remarks. The first case corresponds to $\mathcal{E} = Q$, the quotient bundle on $\mathbb{P}^4 = \mathbb{P}(W)$, for which $H^0(\mathcal{E}) = W$ and $H^0(\mathcal{E}^\vee(1)) = \wedge^2 W^\vee$. Obviously \mathcal{E} cannot be generated by a proper subspace of global sections, but $\mathcal{E}^\vee(1)$

can: by general principles $4 + 4 = 8$ general sections always suffice. Rather unexpectedly, if h is the hyperplane class, we can compute that

$$s(\mathcal{E}^\vee(1)) = \frac{1 - 2h}{(1 - h)^5} = 1 + 3h + 5h^2 + 5h^3 + \mathbf{0}h^4.$$

This means that 7 general sections (but no less) will suffice to generate $\mathcal{E}^\vee(1)$ at every point. Concretely, this means that we obtain a matrix of constant rank four by picking 7 (or more) general linear combinations of the columns of M_{Quot} . And we get in this way all such matrices with associated vector bundle Q . Note also the following consequence of the previous discussion: if we restrict such a matrix to a smaller space of variables, then it necessarily becomes decomposable.

The second case corresponds to $\mathcal{E} = \mathcal{E}_P$, the Drézet bundle on $\mathbb{P}^2 = \mathbb{P}(V)$ defined by the orthogonal P to a non-degenerate tensor in S^2V . In this case $H^0(\mathcal{E}) = P$ and $H^0(\mathcal{E}^\vee(1)) = \text{Syz}_1(P)$ are both five-dimensional, so we need their whole spaces of global sections to generate \mathcal{E} and $\mathcal{E}^\vee(1)$. We represented this case by an explicit skew-symmetric matrix in three variables, but we know that any generic such matrix would be equivalent to this one.

Proof. We have seen that the rank four vector bundle \mathcal{E} associated to our matrix must be either the quotient bundle Q on \mathbb{P}^4 , leading to M_{Quot} , or the above Drézet bundle \mathcal{E}_P on \mathbb{P}^2 . So the only thing we have to prove is that the skew-symmetric matrix M_{Drez} has constant rank four and defines the same Drézet bundle.

Seen as a space of skew-symmetric forms in 5 variables, M_{Drez} is generated by $\omega_1 = e_1 \wedge e_2 + e_3 \wedge e_4$, $\omega_2 = e_2 \wedge e_3 + e_4 \wedge e_5$, $\omega_3 = e_1 \wedge e_5 + e_2 \wedge e_4$. It is easy to check that no nonzero linear combination ω of $\omega_1, \omega_2, \omega_3$ satisfies the equation $\omega \wedge \omega = 0$, which would mean it has rank two. Moreover, recall that the map $\omega \mapsto \omega \wedge \omega$ is precisely the defining morphism of the associated Steiner bundle \mathcal{E} . A straightforward computation shows that the unique relation between the wedge products of the ω_i 's is

$$\omega_1 \wedge \omega_2 - \frac{1}{2}\omega_3 \wedge \omega_3 = 0.$$

Since the tensor $t = u_1u_2 - \frac{1}{2}u_3^2$ has rank three, we conclude that $\mathcal{E} = \mathcal{E}_P$, which completes the proof. \square

4.3. Matrices of constant rank five. The previous analysis can be extended to matrices of linear forms of constant rank five. Let \mathcal{E} be the rank five vector bundle on \mathbb{P}^n defined by such a matrix, and suppose that \mathcal{E} is indecomposable. Now $c_1(\mathcal{E}) + c_1(\mathcal{E}^\vee(1)) = 5$, so again, up to exchanging the two vector bundles, we may suppose that $c_1(\mathcal{E}) \leq 2$. If $c_1(\mathcal{E}) = 1$, then \mathcal{E} must be the quotient bundle Q on \mathbb{P}^5 .

So suppose $c_1(\mathcal{E}) = 2$. The condition $e = 5 \geq n + 2$, implies $n \leq 3$. Moreover, as in the previous case, Propositions 25 and 26 imply that \mathcal{E} is a Drézet bundle \mathcal{E}_P on $\mathbb{P}^n = \mathbb{P}(V)$ for $P \subset S^2V^\vee$ of dimension $e + 1 = 6$.

If $n = 2$, this implies that $P = S^2V^\vee$ and then \mathcal{E} is the universal Drézet bundle discussed in Example 24.

Let us show that $n = 3$ is impossible for an ordinary Drézet bundle. First note that $s(\mathcal{E}^\vee(1)) = (1 + 3h)/(1 + h)^6$ and therefore $s_3(\mathcal{E}^\vee(1)) \neq 0$. This implies that $\mathcal{E}^\vee(1)$, if generated by global sections, has at least $e + n = 8$ independent sections. Since $\chi(\mathcal{E}^\vee(1)) = 4$ and $h^q(\mathcal{E}^\vee(1)) = 0$ for $q > 1$, this means that $h^1(\mathcal{E}^\vee(1)) \geq 4$. Therefore, if $\mathcal{E} = \mathcal{E}_P$ there must exist a four-dimensional vector space of cubics, all apolar to P .

Example 31. The six-dimensional space of quadrics generated by the square-free monomials $P = \langle q_{ij} = x_i x_j, i < j \rangle$ is apolar to the four dimensional space of Fermat type cubic tensors $C = c_0 e_0^3 + c_1 e_1^3 + c_2 e_2^3 + c_3 e_3^3$. In this case nevertheless, P has four base points. As a consequence we do not get a constant rank matrix of linear forms, but we do get a 6×8 matrix of linear forms of generic rank 5. The eight obvious syzygies are $x_i q_{jk} = x_j q_{ik} = x_k q_{ij}$ for $i < j < k$, and we get the matrix

$$\begin{pmatrix} 0 & -x_2 & 0 & x_1 & 0 & 0 \\ x_3 & 0 & 0 & -x_1 & 0 & 0 \\ x_4 & 0 & -x_2 & 0 & 0 & 0 \\ -x_4 & 0 & 0 & 0 & x_1 & 0 \\ 0 & -x_4 & x_3 & 0 & 0 & 0 \\ 0 & 0 & -x_3 & 0 & 0 & x_1 \\ 0 & 0 & 0 & 0 & -x_3 & x_2 \\ 0 & 0 & 0 & x_4 & 0 & -x_2 \end{pmatrix}$$

Tautologically, the kernel is spanned outside the four base points by $\sum_{i < j} x_i x_j q_{ij}$.

Let us show this cannot happen when we do not admit base points.

In order to prove our next statements, we will need to recall a couple of pure algebraic results which will play an important role in the proofs of Lemmas 34 and 37.

Given a standard graded Artinian K -algebra $A = R/I$ where $R = K[x_0, x_1, \dots, x_n]$ and I is a homogeneous ideal of R , we denote its Hilbert function by $HF_A : \mathbb{N} \rightarrow \mathbb{N}$, where $HF_A(j) = \dim_K A_j = \dim_K [R/I]_j$. Since A is Artinian, its Hilbert function is captured in its h -vector $h = (h_0, h_1, \dots, h_d)$ where $h_i = HF_A(i) > 0$ and d is the last index with this property. The integer d is called the *socle degree* of A .

Given integers $n, r \geq 1$, define the r -th binomial expansion of n as

$$n = \binom{m_r}{r} + \binom{m_{r-1}}{r-1} + \dots + \binom{m_e}{e}$$

where $m_r > m_{r-1} > \dots > m_e \geq e \geq 1$ are uniquely determined integers (see [6, Lemma 4.2.6]). Write

$$n^{<r>} = \binom{m_r + 1}{r + 1} + \binom{m_{r-1} + 1}{r} + \dots + \binom{m_e + 1}{e + 1},$$

$$n_{\langle r \rangle} = \binom{m_r - 1}{r} + \binom{m_{r-1} - 1}{r-1} + \cdots + \binom{m_e - 1}{e}.$$

The numerical functions $H : \mathbb{N} \rightarrow \mathbb{N}$ that are Hilbert functions of standard graded K -algebras were characterized by Macaulay [6]. Given a numerical function $H : \mathbb{N} \rightarrow \mathbb{N}$, the following conditions are equivalent:

- (i) there exists a standard graded K -algebra A with H as Hilbert function,
- (ii) $H(0) = 1$ and H satisfies the so-called **Macaulay's inequalities**:

$$H(t+1) \leq H(t)^{\langle t \rangle} \quad \forall t \geq 1.$$

Notice that condition (ii) imposes strong restrictions on the Hilbert function of a standard graded K -algebra, in particular it bounds its growth.

Example 32. Let A be a standard graded Artinian K -algebra with h -vector $h = (h_0, h_1, \dots, h_d)$. If $h_3 \leq 3$ then $h_t \leq 3$ for all $t \geq 3$.

Proposition 33. *Let R/I_1 and R/I_2 be two standard graded Artinian K -algebras with h -vectors $(h_0^1, h_1^1, \dots, h_d^1)$ and $(h_0^2, h_1^2, \dots, h_c^2)$, respectively. Assume that I_1 and I_2 are linked by an Artinian complete intersection K -algebra R/J with h -vector $(1, n+1, h_2, \dots, h_{r-2}, n+1, 1)$. Then*

$$h_i - h_i^1 = h_{r-i}^2 \quad \text{for all } i.$$

Proof. Let us denote by K_2 the canonical module of R/I_2 . Since I_1 and I_2 are linked via J , by definition of linkage we have $I_2 = [J : I_1]$. Moreover there is an exact sequence [29, Proposition 2.1.1]:

$$0 \rightarrow J \rightarrow I_1 \rightarrow K_2(-r) \rightarrow 0.$$

Therefore we get, by definition of the canonical module,

$$\begin{aligned} h_i - h_i^1 &= \dim[K_2(-r)]_i \\ &= \dim[K_2]_{-r+i} \\ &= \dim[\text{Hom}(R/I_2, K)]_{-r+i} \\ &= \dim[R/I_2]_{r-i} \\ &= h_{r-i}^2 \end{aligned}$$

which proves the claim. \square

Lemma 34. *For any six-dimensional subspace $P \subset S^2V^\vee$, consisting of quadrics with no non-trivial common zero, the rank five Drézet bundle on $\mathbb{P}V = \mathbb{P}^3$ verifies $h^1(\mathcal{E}^\vee(1)) \leq 3$. As a consequence $\mathcal{E}^\vee(1)$ cannot be generated by global sections.*

Proof. From the exact cohomology sequence associated to the exact sequence

$$0 \rightarrow \mathcal{E}^\vee(1) \rightarrow P \otimes \mathcal{O}_{\mathbb{P}^3}(1) \rightarrow \mathcal{O}_{\mathbb{P}^3}(3) \rightarrow 0$$

we immediately deduce that $H^1(\mathcal{E}^\vee(1)) \cong (R/I)_3$ where $I \subset K[x, y, z, t]$ is the ideal generated by our six quadrics. Therefore, we only need to check that $\dim(R/I)_3 \leq 3$. Inside I we choose 4 quadrics Q_1, \dots, Q_4 defining a complete intersection ideal $J \subset R$. Let us call h_t the Hilbert function of R/J

and h'_t the Hilbert function of R/I . So, we have $(h_t) = (1, 4, 6, 4, 1, 0, \dots)$ and $(h'_t) = (1, 4, 4, h'_3, h'_4, 0, \dots)$. Let us call I_1 the ideal directly linked to I by means of the complete intersection J and $(h''_t) = (1, h''_1, h''_2, h''_3, h''_4, 0, \dots)$ its Hilbert function. The Hilbert function of R/I_1 is determined by the Hilbert functions of R/I and R/J [10, Theorem 3] (see also Proposition 33), and must verify Macaulay's inequalities (see [6]). In particular, the fact that $h''_2 = h_2 - h'_2 = 6 - 4 = 2$ implies that $h''_1 = h_3 - h'_3 \geq 2$. Thus $h'_3 < 3$, as claimed. \square

The conclusion of this discussion is the following statement:

Proposition 35. *Any indecomposable matrix of linear forms of constant rank five comes from an equivariant vector bundle. More precisely, the associated bundle \mathcal{E} must be either:*

- (1) *the quotient bundle Q on \mathbb{P}^5 , or*
- (2) *the homogeneous universal Drézet bundle on \mathbb{P}^2 , which fits into an extension $0 \rightarrow Q(-1) \rightarrow \mathcal{E} \rightarrow S^2Q \rightarrow 0$.*

In the first case, $H^0(\mathcal{E}) = V$ and $H^0(\mathcal{E}^\vee(1)) = \wedge^2 V^\vee$ have dimension 6 and 15, hence a 6×15 matrix of constant rank five given by the natural contraction map $V \otimes \wedge^2 V^\vee \rightarrow V^\vee$. Any constant rank matrix defining the same vector bundle is then obtained by choosing at least 9 general linear combinations of the 15 columns.

In the second case, that we first met in Examples 10 and 24, we have $H^0(\mathcal{E}) = S^2V$ and $H^0(\mathcal{E}^\vee(1)) = S_{21}V^\vee \subset S^2V^\vee \otimes V^\vee$, of dimensions 6 and 8. The corresponding matrix of constant rank five is given by the natural contraction map $S^2V \otimes S_{21}V^\vee \rightarrow V^\vee$ and appears in Example 10. Any constant rank matrix defining the same vector bundle is then obtained by choosing at least 7 general linear combinations of the 8 columns.

5. CLASSIFICATION FOR $c_1 = 3$ AND MATRICES OF CONSTANT RANK SIX

5.1. Bundles with $c_1 = 3$. Globally generated vector bundles \mathcal{E} with $c_1(\mathcal{E}) = 3$ on \mathbb{P}^n have been classified. Our goal is to extract from this classification the list of indecomposable vector bundles such that $\mathcal{E}^\vee(1)$ is also generated. The indecomposability condition imposes that $e \geq n + 2$, and also $h^0(\mathcal{E}^\vee) = h^0(\mathcal{E}^\vee(-1)) = 0$. Moreover we may suppose that $e \geq 6$, since otherwise $c_1(\mathcal{E}^\vee(1)) < 3$ and therefore $\mathcal{E}^\vee(1)$ must have been classified previously.

Under these hypothesis, [2, Theorem (v)] implies that if $n \geq 5$, \mathcal{E} must be either a Steiner or a Drezet bundle, or given by an exact sequence of type

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(-1) \oplus \mathcal{O}_{\mathbb{P}^n}(-2) \rightarrow \mathcal{O}_{\mathbb{P}^n}^{\oplus e+2} \rightarrow \mathcal{E} \rightarrow 0.$$

On $\mathbb{P}^4 = \mathbb{P}V$, a specific indecomposable and globally generated bundle is $\mathcal{E} = \wedge^2 Q = \mathcal{E}^\vee(1)$. In this case $H^0(\mathcal{E}) = H^0(\mathcal{E}^\vee(1)) = \wedge^2 V$ is ten-dimensional, and the associated 10×10 matrix of constant rank 6 is the

matrix of the symmetric bilinear form $\wedge^2 V \otimes \wedge^2 V \rightarrow \wedge^4 V \simeq V^\vee$. In the basis $e_i \wedge e_j$, $i < j$ ordered lexicographically we get the symmetric matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_5 & -x_4 & x_3 \\ 0 & 0 & 0 & 0 & 0 & -x_5 & x_4 & 0 & 0 & -x_2 \\ 0 & 0 & 0 & 0 & x_5 & 0 & -x_3 & 0 & x_2 & 0 \\ 0 & 0 & 0 & 0 & -x_4 & x_3 & 0 & -x_2 & 0 & 0 \\ 0 & 0 & x_5 & -x_4 & 0 & 0 & 0 & 0 & 0 & x_1 \\ 0 & -x_5 & 0 & x_3 & 0 & 0 & 0 & 0 & -x_1 & 0 \\ 0 & x_4 & -x_3 & 0 & 0 & 0 & 0 & x_1 & 0 & 0 \\ x_5 & 0 & 0 & -x_2 & 0 & 0 & x_1 & 0 & 0 & 0 \\ -x_4 & 0 & x_2 & 0 & 0 & -x_1 & 0 & 0 & 0 & 0 \\ x_3 & -x_2 & 0 & 0 & x_1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Note that $s(\wedge^2 Q) = (1 - h)^5 / (1 - 2h)$, hence $s_4(\wedge^2 Q) = 1 \neq 0$. As a consequence, this matrix cannot be reduced to a smaller one without dropping rank somewhere.

Otherwise, by [2, Theorem (iv)], \mathcal{E} must be an extension by a trivial vector bundle, of a rank four vector bundle \mathcal{F} which must belong to one of the types listed in [2, Theorem (ii)]. Since then $h^0(\mathcal{E}(-1)) = h^0(\mathcal{F}(-1))$ and $h^0(\mathcal{E}(-2)) = h^0(\mathcal{F}(-2))$, types (1-4) can be eliminated. Moreover, in type (7) the vector bundle $\mathcal{F} = Q^\vee(1)$ has no non-trivial extension since the Bott-Borel-Weil theorem implies that $H^1(\mathcal{F}^\vee) = 0$; so this type can be eliminated as well. We thus remain with the same three types as on \mathbb{P}^n for $n \geq 5$.

On \mathbb{P}^3 , a full classification was given in [27], and as in the previous case, a globally generated bundle \mathcal{E} of rank $e > 3$ must be an extension by a trivial bundle, of a rank vector three bundle \mathcal{F} of one of the types listed in [27, Theorem (ii)]. For the same reasons as before we can eliminate cases (2-3-4-6-8) in this list, and we remain with the same three types as before (Drézet, Steiner and mixed) plus case (9), which we call the Tango case.

Finally on \mathbb{P}^2 , the only additional possibility is to consider an extension of the tangent bundle $T = Q(1)$ by a trivial bundle. But since $h^1(\Omega^1) = 1$, only an extension by a trivial line bundle can yield an indecomposable vector bundle \mathcal{E} , in fact still homogeneous. But a quick computation in this case shows that $h^0(\mathcal{E}^\vee(1)) = 3$, so that $\mathcal{E}^\vee(1)$ cannot be generated by global sections.

We have proved:

Proposition 36. *Suppose that \mathcal{E} is an indecomposable vector bundle of rank $e \geq 6$ on \mathbb{P}^n , with $c_1(\mathcal{E}) = 3$. Suppose that \mathcal{E} and $\mathcal{E}^\vee(1)$ are both globally generated. Then either:*

- (1) \mathcal{E} is a Steiner bundle, or
- (2) \mathcal{E} is a Drézet bundle, or
- (3) \mathcal{E} fits into a sequence $0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(-1) \oplus \mathcal{O}_{\mathbb{P}^n}(-2) \rightarrow \mathcal{O}_{\mathbb{P}^n}^{\oplus e+2} \rightarrow \mathcal{E} \rightarrow 0$,
- (4) or $n = 4$ and $\mathcal{E} = \wedge^2 Q$.

We have already discussed the last case. Let us consider the other types of vector bundles of rank six.

5.1.1. *The Steiner case.* A rank six Steiner bundle \mathcal{E} on $\mathbb{P}^n = \mathbb{P}V$, for $n \leq 4$, is given by an exact sequence

$$0 \rightarrow A_3 \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \rightarrow B_9 \otimes \mathcal{O}_{\mathbb{P}^n} \rightarrow \mathcal{E} \rightarrow 0.$$

If \mathcal{E} is uniform, by Proposition 7 we need $3+2n-2 \leq 6$, hence $n = 2$. Then \mathcal{E} must be the sum of three copies of Q . In particular, it is not indecomposable.

5.1.2. *The Drézet case.* Consider a rank six Drézet bundle \mathcal{E}_P on $\mathbb{P}^n = \mathbb{P}V$, for $n \leq 4$, given by an exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(-3) \rightarrow P^\vee \otimes \mathcal{O}_{\mathbb{P}^n} \rightarrow \mathcal{E}_P \rightarrow 0.$$

Here $P \subset S^3V^\vee$ is a seven-dimensional space of cubics, and we wonder whether $\mathcal{E}_P^\vee(1)$ can be generated by global sections.

Lemma 37. *If $n = 3, 4$, $\mathcal{E}_P^\vee(1)$ cannot be generated by global sections.*

If $n = 2$, then $\mathcal{E}_P^\vee(1)$ is generated by global sections if and only if it is also a Drézet bundle, and in this case \mathcal{E}_P is not ordinary.

Proof. The expected dimension of $H^0(\mathcal{E}_P^\vee(1))$ is 0 for $n = 4$ and $n = 3$; and 6 for $n = 2$. For $n = 4$, if \mathcal{E}_P is an ordinary Drézet bundle we have $h^0(\mathcal{E}_P^\vee(1)) = 0$ while for any seven-dimensional subspace $P \subset S^3V^\vee$ consisting of cubics without common zeros we have $h^1(\mathcal{E}_P^\vee(1)) \leq 44$, $h^0(\mathcal{E}_P^\vee(1)) \leq 9$ and hence we do not have enough linear syzygies.

For $n = 3$, if \mathcal{E}_P is an ordinary Drézet bundle we have again $H^0\mathcal{E}_P^\vee(1) = 0$. Otherwise we argue as in Lemma 34. We denote by I the ideal generated by the seven cubics and by J a complete intersection Artinian ideal generated by four cubics in I . We use J to link I to I_1 . We denote by h_t, h'_t and h''_t the Hilbert functions of J, I and I_1 , respectively. By Proposition 33 we know that for $i \geq 0$, $h''_i = h_{8-i} - h'_{8-i}$. In particular, we have $h''_5 = h_3 - h'_3 = 16 - 13 = 3$. Applying Macaulay's inequalities we get $h''_4 = h_4 - h'_4 = 19 - h'_4 \geq 3$ and by Gotzmann persistence theorem [21] we have $h''_4 > 3$. Therefore, we get that $h^1(\mathcal{E}_P^\vee(1)) = h'_4 = h_4 - h''_4 \leq 15$ and, hence, $h^0(\mathcal{E}_P^\vee(1)) \leq 8$ and we do not have enough linear syzygies.

For $n = 2$, a computation shows that $s_2(\mathcal{E}_P^\vee(1)) = 0$. This implies that if $\mathcal{E}_P^\vee(1)$ is generated by global sections, it can be generated by $e + n - 1 = 7$ sections, and we get an exact sequence $0 \rightarrow L \rightarrow \mathcal{O}_{\mathbb{P}^2}^{\oplus 7} \rightarrow \mathcal{E}_P^\vee(1) \rightarrow 0$. Adjusting Chern classes yields $L = \mathcal{O}_{\mathbb{P}^2}(-3)$. As a consequence $h^1(\mathcal{E}_P^\vee(1)) = h^2(L) = 1$, which means that \mathcal{E}_P is not ordinary. \square

Note that we get an exact complex

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(-3) \rightarrow P^\vee \otimes \mathcal{O}_{\mathbb{P}^n} \xrightarrow{M} Q \otimes \mathcal{O}_{\mathbb{P}^n}(1) \rightarrow \mathcal{O}_{\mathbb{P}^n}(4) \rightarrow 0,$$

with $P^\vee = H^0(\mathcal{E}_P)$ and $Q^\vee = H^0(\mathcal{E}_P^\vee(1))$. According to [7], one can then identify P and Q , and M is then skew-symmetric. In other words, the bundle \mathcal{E}_P must be Pfaffian. The corresponding 7×7 matrix of constant rank 6 is

then obtained by restricting the universal skew-symmetric 7×7 matrix to a projective plane that does not meet the codimension three locus of matrices of rank at most four. Finally, P is generated by the seven Pfaffian minors of this matrix.

Example 38. Consider the smooth plane quartic C whose equation is given by the tensor $e_0^3 e_1 + e_1^3 e_2 + e_2^3 e_0$. The seven dimensional space of cubics apolar to this quartic tensor is

$$P = \langle x_0^3 - x_1^2 x_2, x_1^3 - x_2^2 x_0, x_2^3 - x_0^2 x_1, x_0^2 x_2, x_1^2 x_0, x_2^2 x_1, x_0 x_1 x_2 \rangle.$$

Denote the seven generators of P by $A_0, A_1, A_2, B_0, B_1, B_2, E$. They admit seven linear syzygies, and $Syz_1(P)$ is generated by $x_1 B_0 - x_0 E, x_2 B_1 - x_1 E, x_0 B_2 - x_2 E, x_2 B_2 - x_0 B_1 - x_1 A_2, x_0 B_0 - x_1 B_2 - x_2 A_0, x_1 B_1 - x_2 B_0 - x_0 A_1, x_0 A_0 + x_1 A_1 + x_2 A_2$. Indexing the columns by $A_2, A_1, A_0, B_0, B_1, B_2, E$ these seven syzygies yield the skew-symmetric matrix

$$\begin{pmatrix} 0 & 0 & 0 & x_1 & 0 & 0 & -x_0 \\ 0 & 0 & 0 & 0 & x_2 & 0 & -x_1 \\ 0 & 0 & 0 & 0 & 0 & x_0 & -x_2 \\ -x_1 & 0 & 0 & 0 & -x_0 & x_2 & 0 \\ 0 & -x_2 & 0 & x_0 & 0 & -x_1 & 0 \\ 0 & 0 & -x_0 & -x_2 & x_1 & 0 & 0 \\ x_0 & x_1 & x_2 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Using a normal form for plane quartics it is in principle possible to obtain a normal form for all 7×7 matrices of constant rank 6 of linear forms in three variables. Note that the dimension of $\mathbb{P}(S^4 V)/PGL(V)$ is 6, and that this is also the dimension of $G(3, \wedge^2 A_7)/PGL(A_7)$.

We can summarize this discussion as follows.

Proposition 39. *All 7×7 matrices of constant rank 6 of linear forms in three variables are given by Drézet bundles on \mathbb{P}^2 which are not ordinary. These Drézet bundles are in fact Pfaffian, and defined by linear systems of cubics which are apolar to a given quartic curve in the dual projective plane.*

5.1.3. *The mixed case.* Consider now a vector bundle \mathcal{E} on $\mathbb{P}^n = \mathbb{P}V$, for $n \leq 4$, defined by an exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(-1) \oplus \mathcal{O}_{\mathbb{P}^n}(-2) \rightarrow A^\vee \otimes \mathcal{O}_{\mathbb{P}^n} \rightarrow \mathcal{E} \rightarrow 0,$$

with A of dimension eight. Supposing that $h^0(\mathcal{E}^\vee) = 0$, the dual sequence implies that A embeds into $V^\vee \oplus S^2 V^\vee$.

Suppose first that $n = 2$, so that A is a hyperplane in $V^\vee \oplus S^2 V^\vee$. Denote by $(a_0, q_0) \in V \oplus S^2 V$ a generator of its orthogonal. The fiber of \mathcal{E} at $[v]$ can then be identified with $(V \oplus S^2 V)/\langle (v, 0), (0, v^2), (a_0, q_0) \rangle$. In particular there is a natural morphism from Q to \mathcal{E} , which is an injective morphism of vector bundles when q_0 has rank at least two. Under this hypothesis, we get an exact sequence

$$0 \rightarrow Q \rightarrow \mathcal{E} \rightarrow \mathcal{E}_H \rightarrow 0,$$

where \mathcal{E}_H is the rank four Drézet bundle defined by q_0 , so that H is the orthogonal hyperplane to q_0 in S^2V^\perp . Such bundles were discussed in Example 24.

Lemma 40. *This exact sequence splits if and only if a_0 belongs to the span of q_0 . In particular, it always splits when q_0 has rank three, but not necessarily when q_0 has rank two.*

Proof. A splitting of the exact sequence must be induced by a morphism $\iota : S^2V \rightarrow V$ such that $\iota(v^2)$ is a multiple of v for all v , and $\iota(q_0) = a_0$. The first condition implies that ι is the contraction map by some linear form θ . Then the second condition means that $a_0 = \theta \rfloor q_0$, and the existence of such a θ precisely means that a_0 belongs to the span of q_0 . \square

As a consequence, \mathcal{E} is decomposable unless q_0 has rank two and a_0 does not belong to its span. In this case we know by the second part of Example 24 that $\mathcal{F}^\vee(1)$ is not generated by global sections, but this does not a priori prevent $\mathcal{E}^\vee(1)$ to be generated. And this is exactly what happens!

Proposition 41. *Suppose q_0 has rank two, and a_0 does not belong to its span. Then $\mathcal{E}^\vee(1)$ is generated by global sections.*

Proof. As above we can choose a basis such that $(a_0, q_0) = (e_0, e_1e_2)$. We first claim that $h^0(\mathcal{E}^\vee(1)) = 8$, the expected value. Indeed, the exact sequence

$$0 \rightarrow H^0(\mathcal{E}^\vee(1)) \rightarrow A \otimes V^\vee \rightarrow S^2V^\vee \oplus S^3V^\vee \rightarrow H^1(\mathcal{E}^\vee(1)) \rightarrow 0$$

allows to interpret $H^0(\mathcal{E}^\vee(1))$ as a space of linear syzygies between vectors in A . Choose a basis of A given by the vectors $(-x_0, m)$, $(x_1, 0)$, $(x_2, 0)$, and $(0, n)$ for the five monomials n distinct from $m = x_1x_2$ or x_2^2 . Then the linear syzygies of A are generated by the tautological linear syzygy $x_2(x_1, 0) - x_1(x_2, 0)$, and the syzygies of the form

$$\sum_{n \neq m} \ell_n(0, n) + \ell_m(-x_0, m) + \ell_1(x_1, 0) + \ell_2(x_2, 0) = 0$$

where $\sum_n \ell_n n$ is a non-trivial syzygy between the six degree two monomials in x_0, x_1, x_2 ; this imposes that ℓ_m has no term in x_0 , so there are seven such syzygies, which makes eight with the tautological one.

As a consequence, $h^1(\mathcal{E}^\vee(1)) = 0$, and we are allowed to use the snake Lemma exactly as we did for Drézet bundles in the proof of Lemma 20. We conclude that $\mathcal{E}^\vee(1)$ is generated at $[v]$ if and only if the natural morphism

$$S^2V/\langle v^2 \rangle \oplus S^3V/\langle v^3 \rangle \rightarrow \text{Hom}(A, V/\langle v \rangle)$$

is injective. If this is not the case, there exists tensors $Q \in S^2V$ and $C \in S^3V$, not both powers of v , such that for any (ℓ, κ) in A the contraction $\ell \rfloor Q + \kappa \rfloor C$ gives a vector in $\langle v \rangle$. Let us make an explicit computation. Using our preferred basis of A we rewrite these conditions as

$$x_0 \rfloor Q = x_1x_2 \rfloor C, \quad x_1 \rfloor Q = x_2 \rfloor Q = n \rfloor C = 0 \quad \text{mod } v$$

for n any monomial other than x_1x_2 . In particular there must exist scalars s_i such that $x_i^2 \rfloor C = s_i v$, and we deduce that

$$C = \frac{1}{6} \sum_i s_i v_i e_i^3 + \frac{1}{2} \sum_{i \neq j} s_i v_j e_i^2 e_j + t e_0 e_1 e_2$$

for some t . There must also exist r_1, r_2 such that $x_0 x_i \rfloor C = r_i v$, which gives

$$t = r_1 v_2, \quad s_0 v_1 = r_1 v_0, \quad s_1 v_0 = r_1 v_1,$$

$$t = r_2 v_1, \quad s_0 v_2 = r_2 v_0, \quad s_2 v_0 = r_2 v_2.$$

This implies that $s_0 v_1^2 = s_1 v_0^2$ and $s_0 v_2^2 = s_2 v_0^2$.

If (s_0, s_1, s_2) is colinear to (v_0^2, v_1^2, v_2^2) , then C reduces to $t' e_0 e_1 e_2$ modulo v^3 , for some scalar t' . Then we need $t' e_1$ and $t' e_2$ to be both multiples of v , which implies that $t' = 0$. Finally, we remain with the conditions that $x_i \rfloor Q$ be multiples of v for each i , which is only possible if Q is a multiple of v^2 .

The other possibility is that $s_0 = v_0 = 0$, which implies that $r_1 v_1 = r_2 v_2 = 0$. If $v_1 v_2 \neq 0$, we get $r_1 = r_2 = 0$ and we deduce that C has no term involving e_0 , and that we can write

$$C = \frac{1}{6} (s_1 v_1 e_1^3 + s_2 v_2 e_2^3) + \frac{1}{2} (s_1 v_2 e_1^2 e_2 + s_2 v_1 e_1 e_2^2).$$

We deduce that $x_0 \rfloor Q = x_1 x_2 \rfloor C = s_1 v_2 e_1 + s_2 v_1 e_2$ modulo v . Since the three contractions of Q by x_0, x_1, x_2 do not involve e_0 , this must also be the case of Q , so that in fact $x_0 \rfloor Q = 0$. So $s_1 v_2 e_1 + s_2 v_1 e_2$ must be a multiple of $v = v_1 e_1 + v_2 e_2$, which amounts to $s_1 v_2^2 = s_2 v_1^2$ and implies that C is a multiple of v^3 . Finally, the three contractions of Q by x_0, x_1, x_2 must then be multiples of v , and thus Q is a multiple of v^2 . This concludes the proof. \square

We immediately get the associated 8×8 matrix of constant rank 6, using the same basis as above for A and its linear syzygies:

$$M = \begin{pmatrix} 0 & x_2 & -x_1 & 0 & 0 & 0 & 0 & 0 \\ -x_2 & 0 & x_0 & 0 & 0 & -x_1 & 0 & 0 \\ x_1 & -x_0 & 0 & 0 & x_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_1 & 0 & 0 & -x_0 & 0 \\ 0 & 0 & 0 & x_2 & 0 & 0 & 0 & -x_0 \\ 0 & 0 & 0 & 0 & x_0 & 0 & -x_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_0 & 0 & -x_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & x_2 & -x_1 \end{pmatrix}.$$

This matrix clearly exhibits the fact that \mathcal{E} is an extension of the quotient bundle, corresponding to the 3×3 NorthWest block, by a non-generic Drézet bundle corresponding to the 5×5 SouthEast block. The former has constant rank two. The latter has bounded rank four, and we know that the rank does drop to three on the line $(x_0 = 0)$; but remarkably, this is cured by the two extra nonzero entries of the matrix, which keep the rank constant.

Note also that there is an exact complex

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-2) \rightarrow \mathcal{O}_{\mathbb{P}^2}^{\oplus 8} \xrightarrow{M} \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 8} \rightarrow \mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2}(3) \rightarrow 0.$$

Proposition 42. *Suppose q_0 has rank one, and that a_0 does not belong to its span. Then $\mathcal{E}^\vee(1)$ is not generated by global sections.*

Proof. An explicit computation, choosing $(a_0, q_0) = (e_0, e_2^2)$, shows that again $\mathcal{E}^\vee(1)$ has eight sections. This allows to use the same criterion for global generation as in the previous Lemma. And it is easy to exhibit a counter-example: choose $v = e_2$, $C = e_2^2 e_0$ and $Q = \frac{1}{2}e_0^2$. Indeed, if n is a degree two monomial distinct from x_2^2 it is clear that $n \rfloor c$ is a multiple of e_2 , and it is even clearer that $x_1 \rfloor Q = x_2 \rfloor Q = 0$. Finally, $(x_0, -\frac{1}{2}x_2^2)$ is in A , and applying it to (Q, C) we get $x_0 \rfloor Q - \frac{1}{2}x_2^2 \rfloor C = 0$. \square

Now that we know precisely what can happen on \mathbb{P}^2 , it is easy to upgrade to \mathbb{P}^3 and \mathbb{P}^4 . So again suppose that the vector bundle \mathcal{E} is defined by an exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3}(-2) \rightarrow A^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \rightarrow \mathcal{E} \rightarrow 0,$$

with A of dimension eight. Consider a hyperplane $H \subset \mathbb{P}^3$ and denote by \mathcal{F} the restriction of \mathcal{E} to H ; it is also defined by the previous sequence, restricted to H . Moreover if $\mathcal{E}^\vee(1)$ is generated by global sections, then $\mathcal{F}^\vee(1)$ has the same property, so by the previous discussions we know that either \mathcal{F} splits, or is a non-trivial extension of a Drézet bundle by the quotient bundle. Moreover we have seen that $h^0(\mathcal{F}^\vee(1)) = 8$ in both cases. But the exact sequence $0 \rightarrow \mathcal{E}^\vee \rightarrow \mathcal{E}^\vee(1) \rightarrow \mathcal{F}^\vee(1) \rightarrow 0$ implies that if \mathcal{E} is indecomposable, $h^0(\mathcal{E}^\vee(1)) \leq h^0(\mathcal{F}^\vee(1)) = 8$. Since $8 < 6 + 3$ and $s_3(\mathcal{E}^\vee(1)) = 12 \neq 0$, we do not have enough sections for $\mathcal{E}^\vee(1)$ to be globally generated. A fortiori we cannot have such bundles on \mathbb{P}^4 , and we have proved:

Proposition 43. *On \mathbb{P}^3 and \mathbb{P}^4 , there is no rank six indecomposable bundle \mathcal{E} of mixed type such that $\mathcal{E}^\vee(1)$ is globally generated.*

5.1.4. *The Tango case.* Finally, there is a family of vector bundles on \mathbb{P}^3 connected with the Tango bundle, which can be defined by an exact sequence of type

$$0 \rightarrow Q(-1) \oplus \mathcal{O}_{\mathbb{P}^3}(-1) \rightarrow A_{e+4} \otimes \mathcal{O}_{\mathbb{P}^3} \rightarrow \mathcal{E} \rightarrow 0.$$

(Remember that the Tango bundle \mathcal{F}^\vee fits into a sequence $0 \rightarrow Q(-1) \rightarrow B_5 \otimes \mathcal{O}_{\mathbb{P}^3} \rightarrow \mathcal{F}^\vee \rightarrow 0$.) Since $h^0(Q^\vee(1)) = 6$ and $h^0(\mathcal{O}_{\mathbb{P}^3}(1)) = 4$, we must have $h^0(\mathcal{E}^\vee) > 0$ as soon as $e > 6$, and then \mathcal{E} has a trivial factor. Moreover if $e = 6$, necessarily $A_{10}^\vee \simeq H^0(Q^\vee(1)) \oplus H^0(\mathcal{O}_{\mathbb{P}^3}(1))$ and then \mathcal{E} has to decompose as $\wedge^2 Q \oplus Q$. As a consequence there is no indecomposable vector bundle of rank at least six in this family of vector bundles.

Remark. Some extra work would be required to get a full classification for $c_1 = 3$. This classification would necessarily include Example 12.

5.2. Matrices of constant rank six. We have just classified those matrices of constant rank 6 whose associated vector bundle \mathcal{E} has $c_1(\mathcal{E}) = 3$. There remains to understand the cases where $c_1(\mathcal{E}) < 3$, and \mathcal{E} indecomposable.

For $c_1(\mathcal{E}) = 1$, we know that \mathcal{E} must be the quotient bundle Q on $\mathbb{P}^6 = \mathbb{P}V$, and that the possible matrices are submatrices of the 7×21 matrix encoding, in some chosen basis, the contraction map $V \otimes \wedge^2 V^\vee \rightarrow V^\vee$.

For $c_1(\mathcal{E}) = 2$, we know from Propositions 25 and 26 that \mathcal{E} must be a Drézet bundle on \mathbb{P}^n , with $2 \leq n \leq 4$. Since $S^2\mathbb{C}^3$ has only dimension six, there is no such indecomposable rank six Drézet bundle on \mathbb{P}^2 . The case of \mathbb{P}^4 is also easy to exclude:

Lemma 44. *On \mathbb{P}^4 , $\mathcal{E}_P^\vee(1)$ cannot be generated by global sections.*

Proof. The expected dimension of $H^0(\mathcal{E}_P^\vee(1))$ is zero, while we would need at least ten sections to generate it everywhere. If \mathcal{E}_P is an ordinary Drézet bundle we have $H^0\mathcal{E}_P^\vee(1) = 0$ while for any seven-dimensional subspace $P \subset S^2V^\vee$ consisting of quadrics without common zeros we have $H^1\mathcal{E}_P^\vee(1) \leq 9$ and hence $H^0\mathcal{E}_P^\vee(1) \leq 9$. \square

So we can focus on $\mathbb{P}^3 = \mathbb{P}V$, and consider an exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-2) \rightarrow P^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \rightarrow \mathcal{E}_P \rightarrow 0,$$

for some dimension seven subspace P of S^2V^\vee , with no non-trivial common zero. An easy computation yields:

Lemma 45. *For $k \geq -1$, $\mathcal{E}_P(k)$ has no higher cohomology. Moreover*

$$h^0(\mathcal{E}_P(k)) = (k+1)(k^2 + 6k + 7).$$

Dualizing the defining sequence of \mathcal{E}_P , we get:

Lemma 46.

- (1) For $k \geq 0$, $h^0(\mathcal{E}_P^\vee(-k)) = 0$.
- (2) For $k \geq 3$, $h^1(\mathcal{E}_P^\vee(-k)) = 0$.
- (3) For $0 \leq k \leq 2$ and $q \neq 1$, $h^q(\mathcal{E}_P^\vee(-k)) = 0$.
- (4) $h^1(\mathcal{E}_P^\vee) = 3$, $h^1(\mathcal{E}_P^\vee(-1)) = 4$, $h^1(\mathcal{E}_P^\vee(-2)) = 1$.

Denote by $\mathfrak{sl}(\mathcal{E}_P)$ the vector bundle of traceless endomorphisms of \mathcal{E}_P .

Proposition 47. *\mathcal{E}_P is simple, and $h^q(\mathfrak{sl}(\mathcal{E}_P)) = \delta_{q,1}21$.*

Proof. Using the previous lemmas we get the long exact sequence

$$0 \rightarrow H^0(\text{End}(\mathcal{E}_P)) \rightarrow H^1(\mathcal{E}_P^\vee(-2)) \rightarrow P^\vee \otimes H^1(\mathcal{E}_P^\vee) \rightarrow H^1(\text{End}(\mathcal{E}_P)) \rightarrow 0,$$

and $H^q(\text{End}(\mathcal{E}_P)) = 0$ for $q > 1$. Since $H^0(\text{End}(\mathcal{E}_P))$ contains the homotheties and $H^1(\mathcal{E}_P^\vee(-2))$ is one dimensional, we must have $H^0(\text{End}(\mathcal{E}_P)) = \mathbb{C}$, i.e., \mathcal{E}_P is simple. Then we deduce that $H^1(\text{End}(\mathcal{E}_P)) \simeq P^\vee \otimes H^1(\mathcal{E}_P^\vee)$ has dimension $7 \times 3 = 21$. \square

Lemma 48. *For P generic, \mathcal{E}_P is ordinary and $h^0(\mathcal{E}_P^\vee(1)) = 8$.*

Proof. By Proposition 19, $H^0(\mathcal{E}_P^\vee(1))$ coincides with the space $Syz_1(P)$ of linear syzygies between the quadrics in P , that is, the kernel of the multiplication map $P \otimes V^\vee \rightarrow S^3V^\vee$. By Theorem 22, for P generic this morphism has maximal rank, which means here that it is surjective. So its kernel has dimension $7 \times 4 - 20 = 8$. \square

Given a rank six vector bundle on \mathbb{P}^3 , we need in principle at least $6+3 = 9$ sections to generate it everywhere; so 8 sections should not suffice. But note that the identity

$$s(\mathcal{E}_P^\vee(1)) = (1 + 3h)/(1 + h)^7$$

implies $s_3(\mathcal{E}_P^\vee(1)) = 0$, which preserves the hope that $\mathcal{E}_P^\vee(1)$ be generated by global sections. Indeed, we have:

Proposition 49. *For P general, $\mathcal{E}_P^\vee(1)$ is generated by global sections.*

Proof. For P general, \mathcal{E}_P is ordinary and we can apply Lemma 20, according to which if $\mathcal{E}_P^\vee(1)$ is not globally generated at $[v]$, there must exist a cubic tensor $\theta \in S^3V$, independent from v^3 , such that the contractions $P] \theta \subset \mathbb{C}v$.

This suggests to stratify S^3V by the rank ρ of $\iota(\theta)$. If the latter condition is realized, $[\ell]$ must be contained in the projectivized image, a copy of $\mathbb{P}^{\rho-1}$, and N^\perp must be contained in a copy of the Grassmannian $Gr(7, 11 - \rho)$, of dimension $7 \times (4 - \rho)$. Hence a total amount of $27 - 6\rho$ parameters, which is smaller than the dimension 21 of $Gr(7, 10)$ as soon as $\rho \geq 2$. So for N^\perp generic $\rho \geq 2$ is impossible. But suppose that $\rho = 1$. Then θ must be the cube v^3 of a single vector, and for ℓ to belong to the image of $G(\theta)$ it must be proportional to v , which means that $\bar{\theta} = 0$ and concludes the proof. \square

When $\mathcal{E}_P^\vee(1)$ is generated, consider the exact sequence

$$0 \longrightarrow \mathcal{G}_P(-2) \longrightarrow H^0(\mathcal{E}_P^\vee(1)) \otimes \mathcal{O}_{\mathbb{P}^3} \longrightarrow \mathcal{E}_P^\vee(1) \longrightarrow 0.$$

The rank two bundle $\mathcal{G}_P \simeq \mathcal{G}_P^\vee$ has Chern numbers $c_1 = 0$ and $c_2 = 3$.

Lemma 50. *\mathcal{G}_P is stable.*

Proof. Since \mathcal{G}_P has rank two and zero first Chern class, according to Hoppe's criterion for stability [24] we just need to check that $h^0(\mathcal{G}_P(-k)) = 0$ for every $k \geq 0$. Because of the isomorphism $\mathcal{G}_P \simeq \mathcal{G}_P^\vee$ we have an exact sequence

$$0 \longrightarrow \mathcal{E}_P(-k-3) \longrightarrow H^0(\mathcal{E}_P^\vee(1))^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(-k-2) \longrightarrow \mathcal{G}_P(-k) \longrightarrow 0.$$

By Lemma 46, $h^1(\mathcal{E}_P(-k-3)) = 0$ for $k \geq 0$, hence the claim. \square

A *mathematical instanton* on \mathbb{P}^3 is a rank two stable vector bundle \mathcal{F} with $c_1(\mathcal{F}) = 0$ and $h^1(\mathcal{F}(-2)) = 0$. Its *charge* is defined as $c_2(\mathcal{F})$.

Lemma 51. *\mathcal{G}_P is a mathematical instanton of charge three.*

Proof. Lemma 50 addresses the stability condition, while the vanishing of $h^1(\mathcal{G}_P(-2))$ follows from the surjectivity of the evaluation morphism for $\mathcal{E}_P^\vee(1)$. \square

Let us dualize the defining sequence of \mathcal{G}_P , twist it by $\mathcal{O}_{\mathbb{P}^3}(1)$ and concatenate it with the defining sequence of \mathcal{E}_P . Since \mathcal{G}_P is self-dual, we get the free resolution

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-5) \rightarrow \mathcal{O}_{\mathbb{P}^3}(-3)^{\oplus 7} \rightarrow \mathcal{O}_{\mathbb{P}^3}(-2)^{\oplus 8} \rightarrow \mathcal{G}_P \rightarrow 0.$$

The matrix of linear forms defined by \mathcal{E}_P is exactly the one that appears in the middle arrow. Note that this exact sequence already appears in [30, Proposition 4.2]. In [22] it was also observed that via Serre's correspondence, $\mathcal{G}_P(2)$ is associated to elliptic curves of degree seven in \mathbb{P}^3 .

Stable rank two vector bundles \mathcal{G} on \mathbb{P}^3 with Chern classes $c_1 = 0$ and $c_2 = 3$ have been extensively studied. The main result of [20] is that their moduli space $\mathcal{M}(3)$ is the disjoint union of two irreducible components $\mathcal{M}_\alpha(3)$, where $\alpha \in \mathbb{Z}/2\mathbb{Z}$ is the Atiyah-Rees invariant, defined as the parity of $h^1(\mathcal{G}(-2))$. In particular, instanton bundles are parametrized by an open subset of $\mathcal{M}_0(3)$. Both components are smooth, rational, of dimension 21.

Theorem 52. *The exists an irreducible family of 7×8 matrices of linear forms in four variables, of constant rank 6, whose associated vector bundles are parametrized by the open subset of bundles \mathcal{G} inside the moduli space of mathematical instantons of charge 3 and Atiyah-Rees invariant 0 on \mathbb{P}^3 , such that*

- (1) \mathcal{G} has natural cohomology,
- (2) $\mathcal{G}(2)$ is globally generated.

Recall that a vector bundle \mathcal{G} on \mathbb{P}^n is said to have natural cohomology if for any integer k at most one cohomology group $H^i(\mathbb{P}^n, \mathcal{G}(k))$ is non-zero for $0 \leq i \leq n$.

Proof. It follows from Lemme 1.1, Proposition 1.1.1 and sections 1.3 and 1.4 in [22] that an instanton bundle \mathcal{G} gives a matrix of constant rank if and only if it has natural cohomology and $\mathcal{G}(2)$ is generated by global sections. \square

We summarize the relations between the instanton bundle \mathcal{G}_P , the rank six bundle \mathcal{E}_P and the matrix M of linear forms, by the following diagram, where the diagonal short exact sequences are exact:

$$\begin{array}{ccccc}
 \mathcal{O}_{\mathbb{P}^3}(-2) & & & & \mathcal{G}_P(3) \\
 & \searrow P & & & \nearrow \\
 & & \mathcal{O}_{\mathbb{P}^3}^{\oplus 7} & \xrightarrow{M} & \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 8} \\
 & & \searrow & & \nearrow \\
 & & & \mathcal{E}_P &
 \end{array}$$

Since P has codimension three, the initial data for the construction is essentially a tensor in $\mathbb{C}^3 \otimes S^2\mathbb{C}^4$, from which it is easy construct the 7×8

matrix explicitly. In fact if $N = P^\perp \subset S^2V$ is a net of quadrics, then

$$H^0(\mathcal{E}) = S^2V/N, \quad H^0(\mathcal{E}^\vee(1)) = \text{Ker}(N^\perp \otimes V^\vee \rightarrow S^3V^\vee),$$

and the natural duality between N^\perp and S^2V/N yields the desired matrix.

Remark. The study of nets of quadrics in \mathbb{P}^3 is a classical topic, see for example [34]. The set of singular quadrics in such a net is parametrized by a plane quartic with a symmetric determinantal representation. It was already known to Hesse that a general plane quartic admits 36 non-equivalent such determinantal representations, corresponding to its even theta characteristics. This means that a general plane quartic is the characteristic curve of 36 distinct nets of quadrics.

Example 53. Consider the net of quadrics

$$N = \langle e_0^2 - e_1^2 + e_2^2, e_1^2 + e_2^2 + e_3^2, e_0e_2 + e_0e_3 + e_1e_3 \rangle \subset S^2V.$$

The characteristic curve is the plane quartic of equation

$$\lambda^4 - 2\lambda^2\mu^2 + \mu\nu(\mu^2 - \nu^2) = 0,$$

a smooth curve. The dual space of quadrics is

$$P = \langle x_0^2 + x_1^2 - x_3^2, x_0^2 - x_2^2 + x_3^2, x_0x_1, x_1x_2, x_2x_3, x_0(x_2 - x_3), (x_0 - x_1)x_3 \rangle.$$

Denote this basis of P by Q_0, Q_1, A, B, C, D, E . The eight syzygies are

$$\begin{aligned} x_2A &= x_0B, \quad x_3B = x_1C, \quad x_1D = (x_2 - x_3)A, \quad x_2E = (x_0 - x_1)C, \\ x_0(Q_0 - Q_1) &= x_1A - x_0C + (x_2 + 2x_3)D, \\ x_1Q_1 &= x_0(A + C) - x_2B - x_3(D + E), \\ x_2Q_0 &= x_3A + x_1B - x_3C + x_0(D + E), \\ x_3(Q_0 + Q_1) &= (2x_3 + x_2)A - x_2C - x_1D + (2x_0 - x_1)E. \end{aligned}$$

We get the following 7×8 matrix of constant rank 6:

$$\begin{pmatrix} 0 & 0 & x_2 & -x_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_3 & -x_1 & 0 & 0 \\ 0 & 0 & x_3 - x_2 & 0 & 0 & x_1 & 0 \\ 0 & 0 & 0 & 0 & x_1 - x_0 & 0 & x_2 \\ x_0 & -x_0 & -x_1 & 0 & x_0 & -x_2 - 2x_3 & 0 \\ 0 & x_1 & -x_0 & x_2 & -x_0 & x_3 & x_3 \\ x_2 & 0 & -x_3 & -x_1 & x_3 & -x_0 & -x_0 \\ x_3 & x_3 & -x_2 - 2x_3 & 0 & x_2 & x_1 & 2x_0 + x_1 \end{pmatrix}$$

Our final classification is the following:

Theorem 54. *Given an indecomposable matrix of linear forms of constant rank 6, the associated bundle \mathcal{E} must be, up to switching with $\mathcal{E}^\vee(1)$, of one the following types:*

- (1) ($c_1 = 1$) The quotient bundle Q on \mathbb{P}^6 .
- (2) ($c_1 = 2$) A Drézet bundle on \mathbb{P}^3 defined by a net of quadrics.

- (3) ($c_1 = 3$) The second exterior power $\wedge^2 Q$ on \mathbb{P}^4 .
- (4) ($c_1 = 3$) A Drézet bundle on \mathbb{P}^2 defined by the cubics apolar to a plane quartic.
- (5) ($c_1 = 3$) A non-trivial extension of the quotient bundle Q on \mathbb{P}^2 by a rank 4 non-generic Drézet bundle with $c_1 = 2$.

6. MATRICES OF CONSTANT RANK VIA EVALUATION MORPHISMS

Our examples suggest a general procedure to construct spaces of matrices of bounded rank from a vector bundle \mathcal{F} on $\mathbb{P}^n = \mathbb{P}V$ generated by global sections. Such \mathcal{F} will be the cokernel sheaf associated to the corresponding space of matrices of bounded rank e . Such a bundle is generated by any generic space of sections of dimension at least $n + f$, where f is the rank of \mathcal{F} . So consider $\Sigma_{e+f} \subset H^0(\mathbb{P}^n, \mathcal{F})$ some generating space of sections, of dimension $e + f$. The kernel of the evaluation map is a vector bundle \mathcal{K}_e of rank e , fitting into the exact sequence

$$0 \rightarrow \mathcal{K}_e \rightarrow \Sigma_{e+f} \otimes \mathcal{O}_{\mathbb{P}^n} \rightarrow \mathcal{F} \rightarrow 0.$$

If $\mathcal{E} := \mathcal{K}_e(1)$ has enough global sections it will give rise to a space of bounded rank e . If \mathcal{E} is generated by global sections, it will give rise to a space of constant rank e . The composition

$$H^0(\mathbb{P}^n, \mathcal{K}_e(1)) \otimes \mathcal{O}_{\mathbb{P}^n} \rightarrow \mathcal{K}_e(1) \rightarrow \Sigma_{e+f} \otimes \mathcal{O}_{\mathbb{P}^n}(1)$$

yields a space of matrices of linear forms of rank at most e , and size $a_e \times (e+f)$ for $a_e = h^0(\mathcal{K}_e(1))$. This number is potentially large:

$$a_e \geq (n+1)(e+f) - h^0(\mathcal{F}(1)),$$

and equality holds when $h^1(\mathcal{F}(1)) = 0$, which is expected to be a condition for $\mathcal{K}_e(1)$ to have nontrivial sections. We will impose $a_e > e$ so that the rank being bounded by e is a non-trivial condition. This leads to the conditions

$$\frac{1}{n}(h^0(\mathcal{F}(1)) - (n+1)f) < r \leq h^0(\mathcal{F}) - f.$$

For example, suppose $\mathcal{F} = Q^\vee(1)$ is the twisted dual of the quotient bundle, so $f = n$. If we take all the sections of \mathcal{F} , that is for $e = \binom{n+1}{2}$, then $\mathcal{K}_e = \wedge^2 Q^\vee$ and $\mathcal{K}_e(1) = \wedge^{n-2} Q$ is generated by global sections. In general, Σ_{e+n} is a subspace of $\wedge^2 V^\vee$, and \mathcal{K}_e is a subbundle of $\wedge^2 Q^\vee$, with trivial quotient. The expected conditions for \mathcal{K}_e to admit global sections become

$$\frac{(n-1)(n+1)}{3} < e \leq \binom{n}{2}.$$

One expects that $\mathcal{K}_e(1)$ remains generated by global sections when e is close to $\binom{n}{2}$. How close? We have a diagram

$$\begin{array}{ccccc}
& & Q^\vee \otimes \Sigma_{e+n} & \xrightarrow{g} & \mathcal{G} \\
& & \downarrow & & \downarrow \\
H^0(\mathcal{K}_e(1)) \otimes \mathcal{O}_{\mathbb{P}^n} & \longrightarrow & V^\vee \otimes \Sigma_{e+n} \otimes \mathcal{O}_{\mathbb{P}^n} & \longrightarrow & S_{2,1}V^\vee \otimes \mathcal{O}_{\mathbb{P}^n} \\
\downarrow f & & \downarrow & & \downarrow \\
\mathcal{K}_e(1) & \longrightarrow & \mathcal{O}(1) \otimes \Sigma_{e+n} & \longrightarrow & Q^\vee(2)
\end{array}$$

where \mathcal{G} , which has rank $\frac{n(n^2+3n-1)}{3}$, is defined by the rightmost vertical exact sequence.

By the snake lemma, if the sequence of trivial bundles is exact, f is surjective exactly when g is surjective, or equivalently when g^t is injective. In order to decide whether it is, note that there is an exact sequence

$$0 \rightarrow \mathcal{G}^\vee \rightarrow \wedge^2 V \otimes Q \rightarrow \wedge^3 Q \rightarrow 0.$$

To see this, over a point $[v]$ choose a splitting $V = H \oplus \ell$ as before with $\ell = \mathbb{C}v$. then over the point $\mathcal{G} \cong S_{2,1}H^\vee \oplus S^2H^\vee \otimes \ell^\vee$, and thus \mathcal{G}^\vee maps to $\wedge^2 V \otimes Q \cong \mathcal{G}^\vee \oplus \wedge^3 H$, and the latter term is the fiber of $\wedge^3 Q$.

The above discussion implies:

Lemma 55. *The evaluation morphism $H^0(\mathcal{K}_e(1)) \otimes \mathcal{O}_{\mathbb{P}^n} \rightarrow \mathcal{K}_e(1)$ is surjective exactly when the composition*

$$\Sigma_{e+n}^\perp \otimes Q \hookrightarrow \wedge^2 V \otimes Q \rightarrow \wedge^3 Q$$

is everywhere injective.

Theorem 56. *For $n \geq 6$ and $\frac{n(n-1)}{2} \geq e > \frac{(n-1)^2}{2}$, a general subspace $\Sigma_{e+n} \subset \wedge^2 V$ of dimension $e+n$ defines a matrix of linear forms of constant rank e of size $a_e \times b_e$, where*

$$a_e = e + n, \quad b_e = e(n+1) - 2 \binom{n+1}{3}.$$

The corresponding rank e vector bundle \mathcal{E} has $c_1(\mathcal{E}) = e - n + 1$.

Proof. Set $\ell = \binom{n+1}{2} - (n+e)$, the codimension of $\Sigma = \Sigma_{e+n}$, and let $\Theta_\ell = \Sigma_{k+n}^\perp$, which is a generic ℓ -dimensional subspace of $\wedge^2 V$.

First consider the case $\ell = 1$. Suppose $\Theta_1 \otimes Q \rightarrow \wedge^3 Q$ is not injective at some point $[v]$, and that $\omega \otimes \bar{u}$ is sent to zero for a generator ω of Θ_1 and a vector $u \in V$, not colinear to v . This means that $\omega \wedge u$ belongs to the kernel of the projection $\wedge^3 V \rightarrow \wedge^3 Q$, that is $v \wedge (\wedge^2 V)$. But then, since u and v are independent, this implies that $\omega = u \wedge \alpha + v \wedge \beta$ for some vectors α, β . In particular ω has rank at most four, so Θ_1 cannot be generic for $n \geq 6$.

Now consider the general case, and suppose that at some point $[v]$, an element $\omega_1 \otimes \bar{u}_1 + \cdots + \omega_\ell \otimes \bar{u}_\ell$ of $\Theta_\ell \otimes Q$ is sent to zero in $\wedge^3 Q$. Since $\ell < n$ we may suppose $\omega_1, \dots, \omega_\ell$ to be a basis of Θ_ℓ , and $\bar{u}_1, \dots, \bar{u}_\ell$ to be

independent in Q – otherwise we are reduced to considering spaces of smaller dimension. As before, this means that $\omega_1 \wedge u_1 + \cdots + \omega_\ell \wedge u_\ell = v \wedge \sigma$ for some $\sigma \in \wedge^2 V$. Completing u_1, \dots, u_ℓ, v into a basis of V by vectors w_1, \dots, w_m , and decomposing into this basis, we immediately see that $\omega_1, \dots, \omega_\ell$ cannot contain any term of the form $w_i \wedge w_j$. In other words,

$$\Theta_\ell \subset A_{\ell+1} \wedge V, \quad A_{\ell+1} := \langle u_1, \dots, u_\ell, v \rangle.$$

We claim this forces Θ_ℓ to be non-generic, just by counting dimensions. Indeed, Θ_ℓ must belong to a subvariety Z of the Grassmannian $Gr(\ell, \wedge^2 V)$ which is dominated by the total space of the Grassmann bundle $Gr(\ell, A_{\ell+1} \wedge V)$ over $Gr(\ell + 1, V)$, where $A_{\ell+1}$ denotes the tautological bundle on the latter Grassmannian. Our conclusion holds as soon as $\dim Z < \dim Gr(\ell, \wedge^2 V)$, and it is a straightforward to check that this is the case in the indicated range. \square

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