Observers and Disturbance Rejection Control for a Heat Equation

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Hongyinping Feng[a], Cheng-Zhong Xu[⋆], and Peng-Fei Yao[c,d]

***Abstract*—The paper is concerned with active disturbance rejection control of a heat equation. The considered heat equation satisfies the Dirichlet boundary condition on one part of the boundary. On the other part of the boundary is located a Neumann boundary control. The heat equation system suffers from both a model uncertainty in the heat flow modeling and an unknown external disturbance. Our control approach is based on the design of an exponentially converging observer to estimates both the state and the unknown uncertainty. The estimated state and the estimated uncertainty are used to build a stabilizing feedback control law such that the closed-loop system is exponentially stabilized and the external disturbance is rejected.**

***Index Terms*—Heat equation, nonlinear boundary condition, observer, stabilization, disturbance rejection control.**

I. Introduction

In the past two decades, many control approaches have been developed to cope with the disturbances in the context of partial differential equation (PDE) control. In [14] the principle of internal model has been implemented to reject the disturbance generated by an exosystem. Stabilization of a wave equation in the face of harmonic disturbances has been considered by using an adaptive control method [11]. This method has been extended to solve the problem of the output regulation [10]. The sliding mode control has been proposed by using an observer to stabilize an infinite-dimensional system corrupted by input disturbances [9]. More recently, the classical proportional and integral (PI) control has been extended to stabilize nonlinear PDE systems for rejecting constant unknown disturbances [17]. However, when a system suffers from model uncertainties and external disturbances at the same time, the stabilization with disturbance rejection becomes a challenging problem. Nevertheless, the active disturbance rejection control (ADRC) gives a solution to the problem by proposing an extended state observer to estimate both the state and the disturbance and then canceling the disturbance via a stabilizing feedback control law (see [12]). For a short review on the approach, the reader is referred to [6] and [7].

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Now let us explain the idea of the ADRC in a semigroup setting. Suppose that A is the generator of an exponentially stable C_0-semigroup $(e^{tA})_{t\geqslant 0}$ on a Hilbert space X. Let U be the control Hilbert space. Consider the system described by

$$\begin{cases} \dot{x}(t) = Ax(t) + B[f(t) + u(t)], \\ x(0) = x_0, \end{cases} \tag{1}$$

where $B : U \to X$ is the linear control operator, $u : (0,\infty) \to U$ is the control signal, $f : (0,\infty) \to U$ is an unknown disturbance supposed continuous for the moment, and $x_0 \in X$ is the initial state. Notice that the disturbance and the control are matched in the same channel. If a continuous function $\hat{f} : (0,\infty) \to U$ approximates f such that

$$\lim_{t\to\infty} \|f(t) - \hat{f}(t)\|_U = 0, \tag{2}$$

or

$$(f - \hat{f}) \in L^2([0,\infty); U), \tag{3}$$

then the control signal can be chosen as

$$u(t) = -\hat{f}(t) \tag{4}$$

such that the negative impact of the disturbance is asymptotically canceled. Indeed, under the control (4) the controlled system is governed by

$$\begin{cases} \dot{x}(t) = Ax(t) + B[f(t) - \hat{f}(t)], \\ x(0) = x_0. \end{cases} \tag{5}$$

If B is admissible for the semigroup e^{tA} (see [20] or [19] for definition), then the solution of system (5) tends to zero as $t \to \infty$ (see [6]): $\lim_{t\to\infty} \|x(t)\|_X = 0$. Hence the stabilization and the disturbance rejection can be achieved by estimating the disturbance.

In the paper we consider an initial boundary heat equation model that suffers from an uncertainty in the heat flow modeling and an external disturbance. We design an observer enabling us to estimate simultaneously both the model uncertainty and the disturbance by using the measurements of the output and the control. Based on the ADRC principle a dynamic output feedback control law is built to cancel the effect of the disturbance and stabilize exponentially the control plant. Meanwhile our control law guarantees all the state of the closed-loop system uniformly bounded in time if the external disturbance is bounded. The disturbance rejection feedback control laws that we propose here are notably robust and should have potential applications in process control.

The ADRC control design has been investigated in the recent article [22] for a heat equation under nonlinearities and

unknown disturbance on the boundary. However the situation was different from ours for the controller design since there the disturbance and the control were not in the same channel. As the control appears in the same channel as the disturbance in our paper, an extra step is required in the observer design to separate the effect of the disturbance from that of the control. The contribution of our paper is an another novel result toward the application of the ADRC control design to the heat equation system. In the future we endeavor to extend the ADRC control strategy to the heat equation system with more nonlinear uncertainties.

The paper is organized as follows: The mains results are presented in Section II; their proofs are given in Section III and Section IV contains numerical simulations.

II. Main results

In the paper, we study an initial boundary heat equation where the disturbance and the control appear in the same channel. Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded connected open set with a smooth C^2-boundary $\Gamma = \overline{\Gamma_0} \cup \overline{\Gamma_1}$, with both $\Gamma_0 \neq \emptyset$ and $\Gamma_1 \neq \emptyset$ being relatively open in Γ and $\Gamma_0 \cap \Gamma_1 = \emptyset$. Let ν be the unit outward normal vector to Γ, i.e., $\nu(x) = (\nu_1(x), \cdots, \nu_n(x))\ \forall x \in \Gamma$, and let $\mathbb{R}_+ = (0, \infty)$. The system we consider is governed by the heat equation:

$$\begin{cases} w_t(x,t) = \Delta w(x,t), \ \ (x,t) \in \Omega \times \mathbb{R}_+, \\ w(x,t) = 0, \ \ (x,t) \in \Gamma_0 \times \mathbb{R}_+, \\ \partial_\nu w(x,t) = F(w(x,t),t) + u(x,t), \ (x,t) \in \Gamma_1 \times \mathbb{R}_+, \\ w(x,0) = w_0(x), \ \ x \in \Omega, \\ y(x,t) = w(x,t), \ \ x \in \Gamma_1 \times \mathbb{R}_+, \end{cases} \tag{6}$$

where $w(\cdot,t) : \Omega \to \mathbb{R}$ denotes the state at time t, u the (input) control, y the (output) measurement, w_0 the initial state, and $F : L^2(\Omega) \times \mathbb{R} \to L^2(\Gamma_1)$ is an unknown nonlinear mapping that we suppose locally Lipschitz continuous. In (6), $w_t(x,t)$ denotes the partial derivative of $w(x,t)$ with respect to t, or $w_t(x,t) = \dfrac{\partial w(x,t)}{\partial t}$, the Laplacian operator Δ defined by

$$\Delta w(x,t) = \sum_{i=1}^{n} \frac{\partial^2 w(x,t)}{\partial x_i^2},$$

and $\partial_\nu w(x,t)$ is the normal derivative of w at the boundary defined as $\partial_\nu w(x,t) = \sum_{i=1}^n \nu_i(x) w_{x_i}(x,t)$.

Furthermore we assume that the unknown term F in (6) called total disturbance is modeled as $F(w,t) = \xi(w) + \eta(t)$, where ξ denotes the unknown part of the heat flow model and η the unknown external disturbance. The unknown part F and the control u influence the trajectory of the system via the Neumann condition that represents physically the boundary heat flux to the system.

It is easy to see that there exists a function F such that the system (6) without control (i.e. $u = 0$) is unstable with respect to the equilibrium stationary solution. Generally speaking, the solutions of the control plant (6) without control may blow up in finite time with some boundary source term F (see [13]). The objective of the present paper is to design an observer that recovers the unknown term F in system (6) and then to cancel it by a feedback control law such that the state of the closed-loop system is exponentially stable, see (14) below.

For convenience, we write $u \in L^2_{loc}(\mathbb{R}_+;\, L^2(\Gamma_1))$ if

$$\int_0^T \int_{\Gamma_1} |u(x,t)|^2 d\Gamma dt < \infty \quad \forall\, T > 0.$$

We write $u \in H^1_{loc}(\mathbb{R}_+; L^2(\Gamma_1))$ if $u, u_t \in L^2_{loc}(\mathbb{R}_+;\, L^2(\Gamma_1))$. Let us set $H^1_{\Gamma_0}(\Omega) = \{h \in H^1(\Omega) \mid h(x) = 0\ \forall\ x \in \Gamma_0\}$. Throughout the paper are supposed satisfied the following conditions: Assumption I and Assumption II.

Assumption I. Let $\eta \in L^2_{loc}(\mathbb{R}_+;\, L^2(\Gamma_1))$. We suppose that the total disturbance is constituted of model uncertainty and external disturbance as follows

$$\begin{aligned} F: \quad & L^2(\Omega) \times \mathbb{R}_+ \to L^2(\Gamma_1) \\ & (w,t) \mapsto F(w,t) = \xi(w) + \eta(t), \end{aligned}$$

where $\xi : L^2(\Omega) \to L^2(\Gamma_1)$ is locally Lipschitz continuous, i.e., for each $\delta > 0$ there is a positive constant $L > 0$ such that

$$\|\xi(w_1) - \xi(w_2)\|_{L^2(\Gamma_1)} \leqslant L \|w_1 - w_2\|_{L^2(\Omega)}\ \forall\ \|w_i\| < \delta.$$

Assumption II. There exists a real constant $k > 0$ such that $\|\xi(w)\|_{L^2(\Gamma_1)} \leqslant k\left(1 + \|w\|_{L^2(\Omega)}\right)$.

The first contribution of our paper is the following theorem whose proof is postponed to Appendix.

Theorem 1: i) Let Assumption I be satisfied. Then for any $u \in L^2_{loc}(\mathbb{R}_+; L^2(\Gamma_1))$ and $w_0 \in L^2(\Omega)$, the system (6) admits a unique local mild solution $w \in C([0,T]; L^2(\Omega))$ for some $T > 0$. ii) If further Assumption II is satisfied, then the unique mild solution $w \in C([0,T]; L^2(\Omega))$ exists for all $T > 0$. iii) If $\eta, u \in H^1_{loc}(\mathbb{R}_+; L^2(\Gamma_1))$ and $w_0 \in H^1_{\Gamma_0}(\Omega)$ in addition of Assumptions I-II, then the unique mild solution $w \in C([0,\infty); L^2(\Omega)) \cap C^1(\mathbb{R}_+; L^2(\Omega))$ is a classical solution to the system (6) on $(0,\infty)$.

Now we present an observer to recover the total disturbance $F(w,t)$ from measuring the output $y(x,t) = w(x,t)|_{\Gamma_1}$ and the control $u(x,t)$. As the disturbance signal and the control action u appear in the same channel, we need synthesizing an observer to separate the uncertainty from the control action Indeed, to separate the uncertainty from the control action, we consider the candidate observer described by the following PDEs:

$$\begin{cases} v_t(x,t) = \Delta v(x,t),\ x \in \Omega,\ t > 0, \\ v(x,t)|_{\Gamma_0} = 0,\ \partial_\nu v(x,t)|_{\Gamma_1} = u(x,t),\ t > 0, \\ v(x,0) = v_0(x),\ x \in \Omega, \end{cases} \tag{7}$$

and

$$\begin{cases} \tilde{p}_t(x,t) = \Delta \tilde{p}(x,t),\ x \in \Omega,\ t > 0, \\ \tilde{p}(x,t)|_{\Gamma_0} = 0,\ \tilde{p}(x,t)|_{\Gamma_1} = y(x,t) - v(x,t)|_{\Gamma_1}, \\ \tilde{p}(x,0) = \tilde{p}_0(x),\ x \in \Omega, \end{cases} \tag{8}$$

where v_0 and $\tilde{p}_0$ are the initial states that may be chosen arbitrarily. Notice that the output measurement $y(x,t) = w(x,t)|_{\Gamma_1}$ is known from the sensor as well as the control signal $u(x,t)$. These measurements are utilized in the observer.

In the spirit of the ADRC control design, beside the observer for the model uncertainty a state observer is also proposed as follows:

$$\begin{cases} \tilde{w}_t(x,t) = \Delta\tilde{w}(x,t), \ \ \tilde{w}(x,t)|_{\Gamma_0} = 0, \\ \partial_\nu\tilde{w}(x,t)|_{\Gamma_1} = \partial_\nu\tilde{p}(x,t)|_{\Gamma_1} + u(x,t), \\ \tilde{w}(x,0) = \tilde{w}_0(x), \end{cases} \tag{9}$$

where $\partial_\nu\tilde{p}(x,t)$ obtained from (8) replaces the unknown part F and $\tilde{w}$ is the estimated state of w.

Theorem 2: Let η, $u \in H^1_{loc}(\mathbb{R}_+; L^2(\Gamma_1))$ and let Assumptions I-II be satisfied. Then the observer (7)-(9) for the system (6) is well-posed: for each initial state $(w_0, v_0, \tilde{p}_0, \tilde{w}_0) \in H^1_{\Gamma_0}(\Omega) \times [L^2(\Omega)]^3$, the system (7)-(9) and (6) admits a unique solution $(w, v, \tilde{p}, \tilde{w}) \in C([0,\infty); [L^2(\Omega)]^4)$ and there are positive constants L and ω such that

$$\|w(\cdot,t) - \tilde{w}(\cdot,t)\|_{L^2(\Omega)} \leqslant Le^{-\omega t}\left[\|w_0 - \tilde{w}_0\|_{L^2(\Omega)} + \|w_0 - v_0 - \tilde{p}_0\|_{L^2(\Omega)}\right] \ \forall\, t \geqslant 0. \tag{10}$$

Moreover the uncertainty recovered by the observer has exponential convergence: for any $t_0 > 0$ there exists a real number $L_1 > 0$ such that

$$\begin{aligned} &\|\partial_\nu\tilde{p}(\cdot,t) - F(w(\cdot,t),t)\|_{L^2(\Gamma_1)} \\ &\leqslant L_1 e^{-\omega t/2}\|w_0 - v_0 - \tilde{p}_0\|_{L^2(\Omega)} \ \ \forall\, t \geqslant t_0 \end{aligned} \tag{11}$$

where $\omega > 0$ is the same as in (10).

Once the total disturbance is recovered, we design a feedback control law to stabilize system (6). It is sufficient to cancel the total disturbance because stabilizing the system without uncertainty is trivial (if no optimal convergence rate is attempted). In view of (11), the controller is designed as

$$u(x,t) = -\partial_\nu\tilde{p}(x,t), \quad x \in \Gamma_1, \ t \geqslant 0, \tag{12}$$

where $\tilde{p}$ is the solution of the system (8). Under the feedback control law (12), the closed-loop system is described by

$$\begin{cases} w_t(x,t) = \Delta w(x,t), \ w(x,t)|_{\Gamma_0} = 0, \\ \partial_\nu w(x,t)|_{\Gamma_1} = F(w(x,t),t) - \partial_\nu\tilde{p}(x,t)|_{\Gamma_1}, \\ v_t(x,t) = \Delta v(x,t), \ v(x,t)|_{\Gamma_0} = 0, \\ \partial_\nu v(x,t)|_{\Gamma_1} = -\partial_\nu\tilde{p}(x,t)|_{\Gamma_1}, \\ \tilde{p}_t(x,t) = \Delta\tilde{p}(x,t), \ \tilde{p}(x,t)|_{\Gamma_0} = 0, \\ \tilde{p}(x,t)|_{\Gamma_1} = w(x,t)|_{\Gamma_1} - v(x,t)|_{\Gamma_1}, \end{cases} \tag{13}$$

completed by the initial condition

$$(w(x,0), v(x,0), \tilde{p}(x,0)) = (w_0(x), v_0(x), \tilde{p}_0(x)).$$

Our main result can be stated as follows.

Theorem 3: Let $\eta \in H^1_{loc}(\mathbb{R}_+; L^2(\Gamma_1))$. Suppose that $F : L^2(\Omega) \times \mathbb{R}_+ \to L^2(\Gamma_1)$ satisfies Assumptions I-II. Then, for every initial condition $(w_0, v_0, \tilde{p}_0) \in H^1_{\Gamma_0}(\Omega) \times [L^2(\Omega)]^2$ such that $v_0 + \tilde{p}_0 \in H^1_{\Gamma_0}(\Omega)$ and $w_0|_{\Gamma_1} = (v_0 + \tilde{p}_0)|_{\Gamma_1}$, the system (13) admits a unique classical solution $(w(t), v(t), \tilde{p}(t)) \in C([0,\infty); [L^2(\Omega)]^3)$ such that the following statements hold:

(i) The plant system is exponentially stabilized: there are positive constants $L_2 > 0$ and $\omega_2 > 0$ such that

$$\|w(\cdot,t)\|_{L^2(\Omega)} \leqslant L_2 e^{-\omega_2 t}\left(\|w_0 - v_0 - \tilde{p}_0\|_{L^2(\Omega)} + \|w_0\|_{L^2(\Omega)}\right) \ \forall\, t \geqslant 0. \tag{14}$$

(ii) If we assume further that

$$\sup_{t\in[0,\infty)} \|F(0,t)\|_{L^2(\Gamma_1)} < +\infty, \tag{15}$$

then all the states of the closed-loop system are uniformly bounded in time, i.e.,

$$\sup_{t\in[0,\infty)} \|(w(\cdot,t), v(\cdot,t), \tilde{p}(\cdot,t))\|_{[L^2(\Omega)]^3} < +\infty. \tag{16}$$

(iii) The uncertainty estimated by $\frac{\partial\tilde{p}(\cdot,t)}{\partial\nu}\Big|_{\Gamma_1}$ has exponential convergence: given $t_0 > 0$ there is a real number $L_3 > 0$ such that

$$\begin{aligned} &\|\partial_\nu\tilde{p}(\cdot,t) - F(w(\cdot,t),\cdot,t)\|_{L^2(\Gamma_1)} \leqslant \\ &L_3 e^{-\frac{\omega_2 t}{2}}\left[\|w_0 - v_0 - \tilde{p}_0\|_{L^2(\Omega)} + \|w_0\|_{L^2(\Omega)}\right] \ \forall\, t \geqslant t_0 \end{aligned} \tag{17}$$

where $\omega_2 > 0$ is the same as in (14).

III. Proofs of the main results

Although the term $F(w(x,t),t)$ is unknown, the well-posedness of the system (6) can be easily proved if some conditions are imposed on the total disturbance $F(w(x,t),t)$ and the initial state w_0. As we are mainly interested in the observer design, the proof of the well-posedness for the open-loop system (6) is postponed to the Appendix. Due to lack of space in this note, we give only essential ideas to the proof and the detail is referred to the related references. When the uncertainty and the disturbance are exactly known, many results known about existence and blow-up of the solution to the heat equation with a nonlinear boundary condition. Some of these results may be found in [8], [13], [1] and the reference therein.

To prove the main results we first consider the following system:

$$\begin{cases} \Phi_t(x,t) = \Delta\Phi(x,t), \ \Phi(x,t)|_{\Gamma_0} = 0, \\ \partial_\nu\Phi(x,t)|_{\Gamma_1} = u(t), \ \Phi(x,0) = \Phi_0(x) \ \forall\, x \in \Omega, \end{cases} \tag{18}$$

where $u \in L^2_{loc}(\mathbb{R}_+; L^2(\Gamma_1))$. We set $X = L^2(\Omega)$ throughout the paper where X is equipped with the inner product $< f, g >_X = \int_\Omega f(x)\overline{g(x)}dx$. Let us define A by

$$\begin{aligned} D(A) &= \left\{p \in H^2(\Omega) \ \mid p|_{\Gamma_0} = 0, \, \partial_\nu p|_{\Gamma_1} = 0\right\}, \\ Ap &= \Delta p \ \forall\, p \in D(A). \end{aligned} \tag{19}$$

It is easy to see that A is the generator of an exponentially stable analytic semigroup on X. Hence $0 \in \rho(A)$ resolvent set of A. It is well known (cf. [15, p.668]) that

$$D((-A)^{1/2}) = H^1_{\Gamma_0}(\Omega), \tag{20}$$

and that $(-A)^{1/2}$ is an canonical isomorphism from $H^1_{\Gamma_0}(\Omega)$ onto $L^2(\Omega)$. We consider $L^2(\Omega)$ as the pivot space. Then the following Gelfand triple compact inclusions are valid:

$$\begin{aligned} H^1_{\Gamma_0}(\Omega) &= D((-A)^{1/2}) \hookrightarrow L^2(\Omega) \\ &= L^2(\Omega)' \hookrightarrow D((-A)^{1/2})' = H^{-1}_{\Gamma_0}(\Omega) \end{aligned} \tag{21}$$

where $H_{\Gamma_0}^{-1}(\Omega)$ is the dual space of $H_{\Gamma_0}^{1}(\Omega)$ with the pivot space $L^2(\Omega)$. The operator A has an extension still noted $A \in \mathcal{L}(H_{\Gamma_0}^{1}(\Omega), H_{\Gamma_0}^{-1}(\Omega))$ defined by

$$\langle Ax, z\rangle_{H_{\Gamma_0}^{-1}(\Omega),H_{\Gamma_0}^{1}(\Omega)} = -\langle(-A)^{1/2}x, (-A)^{1/2}z\rangle_X \quad \forall\, x, z \in H_{\Gamma_0}^{1}(\Omega). \tag{22}$$

Notice that $X_{-1/2} = H_{\Gamma_0}^{-1}(\Omega)$ is the completion of X with respect to the norm $\|f\|_{-1/2} = \left\|(-A)^{-1/2}f\right\|_X$. Similarly A admits also an extension $A \in \mathcal{L}(X, X_{-1})$ where X_{-1} is the completion of X with respect to the norm $\|f\|_{-1} = \|A^{-1}f\|$ (see [20] for more details). Moreover A is the generator of an exponentially stable analytic semigroup on Z and $\left\|(-A)^{\alpha}e^{tA}\right\|_{\mathcal{L}(Z)} \leqslant M_\alpha t^{-\alpha}e^{-\mu t}$ for some constants M_α, $\mu > 0$ and $\forall\, \alpha \geqslant 0$, $Z = X$, $X_{-1/2}$ or X_{-1} [16, p.74]. Finally $D((-A)^{\alpha})$ denotes the Hilbert space normed as $\|f\|_\alpha = \|(-A)^{\alpha}f\|_X\ \forall\, \alpha \geqslant 0$ and $f \in D((-A)^{\alpha})$.

Define the Neumann map $\Upsilon \in \mathcal{L}(L^2(\Gamma_1),\, H_{\Gamma_0}^{3/2}(\Omega))$ ([15, p. 668]) by $\Upsilon u = \phi$ if and only if

$$\Delta\phi = 0, \quad \phi|_{\Gamma_0} = 0, \quad \partial_\nu\phi|_{\Gamma_1} = u. \tag{23}$$

It is important to note that $H_{\Gamma_0}^{\frac{3}{2}}(\Omega) \subset D\left((-A)^{\frac{3}{4}-\sigma}\right)\ \forall\, \sigma > 0$ with continuous embedding. By using the Neumann map the system (18) is written as

$$\Phi_t(t) = A\Phi(t) + Bu(t) \quad \text{in} \quad H_{\Gamma_0}^{-1}(\Omega), \tag{24}$$

where $B \in \mathcal{L}(L^2(\Gamma_1), H_{\Gamma_0}^{-1}(\Omega))$ is given by

$$Bu = -A\Upsilon u\ \ \forall\, u \in L^2(\Gamma_1), \tag{25}$$

because $\Phi_t(t) = \Delta\Phi - \Delta\phi = \Delta(\Phi - \phi) = A(\Phi - \phi) = A\Phi - A\Upsilon u$. Notice that the extension of A has been used.

Set $\mathbb{T}_t = e^{tA}$. The following result is well known and the reader may be referred to [22] for a proof.

Proposition 1: The control operator B is admissible for the semigroup $\mathbb{T}_t$.

To prove Theorem 2, we study the following useful system on the Hilbert space $\mathcal{X} = X^2$:

$$\begin{cases} \varphi_t(x,t) = \Delta\varphi(x,t), \quad \varphi(x,t)|_\Gamma = 0, \\ \phi_t(x,t) = \Delta\phi(x,t), \quad \phi(x,t)|_{\Gamma_0} = 0, \\ \partial_\nu\phi(x,t)|_{\Gamma_1} = \partial_\nu\varphi(x,t)|_{\Gamma_1}, \end{cases} \tag{26}$$

from the initial condition $(\varphi(\cdot,0), \phi(\cdot,0) = (\varphi_0, \phi_0) \in \mathcal{X}$, where $\mathcal{X}$ is equipped with the inner product

$$\langle (p_1,q_1),(p_2,q_2)\rangle_{\mathcal{X}} = \int_\Omega [\alpha p_1(x)\overline{p_2(x)} + q_1(x)\overline{q_2(x)} - p_1(x)\overline{q_2(x)} - q_1(x)\overline{p_2(x)}]dx, \tag{27}$$

with α a positive constant such that $\alpha > 1$. Given a real number δ satisfying $1 < \delta < \alpha$, by Young's inequality we have

$$\|(p,q)\|_{\mathcal{X}}^2 \geqslant c_1\left[\|p\|_X^2 + \|q\|_X^2\right]\ \forall\, (p,q) \in \mathcal{X}, \tag{28}$$

where $c_1 = \min\left\{\alpha - \delta, 1 - \delta^{-1}\right\} > 0$.

The inequality (28) implies that the inner product (27) induces an another equivalent norm to the usual one on $\mathcal{X}$. Define the operator $\mathcal{A}$ by

$$\begin{cases} D(\mathcal{A}) = \left\{ \begin{pmatrix} p \\ q \end{pmatrix} \in [H^2(\Omega)]^2 \;\middle|\; \begin{matrix} p|_\Gamma = 0 \\ q|_{\Gamma_0} = 0 \\ \partial_\nu p|_{\Gamma_1} = \partial_\nu q|_{\Gamma_1} \end{matrix} \right\}, \\ \mathcal{A}(p,q) = (\Delta p, \Delta q) \quad \forall\, (p,q) \in D(\mathcal{A}). \end{cases} \tag{29}$$

Then the system (26) is written abstractly as

$$\begin{cases} (\varphi_t(\cdot,t), \phi_t(\cdot,t)) = \mathcal{A}(\varphi(\cdot,t), \phi(\cdot,t)), \\ (\varphi(\cdot,0), \phi(\cdot,0)) = (\varphi_0, \phi_0). \end{cases} \tag{30}$$

Lemma 1: The operator $\mathcal{A}$ defined by (29) generates an exponentially stable analytic semigroup $e^{t\mathcal{A}}$ on $\mathcal{X}$: there exist two positive constants M and μ such that

$$\|e^{\mathcal{A}t}\| \leqslant Me^{-\mu t}\ \ \forall\, t \geqslant 0. \tag{31}$$

For each $(\varphi_0, \phi_0) \in \mathcal{X}$ system (30) has a unique classical solution on $(0,\infty)$. Moreover, for each integer $m \in \mathbb{N}$ there exists a real number $M_m > 0$ such that

$$\|\mathcal{A}^m e^{t\mathcal{A}}\| \leqslant M_m t^{-m}e^{-\mu t}\ \ \forall\, t > 0. \tag{32}$$

Proof of Lemma 1: By [4, p.101,Theorem 4.6], $\mathcal{A}$ generates a bounded analytic semigroup on $\mathcal{X}$ if and only if the operators $e^{\pm i\vartheta}\mathcal{A}$ generate bounded C_0 semigroups on $\mathcal{X}$ for some $\vartheta \in (0, \frac{\pi}{2})$. For any $\vartheta \in (0, \frac{\pi}{2})$, we set

$$\mathcal{A}_\vartheta = e^{\pm i\vartheta}\mathcal{A}. \tag{33}$$

We prove that, for some $\vartheta \in (0, \pi/2)$, the operator $\mathcal{A}_\vartheta$ generates a C_0-semigroup of contractions on $\mathcal{X}$. Indeed, $D(\mathcal{A}_\vartheta) = D(\mathcal{A})$ is dense in $\mathcal{X}$ since $C_0^\infty(\Omega) \subset D(\mathcal{A})$. For any $(p,q) \in D(\mathcal{A}_\vartheta) = D(\mathcal{A})$, by the divergence theorem we have

$$\begin{aligned} &\langle \mathcal{A}_\vartheta(p,q),(p,q)\rangle_{\mathcal{X}} = e^{\pm i\vartheta}\langle(\Delta p, \Delta q),(p,q)\rangle_{\mathcal{X}} \\ &= e^{\pm i\vartheta}\Big[\int_\Omega (\alpha\Delta p\overline{p} + \Delta q\overline{q} - \Delta p\overline{q} - \Delta q\overline{p})\,dx\Big] \\ &= e^{\pm i\vartheta}\Big[\int_{\Gamma_1}(\partial_\nu q - \partial_\nu p)\,\bar{q}d\Gamma - \int_\Omega\left(\alpha\left|\nabla p\right|^2 + \left|\nabla q\right|^2\right)dx \\ &+ \int_\Omega(\nabla p\cdot\nabla\overline{q} + \nabla q\cdot\nabla\overline{p})\,dx\Big] = e^{\pm i\vartheta}\Big[-\alpha\|\nabla p\|_{L^2(\Omega)}^2 \\ &-\|\nabla q\|_{L^2(\Omega)}^2 + 2\Re e\left(\int_\Omega \nabla p\nabla\overline{q}dx\right)\Big]. \end{aligned} \tag{34}$$

As

$$\begin{aligned} &\Re e\left[e^{\pm i\vartheta}2\Re e\left(\int_\Omega \nabla p\nabla\overline{q}dx\right)\right] \\ &\leqslant \cos\vartheta\left(\|\nabla p\|_X^2 + \|\nabla q\|_X^2\right), \end{aligned} \tag{35}$$

by simple computations, it follow from (35) and (34) that

$$\Re e\,\langle\mathcal{A}_\vartheta(p,q),(p,q)\rangle_{\mathcal{X}} \leqslant -(\alpha-1)\cos\vartheta\ \|\nabla p\|_X^2 \leqslant 0. \tag{36}$$

Therefore the operator $\mathcal{A}_\vartheta$ is dissipative in $\mathcal{X}\ \forall\, \vartheta \in (0, \pi/2)$.

Notice that $0 \in \rho(\mathcal{A}_\vartheta)$ if and only if $0 \in \rho(\mathcal{A})$. If $0 \in \rho(\mathcal{A}_\vartheta)$, by the Lumer-Phillips theorem [16, Theorem 1.4.3] $\mathcal{A}_\vartheta$ generates a C_0 semigroup of contractions on $\mathcal{X}$. So it is sufficient to prove $\mathcal{A}^{-1} \in \mathcal{L}(\mathcal{X})$.

Indeed, for any $(\hat{p},\hat{q})\in\mathcal{X}$ we solve want to the equation

$$\mathcal{A}(p,q)=(\hat{p},\hat{q}) \tag{37}$$

or, the PDEs

$$\begin{cases}\Delta p(x)=\hat{p}(x), \ \ x\in\Omega,\\ p(x)|_{\Gamma}=0,\end{cases} \tag{38}$$

and

$$\begin{cases}\Delta q(x)=\hat{q}(x), \ \ x\in\Omega,\\ q(x)|_{\Gamma_0}=0,\ \partial_\nu q(x)|_{\Gamma_1}=\partial_\nu p(x)|_{\Gamma_1}.\end{cases} \tag{39}$$

From the theory of elliptic equations [2, p.181, Théorème IX.25], equation (38) has a unique solution $p\in H^2(\Omega)\cap H_0^1(\Omega)$ such that $\|p\|_{H^2(\Omega)}\leqslant c_1\|\hat{p}\|_{L^2(\Omega)}$ for some constant $c_1>0$. Moreover $\partial_\nu p\in H^{1/2}(\Gamma_1)$ satisfies the inequality $\|\partial_\nu p\|_{H^{1/2}(\Gamma_1)}\leqslant c_2\|p\|_{H^2(\Omega)}$ for some $c_2>0$. By the theory of elliptic equations [19, p.429, Proposition 13.6.16], equation (39) has a unique solution $q\in H^2(\Omega)\cap H^1_{\Gamma_0}(\Omega)$ such that $\|q\|_{H^2(\Omega)}\leqslant c_3\left(\|\partial_\nu p\|_{H^{\frac{1}{2}}(\Gamma_1)}+\|\hat{q}\|_{L^2(\Omega)}\right)$ for some $c_3>0$. As the injection $H^2(\Omega)\hookrightarrow L^2(\Omega)$ is continuous, there exists some constant $c_4>0$ such that

$$\|p\|^2_{L^2(\Omega)}+\|q\|^2_{L^2(\Omega)}\leqslant c_4\left(\|\hat{p}\|^2_{L^2(\Omega)}+\|\hat{q}\|^2_{L^2(\Omega)}\right).$$

Hence $\mathcal{A}^{-1}$ is bounded. Therefore $\mathcal{A}$ is the infinitesimal generator of a bounded analytic semigroup on $\mathcal{X}$ (see [4, p.101,Theorem 4.6]). Since $0\in\rho(\mathcal{A})$, the analytic semigroup $e^{t\mathcal{A}}$ is exponentially stable and all the rest follows from [16, Theorem 6.13, p.74]. □

Proof of Theorem 2: Consider the linear invertible transformation as follows:

$$\begin{pmatrix}w\\ v\\ e\\ \varepsilon\end{pmatrix}=\begin{pmatrix}I&0&0&0\\ 0&I&0&0\\ I&-I&-I&0\\ I&0&0&-I\end{pmatrix}\begin{pmatrix}w\\ v\\ \tilde{p}\\ \tilde{w}\end{pmatrix}. \tag{40}$$

Let consider the following PDEs

$$\begin{cases}e_t(x,t)=\Delta e(x,t),\ e(x,t)|_{\Gamma}=0,\\ \varepsilon_t(x,t)=\Delta\varepsilon(x,t),\ \varepsilon(x,t)|_{\Gamma_0}=0,\\ \partial_\nu\varepsilon(x,t)|_{\Gamma_1}=\partial_\nu e(x,t)|_{\Gamma_1},\\ (e(x,0),\varepsilon(x,0))=(e_0(x),\varepsilon_0(x)),\quad x\in\Omega.\end{cases} \tag{41}$$

It is easy to see that $(w(x,t),v(x,t),\tilde{p}(x,t),\tilde{w}(x,t))$ is a unique classical solution of the system of PDEs (6)-(9) if and only if $(w(x,t),v(x,t),e(x,t),\varepsilon(x,t))$ is that of the system of PDEs (6), (7) and (41). So it is equivalent to study existence, uniqueness and regularity of the solution to the latter system of PDEs.

By our Lemma 1, for any $(e_0,\varepsilon_0)\in\mathcal{X}$ the system (41) admits a unique classical solution $(e(\cdot,t),\varepsilon(\cdot,t))\in C([0,\infty);\mathcal{X})\cap C^1((0,\infty);\mathcal{X})$ such that

$$\|(e(\cdot,t),\varepsilon(\cdot,t))\|_{\mathcal{X}}\leqslant Me^{-\mu t}\|(e_0,\varepsilon_0)\|_{\mathcal{X}}\ \ \forall\, t>0, \tag{42}$$

where $M>0$ and $\mu>0$ are defined in (31). This proves (10).

The PDE (7) can be written as

$$\begin{cases}v_t(\cdot,t)=Av(\cdot,t)+Bu(\cdot,t)\quad\text{in}\quad H^{-1}_{\Gamma_0}(\Omega),\\ v(\cdot,0)=v_0.\end{cases} \tag{43}$$

By our Proposition 1, B is admissible for the semigroup $\mathbb{T}_t$. As $u\in H^1_{loc}([0,\infty);L^2(\Gamma_1))$, the system (43) has a unique classical solution given by

$$v(\cdot,t)=e^{tA}v_0+\int_0^t e^{(t-s)A}Bu(\cdot,s)ds.$$

Similarly the PDE (6) is written as

$$\begin{cases}w_t(\cdot,t)=Aw(\cdot,t)+B[F(w(\cdot,t),t)+u(\cdot,t)]\\ w(\cdot,0)=w_0\in H^1_{\Gamma_0}(\Omega).\end{cases} \tag{44}$$

Since $\eta,\,u\in H^1_{loc}([0,\infty);L^2(\Gamma_1))$, by our Theorem 1 the system (44) admits a unique classical solution given by

$$w(\cdot,t)=e^{tA}w_0+\int_0^t e^{(t-s)A}B[F(w(\cdot,s),s)+u(\cdot,s)]ds.$$

Therefore, by the inverse transformation the system of PDEs (6)-(9) admits a unique classical solution $(w,v,\tilde{p},\tilde{w})\in C([0,\infty);X^4)\cap C^1((0,\infty);X^4)$.

From the PDEs (6) and (7)-(8), each classical solution satisfies the following:

$$\partial_\nu e|_{\Gamma_1}=[\partial_\nu w-\partial_\nu v-\partial_\nu\tilde{p}]|_{\Gamma_1}=F(w,t)-\partial_\nu\tilde{p}|_{\Gamma_1}. \tag{45}$$

On the other hand, by exponential stability of the analytic semigroup and by (41), given $t_0>0$ there exists a real constant $L_4>0$ such that, for all $t\geqslant t_0$, the following inequality holds:

$$\|\partial_\nu e(t)\|_{L^2(\Gamma_1)}\leqslant L_4e^{-\mu t/2}\|w_0-v_0-\tilde{p}_0\|_{L^2(\Omega)}. \tag{46}$$

So (11) follows from (45) and (46). To complete the proof we have only to prove (46). Indeed, by Lemma 1 and by [19, p.429, Proposition 13.6.16] the following inequalities hold for some $K_i>0, i=1,2,3$:

$$\begin{aligned}\|\partial_\nu e(t)\|_{L^2(\Gamma_1)}&\leqslant K_1\|e(t)\|_{H^2(\Omega)}\\ &\leqslant K_2\left(\|Ae(t)\|_{L^2(\Omega)}+\|e(t)\|_{L^2(\Omega)}\right)\\ &\leqslant K_2\left(K_3t^{-1}e^{-\mu t/2}+Me^{-\mu t}\right)\|e_0\|_{L^2(\Omega)}\\ &\leqslant K2\left(K_3t_0^{-1}+M\right)e^{-\mu t/2}\|e_0\|_{L^2(\Omega)}\ \ \forall\, t\geqslant t_0.\end{aligned}$$

Hence the proof of Theorem 2 is complete. □

Remark 1: If we are only interested in estimating w in (6), another simpler observer than (7)-(8) is described by:

$$\begin{cases}\hat{w}_t(x,t)=\Delta\hat{w}(x,t),\ x\in\Omega,\ t>0,\\ \hat{w}(x,t)|_{\Gamma_0}=0,\ \hat{w}(x,t)|_{\Gamma_1}=w|_{\Gamma_1},\\ \hat{w}(x,0)=\hat{w}_0(x),\end{cases} \tag{47}$$

where $\hat{w}$ is the estimate of state w. However, in this observer (47) the effects of the control and the unknown disturbance F are mixed together. We choose (7)-(8), instead of (47), to enable estimating the unknown part F. It is essential to estimate the unknown F to stabilize the closed-loop, see (13). For the case where the control and the uncertainty are not located in the same channel, the interested reader is referred to [22].

Proof of Theorem 3: Notice that $(w,v,\tilde{p})$ given in (13) is the whole state of the closed-loop system. For the stability

of the closed-loop system, we consider the new state (w, p, e) obtained by the invertible transformation as follows:

$$\begin{pmatrix} w \\ p \\ e \end{pmatrix} = \begin{pmatrix} I & 0 & 0 \\ I & -I & 0 \\ I & -I & -I \end{pmatrix} \begin{pmatrix} w \\ v \\ \tilde{p} \end{pmatrix}. \tag{48}$$

Then the new state satisfies the following PDE:

$$\begin{cases} e_t(x,t) = \Delta e(x,t),\ e(x,t)|_\Gamma = 0, \\ w_t(x,t) = \Delta w(x,t),\ w(x,t)|_{\Gamma_0} = 0, \\ \partial_\nu w(x,t)|_{\Gamma_1} = \partial_\nu e(x,t)|_{\Gamma_1}, \end{cases} \tag{49}$$

and

$$\begin{cases} p_t(x,t) = \Delta p(x,t),\ p(x,t)|_{\Gamma_0} = 0, \\ \partial_\nu p(x,t)|_{\Gamma_1} = F(w(x,t),t), \end{cases} \tag{50}$$

completed by the initial condition

$$\begin{cases} e(x,0) = w_0(x) - v_0(x) - \tilde{p}_0(x), \\ w(x,0) = w_0(x),\ \ p(x,0) = w_0(x) - v_0(x). \end{cases} \tag{51}$$

As mentioned above $(w(x,t), v(x,t), \tilde{p}(x,t))$ is a unique classical solution of (13) if and only if $(w(x,t), p(x,t), e(x,t))$ via (48) is a unique solution of PDEs (49)-(51). Note that the original closed-loop system (13) is a nonlinear system, while the transformed system (49)-(50) having a triangle structure is essentially linear. Indeed, in (50) the nonlinear term F becomes an inhomogeneous term of a linear system.

By Lemma 1, we have

$$\|(e(\cdot,t), w(\cdot,t))\|_{\mathcal{X}} \leqslant Me^{-\mu t}\|(e_0, w_0)\|_{\mathcal{X}}\ \forall\ t \geqslant 0. \tag{52}$$

Thus the inequality (14) is proved. (This implies that the control plant is exponentially stabilized.)

The PDE (50) is written as

$$\begin{cases} p_t(t) = Ap(t) + B[F(w(\cdot,t),t)], \\ p(0) = p_0. \end{cases} \tag{53}$$

By our Lemma 1, $w(\cdot,t) \in C([0,\infty); L^2(\Omega))$ is a classical solution and uniformly bounded in time. We claim that if $(e_0, w_0) \in D((-\mathcal{A})^{1/2})$ (satisfied by the condition imposed on $(w_0, v_0, \tilde{p}_0)$), then $w(\cdot,t)$ is Hölder continuous from $[0,\infty)$ to $L^2(\Omega)$ with exponent $\frac{1}{2}$ (see a proof from Lemma 2 in Appendix). Since $\xi(w)$ is locally Lipschitz continuous from $L^2(\Omega)$ to $L^2(\Gamma_1)$, it follow that $\xi(w(\cdot,t))$ is globally Hölder continuous from $[0,\infty)$ to $L^2(\Gamma_1)$ with exponent $\frac{1}{2}$. By $\eta \in H^1_{loc}(\mathbb{R}_+; L^2(\Gamma_1))$ this implies that $F(w(\cdot,t),t)$ is Hölder continuous from $[0,\infty)$ to $L^2(\Gamma_1)$. By [16, p.113, Corollary 2.11] and [15, p.668], we prove that the PDE (50) or (53) has a unique classical solution on $(0,\infty)$. Hence the closed-loop system has a unique classical solution.

If $\sup_{t\in\mathbb{R}_+}\|F(0,t)\|_{L^2(\Gamma_1)} < \infty$, then $\|\eta(t)\|_{L^2(\Gamma_1)}$ is bounded (see Assumption I). Therefore $\|F(w(\cdot,t),t)\|_{L^2(\Gamma_1)}$ is bounded on $(0,\infty)$. By exponential stability of e^{tA} and the admissibility of B, it follows from [21, Lemma1.1] that for $p_0 = 0$ and some constant $L_1 > 0$

$$\sup_{t\in\mathbb{R}_+}\|p(\cdot,t)\|_{L^2(\Omega)} \leqslant L_1\|F(w(\cdot,t),t)\|_{L^\infty(\mathbb{R}_+;L^2(\Gamma_1))}. \tag{54}$$

So (16) can be obtained by (54), (52), (53) and (48).

Similar to the proof of Theorem 2, we also have (46). Thus (17) can be obtained from (45) and (46). The proof of Theorem 3 is complete. □

Remark 2: The restriction on the initial condition that $w_0 \in H^1_{\Gamma_0}(\Omega)$, $(v_0 + \tilde{p}_0) \in H^1_{\Gamma_0}(\Omega)$ and $(v_0 + \tilde{p}_0)|_{\Gamma_1} = w_0|_{\Gamma_1}$ is needed in our proof that the solution is a classical solution. Without the restriction we can not prove that the solution is a classical one. An example of model uncertainty is the nonlocal uncertainty such as $\xi(w(\cdot,t))(x) = \int_\Omega w(s,t)ds \cdot \beta(x)$ for $\beta \in L^2(\Gamma_1)$.

IV. Numerical simulations

In this section, we present some numerical simulations for the closed-loop system (13) to demonstrate our theoretical results visually. Similar to [22], we consider a 2D example with the space domain

$$\begin{cases} \Omega = \left\{(x_1,x_2) \in \mathbb{R}^2 | 1 < x_1^2 + x_2^2 < 9\right\}, \\ \Gamma_0 = \left\{(x_1,x_2) \in \mathbb{R}^2 | x_1^2 + x_2^2 = 1\right\}, \\ \Gamma_1 = \left\{(x_1,x_2) \in \mathbb{R}^2 | x_1^2 + x_2^2 = 9\right\}. \end{cases} \tag{55}$$

The initial state and the total disturbance are chosen as

$$\begin{cases} w(x_1,x_2,0) = 2\left(\sqrt{x_1^2+x_2^2} - 3\right)\left(\sqrt{x_1^2+x_2^2} - 1\right) \\ \qquad\qquad \times \left(\dfrac{4x_2^3}{(x_1^2+x_2^2)^{\frac{3}{2}}} - \dfrac{3x_2}{\sqrt{x_1^2+x_2^2}}\right), \\ v(x_1,x_2,0) = 0,\ \tilde{p}(x_1,x_2,0) = 0, \\ F(w(x_1,x_2,t),t) = \sin\arctan\dfrac{x_1}{x_2} + \eta(t), \end{cases} \tag{56}$$

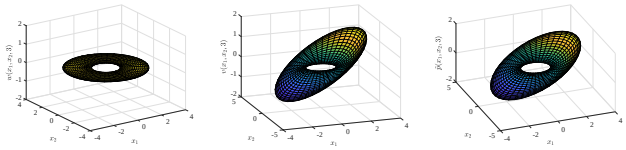
where $\eta(t) = \frac{2}{\pi}\sin^{-1}[\sin(10t - \frac{\pi}{2})]$ is the sawtooth disturbance. The example is a slight modification of the example in [22] (with the boundary condition on Γ_0 changed from Neumann to Dirichlet). Since the classical derivative of the sawtooth disturbance η does not exist, the boundedness assumption for the derivative of total disturbance that is required in the conventional ADRC is never satisfied. For numerical discretization, we first convert the 2D annulus Ω into a rectangle by the polar coordinate transformation, and then convert it back to the original coordinates for numerical values (see [22] for the polar coordinate transformation). The backward Euler method in time and the Chebyshev spectral method for polar variables are used to discretize transformed system. Here, we take the grid size $r_N = 21$ for γ, the grid size $\theta_N = 40$ for θ, and the time step $dt = 10^{-4}$. The numerical algorithm is programmed with Matlab [18].

The the final state of closed-loop system (13) is plotted in Figure 1. The disturbance and its estimation under the polar coordinates are plotted in Figure 2. In order to demonstrate the dynamic evolution of the closed-loop system, the state trace evolution $w(0,x_2,t)$

$$w(0,x_2,t),\quad x_2 \in [-3,-1],\ t \geqslant 0, \tag{57}$$

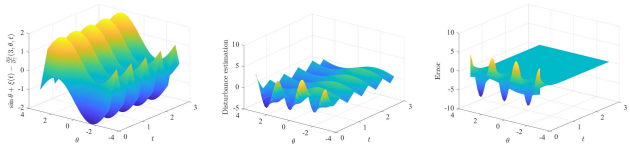
is plotted in Figure 3(b); the disturbance estimation result at $(x_1,x_2) = (0,-3)$ is shown in Figure 3(a).

From these Figures 1-3 we observe that the state of the control plant is stabilized effectively despite the presence of a total disturbance. Notice that $w(x_1, x_2, t)$ converges to zero as $t \to \infty$, but $v(x_1, x_2, t)$ and $\tilde{p}(x_1, x_2, t)$ do not (see Figure 1). Figure 2 shows that the disturbance estimation is effective and the dynamic evolution is smooth. This is the contribution of our paper because here we have removed the derivative boundedness assumption for the total disturbance term from the conventional ADRC. The disturbance trace estimation and the state trace in Figure 3 also validate effectiveness of the proposed control method.



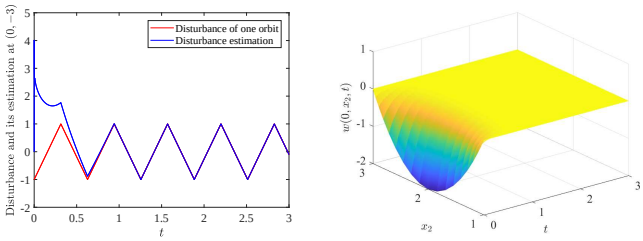
(a) $w(x_1, x_2, 3)$. (b) $v(x_1, x_2, 3)$. (c) $\tilde{p}(x_1, x_2, 3)$.

Fig. 1: The final state of system (13).



(a) Disturbance. (b) Estimation. (c) Estimation error.

Fig. 2: Disturbance and estimated disturbance under polar coordinates.



(a) Disturbance estimation at $(0, -3)$. (b) $w(0, x_2, t), -3 \leqslant x_2 \leqslant -1$.

Fig. 3: Disturbance estimation at the point $(x_1, x_2) = (0,-3)$ and the state trace evolution on the segment $\{0\} \times [-3, -1]$.

APPENDIX

Using the notation introduced in (19) we write the system PDE (6) abstractly as

$$\begin{cases} \dot{w}(t) = Aw(t) + B[F(w(t), t) + u(t)] \\ w(0) = w_0. \end{cases} \tag{58}$$

For the sake of simplicity we set $\mathbb{T}_t = e^{tA}$ and $X = L^2(\Omega)$. Given $T > 0$, we call $w \in C([0, T]; X)$ a mild solution of (58) if it satisfies the following integral equation for every $t \in [0, T]$:

$$w(t) = \mathbb{T}_t w_0 + \int_0^t \mathbb{T}_{t-s} B[F(w(s), s) + u(s))]ds. \tag{59}$$

We call $w \in C([0, T]; X)$ a classical solution of (58) if it is a mild solution and if the following three properties hold:
(a) $w(t)$ is continuously differentiable from $(0, T)$ to X;
(b) $w(t) + A^{-1}B[F(w(t), t) + u(t)] \in D(A)\ \forall\, t \in (0, T)$;
(c) the equation (58) is satisfied $\forall\, t \in (0, T)$.

Proof of Theorem 1: **(i)** Similar to [16, p.184, Theorem 1.2] and [21, Proposition 1.1] the integral equation (59) has a unique local solution $w \in C([0, T]; X)$ as B is admissible for $\mathbb{T}_t$. Indeed, without loss of generality we may take $\eta = 0$. For the sake of simplicity we set $g(t) = \mathbb{T}_{t-t_0} w_0 + \int_{t_0}^t \mathbb{T}_{t-s} Bu(s)ds$. By the admissibility of B, if $u \in L^2_{loc}(\mathbb{R}_+, L^2(\Gamma_1))$ we have $g \in C([0, T]; X)$ $\forall\, T > 0$ (see [20]). Let us take $t_0 \geqslant 0$, $\delta_1 > 0$, and $0 < \Delta t_0 \leqslant 1$ and let us consider the bounded closed set $S_0 = \{w \in C([t_0, t_0 + \Delta t_0]; X) \mid \|w - g\|_\infty \leqslant \delta_1\}$ where $\|w - g\|_\infty = \sup_{t\in[t_0,t_0+\Delta t_0]} \|w(t) - g(t)\|_X$. Define the nonlinear mapping $\Lambda : S_0 \to C([t_0, t_0 + \Delta t_0]; X)$ by $\Lambda(w(t)) = g(t) + \int_{t_0}^t \mathbb{T}_{t-s} B\xi(w(s))ds$. It is easy to check that, for any $w_1, w_2 \in S_0$ and some constants $K_{t_0}, L_{\delta_1} > 0$,

$$\|\Lambda(w_1(t)) - g(t)\|_X \leqslant K_{t_0} \|\xi(w_1(\cdot))\|_{L^2((t_0,t);L^2(\Gamma_1))} \leqslant K_{t_0} M_1 \sqrt{\Delta t_0}, \tag{60}$$

$$\|\Lambda(w_1(t)) - \Lambda(w_2(t))\|_X \leqslant K_{t_0} L_{\delta_1} \sqrt{\Delta t_0} \|w_1 - w_2\|_\infty, \tag{61}$$

where $M_1 = \sup_{w\in S_0, s\in[t_0,t_0+1]} \|\xi(w(s))\|_{L^2(\Gamma_1)}$. By taking $\Delta t_0 = \min\{1, \delta_1^2/(K_{t_0} M_1)^2, 1/(\sqrt{2} K_{t_0} M_1)^2\}$, the mapping Λ becomes a strict contraction from S_0 into S_0. Hence the equation $w(t) = \Lambda(w(t))$ or (59) admits a unique solution $w \in C([t_0, t_0 + \Delta t_0]; X)$. As t_0 is arbitrary, the process can be repeated to extend the solution to the maximal interval of existence $[0, T_{max})$. Moreover, if $T_{max} < \infty$, then $\lim_{t\to T^-_{max}} \|w(t)\| = \infty$.

(ii) Similar to [16, p.199, Theorem 3.3], if Assumption II is satisfied, then the mild solution is global. Indeed, for every $T > 0$, by taking $t_0 = 0$ and by (59) there is a constant $K_T > 0$ such that

$$\|w(t)\|_X \leqslant \|g(t)\|_X + K_T \|\xi(w(\cdot))\|_{L^2((0,t);L^2(\Gamma_1))}. \tag{62}$$

By using Assumption II, (62) implies that

$$\|w(t)\|^2 \leqslant 2 \sup_{t\in[0,T]} \|g(t)\|^2 + 4K_T^2 k^2 \int_0^t \left(1 + \|w(s)\|^2\right) ds.$$

By the Gronwall type lemma, $\|w(t)\|^2 \leqslant M_T e^{4K_T^2 k^2 T}$ holds for some constant $M_T > 0$. As $T > 0$ is arbitrary, the unique solution exists on $(0, \infty)$.

(iii) We claim that if η, $u \in H^1_{loc}(\mathbb{R}_+, L^2(\Gamma_1))$ and $w_0 \in H^1_{\Gamma_0}(\Omega)$, then the unique mild solution $w \in C([0, T]; L^2(\Omega))$ is Hölder continuous from $[0, T]$ into X with exponent $\upsilon = 1/2$, or, $\|w(t) - w(s)\|_X \leqslant H_w |t - s|^{1/2}$ $\forall\, t, s \in [0, T]$ and for some constant $H_w > 0$. By using the claim, we prove that $I_1(t) = \int_0^t A\mathbb{T}_{t-s} B[\xi(w(s)) - \xi(w(t))]ds$ is in $C((0, T); X)$. Indeed, let us fix $\sigma \in (0, 1/4)$. It is well-known that $(-A)^{\frac{3}{4}-\sigma}\Upsilon \in \mathcal{L}(L^2(\Gamma_1), X)$. By $B = -A\Upsilon$, Lipschitz continuity of ξ and Hölder continuity of w, the following inequalities hold true:

$$\|I_1(t)\|_X \leqslant \int_0^t \left\| (-A)^{\frac{5}{4}+\sigma} \mathbb{T}_{t-s} (-A)^{\frac{3}{4}-\sigma} \Upsilon [\xi(w(s)) - \xi(w(t))] \right\|_X ds$$

$$\leqslant \int_0^t M_1(t-s)^{-\frac{5}{4}-\sigma}K_1\|\xi(w(s))-\xi(w(t))\|_{L^2(\Gamma_1)}ds$$
$$\leqslant \int_0^t M_1K_1L_1H_w(t-s)^{-\frac{3}{4}-\sigma}ds = \left(\frac{4M_1K_1L_1H_w}{1-4\sigma}\right)t^{\frac{1}{4}-\sigma}.$$

Hence $I_1 \in C((0,T);X)$.

Without loss of generality set always $\eta = 0$. The mild solution satisfies the integral relation

$$w(t) = \mathbb{T}_t w_0 + \int_0^t \mathbb{T}_{t-s}B[\xi(w(s)) + u(s))]ds. \tag{63}$$

We set $I_2(t) = \mathbb{T}_t w_0 + \int_0^t \mathbb{T}_{t-s}Bu(s)ds$. By analyticity of the semigroup and $u \in H^1_{loc}(\mathbb{R}_+;L^2(\Gamma_1))$, $\dot{I}_2(t) = A\mathbb{T}_t w_0 + \mathbb{T}_t Bu(0) + \int_0^t \mathbb{T}_{t-s}Bu_s(s)ds \in C((0,T);X)$. By direct computations, $\dot{w}(t) = \dot{I}_2(t) + \mathbb{T}_t B\xi(w(t)) + I_1(t)$, where $I_1(t)$ is defined above. Hence $\dot{w}(\cdot) \in C((0,T);X)$. By direct computations it is easy to check that $w(t) + A^{-1}B[\xi(w(t)) + u(t)] = \mathbb{T}_t w_0 + A^{-1}\mathbb{T}_t B[u(0) + \xi(w(t))] + A^{-1}\left[\int_0^t \mathbb{T}_{t-s}Bu'(s)ds + I_1(t)\right] \in D(A)\,\forall\, t > 0$ and $\dot{w}(t) = Aw(t) + B[\xi(w(t)) + u(t)]\ \forall\, t > 0$. Hence $w(t)$ is a classical solution on $(0,T)$.

Let us prove the claim. Put $\epsilon(t) = w(t+h) - w(t)\ \forall\, t \in [0,T]$ and $h \geqslant 0$. It is easy to check that

$$\epsilon(t) = \mathbb{T}_t(\mathbb{T}_h - I)w_0 + I_3(t) + I_4(t) + \int_0^h \mathbb{T}_{t+h-s}B\xi(w(s))ds + \int_0^t \mathbb{T}_{t-s}B\epsilon(s)ds, \tag{64}$$

where $I_3(t) = \int_0^h \mathbb{T}_{h-s}Bu(t+s)ds$ and $I_4(t) = (\mathbb{T}_h - I)\int_0^t \mathbb{T}_{t-s}Bu(s)ds$. By the analyticity and $w_0 \in D((-A)^{1/2})$ there is some constant $K_2 > 0$ (depending on T) such that

$$\|\mathbb{T}_t(\mathbb{T}_h - I)w_0\| + \|I_3(t)\| + \|I_4(t)\| + \left\|\int_0^h \mathbb{T}_{t+h-s}B\xi(w(s))ds\right\| \leqslant K_2\sqrt{h}. \tag{65}$$

By the Gronwall type lemma it follows from (64)-(65) that $\|\epsilon(t)\| \leqslant K_2\sqrt{h}$. So the claim is proved.

If further Assumption II is satisfied, then the classical solution is global. □

Lemma 2: Consider the subsystem PDE (49) and (51) with the initial data satisfying the same condition of Theorem 3. Then the solution $(e(t), w(t))$ is Hölder continuous from $[0,\infty)$ to $[L^2(\Omega)]^2$ with exponent $\upsilon = 1/2$.

Proof of Lemma 2: Consider the PDE satisfied by (e,ϵ) where $\epsilon = w - e$ with (e,w) solution of the PDE (49). Thus ϵ satisfies the PDE: $\epsilon_t = \Delta\epsilon$, $\epsilon|_{\Gamma_0} = 0$, $\partial_\nu\epsilon|_{\Gamma_1} = 0$, $\epsilon(0) = v_0 + \tilde{p}_0$. Hence $e(t) = e^{tA_D}e_0$ and $w(t) = e^{tA}(v_0 + \tilde{p}_0) + e^{tA_D}(w_0 - v_0 - \tilde{p}_0)$ where A is defined by (19) and A_D by

$$D(A_D) = H^2(\Omega) \cap H^1_0(\Omega),\ A_D f = \Delta f\ \forall\, f \in D(A_D).$$

As the semigroups are analytic, it is sufficient to take $(v_0 + \tilde{p}_0) \in H^1_{\Gamma_0}(\Omega)$ and $(w_0 - v_0 - \tilde{p}_0) \in H^1_0(\Omega)$ for $(e(t), w(t))$ to be Hölder continuous with exponent $\upsilon = 1/2$. □

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