# The Navier-Stokes equations in mixed-norm time-space parabolic Morrey spaces

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**Abstract**

We discuss the Navier-Stokes equations with forces in the mixed-norm time-space parabolic Morrey spaces of Krylov.

**Keywords :** Navier–Stokes equations, heat equation, parabolic Morrey spaces, mild solutions.

**AMS classification :** 35K55, 35Q30, 76D05.

## 1 Introduction

In this paper, we consider global mild solutions of the Cauchy problem for the incompressible Navier–Stokes equations on the whole space $\mathbb{R}^3$. More precisely, the Navier–Stokes equations we study are

$$\begin{cases} \partial_t \vec{u} = \Delta \vec{u} - \vec{\nabla} p - \vec{u} \cdot \vec{\nabla} \vec{u} + \vec{f} + \operatorname{div} \mathbb{F} \\ \operatorname{div} \vec{u} = 0 \\ \vec{u}(0,.) = 0 \end{cases} \tag{1}$$

where $\vec{f}$ and $\mathbb{F}$ are small enough in some critical (homogeneous) mixed-norm time-space parabolic Morrey spaces of Krylov.

(Following Krylov [Kryl23]), let us recall the definition of mixed-norm time-space parabolic Morrey spaces on $\mathbb{R} \times \mathbb{R}^d$:

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**Definition 1.** *Let $1 < p, q < +\infty$ and $\beta \in (0, \frac{2}{p} + \frac{d}{q})$. The parabolic Morrey spaces $E_{p,q,\beta}$ and $F_{p,q,\beta}$ are the spaces of locally integrable functions $f(t,x)$ on $\mathbb{R} \times \mathbb{R}^d$ such that $\|f\|_{E_{p,q,\beta}} < +\infty$ or $\|f\|_{F_{p,q,\beta}} < +\infty$, where*

$$\|f\|_{E_{p,q,\beta}} = \sup_{\rho>0, t\in\mathbb{R}, x\in\mathbb{R}^d} \rho^{\beta-\frac{2}{p}-\frac{d}{q}} \left( \int_{|t-s|<\rho^2} \left( \int_{|x-y|<\rho} |f(s,y)|^q \, dy \right)^{\frac{p}{q}} ds \right)^{\frac{1}{p}}$$

*and*

$$\|f\|_{F_{p,q,\beta}} = \sup_{\rho>0, t\in\mathbb{R}, x\in\mathbb{R}^d} \rho^{\beta-\frac{2}{p}-\frac{d}{q}} \left( \int_{|x-y|<\rho} \left( \int_{|t-s|<\rho^2} |f(s,y)|^p \, ds \right)^{\frac{q}{p}} dy \right)^{\frac{1}{q}}.$$

Remark: when $p = q$, writing $r = \frac{d+2}{\beta}$ (so that $p < r < +\infty$), we see that $E_{p,p,\beta} = F_{p,p,\beta} = \mathcal{M}^{p,r}_2(\mathbb{R} \times \mathbb{R}^d)$, where $\mathcal{M}^{p,r}_2(\mathbb{R} \times \mathbb{R}^d)$ is the parabolic Morrey space studied in [Lema16] in the context of Navier–Stokes equations.

Krylov [Kryl23] developed a theory of Itô stochastic equations with as singular a drift as possible. Following an observation of Kinzebulatov [Kinz23], he considered a drift in parabolic Morrey spaces. We will show that his results can be easily applied to the Navier—Stokes problem. More precisely, we consider the initial value problem (on $(0, +\infty) \times \mathbb{R}^3$) with initial value $u(0, .) = 0$, that we transform as a problem on $\mathbb{R} \times \mathbb{R}^3$ by extending the Navier-Stokes solution by $\vec{u}(t, .) = 0$ for $t < 0$.

For a function defined on $(0, +\infty) \times \mathbb{R}^d$, we say that $f \in E_{p,q,\beta}$ or $f \in F_{p,q,\beta}$ if the function $f^\#$ defined by $f^\# = f$ for $t > 0$ and $f^\# = 0$ for $t < 0$ is such that $f^\# \in E_{p,q,\beta}$ or $f^\# \in F_{p,q,\beta}$.

Our main theorem is the following one:

**Theorem 1.** *Let $p, q \in (3, \infty)$ with $\frac{2}{p} + \frac{3}{q} > 1$.*

*a) There exists $\epsilon_0 = \epsilon_0(p,q) > 0$ and $C_0 = C_0(p,q) > 0$ such that, if $\vec{f} \in E_{p/3,q/3,3}$ with $\operatorname{div} \vec{f} = 0$ and $\mathbb{F} \in E_{p/2,q/2,2}$ and if*

$$\|\vec{f}\|_{E_{p/3,q/3,3}} + \|\mathbb{F}\|_{E_{p/2,q/2,2}} < \epsilon_0,$$

*then the Navier–Stokes equations (1) have a global mild solution $\vec{u} \in E_{p,q,1}$ and $\vec{\nabla} \otimes \vec{u} \in E_{p/2,q/2,2}$ with*

$$\|\vec{u}\|_{E_{p,q,1}} + \|\vec{\nabla} \otimes \vec{u}\|_{E_{p/2,q/2,2}} \leq C_0(\|\vec{f}\|_{E_{p/3,q/3,3}} + \|\mathbb{F}\|_{E_{p/2,q/2,2}}).$$

*b) There exists $\epsilon_0 = \epsilon_0(p,q) > 0$ and $C_0 = C_0(p,q) > 0$ such that, if $\vec{f} \in F_{p/3,q/3,3}$ with $\operatorname{div} \vec{f} = 0$ and $\mathbb{F} \in F_{p/2,q/2,2}$ and if*

$$\|\vec{f}\|_{F_{p/3,q/3,3}} + \|\mathbb{F}\|_{F_{p/2,q/2,2}} < \epsilon_0,$$

*then the Navier–Stokes equations (1) have a global mild solution* $\vec{u} \in F_{p,q,1}$ *and* $\vec{\nabla} \otimes \vec{u} \in F_{p/2,q/2,2}$ *with*

$$\|\vec{u}\|_{F_{p,q,1}} + \|\vec{\nabla} \otimes \vec{u}\|_{F_{p/2,q/2,2}} \leq C_0(\|\vec{f}\|_{F_{p/3,q/3,3}} + \|\mathbb{F}\|_{F_{p/2,q/2,2}}).$$

When the force $\vec{f}$ is equal to 0, we may lower the values of $p$ and $q$:

**Theorem 2.** *Let* $p, q \in (2, \infty)$ *with* $\frac{2}{p} + \frac{3}{q} > 1$.
*a) There exists* $\epsilon_0 = \epsilon_0(p,q) > 0$ *and* $C_0 = C_0(p,q) > 0$ *such that, if* $\mathbb{F} \in E_{p/2,q/2,2}$ *and*

$$\|\mathbb{F}\|_{E_{p/2,q/2,2}} < \epsilon_0,$$

*then the Navier–Stokes equations (1) (with* $\vec{f} = 0$*) have a global mild solution* $\vec{u} \in E_{p,q,1}$ *and* $\vec{\nabla} \otimes \vec{u} \in E_{p/2,q/2,2}$ *with*

$$\|\vec{u}\|_{E_{p,q,1}} + \|\vec{\nabla} \otimes \vec{u}\|_{E_{p/2,q/2,2}} \leq C_0\|\mathbb{F}\|_{E_{p/2,q/2,2}}.$$

*b) There exists* $\epsilon_0 = \epsilon_0(p,q) > 0$ *and* $C_0 = C_0(p,q) > 0$ *such that, if* $\mathbb{F} \in F_{p/2,q/2,2}$ *and if*

$$\|\mathbb{F}\|_{F_{p/2,q/2,2}} < \epsilon_0,$$

*then the Navier–Stokes equations (1) (with* $\vec{f} = 0$*) have a global mild solution* $\vec{u} \in F_{p,q,1}$ *and* $\vec{\nabla} \otimes \vec{u} \in F_{p/2,q/2,2}$ *with*

$$\|\vec{u}\|_{F_{p,q,1}} + \|\vec{\nabla} \otimes \vec{u}\|_{F_{p/2,q/2,2}} \leq C_0\|\mathbb{F}\|_{F_{p/2,q/2,2}}.$$

As usual, the mild solution $\vec{u}$ will be searched as a fixed point of the transform

$$\vec{u} \mapsto \vec{T}(\vec{f}) + \vec{T}(\mathbb{P}\,\mathrm{div}\,\mathbb{F}) - \vec{T}(\mathbb{P}\,\mathrm{div}(\vec{u} \otimes \vec{u})),$$

where $\vec{T}(\vec{f}) = \int_0^t e^{(t-s)\Delta}\vec{f}(s,.)\,ds$. Let us remark that, if $X = E_{p,q,1}$ or $X = F_{p,q,1}$, then $X$ is a critical space with respect to the scaling of the Navier—Stokes equations: for $\lambda > 0$, $\|\lambda\vec{u}(\lambda^2 t, \lambda x)\|_X = \|\vec{u}\|_X$. $E_{p,q,1}$ is easily compared to other classical spaces; for instance, we have, for Serrin spaces with $\frac{2}{p} + \frac{3}{q} = 1$, $L^p_t((0,+\infty), L^q_x(\mathbb{R}^3)) \subset E_{p,q,1}$, while $E_{p,q,\beta} \subset \mathcal{M}_2^{\min(p,q),5}$ (a critical space discussed in [Lema16]).

# 2 Heat equation

The proof of theorems 1 and 2 will rely on the theory of the heat equation $\partial_t u - \Delta u = f$ on $\mathbb{R} \times \mathbb{R}^d$ with the boundary condition $u = 0$ at infinity. This condition will be defined as follows:

**Definition 2.** *Let $u$ be a tempered distribution on $\mathbb{R}\times\mathbb{R}^d$. Then $u=0$ at infinity if for every $\theta>0$ one has $e^{\theta^2(\partial_t^2-\Delta^2)}u\in L^\infty(\mathbb{R}\times\mathbb{R}^d)$ and $\lim_{\theta\to+\infty}\|e^{\theta^2(\partial_t^2-\Delta^2)}u\|_\infty=0$.*

Easy examples of $u$ with $u=0$ at infinity are $u\in L^p_tL^q_x$ with $1\le p,q\le\infty$, $(p,q)\neq(\infty,\infty)$ or $u=\partial_t v_0+\sum_{j=1}^d v_j$ where $v_j\in L^\infty(\mathbb{R}\times\mathbb{R}^d)$ for $0\le j\le d$: in the first case, we have $\|e^{\theta^2(\partial_t^2-\Delta^2)}u\|_\infty\le C\|u\|_{L^pL^q}\theta^{-(\frac1p+\frac{d}{2q})}$ and, in the second case, $\|e^{\theta^2(\partial_t^2-\Delta^2)}u\|_\infty\le C(\theta^{-1}\|v_0\|_\infty+\sum_{j=1}^d\theta^{-\frac12}\|v_j\|_\infty)$. Easy examples of bounded functions $u$ with $u\neq0$ at infinity are $u=1$, $u=\mathbb{1}_{t>0}$, or $u=\mathbb{1}_{x_d>0}$: we have $\|e^{\theta^2(\partial_t^2-\Delta^2)}u\|_\infty=1$, as $\|e^{\theta^2(\partial_t^2-\Delta^2)}u\|_\infty\le1$ and $\lim_{(t,x_d)\to(+\infty,+\infty)}e^{\theta^2(\partial_t^2-\Delta^2)}u(t,x)=1$.

**Lemma 1.** *Let $u$ be a tempered distribution on $\mathbb{R}\times\mathbb{R}^d$ such that $u=0$ at infinity. If $\partial_t u-\Delta u=0$, then $u=0$.*

*Proof.* Take the Fourier transform (in time and space variables) of the heat equation. We find

$$(i\tau+|\xi|^2)\hat u(\tau,\xi)=0.$$

Thus, $\hat u$ is supported in $\{(0,0)\}$, hence is a sum of derivatives of the Dirac mass, and $u$ is a polynomial. As $e^{\theta^2(\partial_t^2-\Delta^2)}u$ is then a polynomial, and as $e^{\theta^2(\partial_t^2-\Delta^2)}u\in L^\infty(\mathbb{R}\times\mathbb{R}^d)$, we find that $u$ is a constant; in that case, $e^{\theta^2(\partial_t^2-\Delta^2)}u=u$. As $u=0$ at infinity, we have $u=0$. □

**Lemma 2.** *Let $1<p,q<+\infty$ and $\beta\in(0,\frac2p+\frac dq)$. If $u\in E_{p,q,\beta}$ or $u\in F_{p,q,\beta}$, then $u=0$ at infinity.*

*Proof.* We give the proof for $u\in E_{p,q,\beta}$ (the case $u\in F_{p,q,\beta}$ is proved in a similar way). Let $W(t,x)$ be the inverse Fourier transform of $e^{-\tau^2-|\xi|^4}$. $W$ is in the Schwartz class of smooth functions with rapid decay. In particular, $(\sqrt{|t|}+|x|)^{d+2}|W(t,x)|\in L^\infty(\mathbb{R}\times\mathbb{R}^d)$. Writing

$$\begin{aligned}
&\iint|W(t,x)|\,|u(t,x)|\,dt\,dx\\
&=\iint_{\sqrt{|t|}+|x|<1}|W(t,x)|\;|u(t,x)|\,dt\,dx+\sum_{j=0}^{+\infty}\iint_{2^j<\sqrt{|t|}+|x|<2^{j+1}}|W(t,x)|\;|u(t,x)|\,dt\,dx\\
&\le C\|u\|_{E_{p,q,\beta}}(\|\mathbb{1}_{\sqrt{|t|}+|x|<1}W\|_{L_t^{\frac{p}{p-1}}L_x^{\frac{q}{q-1}}}+\sum_{j=0}^{+\infty}2^{j(\frac2p+\frac dq-\beta)}\|\mathbb{1}_{2^j<\sqrt{|t|}+|x|<2^{j+1}}W\|_{L_t^{\frac{p}{p-1}}L_x^{\frac{q}{q-1}}})\\
&\le C'\|u\|_{E_{p,q,\beta}}(1+\sum_{j=0}^{+\infty}2^{-j\beta})=C_1\|u\|_{E_{p,q,\beta}},
\end{aligned}$$

we get

$$\begin{aligned}|e^{\theta^2(\partial_t^2-\Delta^2)}u(t,x)| \leq & \iint W(s,y)|u(t+\theta^2 s, x+\theta y)|\, ds\, dy \\ \leq & C_1\|u(t+\theta., x+\sqrt{\theta}.)\|_{E_{p,q,\beta}} \\ = & C_1\|u(\theta., \sqrt{\theta}.)\|_{E_{p,q,\beta}} \\ = & C_1\|u\|_{E_{p,q,\beta}}\theta^{-\beta/2}.\end{aligned}$$

Thus, $\lim_{\theta\to+\infty}\|e^{\theta^2(\partial_t^2-\Delta^2)}u\|_\infty = 0$. □

The proof of Theorems 1 and 2 will be based on the following results on the linear heat equation:

**Theorem 3.** *Let $p, q \in (3,\infty)$ and $\beta > 0$ with $\frac{2}{p}+\frac{3}{q} > \beta$.*
*a) If $f \in E_{p/3,q/3,\beta+2}(\mathbb{R}\times\mathbb{R}^d)$, then the heat equation*

$$\partial_t u = \Delta u + f \tag{2}$$

*has a unique solution such that $u = 0$ at infinity. Moreover, there exists $C_0 = C_0(p,q,\beta) > 0$ such that, if denoting by $D_x u$ the gradient of $u$ and by $D_x^2 u$ the Hessian of $u$ (with respect to the space variable), we have*

$$\|u\|_{E_{p,q,\beta}}+\|D_x u\|_{E_{p/2,q/2,\beta+1}}+\|D_x^2 u\|_{E_{p/3,q/3,\beta+2}}+\|\partial_t u\|_{E_{p/3,q/3,\beta+2}} \leq C_0\|f\|_{E_{p/3,q/3,\beta+2}}.$$

*b) If $\in F_{p/3,q/3,\beta+2}(\mathbb{R}\times\mathbb{R}^d)$, then the heat equation (2) has a unique solution such that $u = 0$ at infinity. Moreover, there exists $C_0 = C_0(p,q,\beta) > 0$ such that we have*

$$\|u\|_{F_{p,q,\beta}}+\|D_x u\|_{F_{p/2,q/2,\beta+1}}+\|D_x^2 u\|_{F_{p/3,q/3,\beta+2}}+\|\partial_t u\|_{F_{p/3,q/3,\beta+2}} \leq C_0\|f\|_{F_{p/3,q/3,\beta+2}}.$$

**Theorem 4.** *Let $p, q \in (2,\infty)$ and $\beta > 0$ with $\frac{2}{p}+\frac{3}{q} > \beta$. Let $\sigma(D)$ be a Fourier multiplier (in the space variable) such that $\sigma(\xi) = \sigma_0(\frac{\xi}{|\xi|})$ where $\sigma_0$ is a smooth function on the sphere $\mathcal{S}^{d-1}$.*
*a) If $\mathbb{F} \in E_{p/2,q/2,\beta+1}(\mathbb{R}\times\mathbb{R}^d)$, then the heat equation*

$$\partial_t u = \Delta u + \sigma(D)\,\mathrm{div}\,\mathbb{F} \tag{3}$$

*has a unique solution such that $u = 0$ at infinity. Moreover, there exists $C_0 = C_0(p,q,\beta,\sigma) > 0$ such that we have*

$$\|u\|_{E_{p,q,\beta}} + \|D_x u\|_{E_{p/2,q/2,\beta+1}} \leq C_0\|\mathbb{F}\|_{E_{p/2,q/2,\beta+1}}.$$

*b) If $\mathbb{F} \in F_{p/2,q/2,\beta+1}(\mathbb{R}\times\mathbb{R}^d)$, then the heat equation (3) has a unique solution such that $u = 0$ at infinity. Moreover, there exists $C_0 = C_0(p,q,\beta,\sigma) > 0$ such that we have*

$$\|u\|_{F_{p,q,\beta}} + \|D_x u\|_{F_{p/2,q/2,\beta+1}} \leq C_0\|\mathbb{F}\|_{F_{p/2,q/2,\beta+1}}.$$

Theorem 3 has the following corollary on Sobolev–Morrey inequalities:

**Corollary 1.** *Let $p, q \in (3, \infty)$ and $\beta > 0$ with $\frac{2}{p} + \frac{3}{q} > \beta$. Let $u = 0$ at infinity. Then there exists $C_0 = C_0(p, q, \beta) > 0$ such that we have*

$$\|u\|_{E_{p,q,\beta}} + \|D_x u\|_{E_{p/2,q/2,\beta+1}} + \|D_x^2 u\|_{E_{p/3,q/3,\beta+2}} \leq C_0(\|\partial_t u\|_{E_{p/3,q/3,\beta+2}} + \|\Delta u\|_{E_{p/3,q/3,\beta+2}})$$

*and*

$$\|u\|_{F_{p,q,\beta}} + \|D_x u\|_{F_{p/2,q/2,\beta+1}} + \|D_x^2 u\|_{F_{p/3,q/3,\beta+2}} \leq C_0(\|\partial_t u\|_{F_{p/3,q/3,\beta+2}} + \|\Delta u\|_{F_{p/3,q/3,\beta+2}}).$$

Another corollary is the following variation on the theorem of Krylov [Kryl23]:

**Proposition 1.** *Let $p, q \in (3, \infty)$ and $\beta > 0$ with $\frac{2}{p} + \frac{3}{q} > \max(\beta, 1)$. There exists $\epsilon_0 = \epsilon_0(p, q, \beta) > 0$ and $C_0 = C_0(p, q, \beta) > 0$ such that*
*a) If $f \in E_{p/3,q/3,\beta+2}(\mathbb{R} \times \mathbb{R}^d)$, $\vec{b} \in E_{p,q,1}$, $c \in E_{p/2,q/2,2}$, and if $\|\vec{b}\|_{E_{p,q,1}} + \|c\|_{E_{p/2,q/2,2}} < \epsilon_0$, then the heat equation*

$$\partial_t u = \Delta u + \vec{b} \cdot \vec{\nabla} u + cu + f \tag{4}$$

*has a unique solution such that $u = 0$ at infinity. Moreover, we have*

$$\begin{aligned} &\|u\|_{E_{p,q,\beta}} + \|D_x u\|_{E_{p/2,q/2,\beta+1}} + \|D_x^2 u\|_{E_{p/3,q/3,\beta+2}} \\ &\quad \leq C_0(\|f\|_{E_{p/3,q/3,\beta+2}} + \|\vec{b}\|_{E_{p,q,1}} + \|c\|_{E_{p/2,q/2,2}}). \end{aligned}$$

*b) If $f \in F_{p/3,q/3,\beta+2}(\mathbb{R} \times \mathbb{R}^d)$, $\vec{b} \in F_{p,q,1}$, $c \in F_{p/2,q/2,2}$, and if $\|\vec{b}\|_{F_{p,q,1}} + \|c\|_{F_{p/2,q/2,2}} < \epsilon_0$, then the heat equation (4) has a unique solution such that $u = 0$ at infinity. Moreover, we have*

$$\begin{aligned} &\|u\|_{F_{p,q,\beta}} + \|D_x u\|_{F_{p/2,q/2,\beta+1}} + \|D_x^2 u\|_{F_{p/3,q/3,\beta+2}} \\ &\quad \leq C_0(\|f\|_{F_{p/3,q/3,\beta+2}} + \|\vec{b}\|_{F_{p,q,1}} + \|c\|_{F_{p/2,q/2,2}}). \end{aligned}$$

# 3 Anisotropic Hardy-Littlewood maximal function on mixed norm Lebesgue spaces

Let $\vec{p} = (p_1, \ldots, p_n) \in (1, +\infty)^n$. The mixed-norm Lebesgue spaces $L^{\vec{p}}(\mathbb{R}^n)$ is the space of measurable functions on $\mathbb{R}^n$ such that

$$\|f\|_{L^{\vec{p}}} = \left( \int \ldots \left[ \int \left\{ \int |f(x_1, \ldots, x_n)|^{p_1} \, dx_1 \right\}^{\frac{p_2}{p_1}} \right]^{\frac{p_3}{p_2}} \ldots dx_n \right)^{\frac{1}{p_n}} < +\infty.$$

For $\vec{a} = (a_1, \ldots, a_n) \in [1, +\infty)^n$, the anisotropic cylinders $Q_{\vec{a}}(x, r)$ are defined as

$$Q_{\vec{a}}(x, r) = (x_1 - r^{a_1}, x_1 + r^{a_1}) \times \ldots (x_n - r^{a_n}, x_n + r^{a_n})$$

and the anisotropic Hardy-Littlewood maximal function is defined as

$$\mathcal{M}_{\vec{a}} f(x) = \sup_{r>0} \frac{1}{|Q_{\vec{a}}(x, r)|} \int_{Q_{\vec{a}}(x,r)} |f(y)|\, dy.$$

We then have the following boundedness result for the anisotropic Hardy-Littlewood maximal function on mixed norm Lebesgue spaces [Huan19, Huan21]:

**Proposition 2.** *For $\vec{p} = (p_1, \ldots, p_n) \in (1, +\infty)^n$ and $\vec{a} = (a_1, \ldots, a_n) \in [1, +\infty)^n$, there exists a constant $C = C(\vec{p}, \vec{a})$ such that*

$$\|\mathcal{M}_{\vec{a}} f\|_{L^{\vec{p}}} \leq C \|f\|_{L^{\vec{p}}}.$$

# 4 Harmonic analysis on the parabolic space

We endow $X = \mathbb{R} \times \mathbb{R}^d$ with the Lebesgue measure $d\mu(t, x) = dt\, dx$ and the parabolic metric $\rho((t, x), (s, y)) = \sqrt{|t - s|} + |x - y|$. The parabolic ball $B_r(t, x)$ and the parabolic cylinder $C_r(t, x)$ are defined as

$$B_r(t, x) = \{(s, y) \,/\, \rho((t, x), (s, y)) < r\} \text{ and } C_r(t, x) = (t - r^2, t + r^2) \times B(x, r).$$

We have

$$B_r(t, x) \subset C_r(t, x) \subset B_{2r}(t, x).$$

$(X, \rho, \mu)$ is a space of homogeneous type [Coif71] of homogeneous dimension $Q = d + 2$:

$$\mu(B_r(t, x)) = \mu(B_1(0, 0)) r^Q.$$

We associate to this space three useful operators on (non-negative) functions: the Hardy–Littlewood maximal function $\mathcal{M}_f$ and the Riesz potentials $\mathcal{I}_\alpha f$ (where $0 < \alpha < Q$) defined by

$$\mathcal{M}_f(t, x) = \sup_{r>0} \frac{1}{\mu(B_r(t, x))} \int_{B_r(t,x)} |f(s, y)|\, d\mu(s, y)$$

and

$$\mathcal{I}_\alpha f(t, x) = \int_X \frac{1}{\rho((t, x), (s, y))^{Q-\alpha}} f(s, y)\ d\mu(s, y).$$

For $1 < p \leq q$, we define the Morrey space $\mathcal{M}_2^{p,q}$ by the space of measurable functions such that

$$\|f\|_{\mathcal{M}_2^{p,q}} = \sup_{r>0,(t,x)\in X} \mu(B_r(t,x))^{\frac{1}{q}-\frac{1}{p}} \|\mathbb{1}_{B_r(t,x)} f\|_p < +\infty.$$

We have $L^q(\mathbb{R}\times\mathbb{R}^d) \subset \mathcal{M}_2^{p,q}$, $E_{p,p,\beta} = \mathcal{M}_2^{p,\frac{d+2}{\beta}}$ and $E_{p,q,\beta} \subset \mathcal{M}_2^{\min(p,q),\frac{d+2}{\beta}}$.

The main tool we shall use is Hedberg's inequality [Hedb72]:

**Proposition 3.** *Let $1 < p \leq q < +\infty$ and $0 < \alpha < \frac{Q}{q}$. Then there exists a constant $C = C(d,p,q,\alpha)$ such that, for every $f \in \mathcal{M}_2^{p,q}$, we have*

$$\mathcal{I}_\alpha f(t,x) \leq C(\mathcal{M}_f(t,x))^{1-\frac{\alpha q}{Q}} \|f\|_{\mathcal{M}_2^{p,q}}^{\frac{\alpha q}{Q}}.$$

*Proof.* We easily check that, for every $R > 0$, we have

$$\int_{B_R(t,x)} \frac{1}{\rho((t,x),(s,y))^{Q-\alpha}} |f(s,y)|\, d\mu(s,y) \leq CR^\alpha \mathcal{M}_f(t,x)$$

and

$$\int_{X\setminus B_R(t,x)} \frac{1}{\rho((t,x),(s,y))^{Q-\alpha}} |f(s,y)|\, d\mu(s,y) \leq CR^{\alpha-\frac{Q}{q}} \|f\|_{\mathcal{M}_2^{p,q}}.$$

We then take

$$R^{\frac{Q}{q}} = \frac{\|f\|_{\mathcal{M}_2^{p,q}}}{\mathcal{M}_f(t,x)}.$$

□

Applying Proposition 3 to Krylov spaces, we obtain:

**Proposition 4.** *Let $1 < p,q < +\infty$ and $0 < \beta < \frac{2}{p} + \frac{d}{q}$. Then*
*a) $f \mapsto \mathcal{M}_f$ is bounded from $E_{p,q,\beta}$ to $E_{p,q,\beta}$ and from $F_{p,q,\beta}$ to $F_{p,q,\beta}$.*
*b) If $\beta > 1$, $f \mapsto \mathcal{I}_1 f$ is bounded from $E_{p,q,\beta}$ to $E_{\frac{\beta}{\beta-1}p,\frac{\beta}{\beta-1}q,\beta-1}$ and from $F_{p,q,\beta}$ to $F_{\frac{\beta}{\beta-1}p,\frac{\beta}{\beta-1}q,\beta-1}$.*
*c) If $\beta > 2$, $f \mapsto \mathcal{I}_2 f$ is bounded from $E_{p,q,\beta}$ to $E_{\frac{\beta}{\beta-2}p,\frac{\beta}{\beta-2}q,\beta-2}$ and from $F_{p,q,\beta}$ to $F_{\frac{\beta}{\beta-2}p,\frac{\beta}{\beta-2}q,\beta-2}$*

*Proof.* We consider only the case of $E_{p,q,\beta}$, as the proof for $F_{p,q,\beta}$ is similar. Let us estimate $\mathcal{M}_f$ on a cylinder $C_r(t,x)$. We have $\mathcal{M}_f \leq \mathcal{M}_{f_1} + \mathcal{M}_{f_2}$, where $f_1 = \mathbb{1}_{C_{4r}(t,x)} f$ and $f_2 = f - f_1$. By Proposition 2, we know that

$$\|\mathcal{M}_{f_1}\|_{L_t^p L_x^q} \leq C\|f_1\|_{L_t^p L_x^q} \leq C'\|f\|_{E_{p,q,\beta}} r^{\frac{2}{p}+\frac{d}{q}-\beta}.$$

On the other hand, for $(s,z) \in C_r(t,x) \subset B_{2r}(t,x)$, since $B_{4r}(t,x) \subset C_{4r}(t,x)$,

$$\mathcal{M}_{f_2}(s,z) \leq \sup_{\rho>2r} \frac{1}{|B_\rho(s,z)|} \iint_{B_\rho(s,z)} |f(\sigma,y)|\, d\sigma\, dy \leq C' \|f\|_{E_{p,q,\beta}} r^{-\beta}$$

so that

$$\|\mathbb{1}_{C_r(t,x)} \mathcal{M}_{f_2}\|_{L^p_t L^q_x} \leq C \|f\|_{E_{p,q,\beta}} r^{\frac{2}{p}+\frac{d}{q}-\beta}.$$

a) is proved. b) and c) are then direct consequences of Hedberg's inequality (Proposition 3). □

# 5 The heat equation on $\mathbb{R} \times \mathbb{R}^d$.

In this section, we solve the heat equation on $\mathbb{R} \times \mathbb{R}^d$:

$$\begin{cases} \partial_t u = \Delta u + g \\ u = 0 \text{ at infinity} \end{cases} \tag{5}$$

where $g = f \in E_{p,q,\beta}$ or in $F_{p,q,\beta}$ with $2 < \beta < \frac{2}{p} + \frac{d}{q}$ or $g = \sigma(D) \operatorname{div} \mathbb{F}$ where $\mathbb{F} \in E_{p,q,\beta}$ or in $F_{p,q,\beta}$ with $1 < \beta < \frac{2}{p} + \frac{d}{q}$. The solution $u$ of equation (5) is given by the Duhamel formula

$$u = \int_{-\infty}^{t} e^{(t-s)\Delta} g(s,.)\, ds.$$

In order to estimate $u$ and its derivatives, we need some estimates on the size of the kernel of $e^{t\Delta}$ and its derivatives, or on the derivatives of $\sigma(D)e^{t\Delta}$.

**Lemma 3.** *Let $\psi \in \mathcal{S}(\mathbb{R}^d)$ and, for $\theta > 0$, $\psi_\theta(x) = \frac{1}{\theta^d}\psi(\frac{x}{\theta})$. Let $\sigma(D)$ be a Fourier multiplier (in the space variable) such that $\sigma(\xi) = \sigma_0(\frac{\xi}{|\xi|})$ where $\sigma_0$ is a smooth function on the sphere $\mathcal{S}^{d-1}$. Then, for $\alpha \in \mathbb{N}^d$,*

$$|\partial_\alpha \sigma(D)\psi_\theta(x)| \leq C_{\alpha,\psi,\sigma} \frac{1}{(\theta + |x|)^{d+|\alpha|}}.$$

*Proof.* We have

$$\|\partial_\alpha \sigma(D)\psi_\theta\|_\infty = \theta^{-d-|\alpha|} \|\partial_\alpha \sigma(D)\psi\|_\infty$$

and

$$\||x|^{d+|\alpha|} \partial_\alpha \sigma(D)\psi_\theta\|_\infty = \||x|^{d+|\alpha|} \partial_\alpha \sigma(D)\psi\|_\infty.$$

Thus, we may assume $\theta = 1$. We have

$$|\partial_\alpha \sigma(D)\psi_\theta(x)| \leq \frac{1}{(2\pi)^d}\|\sigma_0\|_\infty \int |\xi|^{|\alpha|}|\hat{\psi}(\xi)|\, dx.$$

On the other hand, taking a smooth function $\phi$ such that $\phi(\xi) = 1$ for $|\xi| < 1$ and $\phi(\xi) = 0$ for $|\xi| > 2$, we have for every $R > 0$, for $1 \leq k \leq d$,

$$|x_k|^{d+|\alpha|}|\partial_\alpha \phi(\frac{D}{R})\sigma(D)\psi_\theta(x)| \leq C(R|x|)^{d+\alpha}\|\sigma_0\|_\infty \|\hat{\psi}\|_\infty$$

and

$$\begin{aligned}|x|^2|x_k|^{d+|\alpha|}&|\partial_\alpha(1-\phi(\frac{D}{R}))\sigma(D)\psi_\theta(x)| \\ &\leq \frac{1}{(2\pi)^d}\int |\Delta \partial_k^{d+|\alpha|}\left((1-\phi(\frac{\xi}{R}))\xi^\alpha \sigma(\xi)\hat{\psi}(\xi)\right)|\, d\xi \\ &\leq C\int_{|\xi|>R}\frac{d\xi}{|\xi|^{d+2}} = C'\frac{1}{R^2}\end{aligned}$$

(as $|\partial^\beta(\hat{\psi}(\xi))| \leq C_{\beta,\psi}|\xi|^{-|\beta|}$, $|\partial^\beta(\xi^\alpha\sigma(\xi))| \leq C_{\alpha,\beta,\sigma}|\xi|^{|\alpha|-|\beta|}$ and $|\partial^\beta(1-\phi(\frac{\xi}{R}))| \leq C_{\beta,\phi}|\xi|^{-|\beta|}$). Taking $R = \frac{1}{|x|}$, we get $x_k^{d+|\alpha|}\partial^\alpha\sigma(D)\psi \in L^\infty$. □

A useful result on the heat kernel is its maximal regularity in $L^p_t L^q_x$ or in $L^q_x L^p_t$:

**Proposition 5.** *Let $1 < p, q < +\infty$. Let $\sigma(D)$ be a Fourier multiplier (in the space variable) such that $\sigma(\xi) = \sigma_0(\frac{\xi}{|\xi|})$ where $\sigma_0$ is a smooth function on the sphere $\mathcal{S}^{d-1}$.*
*a) If $h \in L^p_t L^q_x$, then, for $1 \leq i, j \leq d$,*

$$\|\int_{-\infty}^t e^{(t-s)\Delta}\sigma(D)\partial_i\partial_j h\, ds\|_{L^p_t L^q_x} \leq C\|h\|_{L^p_t L^q_x}.$$

*a) If $h \in L^q_x L^p_t$, then, for $1 \leq i, j \leq d$,*

$$\|\int_{\infty}^t e^{(t-s)\Delta}\sigma(D)\partial_i\partial_j h\, ds\|_{L^q_x L^p_t} \leq C\|h\|_{L^q_x L^p_t}.$$

*Proof.* This is a classical result, obtained through the theory of vector valued singular integrals [Bene62]. We sketch the proof given in [Lema02]. Let $W(x) = \frac{1}{(4\pi)^{d/2}}e^{-\frac{|x|^2}{4}}$ and let

$$\Omega(t,x) = \frac{1}{t^{\frac{d+2}{2}}}(\sigma(D)\partial_i\partial_j W)(\frac{x}{\sqrt{t}}) \text{ for } t > 0, \quad = 0 \text{ for } t < 0.$$

We then consider the operator

$$T(g)(t,x) = \int_{\mathbb{R}} \int_{\mathbb{R}^d} \Omega(t-s, x-y) g(s,y)\, ds\, dy. \tag{6}$$

We have

$$(\partial_t - \Delta) T(g) = \sigma(D) \partial_i \partial_j g$$

and, taking the Fourier transform,

$$\widehat{T(g)}(\tau,\xi) = -\frac{\xi_i \xi_j}{i\tau + |\xi|^2} \sigma(\xi) \hat{g}(\tau,\xi).$$

Thus, $T$ is bounded on $L^2(\mathbb{R}\times\mathbb{R}^d)$. $T$ can be seen as Calderón–Zygmund operator on the parabolic space $\mathbb{R}\times\mathbb{R}^d$ (for the parabolic distance): from Lemma 3, we see that $|\Omega(t,x)| \leq C \frac{1}{(\sqrt{|t|}+|x|)^{d+2}}$, $|\partial_i \Omega(t,x)| \leq C \frac{1}{(\sqrt{|t|}+|x|)^{d+3}}$, and $|\partial_t \Omega(t,x)| = |\Delta \Omega(t,x)| \leq C \frac{1}{(\sqrt{|t|}+|x|)^{d+4}}$. Thus, we get that $T$ is bounded on $L^p(\mathbb{R}\times\mathbb{R}^d)$ for $1 < p < +\infty$. Let $T_t$ be the operator

$$T_t(v) = \int_{\mathbb{R}^d} \frac{1}{t} \Omega(t, x-y) v(y)\, dy$$

and $T_x$ be the operator

$$T_x(w) = \int_{\mathbb{R}} \frac{1}{t-s} \Omega(t-s, x) w(s)\, ds$$

We have

$$\partial_t T_t(v) = \int_{\mathbb{R}^d} \frac{1}{t} \Delta\Omega(t, x-y) v(y)\, dy$$

and

$$\partial_k T_x(w) = \int_{\mathbb{R}} \frac{1}{t-s} \partial_k \Omega(t-s, x) w(s)\, ds.$$

From Lemma 3, we find

$$\|T_t\|_{L^q_x \mapsto L^q_x} \leq C \frac{1}{t} \text{ and } \|\partial_t T_t\|_{L^q_x \mapsto L^q_x} \leq C \frac{1}{t^2}$$

so that the continuity $L^q_t(L^q_x) \mapsto L^q_t(L^q_x)$ of $T$ can be extended to the continuity $L^p_t(L^q_x) \mapsto L^p_t(L^q_x)$. Similarly, we have

$$\|T_x\|_{L^p_t \mapsto L^p_t} \leq C \frac{1}{|x|^d} \text{ and } \|\partial_k T_x\|_{L^p_t \mapsto L^p_t} \leq C \frac{1}{|x|^{d+1}}$$

and the continuity $L^p_x(L^p_t) \mapsto L^p_x(L^p_t)$ of $T$ can be extended to the continuity $L^q_x(L^p_t) \mapsto L^q_x(L^p_t)$. □

Similar estimates hold for Krylov spaces:

**Proposition 6.** *Let $1 < p, q < +\infty$ and $0 < \beta < \frac{2}{p} + \frac{d}{q}$. Let $\sigma(D)$ be a Fourier multiplier (in the space variable) such that $\sigma(\xi) = \sigma_0(\frac{\xi}{|\xi|})$ where $\sigma_0$ is a smooth function on the sphere $\mathcal{S}^{d-1}$.*
*a) If $h \in E_{p,q,\beta}$, then, for $1 \leq i, j \leq d$,*

$$\| \int_{-\infty}^{t} e^{(t-s)\Delta} \sigma(D) \partial_i \partial_j h \, ds \|_{E_{p,q,\beta}} \leq C \|h\|_{E_{p,q,\beta}}.$$

*a) If $h \in F_{p,q,\beta}$, then, for $1 \leq i, j \leq d$,*

$$\| \int_{-\infty}^{t} e^{(t-s)\Delta} \sigma(D) \partial_i \partial_j h \, ds \|_{F_{p,q,\beta}} \leq C \|h\|_{F_{p,q,\beta}}.$$

*Proof.* We consider only the case of $E_{p,q,\beta}$, as the proof for $F_{p,q,\beta}$ is similar. Let us estimate $T(h)$ on a cylinder $C_r(t,x)$ (where $T$ is the operator given by (6)). We have $T(h) = T(h_1) + T(h_2)$, where $h_1 = \mathbb{1}_{C_{4r}(t,x)} h$ and $h_2 = h - h_1$. By Proposition 5, we know that

$$\|h_1\|_{L_t^p L_x^q} \leq C \|h_1\|_{L_t^p L_x^q} \leq C' \|h\|_{E_{p,q,\beta}} r^{\frac{2}{p} + \frac{d}{q} - \beta}.$$

On the other hand, for $(s,z) \in C_r(t,x) \subset B_{2r}(t,x)$, since $B_{4r}(t,x) \subset C_{4r}(t,x)$,

$$\begin{aligned}
|h_2(s,z)| \leq & C \iint_{\rho((\sigma,y),(t,x))>4r} \frac{1}{(\sqrt{|\tau - s|} + |z-y|)^{d+2}} |h(\tau,y)| \, d\tau \, dy \\
\leq & C' \iint_{\rho((\sigma,y),(t,x))>4r} \frac{1}{(\sqrt{|\tau - t|} + |x-y|)^{d+2}} |h(\tau,y)| \, d\tau \, dy \\
\leq & C'' \sum_{j=0}^{+\infty} \frac{1}{(4^j r)^{d+2}} \iint_{C_{4^{j+1}r}} |h(\tau,y)| \, d\tau \, dy \\
\leq & C''' \|h\|_{E_{p,q,\beta}} r^{-\beta}
\end{aligned}$$

so that

$$\|\mathbb{1}_{C_r(t,x)} h_2\|_{L_t^p L_x^q} \leq C \| \|_{E_{p,q,\beta}} r^{\frac{2}{p} + \frac{d}{q} - \beta}. \qquad \square$$

We may now easily prove Theorems 3 and 4, Corollary 1 and Proposition 1. Again, we shall consider only the case of $E_{p,q,\beta}$, as the proofs for $F_{p,q,\beta}$ are similar.

## Proof of Theorem 3

*Proof.* The solution $u$ of equation $\partial_t u = \Delta u + f$ is given by $u = \int_{-\infty}^{t} e^{t-s)\Delta} f(s,.)\, ds$. By proposition (6), we already know that

$$\|D_x^2 u\|_{E_{p/3,q/3,\beta+2}} \leq C\|f\|_{E_{p/3,q/3,\beta+2}}.$$

On the other hand, from Lemma 3, we see that

$$|D_x u(t,x)| \leq C \iint \frac{1}{(\sqrt{|t-s|}+|x-y|)^{d+1}} |f(s,y)|\, ds\, dy = C\mathcal{I}_1(|f|)(t,x)$$

and

$$|u(t,x)| \leq C \iint \frac{1}{(\sqrt{|t-s|}+|x-y|)^{d}} |f(s,y)|\, ds\, dy = C\mathcal{I}_2(|f|)(t,x).$$

We then apply Proposition 4 to get

$$\|D_x u\|_{E_{p/2,q/2,\beta+1}} \leq C\|f\|_{E_{p/3,q/3,\beta+2}}$$

and

$$\|u\|_{E_{p,q,\beta}} \leq C\|f\|_{E_{p/3,q/3,\beta+2}}. \qquad \square$$

## Proof of Theorem 4

*Proof.* The solution $u$ of equation $\partial_t u = \Delta u + \sigma(D)\operatorname{div}\mathbb{F}$ is given by $u = \int_{-\infty}^{t} e^{t-s)\Delta}\sigma(D)\operatorname{div}\mathbb{F}(s,.)\, ds$. By proposition (6), we already know that

$$|D_x u\|_{E_{p/2,q/2,\beta+1}} \leq C_0\|\mathbb{F}\|_{E_{p/2,q/2,\beta+1}}.$$

On the other hand, from Lemma 3, we see that

$$|u(t,x)| \leq C \iint \frac{1}{(\sqrt{|t-s|}+|x-y|)^{d+1}} |\mathbb{F}(s,y)|\, ds\, dy = C\mathcal{I}_1(|\mathbb{F}|)(t,x).$$

We then apply Proposition 4 to get

$$\|u\|_{E_{p,q,\beta}} \leq C\|\mathbb{F}\|_{E_{p/2,q/2,\beta+1}}. \qquad \square$$

## Proof of Corollary 1

*Proof.* We just write $\partial_t u - \Delta u = f$ with $f = \partial_t u - \Delta u$. By Theorem 3, we have

$$\|u\|_{E_{p,q,\beta}} + \|D_x u\|_{E_{p/2,q/2,\beta+1}} + \|D_x^2 u\|_{E_{p/3,q/3,\beta+2}} \leq C_0(\|\partial_t u\|_{E_{p/3,q/3,\beta+2}} + \|\Delta u\|_{E_{p/3,q/3,\beta+2}})$$

$\square$

## Proof of Proposition 1

*Proof.* Let $X = \{u \in E_{p,q,\beta} \ / \ D_x u \in E_{p/2,q/2,\beta+1}, D_x^2 u \in E_{p/3,q/3,\beta+2}\}$, normed with

$$\|u\|_X = \|u\|_{E_{p,q,\beta}} + \|D_x u\|_{E_{p/2,q/2,\beta+1}} + \|D_x^2 u\|_{E_{p/3,q/3,\beta+2}}.$$

We are looking for a solution in $X$ of the equation

$$u = T(f) + T(\vec{b} \cdot \vec{\nabla} u) + T(cu)$$

, where $T(f) = \int_{-\infty}^{t} e^{(t-s)\Delta} f(s,.)\, ds$. By Theorem 3, we have

$$\|T(f)\|_X \leq C_1 \|f\|_{E_{p/3,q/3,\beta+2}},$$

$$\|T(\vec{b} \cdot \vec{\nabla} u)\|_X \leq C_2 \|\vec{b} \cdot \vec{\nabla} u\|_{E_{p/3,q/3,\beta+2}} \leq C_2 \|\vec{b}\|_{E_{p,q,1}} \|D_x u\|_{E_{p/2,q/2,\beta+1}},$$

and

$$\|T(cu)\|_X \leq C_3 \|cu\|_{E_{p/3,q/3,\beta+2}} \leq C_3 \|c\|_{E_{p/2,q/2,2}} \|u\|_{p,q,\beta}.$$

For $C_2\|\vec{b}\|_{E_{p,q,1}} + C_3\|c\|_{E_{p/2,q/2,2}} < \frac{1}{2}$, the operator $S(u) = u - T(\vec{b} \cdot \vec{\nabla} u) - T(cu)$ is an isomorphism of $X$, with $\|S^{-1}\|_{X \mapsto X} \leq 2$. The solution $u$ is then given by $u = S^{-1}(T(f))$, with $\|u\|_X \leq 2C_1 \|f\|_{E_{p/3,q/3,\beta+2}}$. □

# 6 The Navier–Stokes equations.

We prove here Theorem 1. The proof of Theorem 2 is similar. Again, we consider only the case of $E_{p,q,\beta}$.

## Proof of Theorem 1

*Proof.* The Navier–Stokes equations we study are

$$\begin{cases} \partial_t \vec{u} = \Delta \vec{u} - \vec{\nabla} p - \vec{u} \cdot \vec{\nabla} \vec{u} + \vec{f} + \operatorname{div} \mathbb{F} \\ \operatorname{div} \vec{u} = 0 \\ \vec{u}(0,.) = 0 \end{cases} \tag{7}$$

where $\vec{f} \in E_{p/3,q/3,3}$ with $\operatorname{div} \vec{f} = 0$ and $\mathbb{F} \in E_{p/2,q/2,2}$. We rewrite (7) as

$$\partial_t \vec{u} - \Delta \vec{u} = \vec{f} + (\mathrm{Id} - \frac{1}{\Delta} \vec{\nabla} \operatorname{div}) \operatorname{div}(\mathbb{F} - \vec{u} \otimes \vec{u}).$$

($\mathrm{Id} - \frac{1}{\Delta} \vec{\nabla} \operatorname{div} = \mathbb{P}$ is the Leray projection operator on solenoidal vector fields.)

Let $X = \{\vec{u} \in E_{p,q,1} \ / \ D_x\vec{u} \in E_{p/2,q/2,2}\}$, normed with

$$\|u\|_X = \|\vec{u}\|_{E_{p,q,1}} + \|D_x\vec{u}\|_{E_{p/2,q/2,2}}.$$

We are looking for a solution in $X$ of the equation

$$\vec{u} = \vec{T}(\vec{f}) + \vec{T}(\mathbb{P}\,\mathrm{div}\,\mathbb{F}) - \vec{T}(\mathbb{P}\,\mathrm{div}(\vec{u}\otimes\vec{u})),$$

where $\vec{T}(\vec{f}) = \int_0^t e^{(t-s)\Delta}\vec{f}(s,.)\,ds$ $(= \int_{-\infty}^t e^{(t-s)\Delta}\vec{f}(s,.)\,ds$ if we extend $\vec{f}$ to $(-\infty,0)$ with $\vec{f}(t,x) = 0$ for $t<0)$. By Theorem 3, we have

$$\|\vec{T}(\vec{f})\|_X \leq C_1\|\vec{f}\|_{E_{p/3,q/3,3}}.$$

By Theorem 4, we have

$$\|\vec{T}(\mathbb{P}\,\mathrm{div}\,\mathbb{F})\|_X \leq C_2\|\mathbb{F}\|_{E_{p/2,q/2,2}}$$

and

$$\|\vec{T}(\mathbb{P}\,\mathrm{div}(\vec{u}\otimes\vec{u}))\|_X \leq C_3\|\vec{u}\otimes\vec{u}\|_{E_{p/2,q/2,2}} \leq C_3\|\vec{u}\|^2_{p,q,1}.$$

For $C_1\|\vec{f}\|_{E_{p/3,q/3,3}} + C_2\|\mathbb{F}\|_{E_{p/2,q/2,2}} < \frac{1}{4C_3}$, we find a solution $\vec{u}\in X$ with $\|\vec{u}\|_X \leq 2(C_1\|\vec{f}\|_{E_{p/3,q/3,3}} + C_2\|\mathbb{F}\|_{E_{p/2,q/2,2}})$. □

# Appendix 1: Parabolic Besov spaces

In this appendix, we describe some results on the heat equation on parabolic Besov spaces that may shed a new light on many of the inequalities we obtained throughout the paper, which are to be viewed more as regularity assertions than as results on existence of solutions.

For $\delta > 0$, let us define the parabolic Besov space $\mathcal{B}^{-\delta}_{[2],\infty,\infty}$ as the space of tempered distributions on $\mathbb{R}\times\mathbb{R}^d$ such that

$$\|u\|_{\mathcal{B}^{-\delta}_{[2],\infty,\infty}} = \sup_{\theta>0}\theta^{\delta/2}\|e^{\theta^2(\partial_t^2-\Delta^2)}u(t,x)\|_{L^\infty(\mathbb{R}\times\mathbb{R}^d)} < +\infty.$$

(In particular, $E_{p,q,\beta} \subset \mathcal{B}^{-\beta}_{[2],\infty,\infty}$ and $F_{p,q,\beta} \subset \mathcal{B}^{-\beta}_{[2],\infty,\infty}$.) [The index [2] indicates the underlying time-space scaling, where $t$ behaves like $|x|^2$.]

It is easy to check that $\mathcal{B}^{-\beta}_{[2],\infty,\infty}$ coincides with the (realization of the) homogeneous Besov space defined through the parabolic Littlewood-Paley decomposition (for instance, see Triebel's book [Trie06]). Let $\phi(\tau,\xi)$ be a smooth function equal to 1 when $\sqrt{|\tau|}+|\xi| < 1$ and to 0 when $\sqrt{|\tau|}+|\xi| > 2$. Let $S_jf$ be defined as the inverse Fourier transform of $\phi(\frac{\tau}{4^j},\frac{\xi}{2^j})\hat{f}(\tau,\xi)$. Then $\mathcal{B}^{-\beta}_{[2],\infty,\infty}$ is the Banach space of tempered distributions $f$ such that $\sup_{j\in\mathbb{Z}} 2^{j\delta}\|S_jf\|_\infty < +\infty$.

We have the following Gagliardo–Nirenberg inequality

**Proposition 7.** *Let $\delta > 0$, $1 < p, q < +\infty$ and $\beta \in (0, \frac{2}{p} + \frac{d}{q})$.*
*a) If $u \in \mathcal{B}^{-\delta}_{[2],\infty,\infty}$ and $\partial_t u, \Delta u \in E_{p,q,\beta}$, then*

$$\|D_x u\|_{E_{\frac{2+\delta}{1+\delta}p, \frac{2+\delta}{1+\delta}q, \frac{1+\delta}{2+\delta}\beta}} \leq C\|u\|^{\frac{1}{2+\delta}}_{\mathcal{B}^{-\delta}_{[2],\infty,\infty}} (\|\partial_t u\|_{E_{p,q,\beta}} + \|\Delta u\|_{E_{p,q,\beta}})^{\frac{1+\delta}{2+\delta}}.$$

*b) If $u \in \mathcal{B}^{-\delta}_{[2],\infty,\infty}$ and $\partial_t u, \Delta u \in F_{p,q,\beta}$, then*

$$\|D_x u\|_{F_{\frac{2+\delta}{1+\delta}p, \frac{2+\delta}{1+\delta}q, \frac{1+\delta}{2+\delta}\beta}} \leq C\|u\|^{\frac{1}{2+\delta}}_{\mathcal{B}^{-\delta}_{[2],\infty,\infty}} (\|\partial_t u\|_{F_{p,q,\beta}} + \|\Delta u\|_{F_{p,q,\beta}})^{\frac{1+\delta}{2+\delta}}.$$

Our final resuts extend Corollary 1 and Theorem 3 to parabolic Besov spaces:

**Theorem 5.** *Let $\delta > 0$. Let $u = 0$ at infinity. Then $u \in \mathcal{B}^{-\delta}_{[2],\infty,\infty}$ if and only if $\partial_t u \in \mathcal{B}^{-\delta-2}_{[2],\infty,\infty}$ and $\Delta u \in \mathcal{B}^{-\delta-2}_{[2],\infty,\infty}$ Moreover, the norms $\|u\|_{\mathcal{B}^{-\delta}_{[2],\infty,\infty}}$ and $\|\partial_t u\|_{\mathcal{B}^{-\delta-2}_{[2],\infty,\infty}} + \|\Delta u\|_{\mathcal{B}^{-\delta-2}_{[2],\infty,\infty}}$ are equivalent.*

**Theorem 6.** *Let $\delta > 2$. If $f \in \mathcal{B}^{-\delta}_{[2],\infty,\infty}$, then the heat equation*

$$\partial_t u = \Delta u + f \tag{8}$$

*has a unique solution such that $u = 0$ at infinity. Moreover, $u \in \mathcal{B}^{-\delta+2}_{[2],\infty,\infty}$ and*

$$\|u\|_{\mathcal{B}^{-\delta+2}_{[2],\infty,\infty}} \leq C\|f\|_{\mathcal{B}^{-\delta}_{[2],\infty,\infty}}.$$

Those results are not very new, as they can be recovered from the voluminous literature on anisotropic Besov spaces We present them for sake of completeness.

## Proof of Proposition 7.

*Proof.* As $u = 0$ at infinity, we may write

$$u = \int_0^\infty (-\partial_\theta)(e^{\theta^2(\partial_t^2 - \Delta^2)}u)\, d\theta$$

and

$$D_x u = \int_0^\infty (-\partial_\theta) D_x(e^{\theta^2(\partial_t^2 - \Delta^2)}u)\, d\theta.$$

For $R>0$, we rewrite this as

$$\begin{aligned}
D_x u =& -\int_0^R (2\theta\partial_t e^{\theta^2\partial_t^2}\sqrt{\theta}D_x e^{-\theta^2\Delta^2})\partial_t u\,\frac{d\theta}{\sqrt{\theta}}\\
&-\int_R^\infty (2\theta^2\partial_t^2 e^{\frac{3}{4}\theta^2\partial_t^2}\sqrt{\theta}D_x e^{-\frac{3}{4}\theta^2\Delta^2})(e^{\frac{\theta^2}{4}(\partial_t^2-\Delta^2)}u)\,\frac{d\theta}{\theta^{3/2}}\\
&+\int_0^R (2e^{\theta^2\partial_t^2}\theta^{3/2}D_x\Delta e^{-\theta^2\Delta^2})\Delta u\,\frac{d\theta}{\sqrt{\theta}}\\
&+\int_R^\infty (2e^{\frac{3}{4}\theta^2\partial_t^2}\theta^{5/2}D_x\Delta^2 e^{-\frac{3}{4}\theta^2\Delta^2})(e^{\frac{\theta^2}{4}(\partial_t^2-\Delta^2)}u)\,\frac{d\theta}{\theta^{3/2}}\\
=&I+II+III+IV.
\end{aligned}$$

Recall that $\mathcal{M}_f$ designs the parabolic Hardy–Littlewood maximal function. If $\psi\in\mathcal{S}(\mathbb{R}\times\mathbb{R}^d)$, we have, for every $\theta>0$,

$$\iint \frac{1}{\theta^{\frac{d+2}{2}}}|\psi(\frac{s}{\theta},\frac{y}{\sqrt{\theta}})||f(t-s,x-y)\,ds\,dy\le C_\psi\mathcal{M}_f(t,x)$$

and

$$\iint \frac{1}{\theta^{\frac{d+2}{2}}}|\psi(\frac{s}{\theta},\frac{y}{\sqrt{\theta}})||f(t-s,x-y)\,ds\,dy\le \|\psi\|_1\|f\|_\infty.$$

Thus, we have

$$|I|\le C\sqrt{R}\mathcal{M}_{\partial_t u}\text{ and }|III|\le C\sqrt{R}\mathcal{M}_{\Delta u}$$

and

$$|II|+|IV|\le CR^{-\frac{1+\delta}{2}}\|u\|_{\mathcal{B}^{-\delta}_{[2],\infty,\infty}}.$$

Taking $\sqrt{R}=\left(\frac{\|u\|_{\mathcal{B}^{-\delta}_{[2],\infty,\infty}}}{\mathcal{M}_{\partial_t u}(t,x)+\mathcal{M}_{\Delta u}(t,x)}\right)^{\frac{1}{2+\delta}}$, we get

$$|D_x u(t,x)|\le C\|u\|^{\frac{1}{2+\delta}}_{\mathcal{B}^{-\delta}_{[2],\infty,\infty}}(\mathcal{M}_{\partial_t u}(t,x)+\mathcal{M}_{\Delta u}(t,x))^{\frac{1+\delta}{2+\delta}}$$

and we conclude by Proposition 4 a). □

## Proof of Theorem 5.

*Proof.* Writing

$$e^{\theta^2(\partial_t^2-\Delta^2)}(\partial_t u)=\frac{1}{\theta}(\theta\partial_t e^{\frac{3}{4}\theta^2(\partial_t^2-\Delta^2)})(e^{\frac{1}{4}\theta^2(\partial_t^2-\Delta^2)}u)$$

and

$$e^{\theta^2(\partial_t^2-\Delta^2)}(\Delta u)=\frac{1}{\theta}(\theta\Delta e^{\frac{3}{4}\theta^2(\partial_t^2-\Delta^2)})(e^{\frac{1}{4}\theta^2(\partial_t^2-\Delta^2)}u),$$

we find that

$$\|\partial_t u\|_{\mathcal{B}^{-\delta-2}_{[2],\infty,\infty}}+\|\Delta u\|_{\mathcal{B}^{-\delta-2}_{[2],\infty,\infty}}\leq C\|u\|_{\mathcal{B}^{-\delta}_{[2],\infty,\infty}}.$$

As $u=0$ at infinity, we write

$$u=\int_0^\infty(-\partial_\eta)(e^{\eta^2(\partial_t^2-\Delta^2)}u)\,d\eta=2\int_0^{+\infty}\eta(\Delta^2-\partial_t^2)e^{\eta^2(\partial_t^2-\Delta^2)}u\,d\eta.$$

We get

$$\begin{aligned}e^{\theta^2(\partial_t^2-\Delta^2)}u=&2\int_0^{+\infty}(\eta\Delta e^{\frac{\eta^2}{2}(\partial_t^2-\Delta^2)})(e^{(\theta^2+\frac{\eta^2}{2})(\partial_t^2-\Delta^2)}\Delta u)\,d\eta\\&-2\int_0^{+\infty}(\eta\partial_t e^{\frac{\eta^2}{2}(\partial_t^2-\Delta^2)})(e^{(\theta^2+\frac{\eta^2}{2})(\partial_t^2-\Delta^2)}\partial_t u)\,d\eta,\end{aligned}$$

so that

$$\begin{aligned}\|e^{\theta^2(\partial_t^2-\Delta^2)}u\|_\infty\leq&C\int_0^{+\infty}\|e^{(\theta^2+\frac{\eta^2}{2})(\partial_t^2-\Delta^2)}\Delta u\|_\infty+\|e^{(\theta^2+\frac{\eta^2}{2})(\partial_t^2-\Delta^2)}\partial_t u\|_\infty\,d\eta\\\leq&C(\|\partial_t u\|_{\mathcal{B}^{-\delta-2}_{[2],\infty,\infty}}+\|\Delta u\|_{\mathcal{B}^{-\delta-2}_{[2],\infty,\infty}})\int_0^{+\infty}(\theta^2+\frac{\eta^2}{2})^{-\frac{2+\delta}{4}}\,d\eta\\=&C\theta^{-\delta/2}(\|\partial_t u\|_{\mathcal{B}^{-\delta-2}_{[2],\infty,\infty}}+\|\Delta u\|_{\mathcal{B}^{-\delta-2}_{[2],\infty,\infty}})\int_0^{+\infty}(1+\frac{\eta^2}{2})^{-\frac{2+\delta}{4}}\,d\eta\end{aligned}$$

and finally

$$\|u\|_{\mathcal{B}^{-\delta}_{[2],\infty,\infty}}\leq C(\|\partial_t u\|_{\mathcal{B}^{-\delta-2}_{[2],\infty,\infty}}+\|\Delta u\|_{\mathcal{B}^{-\delta-2}_{[2],\infty,\infty}}).$$ □

## Proof of Theorem 6.

*Proof.* We write

$$f=\int_0^\infty(-\partial_\eta)(e^{\eta^2(\partial_t^2-\Delta^2)}f)\,d\eta=-2(\partial_t-\Delta)\int_0^{+\infty}\eta(\Delta+\partial_t)e^{\eta^2(\partial_t^2-\Delta^2)}f\,d\eta.$$

Let

$$u=-2\int_0^{+\infty}\eta(\Delta+\partial_t)e^{\eta^2(\partial_t^2-\Delta^2)}f\,d\eta.$$

We have

$$\begin{aligned}
\|e^{\theta^2(\partial_t^2-\Delta^2)}u\|_\infty \leq & 2\int_0^{+\infty} \|(\eta\Delta e^{\frac{\eta^2}{2}(\partial_t^2-\Delta^2)})(e^{(\theta^2+\frac{\eta^2}{2})(\partial_t^2-\Delta^2)}f)\|_\infty\, d\eta \\
& + 2\int_0^{+\infty} \|(\eta\partial_t e^{\frac{\eta^2}{2}(\partial_t^2-\Delta^2)})(e^{(\theta^2+\frac{\eta^2}{2})(\partial_t^2-\Delta^2)}f)\|_\infty\, d\eta, \\
\leq & C\int_0^{+\infty} \|e^{(\theta^2+\frac{\eta^2}{2})(\partial_t^2-\Delta^2)}f\|_\infty\, d\eta \\
\leq & C\|f\|_{\mathcal{B}^{-\delta-2}_{[2],\infty,\infty}} \int_0^{+\infty} (\theta^2+\frac{\eta^2}{2})^{-\frac{2+\delta}{4}}\, d\eta \\
= & C\theta^{-\delta/2}\|f\|_{\mathcal{B}^{-\delta-2}_{[2],\infty,\infty}} \int_0^{+\infty} (1+\frac{\eta^2}{2})^{-\frac{2+\delta}{4}}\, d\eta
\end{aligned}$$

and finally

$$\|u\|_{\mathcal{B}^{-\delta}_{[2],\infty,\infty}} \leq C\|f\|_{\mathcal{B}^{-\delta-2}_{[2],\infty,\infty}}. \qquad \square$$

# Appendix 2: A remark on the regularity of mild solutions

The reviewer asked the following questions about the regularity of the solution $u$ constructed in Theorem 1: can one expect the constructed solution to be in $L^1_{loc}(\mathbb{R}, C^{0,1})$? The answer is clearly negative, as the force is not regular enough to grant such a regularity. In this appendix, we give an example of a not so regular solution.

We start from a well-known fact: the map $\mathbb{F} \mapsto \int_0^t e^{(t-s)\Delta}\mathbb{P}\operatorname{div}\mathbb{F}\, ds$ maps $L^\infty((0,+\infty), L^{p_0,\infty}(\mathbb{R}^3))$ to $L^\infty((0,+\infty), L^{p_1,\infty}(\mathbb{R}^3))$ for $1 < p_0 < 3$ and $\frac{1}{p_1} = \frac{1}{p_0} - \frac{1}{3}$ (see for instance [Lema02]). Thus, if $\|\mathbb{F}\|_{L^\infty L^{3/2,\infty}}$ is small enough, the Navier–Stokes problem

$$\partial_t \vec{u} = \Delta\vec{u} + \mathbb{P}\operatorname{div}(\mathbb{F} - \vec{u}\otimes\vec{u})$$

will have a solution in $L^\infty((0,+\infty), L^{3,\infty}(\mathbb{R}^3))$. Similarly, if $3/2 < r < 2$ and $\|\mathbb{F}\|_{L^\infty L^{3/2,\infty}} + \|\mathbb{F}\|_{L^\infty L^{r,\infty}}$ is small enough, the Navier–Stokes problem will have a solution $\vec{u}$ in $L^\infty((0,+\infty), L^{3,\infty}(\mathbb{R}^3)) \cap L^\infty((0,+\infty), L^{\frac{3r}{3-r},\infty}(\mathbb{R}^3))$. Moreover, we have

$$\vec{u} - \int_0^t e^{(t-s)\Delta}\mathbb{P}\operatorname{div}\mathbb{F}\, ds = -\int_0^t e^{(t-s)\Delta}\mathbb{P}\operatorname{div}(\vec{u}\otimes\vec{u})\, ds$$

and we find that $\vec{u} - \int_0^t e^{(t-s)\Delta}\mathbb{P}\,\mathrm{div}\,\mathbb{F}\,ds$ belongs to $L^\infty((0,+\infty), L^{\frac{r}{2-r},\infty}(\mathbb{R}^3))$ (with $\frac{r}{2-r} > \frac{3r}{3-r}$).

We may remark that, if $\mathbb{F} \in L^\infty L^{3/2,\infty} \cap L^\infty L^{r,\infty}$ and $\mathbb{F}$ is supported in $[0,1]\times\mathbb{R}^3$, then $\mathbb{F} \in E_{p/2,q/2,2}$ for $p > 2$, $3 < q < 2r$ and $\frac{2}{p} + \frac{3}{q} > 1$. Indeed, we have $\mathbb{F} \in L^\infty_t L^{q/2} \cap L^{p/2}_t L^{q/2}_x$, so that

$$\rho^{2-2\frac{2}{p}-3\frac{2}{q}} \left( \int_{|t-s|<\rho^2} \Big( \int_{|x-y|<\rho} |\mathbb{F}(s,y)|^{q/2}\,dy \Big)^{\frac{p}{q}}\,ds \right)^{\frac{2}{p}} \leq C\rho^{2(1-\frac{3}{q})} \|\mathbb{F}\|_{L^\infty L^{q/2}}$$

(which is bounded for $0 < \rho \leq 1$) and

$$\rho^{2-2\frac{2}{p}-3\frac{2}{q}} \left( \int_{|t-s|<\rho^2} \Big( \int_{|x-y|<\rho} |\mathbb{F}(s,y)|^{q/2}\,dy \Big)^{\frac{p}{q}}\,ds \right)^{\frac{2}{p}} \leq C\rho^{2(1-\frac{2}{p}-\frac{3}{q})} \|\mathbb{F}\|_{L^{p/2} L^{q/2}}$$

(which is bounded for $1 < \rho$).

We now give an example of a mild solution such that $\vec{u}$ is not (locally in time) $L^1 C^{0,1}$. Let $\varphi \in \mathcal{D}(\mathbb{R}^3)$ be equal to 1 on a neighborhood of 0 and with support in $B(0,1)$, let $0 < \epsilon < 1/2$ and let $\vec{f}(t,x) = \lambda \mathbb{1}_{(0,1)}(t) \Delta\vec{\nabla} \wedge (|x|^\epsilon \varphi)$, with $\lambda > 0$ small enough. We have $\mathrm{div}\,\vec{f} = 0$. We write $\vec{f}$ in the divergence form

$$\vec{f} = \sum_{j=1}^3 \partial_j (\lambda \mathbb{1}_{(0,1)}(t) \partial_j \vec{\nabla} \wedge (|x|^\epsilon \varphi)) = \mathrm{div}\,\mathbb{F}.$$

$\mathbb{F}$ is supported in $[0,1]\times\mathbb{R}^3$; moreover, $\mathbb{F} \in L^\infty L^{\rho,\infty}$ for every $\rho \in (1, \frac{3}{2-\epsilon}]$, hence $\mathbb{F} \in L^\infty L^{3/2,\infty} \cap L^\infty L^{\frac{3}{2-\epsilon},\infty}$. If $\lambda$ is small enough, the Navier–Stokes problem have a solution $\vec{u}$ such that

$$\vec{u} \in L^\infty((0,+\infty), L^{3,\infty}(\mathbb{R}^3)) \cap L^\infty((0,+\infty), L^{\frac{3}{1-\epsilon},\infty}(\mathbb{R}^3))$$

and

$$\vec{u} - \int_0^t e^{(t-s)\Delta}\mathbb{P}\,\mathrm{div}\,\mathbb{F}\,ds \in L^\infty((0,+\infty), L^{\frac{3}{1-2\epsilon},\infty}(\mathbb{R}^3)).$$

For $0 < t < 1$, we have

$$\int_0^t e^{(t-s)\Delta}\mathbb{P}\,\mathrm{div}\,\mathbb{F}\,ds = \lambda(\vec{\nabla} \wedge (|x|^\epsilon\varphi) - e^{t\Delta}(\vec{\nabla} \wedge (|x|^\epsilon\varphi))).$$

If locally $\vec{u}$ were $L^1 C^{0,1}$, then$\mathbb{1}_{(1/2,1)}(t)\mathbb{1}_{B(0,1)}(x)\vec{u}$ would be $L^1 L^{\frac{3}{1-2\epsilon},\infty}$. This would give that $\vec{\nabla} \wedge (|x|^\epsilon\varphi) \in L^{\frac{3}{1-2\epsilon},\infty}$. But $\vec{\nabla} \wedge (|x|^\epsilon\varphi) \in L^{r,\infty}$ if and only if $r \leq \frac{3}{1-\epsilon}$.

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