The Navier-Stokes equations in mixed-norm time-space parabolic Morrey spaces

Pierre Gilles Lemarié-Rieusset

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Abstract

We discuss the Navier-Stokes equations with forces in the mixed-norm time-space parabolic Morrey spaces of Krylov.

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1 Introduction

In this paper, we consider global mild solutions of the Cauchy problem for the incompressible Navier–Stokes equations on the whole space $\mathbb{R}^3$. More precisely, the Navier–Stokes equations we study are

$$\begin{cases} \partial_t \vec{u} = \Delta \vec{u} - \vec{\nabla} p - \vec{u} \cdot \vec{\nabla} \vec{u} + \vec{f} + \operatorname{div} \mathbb{F} \\ \operatorname{div} \vec{u} = 0 \\ \vec{u}(0,.) = 0 \end{cases} \tag{1}$$

where $\vec{f}$ and $\mathbb{F}$ are small enough in some critical (homogeneous) mixed-norm time-space parabolic Morrey spaces of Krylov.

(Following Krylov [Kryl23]), let us recall the definition of mixed-norm time-space parabolic Morrey spaces on $\mathbb{R} \times \mathbb{R}^d$:

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Definition 1. *Let $1 < p, q < +\infty$ and $\beta \in (0, \frac{2}{p} + \frac{d}{q})$. The parabolic Morrey spaces $E_{p,q,\beta}$ and $F_{p,q,\beta}$ are the spaces of locally integrable functions $f(t,x)$ on $\mathbb{R} \times \mathbb{R}^d$ such that $\|f\|_{E_{p,q,\beta}} < +\infty$ or $\|f\|_{F_{p,q,\beta}} < +\infty$, where*

$$\|f\|_{E_{p,q,\beta}} = \sup_{\rho>0, t\in\mathbb{R}, x\in\mathbb{R}^d} \rho^{\beta-\frac{2}{p}-\frac{d}{q}} \left(\int_{|t-s|<\rho^2} \left(\int_{|x-y|<\rho} |f(s,y)|^q \, dy \right)^{\frac{p}{q}} ds \right)^{\frac{1}{p}}$$

and

$$\|f\|_{F_{p,q,\beta}} = \sup_{\rho>0, t\in\mathbb{R}, x\in\mathbb{R}^d} \rho^{\beta-\frac{2}{p}-\frac{d}{q}} \left(\int_{|x-y|<\rho} \left(\int_{|t-s|<\rho^2} |f(s,y)|^p \, ds \right)^{\frac{q}{p}} dy \right)^{\frac{1}{q}}.$$

Remark: when $p = q$, writing $r = \frac{d+2}{\beta}$ (so that $p < r < +\infty$), we see that $E_{p,p,\beta} = F_{p,p,\beta} = \mathcal{M}^{p,r}_2(\mathbb{R} \times \mathbb{R}^d)$, where $\mathcal{M}^{p,r}_2(\mathbb{R} \times \mathbb{R}^d)$ is the parabolic Morrey space studied in [Lema16] in the context of Navier–Stokes equations.

For a function defined on $(0, +\infty) \times \mathbb{R}^d$, we say that $f \in E_{p,q,\beta}$ or $f \in F_{p,q,\beta}$ if the function $f^\#$ defined by $f^\# = f$ for $t > 0$ and $f^\# = 0$ for $t < 0$ is such that $f^\# \in E_{p,q,\beta}$ or $f^\# \in F_{p,q,\beta}$.

Our main theorem is the following one:

Theorem 1. *Let $p, q \in (3, \infty)$ with $\frac{2}{p} + \frac{3}{q} > 1$.*
a) There exists $\epsilon_0 = \epsilon_0(p,q) > 0$ and $C_0 = C_0(p,q) > 0$ such that, if $\vec{f} \in E_{p/3,q/3,3}$ with $\operatorname{div} \vec{f} = 0$ and $\mathbb{F} \in E_{p/2,q/2,2}$ and if

$$\|\vec{f}\|_{E_{p/3,q/3,3}} + \|\mathbb{F}\|_{E_{p/2,q/2,2}} < \epsilon_0,$$

then the Navier–Stokes equations (1) have a global mild solution $\vec{u} \in E_{p,q,1}$ and $\vec{\nabla} \otimes \vec{u} \in E_{p/2,q/2,2}$ with

$$\|\vec{u}\|_{E_{p,q,1}} + \|\vec{\nabla} \otimes \vec{u}\|_{E_{p/2,q/2,2}} \leq C_0(\|\vec{f}\|_{E_{p/3,q/3,3}} + \|\mathbb{F}\|_{E_{p/2,q/2,2}}).$$

b) There exists $\epsilon_0 = \epsilon_0(p,q) > 0$ and $C_0 = C_0(p,q) > 0$ such that, if $\vec{f} \in F_{p/3,q/3,3}$ with $\operatorname{div} \vec{f} = 0$ and $\mathbb{F} \in F_{p/2,q/2,2}$ and if

$$\|\vec{f}\|_{F_{p/3,q/3,3}} + \|\mathbb{F}\|_{F_{p/2,q/2,2}} < \epsilon_0,$$

then the Navier–Stokes equations (1) have a global mild solution $\vec{u} \in F_{p,q,1}$ and $\vec{\nabla} \otimes \vec{u} \in F_{p/2,q/2,2}$ with

$$\|\vec{u}\|_{F_{p,q,1}} + \|\vec{\nabla} \otimes \vec{u}\|_{F_{p/2,q/2,2}} \leq C_0(\|\vec{f}\|_{F_{p/3,q/3,3}} + \|\mathbb{F}\|_{F_{p/2,q/2,2}}).$$

When the force $\vec{f}$ is equal to 0, we may lower the values of p and q:

Theorem 2. *Let $p, q \in (2, \infty)$ with $\frac{2}{p} + \frac{3}{q} > 1$.*
a) There exists $\epsilon_0 = \epsilon_0(p,q) > 0$ and $C_0 = C_0(p,q) > 0$ such that, if $\mathbb{F} \in E_{p/2,q/2,2}$ and

$$\|\mathbb{F}\|_{E_{p/2,q/2,2}} < \epsilon_0,$$

then the Navier–Stokes equations (1) (with $\vec{f} = 0$) have a global mild solution $\vec{u} \in E_{p,q,1}$ and $\vec{\nabla} \otimes \vec{u} \in E_{p/2,q/2,2}$ with

$$\|\vec{u}\|_{E_{p,q,1}} + \|\vec{\nabla} \otimes \vec{u}\|_{E_{p/2,q/2,2}} \leq C_0 \|\mathbb{F}\|_{E_{p/2,q/2,2}}.$$

b) There exists $\epsilon_0 = \epsilon_0(p,q) > 0$ and $C_0 = C_0(p,q) > 0$ such that, if $\mathbb{F} \in F_{p/2,q/2,2}$ and if

$$\|\mathbb{F}\|_{F_{p/2,q/2,2}} < \epsilon_0,$$

then the Navier–Stokes equations (1) (with $\vec{f} = 0$) have a global mild solution $\vec{u} \in F_{p,q,1}$ and $\vec{\nabla} \otimes \vec{u} \in F_{p/2,q/2,2}$ with

$$\|\vec{u}\|_{F_{p,q,1}} + \|\vec{\nabla} \otimes \vec{u}\|_{F_{p/2,q/2,2}} \leq C_0 \|\mathbb{F}\|_{F_{p/2,q/2,2}}.$$

2 Heat equation

The proof of theorems 1 and 2 will rely on the theory of the heat equation $\partial_t u - \Delta u = f$ on $\mathbb{R} \times \mathbb{R}^d$ with the boundary condition $u = 0$ at infinity. This condition will be defined as follows:

Definition 2. *Let u be a tempered distribution on $\mathbb{R} \times \mathbb{R}^d$. Then $u = 0$ at infinity if for every $\theta > 0$ one has $e^{\theta^2(\partial_t^2 - \Delta^2)} u \in L^\infty(\mathbb{R} \times \mathbb{R}^d)$ and $\lim_{\theta \to +\infty} \|e^{\theta^2(\partial_t^2 - \Delta^2)} u\|_\infty = 0$.*

Lemma 1. *Let u be a tempered distribution on $\mathbb{R} \times \mathbb{R}^d$ such that $u = 0$ at infinity. If $\partial_t u - \Delta u = 0$, then $u = 0$.*

Proof. Take the Fourier transform (in time and space variables) of the heat equation. We find

$$(i\tau + |\xi|^2)\hat{u}(\tau, \xi) = 0.$$

Thus, $\hat{u}$ is supported in $\{(0,0)\}$, hence is a sum of derivatives of the Dirac mass, and u is a polynomial. As $e^{\theta^2(\partial_t^2 - \Delta^2)} u$ is then a polynomial, and as $e^{\theta^2(\partial_t^2 - \Delta^2)} u \in L^\infty(\mathbb{R} \times \mathbb{R}^d)$, we find that u is a constant; in that case, $e^{\theta^2(\partial_t^2 - \Delta^2)} u = u$. As $u = 0$ at infinity, we have $u = 0$. □

Lemma 2. *Let $1 < p, q < +\infty$ and $\beta \in (0, \frac{2}{p} + \frac{d}{q})$. If $u \in E_{p,q,\beta}$ or $u \in F_{p,q,\beta}$, then $u = 0$ at infinity.*

Proof. We give the proof for $u \in E_{p,q,\beta}$ (the case $u \in F_{p,q,\beta}$ is proved in a similar way). Let $W(t,x)$ be the inverse Fourier transform of $e^{-\tau^2-|\xi|^4}$. W is in the Schwartz class of smooth functions with rapid decay. In particular, $(\sqrt{|t|}+|\xi|^{d+2}|W(t,x)| \in L^\infty(\mathbb{R}\times\mathbb{R}^d)$. Writing

$$\begin{aligned}
&\iint |W(t,x)|\,|u(t,x)|\,dt\,dx\\
=&\iint_{\sqrt{|t|+|x|<1}} |W(t,x)|\ \ |u(t,x)|\,dt\,dx+\sum_{j=0}^{+\infty}\iint_{2^j<\sqrt{|t|+|x|<2^{j+1}}} |W(t,x)|\ \ |u(t,x)|\,dt\,dx\\
\le& C\|u\|_{E_{p,q,\beta}}(\|\mathbb{1}_{\sqrt{|t|+|x|<1}}W\|_{L_t^{\frac{p}{p-1}}L_x^{\frac{q}{q-1}}}+\sum_{j=0}^{+\infty}2^{j(\frac{2}{p}+\frac{d}{q}-\beta)}\|\mathbb{1}_{2^j<\sqrt{|t|+|x|<2^{j+1}}}W\|_{L_t^{\frac{p}{p-1}}L_x^{\frac{q}{q-1}}})\\
\le& C'\|u\|_{E_{p,q,\beta}}(1+\sum_{j=0}^{+\infty}2^{-j\beta}) = C_1\|u\|_{E_{p,q,\beta}},
\end{aligned}$$

we get

$$\begin{aligned}
|e^{\theta^2(\partial_t^2-\Delta^2)}u(t,x)| \le& \iint W(s,y)|u(t+\theta^2 s, x+\theta y)|\,ds\,dy\\
\le& C_1\|u(t+\theta.,x+\sqrt{\theta}.)\|_{E_{p,q,\beta}}\\
=& C_1\|u(\theta.,\sqrt{\theta}.)\|_{E_{p,q,\beta}}\\
=& C_1\|u\|_{E_{p,q,\beta}}\theta^{-\beta/2}.
\end{aligned}$$

Thus, $\lim_{\theta\to+\infty}\|e^{\theta^2(\partial_t^2-\Delta^2)}u\|_\infty = 0$. □

The proof of Theorems 1 and 2 will be based on the following results on the linear heat equation:

Theorem 3. *Let $p,q \in (3,\infty)$ and $\beta > 0$ with $\frac{2}{p}+\frac{3}{q} > \beta$.*
a) If $f \in E_{p/3,q/3,\beta+2}(\mathbb{R}\times\mathbb{R}^d)$, then the heat equation

$$\partial_t u = \Delta u + f \tag{2}$$

has a unique solution such that $u = 0$ at infinity. Moreover, there exists $C_0 = C_0(p,q,\beta) > 0$ such that, if denoting by $D_x u$ the gradient of u and by $D_x^2 u$ the Hessian of u (with respect to the space variable), we have

$$\|u\|_{E_{p,q,\beta}} + \|D_x u\|_{E_{p/2,q/2,\beta+1}} + \|D_x^2 u\|_{E_{p/3,q/3,\beta+2}} \le C_0\|f\|_{E_{p/3,q/3,\beta+2}}.$$

b) If $\in F_{p/3,q/3,\beta+2}(\mathbb{R}\times\mathbb{R}^d)$, then the heat equation (2) has a unique solution such that $u = 0$ at infinity. Moreover, there exists $C_0 = C_0(p,q,\beta) > 0$ such that we have

$$\|u\|_{F_{p,q,\beta}} + \|D_x u\|_{F_{p/2,q/2,\beta+1}} + \|D_x^2 u\|_{F_{p/3,q/3,\beta+2}} \le C_0\|f\|_{F_{p/3,q/3,\beta+2}}.$$

Theorem 4. *Let $p, q \in (2, \infty)$ and $\beta > 0$ with $\frac{2}{p} + \frac{3}{q} > \beta$. Let $\sigma(D)$ be a Fourier multiplier (in the space variable) such that $\sigma(\xi) = \sigma_0(\frac{\xi}{|\xi|})$ where σ_0 is a smooth function on the sphere $\mathcal{S}^{d-1}$.*
a) If $\mathbb{F} \in E_{p/2,q/2,\beta+1}(\mathbb{R} \times \mathbb{R}^d)$, then the heat equation

$$\partial_t u = \Delta u + \sigma(D) \operatorname{div} \mathbb{F} \tag{3}$$

has a unique solution such that $u = 0$ at infinity. Moreover, there exists $C_0 = C_0(p, q, \beta, \sigma) > 0$ such that we have

$$\|u\|_{E_{p,q,\beta}} + \|D_x u\|_{E_{p/2,q/2,\beta+1}} \leq C_0 \|\mathbb{F}\|_{E_{p/2,q/2,\beta+1}}.$$

b) If $\mathbb{F} \in F_{p/2,q/2,\beta+1}(\mathbb{R} \times \mathbb{R}^d)$, then the heat equation (3) has a unique solution such that $u = 0$ at infinity. Moreover, there exists $C_0 = C_0(p, q, \beta, \sigma) > 0$ such that we have

$$\|u\|_{F_{p,q,\beta}} + \|D_x u\|_{F_{p/2,q/2,\beta+1}} \leq C_0 \|\mathbb{F}\|_{F_{p/2,q/2,\beta+1}}.$$

Theorem 3 has the following corollary on Sobolev–Morrey inequalities:

Corollary 1. *Let $p, q \in (3, \infty)$ and $\beta > 0$ with $\frac{2}{p} + \frac{3}{q} > \beta$. Let $u = 0$ at infinity. Then there exists $C_0 = C_0(p, q, \beta) > 0$ such that we have*

$$\|u\|_{E_{p,q,\beta}} + \|D_x u\|_{E_{p/2,q/2,\beta+1}} + \|D_x^2 u\|_{E_{p/3,q/3,\beta+2}} \leq C_0(\|\partial_t u\|_{E_{p/3,q/3,\beta+2}} + \|\Delta u\|_{E_{p/3,q/3,\beta+2}})$$

and

$$\|u\|_{F_{p,q,\beta}} + \|D_x u\|_{F_{p/2,q/2,\beta+1}} + \|D_x^2 u\|_{F_{p/3,q/3,\beta+2}} \leq C_0(\|\partial_t u\|_{F_{p/3,q/3,\beta+2}} + \|\Delta u\|_{F_{p/3,q/3,\beta+2}}).$$

Another corollary is the following theorem of Krylov [Kryl23]:

Proposition 1. *Let $p, q \in (3, \infty)$ and $\beta > 0$ with $\frac{2}{p} + \frac{3}{q} > \max(\beta, 1)$. There exists $\epsilon_0 = \epsilon_0(p, q, \beta) > 0$ and $C_0 = C_0(p, q, \beta) > 0$ such that*
a) If $f \in E_{p/3,q/3,\beta+2}(\mathbb{R} \times \mathbb{R}^d)$, $\vec{b} \in E_{p,q,1}$, $c \in E_{p/2,q/2,2}$, and if $\|\vec{b}\|_{E_{p,q,1}} + \|c\|_{E_{p/2,q/2,2}} < \epsilon_0$, then the heat equation

$$\partial_t u = \Delta u + \vec{b} \cdot \vec{\nabla} u + cu + f \tag{4}$$

has a unique solution such that $u = 0$ at infinity. Moreover, we have

$$\begin{aligned} &\|u\|_{E_{p,q,\beta}} + \|D_x u\|_{E_{p/2,q/2,\beta+1}} + \|D_x^2 u\|_{E_{p/3,q/3,\beta+2}} \\ &\quad \leq C_0(\|f\|_{E_{p/3,q/3,\beta+2}} + \|\vec{b}\|_{E_{p,q,1}} + \|c\|_{E_{p/2,q/2,2}}). \end{aligned}$$

b) If $f \in F_{p/3,q/3,\beta+2}(\mathbb{R} \times \mathbb{R}^d)$, $\vec{b} \in F_{p,q,1}$, $c \in F_{p/2,q/2,2}$, and if $\|\vec{b}\|_{F_{p,q,1}} + \|c\|_{F_{p/2,q/2,2}} < \epsilon_0$, then the heat equation (4) has a unique solution such that $u = 0$ at infinity. Moreover, we have

$$\begin{aligned} &\|u\|_{F_{p,q,\beta}} + \|D_x u\|_{F_{p/2,q/2,\beta+1}} + \|D_x^2 u\|_{F_{p/3,q/3,\beta+2}} \\ &\quad \leq C_0(\|f\|_{F_{p/3,q/3,\beta+2}} + \|\vec{b}\|_{F_{p,q,1}} + \|c\|_{F_{p/2,q/2,2}}). \end{aligned}$$

3 Anisotropic Hardy-Littlewood maximal function on mixed norm Lebesgue spaces

Let $\vec{p} = (p_1, \dots, p_n) \in (1, +\infty)^n$. The mixed-norm Lebesgue spaces $L^{\vec{p}}(\mathbb{R}^n)$ is the space of measurable functions on $\mathbb{R}^n$ such that

$$\|f\|_{L^{\vec{p}}} = \left(\int \dots \left[\int \left\{ \int |f(x_1, \dots, x_n)|^{p_1} \, dx_1 \right\}^{\frac{p_2}{p_1}} \right]^{\frac{p_3}{p_2}} \dots dx_n \right)^{\frac{1}{p_n}} < +\infty.$$

For $\vec{a} = (a_1, \dots, a_n) \in [1, +\infty)^n$, the anisotropic cylinders $Q_{\vec{a}}(x, r)$ are defined as

$$Q_{\vec{a}}(x, r) = (x_1 - r^{a_1}, x_1 + r^{a_1}) \times \dots (x_n - r^{a_n}, x_n + r^{a_n})$$

and the anisotropic Hardy-Littlewood maximal function is defined as

$$\mathcal{M}_{\vec{a}} f(x) = \sup_{r>0} \frac{1}{|Q_{\vec{a}}(x, r)|} \int_{Q_{\vec{a}}(x,r)} |f(y)| \, dy.$$

We then have the following boundedness result for the anisotropic Hardy-Littlewood maximal function on mixed norm Lebesgue spaces [Huan19, Huan21]:

Proposition 2. *For $\vec{p} = (p_1, \dots, p_n) \in (1, +\infty)^n$ and $\vec{a} = (a_1, \dots, a_n) \in [1, +\infty)^n$, there exists a constant $C = C(\vec{p}, \vec{a})$ such that*

$$\|\mathcal{M}_{\vec{a}} f\|_{L^{\vec{p}}} \leq C \|f\|_{L^{\vec{p}}}.$$

4 Harmonic analysis on the parabolic space

We endow $X = \mathbb{R} \times \mathbb{R}^d$ with the Lebesgue measure $d\mu(t, x) = dt \, dx$ and the parabolic metric $\rho((t, x), (s, y)) = \sqrt{|t - s|} + |x - y|$. The parabolic ball $B_r(t, x)$ and the parabolic cylinder $C_r(t, x)$ are defined as

$$B_r(t, x) = \{(s, y) \ / \ \rho((t, x), (s, y)) < r\} \text{ and } C_r(t, x) = (t - r^2, t + r^2) \times B(x, r).$$

We have

$$B_r(t, x) \subset C_r(t, x) \subset B_{2r}(t, x).$$

(X, ρ, μ) is a space of homogeneous type [Coif71] of homogeneous dimension $Q = d + 2$:

$$\mu(B_r(t, x)) = \mu(B_1(0, 0)) r^Q.$$

We associate to this space three useful operators on (non-negative) functions: the Hardy–Littlewood maximal function $\mathcal{M}_f$ and the Riesz potentials $\mathcal{I}_\alpha f$ (where $0 < \alpha < Q$) defined by

$$\mathcal{M}_f(t,x) = \sup_{r>0} \frac{1}{\mu(B_r(t,x))} \int_{B_r(t,x)} |f(s,y)|\, d\mu(s,y)$$

and

$$\mathcal{I}_\alpha f(t,x) = \int_X \frac{1}{\rho((t,x),(s,y))^{Q-\alpha}} f(s,y)\ d\mu(s,y).$$

For $1 < p \leq q$, we define the Morrey space $\mathcal{M}_2^{p,q}$ by the space of measurable functions such that

$$\|f\|_{\mathcal{M}_2^{p,q}} = \sup_{r>0,(t,x)\in X} \mu(B_r(t,x))^{\frac{1}{q}-\frac{1}{p}} \|\mathbb{1}_{B_r(t,x)}\|_p < +\infty.$$

We have $L^q(\mathbb{R}\times\mathbb{R}^d) \subset \mathcal{M}_2^{p,q}$, $E_{p,p,\beta} = \mathcal{M}^{p,\frac{d+2}{\beta}}$ and $E_{p,q,\beta} \subset \mathcal{M}_2^{\min(p,q),\frac{d+2}{\beta}}$.

The main tool we shall use is Hedberg's inequality [Hedb72]:

Proposition 3. *Let $1 < p \leq q < +\infty$ and $0 < \alpha < \frac{Q}{q}$. Then there exists a constant $C = C(d,p,q,\alpha)$ such that, for every $f \in \mathcal{M}_2^{p,q}$, we have*

$$\mathcal{I}_\alpha f(t,x) \leq C(\mathcal{M}_f(t,x))^{1-\frac{\alpha q}{Q}} \|f\|_{\mathcal{M}_2^{p,q}}^{\frac{\alpha q}{Q}}.$$

Proof. We easily check that, for every $R > 0$, we have

$$\int_{B_R(t,x)} \frac{1}{\rho((t,x),(s,y))^{Q-\alpha}} |f(s,y)|\ d\mu(s,y) \leq CR^\alpha \mathcal{M}_f(t,x)$$

and

$$\int_{X\setminus B_R(t,x)} \frac{1}{\rho((t,x),(s,y))^{Q-\alpha}} |f(s,y)|\ d\mu(s,y) \leq CR^{\alpha-\frac{Q}{q}} \|f\|_{\mathcal{M}_2^{p,q}}.$$

We then take

$$R^{\frac{Q}{q}} = \frac{\|f\|_{\mathcal{M}_2^{p,q}}}{\mathcal{M}_f(t,x)}.$$

□

Applying Proposition 3 to Krylov spaces, we obtain:

Proposition 4. *Let $1 < p,q < +\infty$ and $0 < \beta < \frac{2}{p} + \frac{d}{q}$. Then*
a) $f \mapsto \mathcal{M}_f$ is bounded from $E_{p,q,\beta}$ to $E_{p,q,\beta}$ and from $F_{p,q,\beta}$ to $F_{p,q,\beta}$.
b) If $\beta > 1$, $f \mapsto \mathcal{I}_1 f$ is bounded from $E_{p,q,\beta}$ to $E_{\frac{\beta}{\beta-1}p,\frac{\beta}{\beta-1}q,\beta-1}$ and from $F_{p,q,\beta}$ to $F_{\frac{\beta}{\beta-1}p,\frac{\beta}{\beta-1}q,\beta-1}$.
c) If $\beta > 2$, $f \mapsto \mathcal{I}_2 f$ is bounded from $E_{p,q,\beta}$ to $E_{\frac{\beta}{\beta-2}p,\frac{\beta}{\beta-2}q,\beta-2}$ and from $F_{p,q,\beta}$ to $F_{\frac{\beta}{\beta-2}p,\frac{\beta}{\beta-2}q,\beta-2}$

Proof. We consider only the case of $E_{p,q,\beta}$, as the proof for $F_{p,q,\beta}$ is similar. Let us estimate $\mathcal{M}_f$ on a cylinder $C_r(t,x)$. We have $\mathcal{M}_f \leq \mathcal{M}_{f_1} + \mathcal{M}_{f_2}$, where $f_1 = \mathbb{1}_{C_{4r}(t,x)} f$ and $f_2 = f - f_1$. By Proposition 2, we know that

$$\|\mathcal{M}_{f_1}\|_{L_t^p L_x^q} \leq C\|f_1\|_{L_t^p L_x^q} \leq C'\|f\|_{E_{p,q,\beta}} r^{\frac{2}{p}+\frac{d}{q}-\beta}.$$

On the other hand, for $(s,z) \in C_r(t,x) \subset B_{2r}(t,x)$, since $B_{4r}(t,x) \subset C_{4r}(t,x)$,

$$\mathcal{M}_{f_2}(s,z) \leq \sup_{\rho>2r} \frac{1}{|B_\rho(s,z)|} \iint_{B_\rho(s,z)} |f(\sigma,y)|\, d\sigma\, dy \leq C'\|f\|_{E_{p,q,\beta}} r^{-\beta}$$

so that

$$\|\mathbb{1}_{C_r(t,x)} \mathcal{M}_{f_2}\|_{L_t^p L_x^q} \leq C\|f\|_{E_{p,q,\beta}} r^{\frac{2}{p}+\frac{d}{q}-\beta}.$$

a) is proved. b) and c) are then direct consequences of Hedberg's inequality (Proposition 3). □

5 The heat equation on $\mathbb{R} \times \mathbb{R}^d$.

In this section, we solve the heat equation on $\mathbb{R} \times \mathbb{R}^d$:

$$\begin{cases} \partial_t u = \Delta u + g \\ u = 0 \text{ at infinity} \end{cases} \tag{5}$$

where $g = f \in E_{p,q,\beta}$ or in $F_{p,q,\beta}$ with $2 < \beta < \frac{2}{p} + \frac{d}{q}$ or $g = \sigma(D)\operatorname{div} \mathbb{F}$ where $\mathbb{F} \in E_{p,q,\beta}$ or in $F_{p,q,\beta}$ with $1 < \beta < \frac{2}{p} + \frac{d}{q}$. The solution u of equation (5) is given by the Duhamel formula

$$u = \int_{-\infty}^{t} e^{(t-s)\Delta} g(s,.)\, ds.$$

In order to estimate u and its derivatives, we need some estimates on the size of the kernel of $e^{t\Delta}$ and its derivatives, or on the derivatives of $\sigma(D)e^{t\Delta}$.

Lemma 3. *Let $\psi \in \mathcal{S}(\mathbb{R}^d)$ and, for $\theta > 0$, $\psi_\theta(x) = \frac{1}{\theta^d}\psi(\frac{x}{\theta})$. Let $\sigma(D)$ be a Fourier multiplier (in the space variable) such that $\sigma(\xi) = \sigma_0(\frac{\xi}{|\xi|})$ where σ_0 is a smooth function on the sphere $\mathcal{S}^{d-1}$. Then, for $\alpha \in \mathbb{N}^d$,*

$$|\partial_\alpha \sigma(D)\psi_\theta(x)| \leq C_{\alpha,\psi,\sigma} \frac{1}{(\theta + |x|)^{d+|\alpha|}}.$$

Proof. We have

$$\|\partial_\alpha \sigma(D)\psi_\theta\|_\infty = \theta^{-d-|\alpha|}\|\partial_\alpha \sigma(D)\psi\|_\infty$$

and

$$\||x|^{d+|\alpha|}\partial_\alpha \sigma(D)\psi_\theta\|_\infty = \||x|^{d+|\alpha|}\partial_\alpha \sigma(D)\psi\|_\infty$$

Thus, we may assume $\theta = 1$. We have

$$|\partial_\alpha \sigma(D)\psi_\theta(x)| \leq \frac{1}{(2\pi)^d}\|\sigma_0\|_\infty \int |\xi|^{|\alpha|}|\hat{\psi}(\xi)|\, dx.$$

On the other hand, taking a smooth function ϕ such that $\phi(\xi) = 1$ for $|\xi| < 1$ and $\phi(\xi) = 0$ for $|\xi| > 2$, we have for every $R > 0$, for $1 \leq k \leq d$,

$$|x_k|^{d+|\alpha|}|\partial_\alpha \phi(\frac{D}{R})\sigma(D)\psi_\theta(x)| \leq C(R|x|)^{d+\alpha}\|\sigma_0\|_\infty \|\hat{\psi}\|_\infty$$

and

$$\begin{aligned}
|x|^2|x_k|^{d+|\alpha|}&|\partial_\alpha(1-\phi(\frac{D}{R}))\sigma(D)\psi_\theta(x)| \\
&\leq \frac{1}{(2\pi)^d}\int |\Delta\partial_k^{d+|\alpha|}\left((1-\phi(\frac{\xi}{R}))\xi^\alpha \sigma(\xi)\hat{\psi}(\xi)\right)|\, d\xi \\
&\leq C\int_{|\xi|>R}\frac{d\xi}{|\xi|^{d+2}} = C'\frac{1}{R^2}
\end{aligned}$$

(as $|\partial^\beta(\hat{\psi}(\xi))| \leq C_{\beta,\psi}|\xi|^{-|\beta|}$, $|\partial^\beta(\xi^\alpha\sigma(\xi))| \leq C_{\alpha,\beta,\sigma}|\xi|^{|\alpha|-|\beta|}$ and $|\partial^\beta(1-\phi(\frac{\xi}{R}))| \leq C_{\beta,\phi}|\xi|^{-|\beta|}$). Taking $R = \frac{1}{|x|}$, we get $x_k^{d+|\alpha|}\partial^\alpha\sigma(D)\psi \in L^\infty$. □

A useful result on the heat kernel is its maximal regularity in $L_t^pL_x^q$ or in $L_x^qL_t^p$:

Proposition 5. *Let $1 < p, q < +\infty$. Let $\sigma(D)$ be a Fourier multiplier (in the space variable) such that $\sigma(\xi) = \sigma_0(\frac{\xi}{|\xi|})$ where σ_0 is a smooth function on the sphere $\mathcal{S}^{d-1}$.*
a) If $h \in L_t^pL_x^q$, then, for $1 \leq i, j \leq d$,

$$\|\int_{-\infty}^t e^{(t-s)\Delta}\sigma(D)\partial_i\partial_j h\, ds\|_{L_t^pL_x^q} \leq C\|h\|_{L_t^pL_x^q}.$$

a) If $h \in L_x^qL_t^p$, then, for $1 \leq i, j \leq d$,

$$\|\int_{\infty}^t e^{(t-s)\Delta}\sigma(D)\partial_i\partial_j h\, ds\|_{L_x^qL_t^p} \leq C\|h\|_{L_x^qL_t^p}.$$

Proof. This is a classical result, obtained through the theory of vector valued singular integrals [Bene62]. We sketch the proof given in [Lema02]. Let $W(x) = \frac{1}{(4\pi)^{d/2}} e^{-\frac{|x|^2}{4}}$ and let

$$\Omega(t,x) = \frac{1}{t^{\frac{d+2}{2}}} (\sigma(D)\partial_i\partial_j W)(\frac{x}{\sqrt{t}}) \text{ for } t > 0, \quad = 0 \text{ for } t < 0.$$

We then consider the operator

$$T(g)(t,x) = \int_{\mathbb{R}} \int_{\mathbb{R}^d} \Omega(t-s, x-y) g(s,y)\, ds\, dy. \tag{6}$$

We have

$$(\partial_t - \Delta) T(g) = \sigma(D)\partial_i\partial_j g$$

and, taking the Fourier transform,

$$\widehat{T(g)}(\tau,\xi) = -\frac{\xi_i\xi_j}{i\tau + |\xi|^2} \sigma(\xi)\hat{g}(\tau,\xi).$$

Thus, T is bounded on $L^2(\mathbb{R}\times\mathbb{R}^d)$. T can be seen as Calderón–Zygmund operator on the parabolic space $\mathbb{R}\times\mathbb{R}^d$ (for the parabolic distance): from Lemma 3, we see that $|\Omega(t,x)| \leq C\frac{1}{(\sqrt{|t|}+|x|)^{d+2}}$, $|\partial_i\Omega(t,x)| \leq C\frac{1}{(\sqrt{|t|}+|x|)^{d+3}}$, and $|\partial_t\Omega(t,x)| = |\Delta\Omega(t,x)| \leq C\frac{1}{(\sqrt{|t|}+|x|)^{d+4}}$. Thus, we get that T is bounded on $L^p(\mathbb{R}\times\mathbb{R}^d)$ for $1 < p < +\infty$. Let T_t be the operator

$$T_t(v) = \int_{\mathbb{R}^d} \frac{1}{t}\Omega(t, x-y) v(y)\, dy$$

and T_x be the operator

$$T_x(w) = \int_{\mathbb{R}} \frac{1}{t-s}\Omega(t-s, x) w(s)\, ds$$

We have

$$\partial_t T_t(v) = \int_{\mathbb{R}^d} \frac{1}{t}\Delta\Omega(t, x-y) v(y)\, dy$$

and

$$\partial_k T_x(w) = \int_{\mathbb{R}} \frac{1}{t-s}\partial_k\Omega(t-s, x) w(s)\, ds.$$

From Lemma 3, we find

$$\|T_t\|_{L^q_x \mapsto L^q_x} \leq C\frac{1}{t} \text{ and } \|\partial_t T_t\|_{L^q_x \mapsto L^q_x} \leq C\frac{1}{t^2}$$

so that the continuity $L^q_t(L^q_x) \mapsto L^q_t(L^q_x)$ of T can be extended to the continuity $L^p_t(L^q_x) \mapsto L^p_t(L^q_x)$. Similarly, we have

$$\|T_x\|_{L^p_t \mapsto L^p_t} \leq C\frac{1}{|x|^d} \text{ and } \|\partial_k T_x\|_{L^p_t \mapsto L^p_t} \leq C\frac{1}{|x|^{d+1}}$$

and the continuity $L^p_x(L^p_t) \mapsto L^p_x(L^p_t)$ of T can be extended to the continuity $L^q_x(L^p_t) \mapsto L^q_x(L^p_t)$. □

Similar estimates hold for Krylov spaces:

Proposition 6. *Let $1 < p, q < +\infty$ and $0 < \beta < \frac{2}{p} + \frac{d}{q}$. Let $\sigma(D)$ be a Fourier multiplier (in the space variable) such that $\sigma(\xi) = \sigma_0(\frac{\xi}{|\xi|})$ where σ_0 is a smooth function on the sphere $\mathcal{S}^{d-1}$.*
a) If $h \in E_{p,q,\beta}$, then, for $1 \leq i, j \leq d$,

$$\| \int_{-\infty}^{t} e^{(t-s)\Delta}\sigma(D)\partial_i\partial_j h \, ds\|_{E_{p,q,\beta}} \leq C\|h\|_{E_{p,q,\beta}}.$$

a) If $h \in F_{p,q,\beta}$, then, for $1 \leq i, j \leq d$,

$$\| \int_{-\infty}^{t} e^{(t-s)\Delta}\sigma(D)\partial_i\partial_j h \, ds\|_{F_{p,q,\beta}} \leq C\|h\|_{F_{p,q,\beta}}.$$

Proof. We consider only the case of $E_{p,q,\beta}$, as the proof for $F_{p,q,\beta}$ is similar. Let us estimate $T(h)$ on a cylinder $C_r(t,x)$ (where T is the operator given by (6)). We have $T(h) = T(h_1) + T(h_2)$, where $h_1 = \mathbb{1}_{C_{4r}(t,x)}h$ and $h_2 = h - h_1$. By Proposition 5, we know that

$$\|h_1\|_{L^p_t L^q_x} \leq C\|h_1\|_{L^p_t L^q_x} \leq C'\|h\|_{E_{p,q,\beta}} r^{\frac{2}{p}+\frac{d}{q}-\beta}.$$

On the other hand, for $(s,z) \in C_r(t,x) \subset B_{2r}(t,x)$, since $B_{4r}(t,x) \subset C_{4r}(t,x)$,

$$\begin{aligned}
|h_2(s,z)| \leq & C \iint_{\rho((\sigma,y),(t,x))>4r} \frac{1}{(\sqrt{|\tau - s|} + |z-y|)^{d+2}} |h(\tau,y)| \, d\tau \, dy \\
\leq & C' \iint_{\rho((\sigma,y),(t,x))>4r} \frac{1}{(\sqrt{|\tau - t|} + |x-y|)^{d+2}} |h(\tau,y)| \, d\tau \, dy \\
\leq & C'' \sum_{j=0}^{+\infty} \frac{1}{(4^j r)^{d+2}} \iint_{C_{4^{j+1}r}} |h(\tau,y)| \, d\tau \, dy \\
\leq & C'''\|h\|_{E_{p,q,\beta}} r^{-\beta}
\end{aligned}$$

so that

$$\|\mathbb{1}_{C_r(t,x)}h_2\|_{L^p_t L^q_x} \leq C\|\|_{E_{p,q,\beta}} r^{\frac{2}{p}+\frac{d}{q}-\beta}.$$ □

We may now easily prove Theorems 3 and 4, Corollary 1 and Proposition 1. Again, we shall consider only the case of $E_{p,q,\beta}$, as the proofs for $F_{p,q,\beta}$ are similar.

Proof of Theorem 3

Proof. The solution u of equation $\partial_t u = \Delta u + f$ is given by $u = \int_{-\infty}^{t} e^{t-s)\Delta} f(s,.)\, ds$. By proposition (6), we already know that

$$\|D_x^2 u\|_{E_{p/3,q/3,\beta+2}} \leq C\|f\|_{E_{p/3,q/3,\beta+2}}.$$

On the other hand, from Lemma 3, we see that

$$|D_x u(t,x)| \leq C \iint \frac{1}{(\sqrt{|t-s|}+|x-y|)^{d+1}} |f(s,y)|\, ds\, dy = C\mathcal{I}_1(|f|)(t,x)$$

and

$$|u(t,x)| \leq C \iint \frac{1}{(\sqrt{|t-s|}+|x-y|)^{d}} |f(s,y)|\, ds\, dy = C\mathcal{I}_2(|f|)(t,x).$$

We then apply Proposition 4 to get

$$\|D_x u\|_{E_{p/2,q/2,\beta+1}} \leq C\|f\|_{E_{p/3,q/3,\beta+2}}$$

and

$$\|u\|_{E_{p,q,\beta}} \leq C\|f\|_{E_{p/3,q/3,\beta+2}}. \qquad \square$$

Proof of Theorem 4

Proof. The solution u of equation $\partial_t u = \Delta u + \sigma(D)\operatorname{div}\mathbb{F}$ is given by $u = \int_{-\infty}^{t} e^{t-s)\Delta}\sigma(D)\operatorname{div}\mathbb{F}(s,.)\, ds$. By proposition (6), we already know that

$$|D_x u\|_{E_{p/2,q/2,\beta+1}} \leq C_0\|\mathbb{F}\|_{E_{p/2,q/2,\beta+1}}.$$

On the other hand, from Lemma 3, we see that

$$|u(t,x)| \leq C \iint \frac{1}{(\sqrt{|t-s|}+|x-y|)^{d+1}} |\mathbb{F}(s,y)|\, ds\, dy = C\mathcal{I}_1(|\mathbb{F}|)(t,x).$$

We then apply Proposition 4 to get

$$\|u\|_{E_{p,q,\beta}} \leq C\|\mathbb{F}\|_{E_{p/2,q/2,\beta+1}}. \qquad \square$$

Proof of Corollary 1

Proof. We just write $\partial_t u - \Delta u = f$ with $f = \partial_t u - \Delta u$. By Theorem 3, we have

$$\|u\|_{E_{p,q,\beta}} + \|D_x u\|_{E_{p/2,q/2,\beta+1}} + \|D_x^2 u\|_{E_{p/3,q/3,\beta+2}} \leq C_0(\|\partial_t u\|_{E_{p/3,q/3,\beta+2}} + \|\Delta u\|_{E_{p/3,q/3,\beta+2}})$$

□

Proof of Proposition 1

Proof. Let $X = \{u \in E_{p,q,\beta} \ / \ D_x u \in E_{p/2,q/2,\beta+1}, D_x^2 u \in E_{p/3,q/3,\beta+2}\}$, normed with

$$\|u\|_X = \|u\|_{E_{p,q,\beta}} + \|D_x u\|_{E_{p/2,q/2,\beta+1}} + \|D_x^2 u\|_{E_{p/3,q/3,\beta+2}}.$$

We are looking for a solution in X of the equation

$$u = T(f) + T(\vec{b} \cdot \vec{\nabla} u) + T(cu)$$

, where $T(f) = \int_{-\infty}^{t} e^{(t-s)\Delta} f(s,.)\, ds$. By Theorem 3, we have

$$\|T(f)\|_X \leq C_1 \|f\|_{E_{p/3,q/3,\beta+2}},$$

$$\|T(\vec{b} \cdot \vec{\nabla} u)\|_X \leq C_2 \|\vec{b} \cdot \vec{\nabla} u\|_{E_{p/3,q/3,\beta+2}} \leq C_2 \|\vec{b}\|_{E_{p,q,1}} \|D_x u\|_{E_{p/2,q/2,\beta+1}},$$

and

$$\|T(cu)\|_X \leq C_3 \|cu\|_{E_{p/3,q/3,\beta+2}} \leq C_3 \|c\|_{E_{p/2,q/2,2}} \|u\|_{p,q,\beta}.$$

For $C_2\|\vec{b}\|_{E_{p,q,1}} + C_3\|c\|_{E_{p/2,q/2,2}} < \frac{1}{2}$, the operator $S(u) = u - T(\vec{b} \cdot \vec{\nabla} u) - T(cu)$ is an isomorphism of X, with $\|S^{-1}\|_{X \mapsto X} \leq 2$. The solution u is then given by $u = S^{-1}(T(f))$, with $\|u\|_X \leq 2C_1 \|f\|_{E_{p/3,q/3,\beta+2}}$. □

6 The Navier–Stokes equations.

We prove here Theorem 1. The proof of Theorem 2 is similar. Again, we consider only the case of $E_{p,q,\beta}$.

Proof of Theorem 1

Proof. The Navier–Stokes equations we study are

$$\begin{cases} \partial_t \vec{u} = \Delta \vec{u} - \vec{\nabla} p - \vec{u} \cdot \vec{\nabla} \vec{u} + \vec{f} + \operatorname{div} \mathbb{F} \\ \operatorname{div} \vec{u} = 0 \\ \vec{u}(0,.) = 0 \end{cases} \tag{7}$$

where $\vec{f} \in E_{p/3,q/3,3}$ with $\operatorname{div} \vec{f} = 0$ and $\mathbb{F} \in E_{p/2,q/2,2}$. We rewrite (7) as

$$\partial_t \vec{u} - \Delta \vec{u} = \vec{f} + (\mathrm{Id} - \frac{1}{\Delta} \vec{\nabla} \operatorname{div}) \operatorname{div}(\mathbb{F} - \vec{u} \otimes \vec{u}).$$

($\mathrm{Id} - \frac{1}{\Delta} \vec{\nabla} \operatorname{div} = \mathbb{P}$ is the Leray projection operator on solenoidal vector fields.)

Let $X = \{\vec{u} \in E_{p,q,1} \ / \ D_x \vec{u} \in E_{p/2,q/2,2}\}$, normed with

$$\|u\|_X = \|\vec{u}\|_{E_{p,q,1}} + \|D_x \vec{u}\|_{E_{p/2,q/2,2}}.$$

We are looking for a solution in X of the equation

$$\vec{u} = \vec{T}(\vec{f}) + \vec{T}(\mathbb{P} \operatorname{div} \mathbb{F}) - \vec{T}(\mathbb{P} \operatorname{div}(\vec{u} \otimes \vec{u})),$$

where $\vec{T}(\vec{f}) = \int_0^t e^{(t-s)\Delta} \vec{f}(s,.)\, ds$ ($= \int_{-\infty}^t e^{(t-s)\Delta} \vec{f}(s,.)\, ds$ if we extend $\vec{f}$ to $(-\infty, 0)$ with $\vec{f}(t,x) = 0$ for $t < 0$). By Theorem 3, we have

$$\|\vec{T}(\vec{f})\|_X \leq C_1 \|\vec{f}\|_{E_{p/3,q/3,3}}.$$

By Theorem 4, we have

$$\|\vec{T}(\mathbb{P} \operatorname{div} \mathbb{F})\|_X \leq C_2 \|\mathbb{F}\|_{E_{p/2,q/2,2}}$$

and

$$\|\vec{T}(\mathbb{P} \operatorname{div}(\vec{u} \otimes \vec{u}))\|_X \leq C_3 \|\vec{u} \otimes \vec{u}\|_{E_{p/2,q/2,2}} \leq C_3 \|\vec{u}\|_{p,q,1}^2.$$

For $C_1 \|\vec{f}\|_{E_{p/3,q/3,3}} + C_2 \|\mathbb{F}\|_{E_{p/2,q/2,2}} < \frac{1}{4C_3}$, we find a solution $\vec{u} \in X$ with $\|\vec{u}\|_X \leq 2(C_1 \|\vec{f}\|_{E_{p/3,q/3,3}} + C_2 \|\mathbb{F}\|_{E_{p/2,q/2,2}})$. □

Appendix: Parabolic Besov spaces

In this appendix, we describe some results on the heat equation on parabolic Besov spaces that may shed a new light on many of the inequalities we obtained throughout the paper, which are to be viewed more as regularity assertions than as results on existence of solutions.

For $\delta > 0$, let us define the parabolic Besov space $\mathcal{B}^{-\delta}_{[2],\infty,\infty}$ as the space of tempered distributions on $\mathbb{R} \times \mathbb{R}^d$ such that

$$\|u\|_{\mathcal{B}^{-\delta}_{[2],\infty,\infty}} = \sup_{\theta > 0} \theta^{\delta/2} \|e^{\theta^2(\partial_t^2 - \Delta^2)} u(t,x)\|_{L^\infty(\mathbb{R} \times \mathbb{R}^d)} < +\infty.$$

(In particular, $E_{p,q,\beta} \subset \mathcal{B}^{-\beta}_{[2],\infty,\infty}$ and $F_{p,q,\beta} \subset \mathcal{B}^{-\beta}_{[2],\infty,\infty}$.)

It is easy to check that $\mathcal{B}^{-\beta}_{[2],\infty,\infty}$ coincides with the (realization of the) homogeneous Besov space defined through the parabolic Littlewood-Paley decomposition (for instance, see Triebel's book [Trie06]). Let $\phi(\tau,\xi)$ be a smooth function equal to 1 when $\sqrt{|\tau|}+|\xi|<1$ and to 0 when $\sqrt{|\tau|}+|\xi|>2$. Let S_jf be defined as the inverse Fourier transform of $\phi(\frac{\tau}{4^j},\frac{\xi}{2^j})\hat{f}(\tau,\xi)$. Then $\mathcal{B}^{-\beta}_{[2],\infty,\infty}$ is the Banach space of tempered distributions f such that $\sup_{j\in\mathbb{Z}} 2^{j\delta}\|S_jf\|_\infty<+\infty$.

We have the following Gagliardo–Nirenberg inequality

Proposition 7. *Let $\delta>0$, $1<p,q<+\infty$ and $\beta\in(0,\frac{2}{p}+\frac{d}{q})$.*
a) If $u\in\mathcal{B}^{-\delta}_{[2],\infty,\infty}$ and $\partial_t u,\Delta u\in E_{p,q,\beta}$, then

$$\|D_x u\|_{E_{\frac{2+\delta}{1+\delta}p,\frac{2+\delta}{1+\delta}q,\frac{1+\delta}{2+\delta}\beta}}\leq C\|u\|^{\frac{1}{2+\delta}}_{\mathcal{B}^{-\delta}_{[2],\infty,\infty}}(\|\partial_t u\|_{E_{p,q,\beta}}+\|\Delta u\|_{E_{p,q,\beta}})^{\frac{1+\delta}{2+\delta}}.$$

b) If $u\in\mathcal{B}^{-\delta}_{[2],\infty,\infty}$ and $\partial_t u,\Delta u\in F_{p,q,\beta}$, then

$$\|D_x u\|_{F_{\frac{2+\delta}{1+\delta}p,\frac{2+\delta}{1+\delta}q,\frac{1+\delta}{2+\delta}\beta}}\leq C\|u\|^{\frac{1}{2+\delta}}_{\mathcal{B}^{-\delta}_{[2],\infty,\infty}}(\|\partial_t u\|_{F_{p,q,\beta}}+\|\Delta u\|_{F_{p,q,\beta}})^{\frac{1+\delta}{2+\delta}}.$$

Our final resuts extend Corollary 1 and Theorem 3 to parabolic Besov spaces:

Theorem 5. *Let $\delta>0$. Let $u=0$ at infinity. Then $u\in\mathcal{B}^{-\delta}_{[2],\infty,\infty}$ if and only if $\partial_t u\in\mathcal{B}^{-\delta-2}_{[2],\infty,\infty}$ and $\Delta u\in\mathcal{B}^{-\delta-2}_{[2],\infty,\infty}$ Moreover, the norms $\|u\|_{\mathcal{B}^{-\delta}_{[2],\infty,\infty}}$ and $\|\partial_t u\|_{\mathcal{B}^{-\delta-2}_{[2],\infty,\infty}}+\|\Delta u\|_{\mathcal{B}^{-\delta-2}_{[2],\infty,\infty}}$ are equivalent.*

Theorem 6. *Let $\delta>2$. If $f\in\mathcal{B}^{-\delta}_{[2],\infty,\infty}$, then the heat equation*

$$\partial_t u=\Delta u+f \tag{8}$$

has a unique solution such that $u=0$ at infinity. Moreover, $u\in\mathcal{B}^{-\delta+2}_{[2],\infty,\infty}$ and

$$\|u\|_{\mathcal{B}^{-\delta+2}_{[2],\infty,\infty}}\leq C\|f\|_{\mathcal{B}^{-\delta}_{[2],\infty,\infty}}.$$

Those results are not very new, as they can be recovered from the voluminous literature on anisotropic Besov spaces We present them for sake of completeness.

Proof of Proposition 7.

Proof. As $u = 0$ at infinity, we may write

$$u = \int_0^\infty (-\partial_\theta)(e^{\theta^2(\partial_t^2-\Delta^2)}u)\, d\theta$$

and

$$D_x u = \int_0^\infty (-\partial_\theta) D_x(e^{\theta^2(\partial_t^2-\Delta^2)}u)\, d\theta.$$

For $R > 0$, we rewrite this as

$$\begin{aligned} D_x u =& -\int_0^R (2\theta\partial_t e^{\theta^2\partial_t^2}\sqrt{\theta} D_x e^{-\theta^2\Delta^2})\partial_t u\, \frac{d\theta}{\sqrt{\theta}} \\ &- \int_R^\infty (2\theta^2\partial_t^2 e^{\frac{3}{4}\theta^2\partial_t^2}\sqrt{\theta} D_x e^{-\frac{3}{4}\theta^2\Delta^2})(e^{\frac{\theta^2}{4}(\partial_t^2-\Delta^2)}u)\, \frac{d\theta}{\theta^{3/2}} \\ &+ \int_0^R (2e^{\theta^2\partial_t^2}\theta^{3/2} D_x\Delta e^{-\theta^2\Delta^2})\Delta u\, \frac{d\theta}{\sqrt{\theta}} \\ &+ \int_R^\infty (2e^{\frac{3}{4}\theta^2\partial_t^2}\theta^{5/2} D_x\Delta^2 e^{-\frac{3}{4}\theta^2\Delta^2})(e^{\frac{\theta^2}{4}(\partial_t^2-\Delta^2)}u)\, \frac{d\theta}{\theta^{3/2}} \\ =& I + II + III + IV. \end{aligned}$$

Recall that $\mathcal{M}_f$ designs the parabolic Hardy–Littlewood maximal function. If $\psi \in \mathcal{S}(\mathbb{R}\times\mathbb{R}^d)$, we have, for every $\theta > 0$,

$$\iint \frac{1}{\theta^{\frac{d+2}{2}}} |\psi(\frac{s}{\theta}, \frac{y}{\sqrt{\theta}})| |f(t-s, x-y)\, ds\, dy \le C_\psi \mathcal{M}_f(t,x)$$

and

$$\iint \frac{1}{\theta^{\frac{d+2}{2}}} |\psi(\frac{s}{\theta}, \frac{y}{\sqrt{\theta}})| |f(t-s, x-y)\, ds\, dy \le \|\psi\|_1 \|f\|_\infty.$$

Thus, we have

$$|I| \le C\sqrt{R}\mathcal{M}_{\partial_t u} \text{ and } |III| \le C\sqrt{R}\mathcal{M}_{\Delta u}$$

and

$$|II| + |IV| \le C R^{-\frac{1+\delta}{2}} \|u\|_{\mathcal{B}^{-\delta}_{[2],\infty,\infty}}.$$

Taking $\sqrt{R} = \left(\frac{\|u\|_{\mathcal{B}^{-\delta}_{[2],\infty,\infty}}}{\mathcal{M}_{\partial_t u}(t,x) + \mathcal{M}_{\Delta u}(t,x)} \right)^{\frac{1}{2+\delta}}$, we get

$$|D_x u(t,x)| \le C \|u\|^{\frac{1}{2+\delta}}_{\mathcal{B}^{-\delta}_{[2],\infty,\infty}} (\mathcal{M}_{\partial_t u}(t,x) + \mathcal{M}_{\Delta u}(t,x))^{\frac{1+\delta}{2+\delta}}$$

and we conclude by Proposition 4 a). □

Proof of Theorem 5.

Proof. Writing

$$e^{\theta^2(\partial_t^2-\Delta^2)}(\partial_t u) = \frac{1}{\theta}(\theta\partial_t e^{\frac{3}{4}\theta^2(\partial_t^2-\Delta^2)})(e^{\frac{1}{4}\theta^2(\partial_t^2-\Delta^2)}u)$$

and

$$e^{\theta^2(\partial_t^2-\Delta^2)}(\Delta u) = \frac{1}{\theta}(\theta\Delta e^{\frac{3}{4}\theta^2(\partial_t^2-\Delta^2)})(e^{\frac{1}{4}\theta^2(\partial_t^2-\Delta^2)}u),$$

we find that

$$\|\partial_t u\|_{\mathcal{B}^{-\delta-2}_{[2],\infty,\infty}} + \|\Delta u\|_{\mathcal{B}^{-\delta-2}_{[2],\infty,\infty}} \leq C\|u\|_{\mathcal{B}^{-\delta}_{[2],\infty,\infty}}.$$

As $u = 0$ at infinity, we write

$$u = \int_0^\infty (-\partial_\eta)(e^{\eta^2(\partial_t^2-\Delta^2)}u)\,d\eta = 2\int_0^{+\infty} \eta(\Delta^2-\partial_t^2)e^{\eta^2(\partial_t^2-\Delta^2)}u\,d\eta.$$

We get

$$\begin{aligned} e^{\theta^2(\partial_t^2-\Delta^2)}u =& 2\int_0^{+\infty} (\eta\Delta e^{\frac{\eta^2}{2}(\partial_t^2-\Delta^2)})(e^{(\theta^2+\frac{\eta^2}{2})(\partial_t^2-\Delta^2)}\Delta u)\,d\eta \\ & - 2\int_0^{+\infty} (\eta\partial_t e^{\frac{\eta^2}{2}(\partial_t^2-\Delta^2)})(e^{(\theta^2+\frac{\eta^2}{2})(\partial_t^2-\Delta^2)}\partial_t u)\,d\eta, \end{aligned}$$

so that

$$\begin{aligned} \|e^{\theta^2(\partial_t^2-\Delta^2)}u\|_\infty \leq& C\int_0^{+\infty} \|e^{(\theta^2+\frac{\eta^2}{2})(\partial_t^2-\Delta^2)}\Delta u\|_\infty + \|e^{(\theta^2+\frac{\eta^2}{2})(\partial_t^2-\Delta^2)}\partial_t u\|_\infty\,d\eta \\ \leq& C(\|\partial_t u\|_{\mathcal{B}^{-\delta-2}_{[2],\infty,\infty}} + \|\Delta u\|_{\mathcal{B}^{-\delta-2}_{[2],\infty,\infty}})\int_0^{+\infty} (\theta^2+\frac{\eta^2}{2})^{-\frac{2+\delta}{4}}\,d\eta \\ =& C\theta^{-\delta/2}(\|\partial_t u\|_{\mathcal{B}^{-\delta-2}_{[2],\infty,\infty}} + \|\Delta u\|_{\mathcal{B}^{-\delta-2}_{[2],\infty,\infty}})\int_0^{+\infty} (1+\frac{\eta^2}{2})^{-\frac{2+\delta}{4}}\,d\eta \end{aligned}$$

and finally

$$\|u\|_{\mathcal{B}^{-\delta}_{[2],\infty,\infty}} \leq C(\|\partial_t u\|_{\mathcal{B}^{-\delta-2}_{[2],\infty,\infty}} + \|\Delta u\|_{\mathcal{B}^{-\delta-2}_{[2],\infty,\infty}}).$$ □

Proof of Theorem 6.

Proof. We write

$$f = \int_0^\infty (-\partial_\eta)(e^{\eta^2(\partial_t^2-\Delta^2)}f)\,d\eta = -2(\partial_t-\Delta)\int_0^{+\infty} \eta(\Delta+\partial_t)e^{\eta^2(\partial_t^2-\Delta^2)}f\,d\eta.$$

Let

$$u = -2\int_0^{+\infty} \eta(\Delta + \partial_t)e^{\eta^2(\partial_t^2-\Delta^2)} f\, d\eta.$$

We have

$$\begin{aligned}
\|e^{\theta^2(\partial_t^2-\Delta^2)}u\|_\infty \leq & 2\int_0^{+\infty} \|(\eta\Delta e^{\frac{\eta^2}{2}(\partial_t^2-\Delta^2)})(e^{(\theta^2+\frac{\eta^2}{2})(\partial_t^2-\Delta^2)}f)\|_\infty\, d\eta \\
& + 2\int_0^{+\infty} \|(\eta\partial_t e^{\frac{\eta^2}{2}(\partial_t^2-\Delta^2)})(e^{(\theta^2+\frac{\eta^2}{2})(\partial_t^2-\Delta^2)}f)\|_\infty\, d\eta, \\
\leq & C\int_0^{+\infty} \|e^{(\theta^2+\frac{\eta^2}{2})(\partial_t^2-\Delta^2)}f\|_\infty\, d\eta \\
\leq & C\|f\|_{\mathcal{B}^{-\delta-2}_{[2],\infty,\infty}} \int_0^{+\infty} (\theta^2+\frac{\eta^2}{2})^{-\frac{2+\delta}{4}}\, d\eta \\
= & C\theta^{-\delta/2}\|f\|_{\mathcal{B}^{-\delta-2}_{[2],\infty,\infty}} \int_0^{+\infty} (1+\frac{\eta^2}{2})^{-\frac{2+\delta}{4}}\, d\eta
\end{aligned}$$

and finally

$$\|u\|_{\mathcal{B}^{-\delta}_{[2],\infty,\infty}} \leq C\|f\|_{\mathcal{B}^{-\delta-2}_{[2],\infty,\infty}}. \qquad \square$$

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