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# The Navier–Stokes equations in mixed-norm time-space parabolic Morrey spaces.

Pierre Gilles Lemarié-Rieusset\*

## Abstract

We discuss the Navier–Stokes equations with forces in the mixed-norm time-space parabolic Morrey spaces of Krylov.

**Keywords :** Navier–Stokes equations, heat equation, parabolic Morrey spaces, mild solutions.

**AMS classification :** 35K55, 35Q30, 76D05.

## 1 Introduction

In this paper, we consider global mild solutions of the Cauchy problem for the incompressible Navier–Stokes equations on the whole space  $\mathbb{R}^3$ . More precisely, the Navier–Stokes equations we study are

$$\begin{cases} \partial_t \vec{u} = \Delta \vec{u} - \vec{\nabla} p - \vec{u} \cdot \vec{\nabla} \vec{u} + \vec{f} + \operatorname{div} \mathbb{F} \\ \operatorname{div} \vec{u} = 0 \\ \vec{u}(0, \cdot) = 0 \end{cases} \quad (1)$$

where  $\vec{f}$  and  $\mathbb{F}$  are small enough in some critical (homogeneous) mixed-norm time-space parabolic Morrey spaces of Krylov.

(Following Krylov [Kry123]), let us recall the definition of mixed-norm time-space parabolic Morrey spaces on  $\mathbb{R} \times \mathbb{R}^d$ :

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**Definition 1.** Let  $1 < p, q < +\infty$  and  $\beta \in (0, \frac{2}{p} + \frac{d}{q})$ . The parabolic Morrey spaces  $E_{p,q,\beta}$  and  $F_{p,q,\beta}$  are the spaces of locally integrable functions  $f(t, x)$  on  $\mathbb{R} \times \mathbb{R}^d$  such that  $\|f\|_{E_{p,q,\beta}} < +\infty$  or  $\|f\|_{F_{p,q,\beta}} < +\infty$ , where

$$\|f\|_{E_{p,q,\beta}} = \sup_{\rho > 0, t \in \mathbb{R}, x \in \mathbb{R}^d} \rho^{\beta - \frac{2}{p} - \frac{d}{q}} \left( \int_{|t-s| < \rho^2} \left( \int_{|x-y| < \rho} |f(s, y)|^q dy \right)^{\frac{p}{q}} ds \right)^{\frac{1}{p}}$$

and

$$\|f\|_{F_{p,q,\beta}} = \sup_{\rho > 0, t \in \mathbb{R}, x \in \mathbb{R}^d} \rho^{\beta - \frac{2}{p} - \frac{d}{q}} \left( \int_{|x-y| < \rho} \left( \int_{|t-s| < \rho^2} |f(s, y)|^p ds \right)^{\frac{q}{p}} dy \right)^{\frac{1}{q}}.$$

Remark: when  $p = q$ , writing  $r = \frac{d+2}{\beta}$  (so that  $p < r < +\infty$ ), we see that  $E_{p,p,\beta} = F_{p,p,\beta} = \mathcal{M}_2^{p,r}(\mathbb{R} \times \mathbb{R}^d)$ , where  $\mathcal{M}_2^{p,r}(\mathbb{R} \times \mathbb{R}^d)$  is the parabolic Morrey space studied in [Lema16] in the context of Navier–Stokes equations.

Krylov [Kryl23] developed a theory of Itô stochastic equations with as singular a drift as possible. Following an observation of Kinzebulatov [Kinz23], he considered a drift in parabolic Morrey spaces. We will show that his results can be easily applied to the Navier–Stokes problem. More precisely, we consider the initial value problem (on  $(0, +\infty) \times \mathbb{R}^3$ ) with initial value  $u(0, \cdot) = 0$ , that we transform as a problem on  $\mathbb{R} \times \mathbb{R}^3$  by extending the Navier–Stokes solution by  $\vec{u}(t, \cdot) = 0$  for  $t < 0$ .

For a function defined on  $(0, +\infty) \times \mathbb{R}^d$ , we say that  $f \in E_{p,q,\beta}$  or  $f \in F_{p,q,\beta}$  if the function  $f^\#$  defined by  $f^\# = f$  for  $t > 0$  and  $f^\# = 0$  for  $t < 0$  is such that  $f^\# \in E_{p,q,\beta}$  or  $f^\# \in F_{p,q,\beta}$ .

Our main theorem is the following one:

**Theorem 1.** Let  $p, q \in (3, \infty)$  with  $\frac{2}{p} + \frac{3}{q} > 1$ .

a) There exists  $\epsilon_0 = \epsilon_0(p, q) > 0$  and  $C_0 = C_0(p, q) > 0$  such that, if  $\vec{f} \in E_{p/3, q/3, 3}$  with  $\operatorname{div} \vec{f} = 0$  and  $\mathbb{F} \in E_{p/2, q/2, 2}$  and if

$$\|\vec{f}\|_{E_{p/3, q/3, 3}} + \|\mathbb{F}\|_{E_{p/2, q/2, 2}} < \epsilon_0,$$

then the Navier–Stokes equations (1) have a global mild solution  $\vec{u} \in E_{p, q, 1}$  and  $\vec{\nabla} \otimes \vec{u} \in E_{p/2, q/2, 2}$  with

$$\|\vec{u}\|_{E_{p, q, 1}} + \|\vec{\nabla} \otimes \vec{u}\|_{E_{p/2, q/2, 2}} \leq C_0 (\|\vec{f}\|_{E_{p/3, q/3, 3}} + \|\mathbb{F}\|_{E_{p/2, q/2, 2}}).$$

b) There exists  $\epsilon_0 = \epsilon_0(p, q) > 0$  and  $C_0 = C_0(p, q) > 0$  such that, if  $\vec{f} \in F_{p/3, q/3, 3}$  with  $\operatorname{div} \vec{f} = 0$  and  $\mathbb{F} \in F_{p/2, q/2, 2}$  and if

$$\|\vec{f}\|_{F_{p/3, q/3, 3}} + \|\mathbb{F}\|_{F_{p/2, q/2, 2}} < \epsilon_0,$$

then the Navier–Stokes equations (1) have a global mild solution  $\vec{u} \in F_{p,q,1}$  and  $\vec{\nabla} \otimes \vec{u} \in F_{p/2,q/2,2}$  with

$$\|\vec{u}\|_{F_{p,q,1}} + \|\vec{\nabla} \otimes \vec{u}\|_{F_{p/2,q/2,2}} \leq C_0(\|\vec{f}\|_{F_{p/3,q/3,3}} + \|\mathbb{F}\|_{F_{p/2,q/2,2}}).$$

When the force  $\vec{f}$  is equal to 0, we may lower the values of  $p$  and  $q$ :

**Theorem 2.** Let  $p, q \in (2, \infty)$  with  $\frac{2}{p} + \frac{3}{q} > 1$ .

a) There exists  $\epsilon_0 = \epsilon_0(p, q) > 0$  and  $C_0 = C_0(p, q) > 0$  such that, if  $\mathbb{F} \in E_{p/2,q/2,2}$  and

$$\|\mathbb{F}\|_{E_{p/2,q/2,2}} < \epsilon_0,$$

then the Navier–Stokes equations (1) (with  $\vec{f} = 0$ ) have a global mild solution  $\vec{u} \in E_{p,q,1}$  and  $\vec{\nabla} \otimes \vec{u} \in E_{p/2,q/2,2}$  with

$$\|\vec{u}\|_{E_{p,q,1}} + \|\vec{\nabla} \otimes \vec{u}\|_{E_{p/2,q/2,2}} \leq C_0\|\mathbb{F}\|_{E_{p/2,q/2,2}}.$$

b) There exists  $\epsilon_0 = \epsilon_0(p, q) > 0$  and  $C_0 = C_0(p, q) > 0$  such that, if  $\mathbb{F} \in F_{p/2,q/2,2}$  and if

$$\|\mathbb{F}\|_{F_{p/2,q/2,2}} < \epsilon_0,$$

then the Navier–Stokes equations (1) (with  $\vec{f} = 0$ ) have a global mild solution  $\vec{u} \in F_{p,q,1}$  and  $\vec{\nabla} \otimes \vec{u} \in F_{p/2,q/2,2}$  with

$$\|\vec{u}\|_{F_{p,q,1}} + \|\vec{\nabla} \otimes \vec{u}\|_{F_{p/2,q/2,2}} \leq C_0\|\mathbb{F}\|_{F_{p/2,q/2,2}}.$$

As usual, the mild solution  $\vec{u}$  will be searched as a fixed point of the transform

$$\vec{u} \mapsto \vec{T}(\vec{f}) + \vec{T}(\mathbb{P} \operatorname{div} \mathbb{F}) - \vec{T}(\mathbb{P} \operatorname{div}(\vec{u} \otimes \vec{u})),$$

where  $\vec{T}(\vec{f}) = \int_0^t e^{(t-s)\Delta} \vec{f}(s, \cdot) ds$ . Let us remark that, if  $X = E_{p,q,1}$  or  $X = F_{p,q,1}$ , then  $X$  is a critical space with respect to the scaling of the Navier–Stokes equations: for  $\lambda > 0$ ,  $\|\lambda \vec{u}(\lambda^2 t, \lambda x)\|_X = \|\vec{u}\|_X$ .  $E_{p,q,1}$  is easily compared to other classical spaces; for instance, we have, for Serrin spaces with  $\frac{2}{p} + \frac{3}{q} = 1$ ,  $L_t^p((0, +\infty), L_x^q(\mathbb{R}^3)) \subset E_{p,q,1}$ , while  $E_{p,q,\beta} \subset \mathcal{M}_2^{\min(p,q)}$ <sup>5</sup> (a critical space discussed in [Lema16]).

## 2 Heat equation

The proof of theorems 1 and 2 will rely on the theory of the heat equation  $\partial_t u - \Delta u = f$  on  $\mathbb{R} \times \mathbb{R}^d$  with the boundary condition  $u = 0$  at infinity. This condition will be defined as follows:

**Definition 2.** Let  $u$  be a tempered distribution on  $\mathbb{R} \times \mathbb{R}^d$ . Then  $u = 0$  at infinity if for every  $\theta > 0$  one has  $e^{\theta^2(\partial_t^2 - \Delta^2)}u \in L^\infty(\mathbb{R} \times \mathbb{R}^d)$  and  $\lim_{\theta \rightarrow +\infty} \|e^{\theta^2(\partial_t^2 - \Delta^2)}u\|_\infty = 0$ .

Easy examples of  $u$  with  $u = 0$  at infinity are  $u \in L_t^p L_x^q$  with  $1 \leq p, q \leq \infty$ ,  $(p, q) \neq (\infty, \infty)$  or  $u = \partial_t v_0 + \sum_{j=1}^d v_j$  where  $v_j \in L^\infty(\mathbb{R} \times \mathbb{R}^d)$  for  $0 \leq j \leq d$ : in the first case, we have  $\|e^{\theta^2(\partial_t^2 - \Delta^2)}u\|_\infty \leq C\|u\|_{L^p L^q} \theta^{-(\frac{1}{p} + \frac{d}{2q})}$  and, in the second case,  $\|e^{\theta^2(\partial_t^2 - \Delta^2)}u\|_\infty \leq C(\theta^{-1}\|v_0\|_\infty + \sum_{j=1}^d \theta^{-\frac{1}{2}}\|v_j\|_\infty)$ . Easy examples of bounded functions  $u$  with  $u \neq 0$  at infinity are  $u = 1$ ,  $u = \mathbf{1}_{t>0}$ , or  $u = \mathbf{1}_{x_d>0}$ : we have  $\|e^{\theta^2(\partial_t^2 - \Delta^2)}u\|_\infty = 1$ , as  $\|e^{\theta^2(\partial_t^2 - \Delta^2)}u\|_\infty \leq 1$  and  $\lim_{(t, x_d) \rightarrow (+\infty, +\infty)} e^{\theta^2(\partial_t^2 - \Delta^2)}u(t, x) = 1$ .

**Lemma 1.** Let  $u$  be a tempered distribution on  $\mathbb{R} \times \mathbb{R}^d$  such that  $u = 0$  at infinity. If  $\partial_t u - \Delta u = 0$ , then  $u = 0$ .

*Proof.* Take the Fourier transform (in time and space variables) of the heat equation. We find

$$(i\tau + |\xi|^2)\hat{u}(\tau, \xi) = 0.$$

Thus,  $\hat{u}$  is supported in  $\{(0, 0)\}$ , hence is a sum of derivatives of the Dirac mass, and  $u$  is a polynomial. As  $e^{\theta^2(\partial_t^2 - \Delta^2)}u$  is then a polynomial, and as  $e^{\theta^2(\partial_t^2 - \Delta^2)}u \in L^\infty(\mathbb{R} \times \mathbb{R}^d)$ , we find that  $u$  is a constant; in that case,  $e^{\theta^2(\partial_t^2 - \Delta^2)}u = u$ . As  $u = 0$  at infinity, we have  $u = 0$ .  $\square$

**Lemma 2.** Let  $1 < p, q < +\infty$  and  $\beta \in (0, \frac{2}{p} + \frac{d}{q})$ . If  $u \in E_{p, q, \beta}$  or  $u \in F_{p, q, \beta}$ , then  $u = 0$  at infinity.

*Proof.* We give the proof for  $u \in E_{p, q, \beta}$  (the case  $u \in F_{p, q, \beta}$  is proved in a similar way). Let  $W(t, x)$  be the inverse Fourier transform of  $e^{-\tau^2 - |\xi|^4}$ .  $W$  is in the Schwartz class of smooth functions with rapid decay. In particular,  $(\sqrt{|t|} + |x|)^{d+2}|W(t, x)| \in L^\infty(\mathbb{R} \times \mathbb{R}^d)$ . Writing

$$\begin{aligned} & \iint |W(t, x)| |u(t, x)| dt dx \\ &= \iint_{\sqrt{|t|+|x|}<1} |W(t, x)| |u(t, x)| dt dx + \sum_{j=0}^{+\infty} \iint_{2^j < \sqrt{|t|+|x|} < 2^{j+1}} |W(t, x)| |u(t, x)| dt dx \\ &\leq C\|u\|_{E_{p, q, \beta}} (\|\mathbf{1}_{\sqrt{|t|+|x|}<1} W\|_{L_t^{\frac{p}{p-1}} L_x^{\frac{q}{q-1}}} + \sum_{j=0}^{+\infty} 2^{j(\frac{2}{p} + \frac{d}{q} - \beta)} \|\mathbf{1}_{2^j < \sqrt{|t|+|x|} < 2^{j+1}} W\|_{L_t^{\frac{p}{p-1}} L_x^{\frac{q}{q-1}}}) \\ &\leq C'\|u\|_{E_{p, q, \beta}} (1 + \sum_{j=0}^{+\infty} 2^{-j\beta}) = C_1\|u\|_{E_{p, q, \beta}}, \end{aligned}$$

we get

$$\begin{aligned}
|e^{\theta^2(\partial_t^2 - \Delta^2)}u(t, x)| &\leq \iint W(s, y)|u(t + \theta^2s, x + \theta y)| ds dy \\
&\leq C_1 \|u(t + \theta \cdot, x + \sqrt{\theta} \cdot)\|_{E_{p,q,\beta}} \\
&= C_1 \|u(\theta \cdot, \sqrt{\theta} \cdot)\|_{E_{p,q,\beta}} \\
&= C_1 \|u\|_{E_{p,q,\beta}} \theta^{-\beta/2}.
\end{aligned}$$

Thus,  $\lim_{\theta \rightarrow +\infty} \|e^{\theta^2(\partial_t^2 - \Delta^2)}u\|_{\infty} = 0$ .  $\square$

The proof of Theorems 1 and 2 will be based on the following results on the linear heat equation:

**Theorem 3.** *Let  $p, q \in (3, \infty)$  and  $\beta > 0$  with  $\frac{2}{p} + \frac{3}{q} > \beta$ .*

a) *If  $f \in E_{p/3, q/3, \beta+2}(\mathbb{R} \times \mathbb{R}^d)$ , then the heat equation*

$$\partial_t u = \Delta u + f \quad (2)$$

*has a unique solution such that  $u = 0$  at infinity. Moreover, there exists  $C_0 = C_0(p, q, \beta) > 0$  such that, if denoting by  $D_x u$  the gradient of  $u$  and by  $D_x^2 u$  the Hessian of  $u$  (with respect to the space variable), we have*

$$\|u\|_{E_{p,q,\beta}} + \|D_x u\|_{E_{p/2, q/2, \beta+1}} + \|D_x^2 u\|_{E_{p/3, q/3, \beta+2}} + \|\partial_t u\|_{E_{p/3, q/3, \beta+2}} \leq C_0 \|f\|_{E_{p/3, q/3, \beta+2}}.$$

b) *If  $f \in F_{p/3, q/3, \beta+2}(\mathbb{R} \times \mathbb{R}^d)$ , then the heat equation (2) has a unique solution such that  $u = 0$  at infinity. Moreover, there exists  $C_0 = C_0(p, q, \beta) > 0$  such that we have*

$$\|u\|_{F_{p,q,\beta}} + \|D_x u\|_{F_{p/2, q/2, \beta+1}} + \|D_x^2 u\|_{F_{p/3, q/3, \beta+2}} + \|\partial_t u\|_{F_{p/3, q/3, \beta+2}} \leq C_0 \|f\|_{F_{p/3, q/3, \beta+2}}.$$

**Theorem 4.** *Let  $p, q \in (2, \infty)$  and  $\beta > 0$  with  $\frac{2}{p} + \frac{3}{q} > \beta$ . Let  $\sigma(D)$  be a Fourier multiplier (in the space variable) such that  $\sigma(\xi) = \sigma_0(\frac{\xi}{|\xi|})$  where  $\sigma_0$  is a smooth function on the sphere  $\mathcal{S}^{d-1}$ .*

a) *If  $\mathbb{F} \in E_{p/2, q/2, \beta+1}(\mathbb{R} \times \mathbb{R}^d)$ , then the heat equation*

$$\partial_t u = \Delta u + \sigma(D) \operatorname{div} \mathbb{F} \quad (3)$$

*has a unique solution such that  $u = 0$  at infinity. Moreover, there exists  $C_0 = C_0(p, q, \beta, \sigma) > 0$  such that we have*

$$\|u\|_{E_{p,q,\beta}} + \|D_x u\|_{E_{p/2, q/2, \beta+1}} \leq C_0 \|\mathbb{F}\|_{E_{p/2, q/2, \beta+1}}.$$

b) *If  $\mathbb{F} \in F_{p/2, q/2, \beta+1}(\mathbb{R} \times \mathbb{R}^d)$ , then the heat equation (3) has a unique solution such that  $u = 0$  at infinity. Moreover, there exists  $C_0 = C_0(p, q, \beta, \sigma) > 0$  such that we have*

$$\|u\|_{F_{p,q,\beta}} + \|D_x u\|_{F_{p/2, q/2, \beta+1}} \leq C_0 \|\mathbb{F}\|_{F_{p/2, q/2, \beta+1}}.$$

Theorem 3 has the following corollary on Sobolev–Morrey inequalities:

**Corollary 1.** *Let  $p, q \in (3, \infty)$  and  $\beta > 0$  with  $\frac{2}{p} + \frac{3}{q} > \beta$ . Let  $u = 0$  at infinity. Then there exists  $C_0 = C_0(p, q, \beta) > 0$  such that we have*

$$\|u\|_{E_{p,q,\beta}} + \|D_x u\|_{E_{p/2,q/2,\beta+1}} + \|D_x^2 u\|_{E_{p/3,q/3,\beta+2}} \leq C_0(\|\partial_t u\|_{E_{p/3,q/3,\beta+2}} + \|\Delta u\|_{E_{p/3,q/3,\beta+2}})$$

and

$$\|u\|_{F_{p,q,\beta}} + \|D_x u\|_{F_{p/2,q/2,\beta+1}} + \|D_x^2 u\|_{F_{p/3,q/3,\beta+2}} \leq C_0(\|\partial_t u\|_{F_{p/3,q/3,\beta+2}} + \|\Delta u\|_{F_{p/3,q/3,\beta+2}}).$$

Another corollary is the following variation on the theorem of Krylov [Kryl23]:

**Proposition 1.** *Let  $p, q \in (3, \infty)$  and  $\beta > 0$  with  $\frac{2}{p} + \frac{3}{q} > \max(\beta, 1)$ . There exists  $\epsilon_0 = \epsilon_0(p, q, \beta) > 0$  and  $C_0 = C_0(p, q, \beta) > 0$  such that*

*a) If  $f \in E_{p/3,q/3,\beta+2}(\mathbb{R} \times \mathbb{R}^d)$ ,  $\vec{b} \in E_{p,q,1}$ ,  $c \in E_{p/2,q/2,2}$ , and if  $\|\vec{b}\|_{E_{p,q,1}} + \|c\|_{E_{p/2,q/2,2}} < \epsilon_0$ , then the heat equation*

$$\partial_t u = \Delta u + \vec{b} \cdot \vec{\nabla} u + cu + f \tag{4}$$

*has a unique solution such that  $u = 0$  at infinity. Moreover, we have*

$$\begin{aligned} & \|u\|_{E_{p,q,\beta}} + \|D_x u\|_{E_{p/2,q/2,\beta+1}} + \|D_x^2 u\|_{E_{p/3,q/3,\beta+2}} \\ & \leq C_0(\|f\|_{E_{p/3,q/3,\beta+2}} + \|\vec{b}\|_{E_{p,q,1}} + \|c\|_{E_{p/2,q/2,2}}). \end{aligned}$$

*b) If  $f \in F_{p/3,q/3,\beta+2}(\mathbb{R} \times \mathbb{R}^d)$ ,  $\vec{b} \in F_{p,q,1}$ ,  $c \in F_{p/2,q/2,2}$ , and if  $\|\vec{b}\|_{F_{p,q,1}} + \|c\|_{F_{p/2,q/2,2}} < \epsilon_0$ , then the heat equation (4) has a unique solution such that  $u = 0$  at infinity. Moreover, we have*

$$\begin{aligned} & \|u\|_{F_{p,q,\beta}} + \|D_x u\|_{F_{p/2,q/2,\beta+1}} + \|D_x^2 u\|_{F_{p/3,q/3,\beta+2}} \\ & \leq C_0(\|f\|_{F_{p/3,q/3,\beta+2}} + \|\vec{b}\|_{F_{p,q,1}} + \|c\|_{F_{p/2,q/2,2}}). \end{aligned}$$

### 3 Anisotropic Hardy-Littlewood maximal function on mixed norm Lebesgue spaces

Let  $\vec{p} = (p_1, \dots, p_n) \in (1, +\infty)^n$ . The mixed-norm Lebesgue spaces  $L^{\vec{p}}(\mathbb{R}^n)$  is the space of measurable functions on  $\mathbb{R}^n$  such that

$$\|f\|_{L^{\vec{p}}} = \left( \int \dots \left[ \int \left\{ \int |f(x_1, \dots, x_n)|^{p_1} dx_1 \right\}^{\frac{p_2}{p_1}} \dots dx_n \right]^{\frac{1}{p_n}} < +\infty.$$

For  $\vec{a} = (a_1, \dots, a_n) \in [1, +\infty)^n$ , the anisotropic cylinders  $Q_{\vec{a}}(x, r)$  are defined as

$$Q_{\vec{a}}(x, r) = (x_1 - r^{a_1}, x_1 + r^{a_1}) \times \dots \times (x_n - r^{a_n}, x_n + r^{a_n})$$

and the anisotropic Hardy-Littlewood maximal function is defined as

$$\mathcal{M}_{\vec{a}}f(x) = \sup_{r>0} \frac{1}{|Q_{\vec{a}}(x, r)|} \int_{Q_{\vec{a}}(x, r)} |f(y)| dy.$$

We then have the following boundedness result for the anisotropic Hardy-Littlewood maximal function on mixed norm Lebesgue spaces [Huan19, Huan21]:

**Proposition 2.** *For  $\vec{p} = (p_1, \dots, p_n) \in (1, +\infty)^n$  and  $\vec{a} = (a_1, \dots, a_n) \in [1, +\infty)^n$ , there exists a constant  $C = C(\vec{p}, \vec{a})$  such that*

$$\|\mathcal{M}_{\vec{a}}f\|_{L^{\vec{p}}} \leq C \|f\|_{L^{\vec{p}}}.$$

## 4 Harmonic analysis on the parabolic space

We endow  $X = \mathbb{R} \times \mathbb{R}^d$  with the Lebesgue measure  $d\mu(t, x) = dt dx$  and the parabolic metric  $\rho((t, x), (s, y)) = \sqrt{|t - s|} + |x - y|$ . The parabolic ball  $B_r(t, x)$  and the parabolic cylinder  $C_r(t, x)$  are defined as

$$B_r(t, x) = \{(s, y) / \rho((t, x), (s, y)) < r\} \text{ and } C_r(t, x) = (t - r^2, t + r^2) \times B(x, r).$$

We have

$$B_r(t, x) \subset C_r(t, x) \subset B_{2r}(t, x).$$

$(X, \rho, \mu)$  is a space of homogeneous type [Coif71] of homogeneous dimension  $Q = d + 2$ :

$$\mu(B_r(t, x)) = \mu(B_1(0, 0))r^Q.$$

We associate to this space three useful operators on (non-negative) functions: the Hardy-Littlewood maximal function  $\mathcal{M}_f$  and the Riesz potentials  $\mathcal{I}_\alpha f$  (where  $0 < \alpha < Q$ ) defined by

$$\mathcal{M}_f(t, x) = \sup_{r>0} \frac{1}{\mu(B_r(t, x))} \int_{B_r(t, x)} |f(s, y)| d\mu(s, y)$$

and

$$\mathcal{I}_\alpha f(t, x) = \int_X \frac{1}{\rho((t, x), (s, y))^{Q-\alpha}} f(s, y) d\mu(s, y).$$



For  $1 < p \leq q$ , we define the Morrey space  $\mathcal{M}_2^{p,q}$  by the space of measurable functions such that

$$\|f\|_{\mathcal{M}_2^{p,q}} = \sup_{r>0, (t,x) \in X} \mu(B_r(t,x))^{\frac{1}{q} - \frac{1}{p}} \|\mathbb{1}_{B_r(t,x)} f\|_p < +\infty.$$

We have  $L^q(\mathbb{R} \times \mathbb{R}^d) \subset \mathcal{M}_2^{p,q}$ ,  $E_{p,p,\beta} = \mathcal{M}_2^{p, \frac{d+2}{\beta}}$  and  $E_{p,q,\beta} \subset \mathcal{M}_2^{\min(p,q), \frac{d+2}{\beta}}$ .

The main tool we shall use is Hedberg's inequality [Hed72]:

**Proposition 3.** *Let  $1 < p \leq q < +\infty$  and  $0 < \alpha < \frac{Q}{q}$ . Then there exists a constant  $C = C(d, p, q, \alpha)$  such that, for every  $f \in \mathcal{M}_2^{p,q}$ , we have*

$$\mathcal{I}_\alpha f(t, x) \leq C(\mathcal{M}_f(t, x))^{1 - \frac{\alpha q}{Q}} \|f\|_{\mathcal{M}_2^{p,q}}^{\frac{\alpha q}{Q}}.$$

*Proof.* We easily check that, for every  $R > 0$ , we have

$$\int_{B_R(t,x)} \frac{1}{\rho((t,x), (s,y))^{Q-\alpha}} |f(s,y)| d\mu(s,y) \leq CR^\alpha \mathcal{M}_f(t,x)$$

and

$$\int_{X \setminus B_R(t,x)} \frac{1}{\rho((t,x), (s,y))^{Q-\alpha}} |f(s,y)| d\mu(s,y) \leq CR^{\alpha - \frac{Q}{q}} \|f\|_{\mathcal{M}_2^{p,q}}.$$

We then take

$$R^{\frac{Q}{q}} = \frac{\|f\|_{\mathcal{M}_2^{p,q}}}{\mathcal{M}_f(t,x)}. \quad \square$$

Applying Proposition 3 to Krylov spaces, we obtain:

**Proposition 4.** *Let  $1 < p, q < +\infty$  and  $0 < \beta < \frac{2}{p} + \frac{d}{q}$ . Then*

- a)  $f \mapsto \mathcal{M}_f$  is bounded from  $E_{p,q,\beta}$  to  $E_{p,q,\beta}$  and from  $F_{p,q,\beta}$  to  $F_{p,q,\beta}$ .
- b) If  $\beta > 1$ ,  $f \mapsto \mathcal{I}_1 f$  is bounded from  $E_{p,q,\beta}$  to  $E_{\frac{\beta}{\beta-1}p, \frac{\beta}{\beta-1}q, \beta-1}$  and from  $F_{p,q,\beta}$  to  $F_{\frac{\beta}{\beta-1}p, \frac{\beta}{\beta-1}q, \beta-1}$ .
- c) If  $\beta > 2$ ,  $f \mapsto \mathcal{I}_2 f$  is bounded from  $E_{p,q,\beta}$  to  $E_{\frac{\beta}{\beta-2}p, \frac{\beta}{\beta-2}q, \beta-2}$  and from  $F_{p,q,\beta}$  to  $F_{\frac{\beta}{\beta-2}p, \frac{\beta}{\beta-2}q, \beta-2}$ .

*Proof.* We consider only the case of  $E_{p,q,\beta}$ , as the proof for  $F_{p,q,\beta}$  is similar. Let us estimate  $\mathcal{M}_f$  on a cylinder  $C_r(t,x)$ . We have  $\mathcal{M}_f \leq \mathcal{M}_{f_1} + \mathcal{M}_{f_2}$ , where  $f_1 = \mathbb{1}_{C_{4r}(t,x)} f$  and  $f_2 = f - f_1$ . By Proposition 2, we know that

$$\|\mathcal{M}_{f_1}\|_{L_t^p L_x^q} \leq C \|f_1\|_{L_t^p L_x^q} \leq C' \|f\|_{E_{p,q,\beta}} r^{\frac{2}{p} + \frac{d}{q} - \beta}.$$

On the other hand, for  $(s, z) \in C_r(t, x) \subset B_{2r}(t, x)$ , since  $B_{4r}(t, x) \subset C_{4r}(t, x)$ ,

$$\mathcal{M}_{f_2}(s, z) \leq \sup_{\rho > 2r} \frac{1}{|B_\rho(s, z)|} \iint_{B_\rho(s, z)} |f(\sigma, y)| d\sigma dy \leq C' \|f\|_{E_{p,q,\beta}} r^{-\beta}$$

so that

$$\|\mathbf{1}_{C_r(t,x)} \mathcal{M}_{f_2}\|_{L_t^p L_x^q} \leq C \|f\|_{E_{p,q,\beta}} r^{\frac{2}{p} + \frac{d}{q} - \beta}.$$

a) is proved. b) and c) are then direct consequences of Hedberg's inequality (Proposition 3).  $\square$

## 5 The heat equation on $\mathbb{R} \times \mathbb{R}^d$ .

In this section, we solve the heat equation on  $\mathbb{R} \times \mathbb{R}^d$ :

$$\begin{cases} \partial_t u = \Delta u + g \\ u = 0 \text{ at infinity} \end{cases} \quad (5)$$

where  $g = f \in E_{p,q,\beta}$  or in  $F_{p,q,\beta}$  with  $2 < \beta < \frac{2}{p} + \frac{d}{q}$  or  $g = \sigma(D) \operatorname{div} \mathbb{F}$  where  $\mathbb{F} \in E_{p,q,\beta}$  or in  $F_{p,q,\beta}$  with  $1 < \beta < \frac{2}{p} + \frac{d}{q}$ . The solution  $u$  of equation (5) is given by the Duhamel formula

$$u = \int_{-\infty}^t e^{(t-s)\Delta} g(s, \cdot) ds.$$

In order to estimate  $u$  and its derivatives, we need some estimates on the size of the kernel of  $e^{t\Delta}$  and its derivatives, or on the derivatives of  $\sigma(D)e^{t\Delta}$ .

**Lemma 3.** *Let  $\psi \in \mathcal{S}(\mathbb{R}^d)$  and, for  $\theta > 0$ ,  $\psi_\theta(x) = \frac{1}{\theta^d} \psi(\frac{x}{\theta})$ . Let  $\sigma(D)$  be a Fourier multiplier (in the space variable) such that  $\sigma(\xi) = \sigma_0(\frac{\xi}{|\xi|})$  where  $\sigma_0$  is a smooth function on the sphere  $\mathcal{S}^{d-1}$ . Then, for  $\alpha \in \mathbb{N}^d$ ,*

$$|\partial_\alpha \sigma(D) \psi_\theta(x)| \leq C_{\alpha,\psi,\sigma} \frac{1}{(\theta + |x|)^{d+|\alpha|}}.$$

*Proof.* We have

$$\|\partial_\alpha \sigma(D) \psi_\theta\|_\infty = \theta^{-d-|\alpha|} \|\partial_\alpha \sigma(D) \psi\|_\infty$$

and

$$\| |x|^{d+|\alpha|} \partial_\alpha \sigma(D) \psi_\theta \|_\infty = \| |x|^{d+|\alpha|} \partial_\alpha \sigma(D) \psi \|_\infty.$$

Thus, we may assume  $\theta = 1$ . We have

$$|\partial_\alpha \sigma(D) \psi_\theta(x)| \leq \frac{1}{(2\pi)^d} \|\sigma_0\|_\infty \int |\xi|^{|\alpha|} |\hat{\psi}(\xi)| dx.$$

On the other hand, taking a smooth function  $\phi$  such that  $\phi(\xi) = 1$  for  $|\xi| < 1$  and  $\phi(\xi) = 0$  for  $|\xi| > 2$ , we have for every  $R > 0$ , for  $1 \leq k \leq d$ ,

$$|x_k|^{d+|\alpha|} |\partial_\alpha \phi(\frac{D}{R}) \sigma(D) \psi_\theta(x)| \leq C(R|x|)^{d+\alpha} \|\sigma_0\|_\infty \|\hat{\psi}\|_\infty$$

and

$$\begin{aligned} & |x|^2 |x_k|^{d+|\alpha|} |\partial_\alpha (1 - \phi(\frac{D}{R})) \sigma(D) \psi_\theta(x)| \\ & \leq \frac{1}{(2\pi)^d} \int |\Delta \partial_k^{d+|\alpha|} \left( (1 - \phi(\frac{\xi}{R})) \xi^\alpha \sigma(\xi) \hat{\psi}(\xi) \right)| d\xi \\ & \leq C \int_{|\xi| > R} \frac{d\xi}{|\xi|^{d+2}} = C' \frac{1}{R^2} \end{aligned}$$

(as  $|\partial^\beta(\hat{\psi}(\xi))| \leq C_{\beta,\psi} |\xi|^{-|\beta|}$ ,  $|\partial^\beta(\xi^\alpha \sigma(\xi))| \leq C_{\alpha,\beta,\sigma} |\xi|^{|\alpha|-|\beta|}$  and  $|\partial^\beta(1 - \phi(\frac{\xi}{R}))| \leq C_{\beta,\phi} |\xi|^{-|\beta|}$ ). Taking  $R = \frac{1}{|x|}$ , we get  $x_k^{d+|\alpha|} \partial^\alpha \sigma(D) \psi \in L^\infty$ .  $\square$

A useful result on the heat kernel is its maximal regularity in  $L_t^p L_x^q$  or in  $L_x^q L_t^p$ :

**Proposition 5.** *Let  $1 < p, q < +\infty$ . Let  $\sigma(D)$  be a Fourier multiplier (in the space variable) such that  $\sigma(\xi) = \sigma_0(\frac{\xi}{|\xi|})$  where  $\sigma_0$  is a smooth function on the sphere  $\mathcal{S}^{d-1}$ .*

a) *If  $h \in L_t^p L_x^q$ , then, for  $1 \leq i, j \leq d$ ,*

$$\left\| \int_{-\infty}^t e^{(t-s)\Delta} \sigma(D) \partial_i \partial_j h ds \right\|_{L_t^p L_x^q} \leq C \|h\|_{L_t^p L_x^q}.$$

a) *If  $h \in L_x^q L_t^p$ , then, for  $1 \leq i, j \leq d$ ,*

$$\left\| \int_{-\infty}^t e^{(t-s)\Delta} \sigma(D) \partial_i \partial_j h ds \right\|_{L_x^q L_t^p} \leq C \|h\|_{L_x^q L_t^p}.$$

*Proof.* This is a classical result, obtained through the theory of vector valued singular integrals [Bene62]. We sketch the proof given in [Lema02]. Let  $W(x) = \frac{1}{(4\pi)^{d/2}} e^{-\frac{|x|^2}{4}}$  and let

$$\Omega(t, x) = \frac{1}{t^{\frac{d+2}{2}}} (\sigma(D) \partial_i \partial_j W) \left( \frac{x}{\sqrt{t}} \right) \text{ for } t > 0, \quad = 0 \text{ for } t < 0.$$

We then consider the operator

$$T(g)(t, x) = \int_{\mathbb{R}} \int_{\mathbb{R}^d} \Omega(t-s, x-y)g(s, y) ds dy. \quad (6)$$

We have

$$(\partial_t - \Delta)T(g) = \sigma(D)\partial_i\partial_j g$$

and, taking the Fourier transform,

$$\widehat{T(g)}(\tau, \xi) = -\frac{\xi_i\xi_j}{i\tau + |\xi|^2}\sigma(\xi)\hat{g}(\tau, \xi).$$

Thus,  $T$  is bounded on  $L^2(\mathbb{R} \times \mathbb{R}^d)$ .  $T$  can be seen as Calderón–Zygmund operator on the parabolic space  $\mathbb{R} \times \mathbb{R}^d$  (for the parabolic distance): from Lemma 3, we see that  $|\Omega(t, x)| \leq C \frac{1}{(\sqrt{|t|+|x|})^{d+2}}$ ,  $|\partial_i\Omega(t, x)| \leq C \frac{1}{(\sqrt{|t|+|x|})^{d+3}}$ , and  $|\partial_t\Omega(t, x)| = |\Delta\Omega(t, x)| \leq C \frac{1}{(\sqrt{|t|+|x|})^{d+4}}$ . Thus, we get that  $T$  is bounded on  $L^p(\mathbb{R} \times \mathbb{R}^d)$  for  $1 < p < +\infty$ . Let  $T_t$  be the operator

$$T_t(v) = \int_{\mathbb{R}^d} \frac{1}{t} \Omega(t, x-y)v(y) dy$$

and  $T_x$  be the operator

$$T_x(w) = \int_{\mathbb{R}} \frac{1}{t-s} \Omega(t-s, x)w(s) ds$$

We have

$$\partial_t T_t(v) = \int_{\mathbb{R}^d} \frac{1}{t} \Delta\Omega(t, x-y)v(y) dy$$

and

$$\partial_k T_x(w) = \int_{\mathbb{R}} \frac{1}{t-s} \partial_k\Omega(t-s, x)w(s) ds.$$

From Lemma 3, we find

$$\|T_t\|_{L_x^q \rightarrow L_x^q} \leq C \frac{1}{t} \text{ and } \|\partial_t T_t\|_{L_x^q \rightarrow L_x^q} \leq C \frac{1}{t^2}$$

so that the continuity  $L_t^q(L_x^q) \mapsto L_t^q(L_x^q)$  of  $T$  can be extended to the continuity  $L_t^p(L_x^q) \mapsto L_t^p(L_x^q)$ . Similarly, we have

$$\|T_x\|_{L_t^p \rightarrow L_t^p} \leq C \frac{1}{|x|^d} \text{ and } \|\partial_k T_x\|_{L_t^p \rightarrow L_t^p} \leq C \frac{1}{|x|^{d+1}}$$

and the continuity  $L_x^p(L_t^p) \mapsto L_x^p(L_t^p)$  of  $T$  can be extended to the continuity  $L_x^q(L_t^p) \mapsto L_x^q(L_t^p)$ .  $\square$

Similar estimates hold for Krylov spaces:

**Proposition 6.** *Let  $1 < p, q < +\infty$  and  $0 < \beta < \frac{2}{p} + \frac{d}{q}$ . Let  $\sigma(D)$  be a Fourier multiplier (in the space variable) such that  $\sigma(\xi) = \sigma_0(\frac{\xi}{|\xi|})$  where  $\sigma_0$  is a smooth function on the sphere  $\mathcal{S}^{d-1}$ .*

a) *If  $h \in E_{p,q,\beta}$ , then, for  $1 \leq i, j \leq d$ ,*

$$\left\| \int_{-\infty}^t e^{(t-s)\Delta} \sigma(D) \partial_i \partial_j h \, ds \right\|_{E_{p,q,\beta}} \leq C \|h\|_{E_{p,q,\beta}}.$$

a) *If  $h \in F_{p,q,\beta}$ , then, for  $1 \leq i, j \leq d$ ,*

$$\left\| \int_{-\infty}^t e^{(t-s)\Delta} \sigma(D) \partial_i \partial_j h \, ds \right\|_{F_{p,q,\beta}} \leq C \|h\|_{F_{p,q,\beta}}.$$

*Proof.* We consider only the case of  $E_{p,q,\beta}$ , as the proof for  $F_{p,q,\beta}$  is similar. Let us estimate  $T(h)$  on a cylinder  $C_r(t, x)$  (where  $T$  is the operator given by (6)). We have  $T(h) = T(h_1) + T(h_2)$ , where  $h_1 = \mathbf{1}_{C_{4r}(t,x)} h$  and  $h_2 = h - h_1$ . By Proposition 5, we know that

$$\|h_1\|_{L_t^p L_x^q} \leq C \|h_1\|_{L_t^p L_x^q} \leq C' \|h\|_{E_{p,q,\beta}} r^{\frac{2}{p} + \frac{d}{q} - \beta}.$$

On the other hand, for  $(s, z) \in C_r(t, x) \subset B_{2r}(t, x)$ , since  $B_{4r}(t, x) \subset C_{4r}(t, x)$ ,

$$\begin{aligned} |h_2(s, z)| &\leq C \iint_{\rho((\sigma,y),(t,x)) > 4r} \frac{1}{(\sqrt{|\tau-s|} + |z-y|)^{d+2}} |h(\tau, y)| \, d\tau \, dy \\ &\leq C' \iint_{\rho((\sigma,y),(t,x)) > 4r} \frac{1}{(\sqrt{|\tau-t|} + |x-y|)^{d+2}} |h(\tau, y)| \, d\tau \, dy \\ &\leq C'' \sum_{j=0}^{+\infty} \frac{1}{(4^j r)^{d+2}} \iint_{C_{4^{j+1}r}} |h(\tau, y)| \, d\tau \, dy \\ &\leq C''' \|h\|_{E_{p,q,\beta}} r^{-\beta} \end{aligned}$$

so that

$$\|\mathbf{1}_{C_r(t,x)} h_2\|_{L_t^p L_x^q} \leq C \|h\|_{E_{p,q,\beta}} r^{\frac{2}{p} + \frac{d}{q} - \beta}. \quad \square$$

We may now easily prove Theorems 3 and 4, Corollary 1 and Proposition 1. Again, we shall consider only the case of  $E_{p,q,\beta}$ , as the proofs for  $F_{p,q,\beta}$  are similar.

### Proof of Theorem 3

*Proof.* The solution  $u$  of equation  $\partial_t u = \Delta u + f$  is given by  $u = \int_{-\infty}^t e^{(t-s)\Delta} f(s, \cdot) ds$ . By proposition (6), we already know that

$$\|D_x^2 u\|_{E_{p/3, q/3, \beta+2}} \leq C \|f\|_{E_{p/3, q/3, \beta+2}}.$$

On the other hand, from Lemma 3, we see that

$$|D_x u(t, x)| \leq C \iint \frac{1}{(\sqrt{|t-s|} + |x-y|)^{d+1}} |f(s, y)| ds dy = C\mathcal{I}_1(|f|)(t, x)$$

and

$$|u(t, x)| \leq C \iint \frac{1}{(\sqrt{|t-s|} + |x-y|)^d} |f(s, y)| ds dy = C\mathcal{I}_2(|f|)(t, x).$$

We then apply Proposition 4 to get

$$\|D_x u\|_{E_{p/2, q/2, \beta+1}} \leq C \|f\|_{E_{p/3, q/3, \beta+2}}$$

and

$$\|u\|_{E_{p, q, \beta}} \leq C \|f\|_{E_{p/3, q/3, \beta+2}}. \quad \square$$

### Proof of Theorem 4

*Proof.* The solution  $u$  of equation  $\partial_t u = \Delta u + \sigma(D) \operatorname{div} \mathbb{F}$  is given by  $u = \int_{-\infty}^t e^{(t-s)\Delta} \sigma(D) \operatorname{div} \mathbb{F}(s, \cdot) ds$ . By proposition (6), we already know that

$$\|D_x u\|_{E_{p/2, q/2, \beta+1}} \leq C_0 \|\mathbb{F}\|_{E_{p/2, q/2, \beta+1}}.$$

On the other hand, from Lemma 3, we see that

$$|u(t, x)| \leq C \iint \frac{1}{(\sqrt{|t-s|} + |x-y|)^{d+1}} |\mathbb{F}(s, y)| ds dy = C\mathcal{I}_1(|\mathbb{F}|)(t, x).$$

We then apply Proposition 4 to get

$$\|u\|_{E_{p, q, \beta}} \leq C \|\mathbb{F}\|_{E_{p/2, q/2, \beta+1}}. \quad \square$$

### Proof of Corollary 1

*Proof.* We just write  $\partial_t u - \Delta u = f$  with  $f = \partial_t u - \Delta u$ . By Theorem 3, we have

$$\|u\|_{E_{p, q, \beta}} + \|D_x u\|_{E_{p/2, q/2, \beta+1}} + \|D_x^2 u\|_{E_{p/3, q/3, \beta+2}} \leq C_0 (\|\partial_t u\|_{E_{p/3, q/3, \beta+2}} + \|\Delta u\|_{E_{p/3, q/3, \beta+2}})$$

□

## Proof of Proposition 1

*Proof.* Let  $X = \{u \in E_{p,q,\beta} / D_x u \in E_{p/2,q/2,\beta+1}, D_x^2 u \in E_{p/3,q/3,\beta+2}\}$ , normed with

$$\|u\|_X = \|u\|_{E_{p,q,\beta}} + \|D_x u\|_{E_{p/2,q/2,\beta+1}} + \|D_x^2 u\|_{E_{p/3,q/3,\beta+2}}.$$

We are looking for a solution in  $X$  of the equation

$$u = T(f) + T(\vec{b} \cdot \vec{\nabla} u) + T(cu)$$

, where  $T(f) = \int_{-\infty}^t e^{(t-s)\Delta} f(s, \cdot) ds$ . By Theorem 3, we have

$$\|T(f)\|_X \leq C_1 \|f\|_{E_{p/3,q/3,\beta+2}},$$

$$\|T(\vec{b} \cdot \vec{\nabla} u)\|_X \leq C_2 \|\vec{b} \cdot \vec{\nabla} u\|_{E_{p/3,q/3,\beta+2}} \leq C_2 \|\vec{b}\|_{E_{p,q,1}} \|D_x u\|_{E_{p/2,q/2,\beta+1}},$$

and

$$\|T(cu)\|_X \leq C_3 \|cu\|_{E_{p/3,q/3,\beta+2}} \leq C_3 \|c\|_{E_{p/2,q/2,2}} \|u\|_{p,q,\beta}.$$

For  $C_2 \|\vec{b}\|_{E_{p,q,1}} + C_3 \|c\|_{E_{p/2,q/2,2}} < \frac{1}{2}$ , the operator  $S(u) = u - T(\vec{b} \cdot \vec{\nabla} u) - T(cu)$  is an isomorphism of  $X$ , with  $\|S^{-1}\|_{X \rightarrow X} \leq 2$ . The solution  $u$  is then given by  $u = S^{-1}(T(f))$ , with  $\|u\|_X \leq 2C_1 \|f\|_{E_{p/3,q/3,\beta+2}}$ .  $\square$

## 6 The Navier–Stokes equations.

We prove here Theorem 1. The proof of Theorem 2 is similar. Again, we consider only the case of  $E_{p,q,\beta}$ .

### Proof of Theorem 1

*Proof.* The Navier–Stokes equations we study are

$$\begin{cases} \partial_t \vec{u} = \Delta \vec{u} - \vec{\nabla} p - \vec{u} \cdot \vec{\nabla} \vec{u} + \vec{f} + \operatorname{div} \mathbb{F} \\ \operatorname{div} \vec{u} = 0 \\ \vec{u}(0, \cdot) = 0 \end{cases} \quad (7)$$

where  $\vec{f} \in E_{p/3,q/3,3}$  with  $\operatorname{div} \vec{f} = 0$  and  $\mathbb{F} \in E_{p/2,q/2,2}$ . We rewrite (7) as

$$\partial_t \vec{u} - \Delta \vec{u} = \vec{f} + (\operatorname{Id} - \frac{1}{\Delta} \vec{\nabla} \operatorname{div}) \operatorname{div}(\mathbb{F} - \vec{u} \otimes \vec{u}).$$

( $\operatorname{Id} - \frac{1}{\Delta} \vec{\nabla} \operatorname{div} = \mathbb{P}$  is the Leray projection operator on solenoidal vector fields.)

Let  $X = \{\vec{u} \in E_{p,q,1} / D_x \vec{u} \in E_{p/2,q/2,2}\}$ , normed with

$$\|u\|_X = \|\vec{u}\|_{E_{p,q,1}} + \|D_x \vec{u}\|_{E_{p/2,q/2,2}}.$$

We are looking for a solution in  $X$  of the equation

$$\vec{u} = \vec{T}(\vec{f}) + \vec{T}(\mathbb{P} \operatorname{div} \mathbb{F}) - \vec{T}(\mathbb{P} \operatorname{div}(\vec{u} \otimes \vec{u})),$$

where  $\vec{T}(\vec{f}) = \int_0^t e^{(t-s)\Delta} \vec{f}(s, \cdot) ds$  ( $= \int_{-\infty}^t e^{(t-s)\Delta} \vec{f}(s, \cdot) ds$  if we extend  $\vec{f}$  to  $(-\infty, 0)$  with  $\vec{f}(t, x) = 0$  for  $t < 0$ ). By Theorem 3, we have

$$\|\vec{T}(\vec{f})\|_X \leq C_1 \|\vec{f}\|_{E_{p/3,q/3,3}}.$$

By Theorem 4, we have

$$\|\vec{T}(\mathbb{P} \operatorname{div} \mathbb{F})\|_X \leq C_2 \|\mathbb{F}\|_{E_{p/2,q/2,2}}$$

and

$$\|\vec{T}(\mathbb{P} \operatorname{div}(\vec{u} \otimes \vec{u}))\|_X \leq C_3 \|\vec{u} \otimes \vec{u}\|_{E_{p/2,q/2,2}} \leq C_3 \|\vec{u}\|_{p,q,1}^2.$$

For  $C_1 \|\vec{f}\|_{E_{p/3,q/3,3}} + C_2 \|\mathbb{F}\|_{E_{p/2,q/2,2}} < \frac{1}{4C_3}$ , we find a solution  $\vec{u} \in X$  with  $\|\vec{u}\|_X \leq 2(C_1 \|\vec{f}\|_{E_{p/3,q/3,3}} + C_2 \|\mathbb{F}\|_{E_{p/2,q/2,2}})$ .  $\square$

## Appendix 1: Parabolic Besov spaces

In this appendix, we describe some results on the heat equation on parabolic Besov spaces that may shed a new light on many of the inequalities we obtained throughout the paper, which are to be viewed more as regularity assertions than as results on existence of solutions.

For  $\delta > 0$ , let us define the parabolic Besov space  $\mathcal{B}_{[2],\infty,\infty}^{-\delta}$  as the space of tempered distributions on  $\mathbb{R} \times \mathbb{R}^d$  such that

$$\|u\|_{\mathcal{B}_{[2],\infty,\infty}^{-\delta}} = \sup_{\theta > 0} \theta^{\delta/2} \|e^{\theta^2(\partial_t^2 - \Delta^2)} u(t, x)\|_{L^\infty(\mathbb{R} \times \mathbb{R}^d)} < +\infty.$$

(In particular,  $E_{p,q,\beta} \subset \mathcal{B}_{[2],\infty,\infty}^{-\beta}$  and  $F_{p,q,\beta} \subset \mathcal{B}_{[2],\infty,\infty}^{-\beta}$ .) [The index [2] indicates the underlying time-space scaling, where  $t$  behaves like  $|x|^2$ .]

It is easy to check that  $\mathcal{B}_{[2],\infty,\infty}^{-\beta}$  coincides with the (realization of the) homogeneous Besov space defined through the parabolic Littlewood-Paley decomposition (for instance, see Triebel's book [Trie06]). Let  $\phi(\tau, \xi)$  be a smooth function equal to 1 when  $\sqrt{|\tau|} + |\xi| < 1$  and to 0 when  $\sqrt{|\tau|} + |\xi| > 2$ . Let  $S_j f$  be defined as the inverse Fourier transform of  $\phi(\frac{\tau}{4^j}, \frac{\xi}{2^j}) \hat{f}(\tau, \xi)$ . Then  $\mathcal{B}_{[2],\infty,\infty}^{-\beta}$  is the Banach space of tempered distributions  $f$  such that  $\sup_{j \in \mathbb{Z}} 2^{j\delta} \|S_j f\|_\infty < +\infty$ .

We have the following Gagliardo–Nirenberg inequality



**Proposition 7.** Let  $\delta > 0$ ,  $1 < p, q < +\infty$  and  $\beta \in (0, \frac{2}{p} + \frac{d}{q})$ .

a) If  $u \in \mathcal{B}_{[2],\infty,\infty}^{-\delta}$  and  $\partial_t u, \Delta u \in E_{p,q,\beta}$ , then

$$\|D_x u\|_{E_{\frac{2+\delta}{1+\delta}p, \frac{2+\delta}{1+\delta}q, \frac{1+\delta}{2+\delta}\beta}} \leq C \|u\|_{\mathcal{B}_{[2],\infty,\infty}^{-\delta}}^{1+\delta} (\|\partial_t u\|_{E_{p,q,\beta}} + \|\Delta u\|_{E_{p,q,\beta}})^{\frac{1+\delta}{2+\delta}}.$$

b) If  $u \in \mathcal{B}_{[2],\infty,\infty}^{-\delta}$  and  $\partial_t u, \Delta u \in F_{p,q,\beta}$ , then

$$\|D_x u\|_{F_{\frac{2+\delta}{1+\delta}p, \frac{2+\delta}{1+\delta}q, \frac{1+\delta}{2+\delta}\beta}} \leq C \|u\|_{\mathcal{B}_{[2],\infty,\infty}^{-\delta}}^{1+\delta} (\|\partial_t u\|_{F_{p,q,\beta}} + \|\Delta u\|_{F_{p,q,\beta}})^{\frac{1+\delta}{2+\delta}}.$$

Our final results extend Corollary 1 and Theorem 3 to parabolic Besov spaces:

**Theorem 5.** Let  $\delta > 0$ . Let  $u = 0$  at infinity. Then  $u \in \mathcal{B}_{[2],\infty,\infty}^{-\delta}$  if and only if  $\partial_t u \in \mathcal{B}_{[2],\infty,\infty}^{-\delta-2}$  and  $\Delta u \in \mathcal{B}_{[2],\infty,\infty}^{-\delta-2}$ . Moreover, the norms  $\|u\|_{\mathcal{B}_{[2],\infty,\infty}^{-\delta}}$  and  $\|\partial_t u\|_{\mathcal{B}_{[2],\infty,\infty}^{-\delta-2}} + \|\Delta u\|_{\mathcal{B}_{[2],\infty,\infty}^{-\delta-2}}$  are equivalent.

**Theorem 6.** Let  $\delta > 2$ . If  $f \in \mathcal{B}_{[2],\infty,\infty}^{-\delta}$ , then the heat equation

$$\partial_t u = \Delta u + f \tag{8}$$

has a unique solution such that  $u = 0$  at infinity. Moreover,  $u \in \mathcal{B}_{[2],\infty,\infty}^{-\delta+2}$  and

$$\|u\|_{\mathcal{B}_{[2],\infty,\infty}^{-\delta+2}} \leq C \|f\|_{\mathcal{B}_{[2],\infty,\infty}^{-\delta}}.$$

Those results are not very new, as they can be recovered from the voluminous literature on anisotropic Besov spaces. We present them for sake of completeness.

## Proof of Proposition 7.

*Proof.* As  $u = 0$  at infinity, we may write

$$u = \int_0^\infty (-\partial_\theta)(e^{\theta^2(\partial_t^2 - \Delta^2)}u) d\theta$$

and

$$D_x u = \int_0^\infty (-\partial_\theta)D_x(e^{\theta^2(\partial_t^2 - \Delta^2)}u) d\theta.$$

For  $R > 0$ , we rewrite this as

$$\begin{aligned}
D_x u &= - \int_0^R (2\theta \partial_t e^{\theta^2 \partial_t^2} \sqrt{\theta} D_x e^{-\theta^2 \Delta^2}) \partial_t u \frac{d\theta}{\sqrt{\theta}} \\
&\quad - \int_R^\infty (2\theta^2 \partial_t^2 e^{\frac{3}{4}\theta^2 \partial_t^2} \sqrt{\theta} D_x e^{-\frac{3}{4}\theta^2 \Delta^2}) (e^{\frac{\theta^2}{4}(\partial_t^2 - \Delta^2)} u) \frac{d\theta}{\theta^{3/2}} \\
&\quad + \int_0^R (2e^{\theta^2 \partial_t^2} \theta^{3/2} D_x \Delta e^{-\theta^2 \Delta^2}) \Delta u \frac{d\theta}{\sqrt{\theta}} \\
&\quad + \int_R^\infty (2e^{\frac{3}{4}\theta^2 \partial_t^2} \theta^{5/2} D_x \Delta^2 e^{-\frac{3}{4}\theta^2 \Delta^2}) (e^{\frac{\theta^2}{4}(\partial_t^2 - \Delta^2)} u) \frac{d\theta}{\theta^{3/2}} \\
&= I + II + III + IV.
\end{aligned}$$

Recall that  $\mathcal{M}_f$  designs the parabolic Hardy–Littlewood maximal function. If  $\psi \in \mathcal{S}(\mathbb{R} \times \mathbb{R}^d)$ , we have, for every  $\theta > 0$ ,

$$\iint \frac{1}{\theta^{\frac{d+2}{2}}} |\psi(\frac{s}{\theta}, \frac{y}{\sqrt{\theta}})| |f(t-s, x-y)| ds dy \leq C_\psi \mathcal{M}_f(t, x)$$

and

$$\iint \frac{1}{\theta^{\frac{d+2}{2}}} |\psi(\frac{s}{\theta}, \frac{y}{\sqrt{\theta}})| |f(t-s, x-y)| ds dy \leq \|\psi\|_1 \|f\|_\infty.$$

Thus, we have

$$|I| \leq C\sqrt{R} \mathcal{M}_{\partial_t u} \text{ and } |III| \leq C\sqrt{R} \mathcal{M}_{\Delta u}$$

and

$$|II| + |IV| \leq CR^{-\frac{1+\delta}{2}} \|u\|_{\mathcal{B}_{[2], \infty, \infty}^{-\delta}}.$$

Taking  $\sqrt{R} = \left( \frac{\|u\|_{\mathcal{B}_{[2], \infty, \infty}^{-\delta}}}{\mathcal{M}_{\partial_t u}(t, x) + \mathcal{M}_{\Delta u}(t, x)} \right)^{\frac{1}{2+\delta}}$ , we get

$$|D_x u(t, x)| \leq C \|u\|_{\mathcal{B}_{[2], \infty, \infty}^{-\delta}}^{\frac{1}{2+\delta}} (\mathcal{M}_{\partial_t u}(t, x) + \mathcal{M}_{\Delta u}(t, x))^{\frac{1+\delta}{2+\delta}}$$

and we conclude by Proposition 4 a). □

## Proof of Theorem 5.

*Proof.* Writing

$$e^{\theta^2(\partial_t^2 - \Delta^2)}(\partial_t u) = \frac{1}{\theta} (\theta \partial_t e^{\frac{3}{4}\theta^2(\partial_t^2 - \Delta^2)}) (e^{\frac{1}{4}\theta^2(\partial_t^2 - \Delta^2)} u)$$

and

$$e^{\theta^2(\partial_t^2 - \Delta^2)}(\Delta u) = \frac{1}{\theta}(\theta \Delta e^{\frac{3}{4}\theta^2(\partial_t^2 - \Delta^2)})(e^{\frac{1}{4}\theta^2(\partial_t^2 - \Delta^2)}u),$$

we find that

$$\|\partial_t u\|_{\mathcal{B}_{[2],\infty,\infty}^{-\delta-2}} + \|\Delta u\|_{\mathcal{B}_{[2],\infty,\infty}^{-\delta-2}} \leq C\|u\|_{\mathcal{B}_{[2],\infty,\infty}^{-\delta}}.$$

As  $u = 0$  at infinity, we write

$$u = \int_0^\infty (-\partial_\eta)(e^{\eta^2(\partial_t^2 - \Delta^2)}u) d\eta = 2 \int_0^{+\infty} \eta(\Delta^2 - \partial_t^2)e^{\eta^2(\partial_t^2 - \Delta^2)}u d\eta.$$

We get

$$\begin{aligned} e^{\theta^2(\partial_t^2 - \Delta^2)}u &= 2 \int_0^{+\infty} (\eta \Delta e^{\frac{\eta^2}{2}(\partial_t^2 - \Delta^2)})(e^{(\theta^2 + \frac{\eta^2}{2})(\partial_t^2 - \Delta^2)}\Delta u) d\eta \\ &\quad - 2 \int_0^{+\infty} (\eta \partial_t e^{\frac{\eta^2}{2}(\partial_t^2 - \Delta^2)})(e^{(\theta^2 + \frac{\eta^2}{2})(\partial_t^2 - \Delta^2)}\partial_t u) d\eta, \end{aligned}$$

so that

$$\begin{aligned} \|e^{\theta^2(\partial_t^2 - \Delta^2)}u\|_\infty &\leq C \int_0^{+\infty} \|e^{(\theta^2 + \frac{\eta^2}{2})(\partial_t^2 - \Delta^2)}\Delta u\|_\infty + \|e^{(\theta^2 + \frac{\eta^2}{2})(\partial_t^2 - \Delta^2)}\partial_t u\|_\infty d\eta \\ &\leq C(\|\partial_t u\|_{\mathcal{B}_{[2],\infty,\infty}^{-\delta-2}} + \|\Delta u\|_{\mathcal{B}_{[2],\infty,\infty}^{-\delta-2}}) \int_0^{+\infty} (\theta^2 + \frac{\eta^2}{2})^{-\frac{2+\delta}{4}} d\eta \\ &= C\theta^{-\delta/2}(\|\partial_t u\|_{\mathcal{B}_{[2],\infty,\infty}^{-\delta-2}} + \|\Delta u\|_{\mathcal{B}_{[2],\infty,\infty}^{-\delta-2}}) \int_0^{+\infty} (1 + \frac{\eta^2}{2})^{-\frac{2+\delta}{4}} d\eta \end{aligned}$$

and finally

$$\|u\|_{\mathcal{B}_{[2],\infty,\infty}^{-\delta}} \leq C(\|\partial_t u\|_{\mathcal{B}_{[2],\infty,\infty}^{-\delta-2}} + \|\Delta u\|_{\mathcal{B}_{[2],\infty,\infty}^{-\delta-2}}). \quad \square$$

## Proof of Theorem 6.

*Proof.* We write

$$f = \int_0^\infty (-\partial_\eta)(e^{\eta^2(\partial_t^2 - \Delta^2)}f) d\eta = -2(\partial_t - \Delta) \int_0^{+\infty} \eta(\Delta + \partial_t)e^{\eta^2(\partial_t^2 - \Delta^2)}f d\eta.$$

Let

$$u = -2 \int_0^{+\infty} \eta(\Delta + \partial_t)e^{\eta^2(\partial_t^2 - \Delta^2)}f d\eta.$$

We have

$$\begin{aligned}
\|e^{\theta^2(\partial_t^2 - \Delta^2)}u\|_\infty &\leq 2 \int_0^{+\infty} \|(\eta\Delta e^{\frac{\eta^2}{2}(\partial_t^2 - \Delta^2)})(e^{(\theta^2 + \frac{\eta^2}{2})(\partial_t^2 - \Delta^2)}f)\|_\infty d\eta \\
&\quad + 2 \int_0^{+\infty} \|(\eta\partial_t e^{\frac{\eta^2}{2}(\partial_t^2 - \Delta^2)})(e^{(\theta^2 + \frac{\eta^2}{2})(\partial_t^2 - \Delta^2)}f)\|_\infty d\eta, \\
&\leq C \int_0^{+\infty} \|e^{(\theta^2 + \frac{\eta^2}{2})(\partial_t^2 - \Delta^2)}f\|_\infty d\eta \\
&\leq C \|f\|_{\mathcal{B}_{[2],\infty,\infty}^{-\delta-2}} \int_0^{+\infty} \left(\theta^2 + \frac{\eta^2}{2}\right)^{-\frac{2+\delta}{4}} d\eta \\
&= C\theta^{-\delta/2} \|f\|_{\mathcal{B}_{[2],\infty,\infty}^{-\delta-2}} \int_0^{+\infty} \left(1 + \frac{\eta^2}{2}\right)^{-\frac{2+\delta}{4}} d\eta
\end{aligned}$$

and finally

$$\|u\|_{\mathcal{B}_{[2],\infty,\infty}^{-\delta}} \leq C \|f\|_{\mathcal{B}_{[2],\infty,\infty}^{-\delta-2}}. \quad \square$$

## Appendix 2: A remark on the regularity of mild solutions

The reviewer asked the following questions about the regularity of the solution  $u$  constructed in Theorem 1: can one expect the constructed solution to be in  $L^1_{loc}(\mathbb{R}, C^{0,1})$ ? The answer is clearly negative, as the force is not regular enough to grant such a regularity. In this appendix, we give an example of a not so regular solution.

We start from a well-known fact: the map  $\mathbb{F} \mapsto \int_0^t e^{(t-s)\Delta} \mathbb{P} \operatorname{div} \mathbb{F} ds$  maps  $L^\infty((0, +\infty), L^{p_0, \infty}(\mathbb{R}^3))$  to  $L^\infty((0, +\infty), L^{p_1, \infty}(\mathbb{R}^3))$  for  $1 < p_0 < 3$  and  $\frac{1}{p_1} = \frac{1}{p_0} - \frac{1}{3}$  (see for instance [Lema02]). Thus, if  $\|\mathbb{F}\|_{L^\infty L^{3/2, \infty}}$  is small enough, the Navier–Stokes problem

$$\partial_t \vec{u} = \Delta \vec{u} + \mathbb{P} \operatorname{div}(\mathbb{F} - \vec{u} \otimes \vec{u})$$

will have a solution in  $L^\infty((0, +\infty), L^{3, \infty}(\mathbb{R}^3))$ . Similarly, if  $3/2 < r < 2$  and  $\|\mathbb{F}\|_{L^\infty L^{3/2, \infty}} + \|\mathbb{F}\|_{L^\infty L^{r, \infty}}$  is small enough, the Navier–Stokes problem will have a solution  $\vec{u}$  in  $L^\infty((0, +\infty), L^{3, \infty}(\mathbb{R}^3)) \cap L^\infty((0, +\infty), L^{\frac{3r}{3-r}, \infty}(\mathbb{R}^3))$ . Moreover, we have

$$\vec{u} - \int_0^t e^{(t-s)\Delta} \mathbb{P} \operatorname{div} \mathbb{F} ds = - \int_0^t e^{(t-s)\Delta} \mathbb{P} \operatorname{div}(\vec{u} \otimes \vec{u}) ds$$

and we find that  $\vec{u} - \int_0^t e^{(t-s)\Delta} \mathbb{P} \operatorname{div} \mathbb{F} ds$  belongs to  $L^\infty((0, +\infty), L^{\frac{r}{2-r}, \infty}(\mathbb{R}^3))$  (with  $\frac{r}{2-r} > \frac{3r}{3-r}$ ).

We may remark that, if  $\mathbb{F} \in L^\infty L^{3/2, \infty} \cap L^\infty L^{r, \infty}$  and  $\mathbb{F}$  is supported in  $[0, 1] \times \mathbb{R}^3$ , then  $\mathbb{F} \in E_{p/2, q/2, 2}$  for  $p > 2$ ,  $3 < q < 2r$  and  $\frac{2}{p} + \frac{3}{q} > 1$ . Indeed, we have  $\mathbb{F} \in L_t^\infty L^{q/2} \cap L_t^{p/2} L_x^{q/2}$ , so that

$$\rho^{2-2\frac{2}{p}-3\frac{2}{q}} \left( \int_{|t-s|<\rho^2} \left( \int_{|x-y|<\rho} |\mathbb{F}(s, y)|^{q/2} dy \right)^{\frac{p}{q}} ds \right)^{\frac{2}{p}} \leq C \rho^{2(1-\frac{3}{q})} \|\mathbb{F}\|_{L^\infty L^{q/2}}$$

(which is bounded for  $0 < \rho \leq 1$ ) and

$$\rho^{2-2\frac{2}{p}-3\frac{2}{q}} \left( \int_{|t-s|<\rho^2} \left( \int_{|x-y|<\rho} |\mathbb{F}(s, y)|^{q/2} dy \right)^{\frac{p}{q}} ds \right)^{\frac{2}{p}} \leq C \rho^{2(1-\frac{2}{p}-\frac{3}{q})} \|\mathbb{F}\|_{L^{p/2} L^{q/2}}$$

(which is bounded for  $1 < \rho$ ).

We now give an example of a mild solution such that  $\vec{u}$  is not (locally in time)  $L^1 C^{0,1}$ . Let  $\varphi \in \mathcal{D}(\mathbb{R}^3)$  be equal to 1 on a neighborhood of 0 and with support in  $B(0, 1)$ , let  $0 < \epsilon < 1/2$  and let  $\vec{f}(t, x) = \lambda \mathbf{1}_{(0,1)}(t) \Delta \vec{\nabla} \wedge (|x|^\epsilon \varphi)$ , with  $\lambda > 0$  small enough. We have  $\operatorname{div} \vec{f} = 0$ . We write  $\vec{f}$  in the divergence form

$$\vec{f} = \sum_{j=1}^3 \partial_j (\lambda \mathbf{1}_{(0,1)}(t) \partial_j \vec{\nabla} \wedge (|x|^\epsilon \varphi)) = \operatorname{div} \mathbb{F}.$$

$\mathbb{F}$  is supported in  $[0, 1] \times \mathbb{R}^3$ ; moreover,  $\mathbb{F} \in L^\infty L^{\rho, \infty}$  for every  $\rho \in (1, \frac{3}{2-\epsilon}]$ , hence  $\mathbb{F} \in L^\infty L^{3/2, \infty} \cap L^\infty L^{\frac{3}{2-\epsilon}, \infty}$ . If  $\lambda$  is small enough, the Navier–Stokes problem have a solution  $\vec{u}$  such that

$$\vec{u} \in L^\infty((0, +\infty), L^{3, \infty}(\mathbb{R}^3)) \cap L^\infty((0, +\infty), L^{\frac{3}{1-\epsilon}, \infty}(\mathbb{R}^3))$$

and

$$\vec{u} - \int_0^t e^{(t-s)\Delta} \mathbb{P} \operatorname{div} \mathbb{F} ds \in L^\infty((0, +\infty), L^{\frac{3}{1-2\epsilon}, \infty}(\mathbb{R}^3)).$$

For  $0 < t < 1$ , we have

$$\int_0^t e^{(t-s)\Delta} \mathbb{P} \operatorname{div} \mathbb{F} ds = \lambda (\vec{\nabla} \wedge (|x|^\epsilon \varphi) - e^{t\Delta} (\vec{\nabla} \wedge (|x|^\epsilon \varphi))).$$

If locally  $\vec{u}$  were  $L^1 C^{0,1}$ , then  $\mathbf{1}_{(1/2, 1)}(t) \mathbf{1}_{B(0,1)}(x) \vec{u}$  would be  $L^1 L^{\frac{3}{1-2\epsilon}, \infty}$ . This would give that  $\vec{\nabla} \wedge (|x|^\epsilon \varphi) \in L^{\frac{3}{1-2\epsilon}, \infty}$ . But  $\vec{\nabla} \wedge (|x|^\epsilon \varphi) \in L^{r, \infty}$  if and only if  $r \leq \frac{3}{1-\epsilon}$ .

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