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# ON ALMOST STABLE LINEAR WEINGARTEN HYPERSURFACES

JULIEN ROTH AND ABHITOSH UPADHYAY

ABSTRACT. We prove that generalized linear Weingarten hypersurfaces of the Euclidean space which are almost stable for the associated stability problem are geodesic spheres.

## 1. INTRODUCTION

The stability of hypersurfaces for volume preserving variational problems has a long history since the first result for the stability of constant mean curvature in the Euclidean space by Barbosa and do Carmo [3]. Many authors have been interested in stability problems in various contexts, like for other space forms and/or higher order mean curvatures (see [1, 2, 4, 5, 7, 8, 9, 24] for instance). More recently, Velásquez, de Sousa and de Lima [26] defined the notion of  $(r, s)$ -stability which arises from the second order variational problems for some generalized linear Weingarten hypersurfaces and it is the analogue of what is called the usual stability for the constant mean curvature hypersurfaces or the  $r$ -stability for constant higher order mean curvature hypersurfaces. They prove that the only closed  $(r, s)$ -stable hypersurfaces in the sphere or the hyperbolic space are geodesic hyperspheres. This result was later extended for hypersurfaces of the Euclidean space by da Silva, de Lima and Velásquez [11].

On the other hand, during the last decade, an intensive interest has been brought to the study of hypersurfaces of Euclidean spaces in an anisotropic setting. Many of the classical characterizations of the geodesic hyperspheres have an analogue with the Wulff Shape as characteristic hypersurface, like anisotropic Hopf or Alexandrov-type theorems (see [14, 15, 16, 18, 19, 20, 23]).

In [12], da Silva, de Lima and Velásquez gave an anisotropic analogue to their results in [11] and recover as particular cases the results of [6], [15] and [20]. This gives another characterization of the Wulff shape as the only hypersurface (up to translations and homotheties) which have linearly related anisotropic mean curvatures, without assuming that  $X(M)$  is convex or embedded.

In the present note, we prove that the result of da Silva, de Lima and Velásquez in [12] can be improved with a weaker assumption, namely with almost  $(r, s, F)$ -stability. Precisely, we prove the following:

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**Theorem 1.1.** *Let  $n, r, s$  be three integers so that  $0 \leq r \leq s \leq n - 2$ . Suppose that  $F : \mathbb{S}^n \rightarrow \mathbb{R}_+$  is a smooth function satisfying the following convexity assumption*

$$A_F = (\nabla dF + F \text{Id}_{T_x \mathbb{S}^n})_x > 0,$$

for all  $x \in \mathbb{S}^n$  and let  $X : M^n \rightarrow \mathbb{R}^{n+1}$  be a closed hypersurface with positive anisotropic  $(s+1)$ -th mean curvature  $H_{s+1}^F$ . Assume that the quantity  $\sum_{j=r}^s a_j b_j H_{j+1}^F$

is constant, where  $a_r, \dots, a_s$  are some nonnegative constants (with at least one of them non zero) and  $b_j = (j+1) \binom{n}{j+1}$  for any  $j \in \{r, \dots, s\}$ . Then there exists  $\varepsilon_0 > 0$  depending on  $n, r, s, a, \min_M(H_{s+1}^F), \min_M(H_s^F), \|S_F\|_\infty, F, R$  and  $V(M)$  so that if  $X : M^n \rightarrow \mathbb{R}^{n+1}$  is  $\varepsilon$ -almost  $(r, s, F)$ -stable for  $\varepsilon \in [0, \varepsilon_0]$ , then up to translations and homotheties,  $X(M)$  is the Wulff shape  $W_F$ , where  $R$  denotes the extrinsic radius of the hypersurface and  $a = (a_r, \dots, a_s)$ .

The notion of  $(r, s, F)$ -stability and  $\varepsilon$ -almost stability and will be defined in Section 2.2. See also (24) for the definition of  $H_{s+1}^F$ . This result also extends the result of the first author and Scheuer about almost  $r$ -stable hypersurfaces (see [22]) since one can recover it by taking  $F = 1$  and  $r = s$ . Note also that for  $\varepsilon = 0$ , we recover the result of [12].

## 2. PRELIMINARIES

**2.1. Anisotropic mean curvatures.** We recall the classical facts of anisotropic mean curvatures. Let  $F : \mathbb{S}^n \rightarrow \mathbb{R}_+$  be a smooth function satisfying the following convexity assumption

$$(1) \quad A_F = (\nabla dF + F \text{Id}_{T_x \mathbb{S}^n})_x > 0,$$

at any point  $x \in \mathbb{S}^n$ , in the sense of quadratic forms and where  $\nabla dF$  is the Hessian of  $F$ . Now, we consider the following map

$$\begin{aligned} \phi : \mathbb{S}^n &\longrightarrow \mathbb{R}^{n+1} \\ x &\longmapsto F(x)x + (\text{grad}_{\mathbb{S}^n} F)_x. \end{aligned}$$

The image  $W_F = \phi(\mathbb{S}^n)$  which is a smooth hypersurface of  $\mathbb{R}^{n+1}$  is called the Wulff shape of  $F$  and from the convexity condition (1),  $W_F$  is also convex. Note that if  $F$  is a positive constant  $c$ , the Wulff shape is just the sphere of radius  $c$ .

Let  $(M^n, g)$  be a closed, connected, oriented Riemannian manifold isometrically immersed into  $\mathbb{R}^{n+1}$  by  $X$  and denote by  $\nu$  its Gauss map. Let  $X^T = X - \langle X, \nu \rangle \nu$  be the projection of the position vector  $X$  on the tangent bundle of  $X(M)$ .

The (real-valued) second fundamental form  $B$  of the immersion is defined by

$$B(Y, Z) = \langle \bar{\nabla}_Y \nu, Z \rangle,$$

for any  $Y, Z \in \Gamma(TM)$ , where  $\langle \cdot, \cdot \rangle$  and  $\bar{\nabla}$  are respectively the Riemannian metric and the Riemannian connection of  $\mathbb{R}^{n+1}$ . We also denote by  $S$  the Weingarten operator, which is the  $(1, 1)$ -tensor associated with  $B$  via the metric  $g$ .

We consider  $N_F = \phi(\nu) : M \rightarrow W_F$ , the anisotropic Gauss map of  $M$ . We set  $S_F = -dN_F = -A_F \circ d\nu = A_F \circ S$  which is the anisotropic shape operator, also called  $F$ -Weingarten operator and its eigenvalues  $\kappa_1, \dots, \kappa_n$  are the anisotropic

principal curvatures. Now we recall the definition of the anisotropic mean curvature  $H^F$  and higher order mean curvatures  $H_r^F$ . First, at a point  $x \in M$ , we consider an orthonormal frame  $\{e_1, \dots, e_n\}$  of  $T_x M$ . For all  $k \in \{1, \dots, n\}$ , we set

$$\sigma_r = \sum_{\substack{1 \leq i_1, \dots, i_r \leq n \\ 1 \leq j_1, \dots, j_r \leq n}} \epsilon \begin{pmatrix} i_1 \cdots i_r \\ j_1 \cdots j_r \end{pmatrix} S_{i_1 j_1}^F \cdots S_{i_r j_r}^F,$$

where  $S_{ij}^F$  are the coefficients of the  $F$ -Weingarten operator. The symbols  $\epsilon \begin{pmatrix} i_1 \cdots i_r \\ j_1 \cdots j_r \end{pmatrix}$  are the usual permutation symbols which are zero if the sets  $\{i_1, \dots, i_r\}$  and  $\{j_1, \dots, j_r\}$  are different or if there exist distinct  $p$  and  $q$  with  $i_p = i_q$ . For all other cases,  $\epsilon \begin{pmatrix} i_1 \cdots i_r \\ j_1 \cdots j_r \end{pmatrix}$  is the signature of the permutation  $\begin{pmatrix} i_1 \cdots i_r \\ j_1 \cdots j_r \end{pmatrix}$ . Equivalently, we have

$$\sigma_r = \sum_{1 \leq i_1 < i_2 < \cdots < i_r \leq n} \kappa_{i_1} \kappa_{i_2} \cdots \kappa_{i_r}.$$

The  $r$ -th anisotropic mean curvature of the immersion is defined by

$$H_r^F = \binom{n}{r}^{-1} \sigma_r.$$

We denote simply the anisotropic mean curvature  $H_1^F$  by  $H^F$ . Moreover, for convenience, we set  $H_0^F = 1$  and  $H_{n+1}^F = 0$  by convention. Note that for the Wulff shape,  $\kappa_1 = \kappa_2 = \cdots = \kappa_n$  are constants so all the anisotropic mean curvatures  $H_r^F$  are also constants. For  $r \in \{1, \dots, n\}$ , the symmetric  $(1, 1)$ -tensor associated with  $H_r^F$  is

$$P_r = \frac{1}{r!} \sum_{\substack{1 \leq i, i_1, \dots, i_r \leq n \\ 1 \leq j, j_1, \dots, j_r \leq n}} \epsilon \begin{pmatrix} i \ i_1 \cdots i_r \\ j \ j_1 \cdots j_r \end{pmatrix} S_{i_1 j_1}^F \cdots S_{i_r j_r}^F e_i^* \otimes e_j^*.$$

We have these classical facts about the anisotropic mean curvatures (see [13] for instance).

**Lemma 2.1.** *For any  $r \in \{1, \dots, n\}$ , we have*

- (1)  $P_r$  is divergence-free,
- (2)  $\text{tr}(P_r) = (n - r)\sigma_r$ ,
- (3)  $\text{tr}(P_r S_F) = (r + 1)\sigma_{r+1}$ ,
- (4)  $\text{tr}(P_r S_F^2) = \sigma_1 \sigma_{r+1} - \sigma_{r+2}$ .

**Lemma 2.2.** *Let  $r \in \{1, \dots, n - 1\}$ . If  $H_{r+1} > 0$ , then for all  $j \in \{1, \dots, r\}$ ,*

- (1)  $H_j^F > 0$ ,
- (2)  $(H_j^F)^2 - H_{j+1} H_{j-1} \geq 0$ ,
- (3)  $H^F H_{j+1}^F - H_{j+2}^F \geq 0$ .

Moreover, for inequalities (2) and (3), equality occurs at a point if and only if all the anisotropic principal curvatures are equal at that point. Hence, equality

occurs everywhere if and only if  $M$  is the Wulff shape  $W_F$ , up to translations and homotheties.

We also recall the anisotropic analogue of the classical Hsiung-Minkowski formulas [17]. The proof can be found in [13] and it uses in particular the fact that  $P_r$  is divergence-free.

**Lemma 2.3.** *Let  $r \in \{0, \dots, n-1\}$ . Then, we have*

$$\int_M \left( F(\nu)H_r^F + H_{r+1}^F \langle X, \nu \rangle \right) dv_g = 0.$$

Finally, we recall two useful results in the case of embedded hypersurfaces and so, bound some domains in  $\mathbb{R}^{n+1}$ . First, we have the classical identity obtained directly by the divergence formula

$$(2) \quad \int_M \langle X, \nu \rangle dv_g = (n+1)V(\Omega),$$

where  $V(\Omega)$  is the volume of the domain  $\Omega$  bounded by  $M$  and  $\nu$  is the outer normal unit vector field. Secondly, we have the anisotropic analogue of the well-known Heintze-Karcher inequality (see [13]).

**Lemma 2.4.** *If  $M$  is embedded (so bounds a domain  $\Omega$ ) and  $H^F$  is everywhere positive, then the following inequality holds*

$$\int_M \frac{F(\nu)}{H^F} dv_g \geq \int_M \langle X, \nu \rangle dv_g = (n+1)V(\Omega),$$

with equality iff  $M$  is the Wulff shape  $W_F$  (up to translations and homotheties).

**2.2. The variational problem.** In this section, we describe the stability problem which we will consider in this note. First, we introduce the anisotropic  $r$ -area functionals

$$(3) \quad \mathcal{A}_{r,F} = \left( \int_M F(\nu) \sigma_r dv_g \right),$$

for  $r \in \{0, \dots, n-1\}$  and where  $dv_g$  denotes the Riemannian volume form on  $M$ . Now we consider a variation of the immersion  $X$ . Precisely, suppose  $\varepsilon > 0$  and

$$(4) \quad \mathcal{X}: (-\varepsilon, \varepsilon) \times M \longrightarrow \mathbb{R}^{n+1},$$

such that for all  $t \in (-\varepsilon, \varepsilon)$ ,  $\mathcal{X}_t := \mathcal{X}(t, \cdot)$  is an immersion of  $M$  into  $\mathbb{R}^{n+1}$  and  $\mathcal{X}(0, \cdot) = X$ . We denote by  $\sigma_r(t)$  the corresponding curvature functions,  $\mathcal{A}_{r,F}(t)$  the  $r$ -area of  $\mathcal{X}_t$ ,  $g_t$  the metric on  $M$  induced by  $\mathcal{X}_t$  and finally we set

$$(5) \quad f_t = \left\langle \frac{dX}{dt}, \nu_t \right\rangle,$$

where  $\nu_t$  is the unit normal to  $M$  induced by  $\mathcal{X}_t$ . Now, the first variation formula for  $\mathcal{A}_{r,F}$  is given by

$$(6) \quad \mathcal{A}'_{r,F}(t) = -b_{r+1} \int_M f_t H_{r+1}^F(t) dv_{g_t},$$

where  $b_{r+1} = (r+1) \binom{n}{r+1}$  (see [15]). We also consider the volume functional

$$(7) \quad V(t) = \int_{[0,t] \times M} \mathcal{X}^* dv.$$

It is easy to see ([4, Lemma 2.1]) that  $V$  satisfies

$$(8) \quad V'(t) = \int_M f_t dv_{g_t}.$$

Therefore,  $\mathcal{X}$  preserves the volume if and only if

$$\int_M f_t dv_{g_t} = 0, \quad \forall t.$$

Moreover, according to [4, Lemma 2.2], for any function  $f_0 : M \rightarrow \mathbb{R}$  such that  $\int_M f_0 dv_g = 0$ , there exists a variation of  $X$  preserving the volume and with normal part given by  $f_0$ .

Now, let  $r$  and  $s$  be two integers satisfying  $0 \leq r \leq s \leq n - 2$  and  $a_j, j = r, \dots, s$  some nonnegative real numbers with at least of them one non zero. We consider the following anisotropic  $(r, s, F)$ -area functional  $\mathcal{B}_{r,s,F}$  defined by

$$(9) \quad \mathcal{B}_{r,s,F} = \sum_{j=r}^s a_j \mathcal{A}_{j,F}.$$

These area functionals appear naturally when one considers hypersurfaces with linearly related higher order anisotropic mean curvatures. Indeed, we consider variations of  $M$  that preserve the balanced volume. The Jacobi functionals which are associated with this anisotropic  $(r, s)$ -area is given by

$$\begin{aligned} \mathcal{J}_{r,s,F} : (\varepsilon, \varepsilon) &\longrightarrow \mathbb{R} \\ t &\longmapsto \mathcal{B}_{r,s,F}(t) + \Lambda V(t), \end{aligned}$$

where  $\Lambda$  is a constant to be determined. From (6) and (8), we have

$$\mathcal{J}'_{r,s,F}(t) = \int_M \left( - \sum_{j=r}^s a_j b_j H_{j+1}^F + \Lambda \right) f_t dv_{g_t}.$$

Hence, we have, like in the isotropic context, that  $\sum_{j=r}^s a_j b_j H_{j+1}^F$  is constant if and only if  $X$  it is a critical point of the functional  $\mathcal{J}_{r,s,F}$ , or equivalently, if and only if  $X$  is a critical point of  $\mathcal{B}_{r,s,F}$  for variations that preserve the balanced volume.

At that point we want to point out that the choice of the linear combination in the statement  $\sum_{j=r}^s a_j b_j H_{j+1}^F$  comes from this choice of variational problem  $\mathcal{B}_{r,s,F} = \sum_{j=r}^s a_j \mathcal{A}_{j,F}$ . The coefficient  $b_j$  appear naturally when computing the first variation of this functional.

Now, we give the definition of the anisotropic  $(r, s)$ -stability that we call  $(r, s, F)$ -stability.

**Definition 2.5.** *Let  $n, r, s$  be three integers so that  $0 \leq r \leq s \leq n - 2$  and suppose that  $F : \mathbb{S}^n \rightarrow \mathbb{R}_+$  is a smooth function satisfying the convexity assumption (1) and  $X : M^n \rightarrow \mathbb{R}^{n+1}$  is a closed hypersurface satisfying  $\sum_{j=r}^s a_j b_j H_{j+1}^F = \text{constant}$ . Then,  $X$  is said to be  $(r, s, F)$ -stable if  $\mathcal{B}'_{r,s,F}(0) \geq 0$  for all volume-preserving variations of  $X$ .*

We consider the Jacobi operator  $\mathcal{J}''_{r,s,F}(0)$  defined on the set  $\mathcal{F}$  of smooth functions on  $M$  with  $\int_M f dv_g = 0$ . From the above definition, we have clearly  $\mathcal{B}''_{r,s,F}(0) = \mathcal{J}''_{r,s,F}(0)[f]$  where  $f \in \mathcal{F}$  defines the variation  $\mathcal{X}$ . Therefore, the  $(r, s, F)$ -stability corresponds to the non-negativity of the Jacobi operator.

Now, we can define the notion of  $\varepsilon$ -almost  $(r, s, F)$ -stability for  $\varepsilon \geq 0$  by

$$(10) \quad \mathcal{J}''_{r,s,F}(0)[f] \geq -\varepsilon \int_M f^2 dv_g,$$

for any smooth function  $\int_M f dv_g = 0$ .

Now, we give the second variation formula for this variational problem.

**Lemma 2.6.** *For any variation  $\mathcal{X}$  of  $X$  preserving the balanced volume, the second variation formula of  $\mathcal{B}_{r,s,F}$  at  $t = 0$  is given by*

$$\mathcal{B}''_{r,s,F}(0) = \mathcal{J}''_{r,s,F}(0)[f] = - \sum_{j=r}^s (j+1)a_j \int_M (L_j f + \langle T_j \circ d\nu, d\nu \rangle f) f dv_g,$$

where  $f \in \mathcal{F}$  is the normal part of the variation  $\mathcal{X}$ ,  $T_j = P_j \circ A_F$  and  $L_j = -\text{div}(T_j \nabla(\cdot))$ .

*Proof:* The proof comes directly from the second variation formula for each functional  $\mathcal{A}_{j,F}$ . Indeed, from [15], we have

$$\mathcal{A}''_{j,F}(0) = -(j+1) \int_M (L_j f + \langle T_j \circ d\nu, d\nu \rangle f) f dv_g.$$

Then, we have just to multiply by  $a_j$  and sum from  $r$  to  $s$  to get the result.  $\square$

**2.3. Some lemmas.** We define the following operators by fixing some notations. For any  $f \in \mathcal{C}^\infty$ , we set

$$I_{j,F}[f] = L_j f + \langle T_j \circ d\nu, d\nu \rangle f$$

and

$$\mathcal{R}_{r,s,F} = \sum_{j=r}^s (j+1)a_j I_{j,F}[f].$$

Obviously from this definition and Lemma 2.6, we have

$$\mathcal{J}''_{r,s,F}(0)[f] = - \int_M f \mathcal{R}_{r,s,F}[f] dv_g.$$

Now, we recall the lemma due to He and Li [15].

**Lemma 2.7.** *For any  $j \in \{r, \dots, s\}$ , we have*

- (1)  $I_{j,F}[\langle X, \nu \rangle] = -\langle \text{grad } \sigma_{j+1}, X^T \rangle - (j+1)\sigma_{j+1}$ ,
- (2)  $I_{j,F}[F(\nu)] = -\langle \text{grad } \sigma_{j+1}, (\text{grad}_{\mathbb{S}^n} F) \circ \nu \rangle + \sigma_1 \sigma_{j+1} - (j+2)\sigma_{j+2}$ .

The proof of this lemma follows the idea of [21] and uses Lemma 2.1. Now, we mention the symmetry of  $\mathcal{R}_{r,s,F}$  with respect to  $L_2$ -scalar product by the following lemma.

**Lemma 2.8.** *For any two smooth functions  $f$  and  $h$  over  $M$ , we have*

$$\int_M h \mathcal{R}_{r,s,F}[f] dv_g = \int_M f \mathcal{R}_{r,s,F}[h] dv_g.$$

*Proof:* The proof is fairly standard. First, we compute

$$\begin{aligned} \int_M h L_j f dv_g &= \int_M h \operatorname{div} (T_j \nabla f) dv_g = - \int_M \langle T_j \nabla f, \nabla h \rangle dv_g \\ &= - \int_M \langle T_j \nabla h, \nabla f \rangle dv_g = \int_M f \operatorname{div} (T_j \nabla h) dv_g = \int_M f L_j h dv_g, \end{aligned}$$

where we have used the symmetry of  $T_j$  and the divergence theorem. Hence, from the definition of  $I_{j,F}$  and the above identity, we get

$$\int_M h I_{j,F}[f] dv_g = \int_M f I_{j,F}[h] dv_g.$$

Finally, multiplying by  $a_j b_j$  and taking the sum over  $j$  from  $r$  to  $s$ , we get

$$\int_M h \mathcal{R}_{r,s,F}[f] dv_g = \int_M f \mathcal{R}_{r,s,F}[h] dv_g.$$

This concludes the proof.  $\square$

### 3. PROOF OF THEOREM 1.1

We have divided the proof of in four parts. First, using the almost stability assumption, we show that some specific quantities expressed in terms of higher order mean curvatures are closed for the  $L^1$ -norm. From this, we will deduce that the hypersurface is almost anisotropically umbilical, which implies, by a result of de Rosa and Giofrè, that the hypersurface is close to a Wulff shape and embedded. Finally, we conclude by showing that the generalized linear Weingarten relation for the hypersurface along with the fact that it is embedded implies that the hypersurface is a Wulff shape.

**Step 1: Smallness of  $H^F H_{r+1}^F - H_{r+2}^F$ .**

Suppose  $X : M^n \rightarrow \mathbb{R}^{n+1}$  is  $\varepsilon$ -almost  $(r, s, F)$ -stable. By definition, we have

$$\mathcal{J}_{r,s,F}''(0)[f] \geq -\varepsilon \int_M f^2 dv_g$$

for any function  $f \in \mathcal{F}$ . We choose the particular test function  $f$  defined by

$$f = \alpha F(\nu) + \beta \langle X, \nu \rangle,$$

with  $\alpha = \frac{\int_M \left( \sum_{j=r}^s a_j b_j F(\nu) H_j^F \right) dv_g}{\int_M F(\nu)}$  and  $\beta = \sum_{j=r}^s a_j b_j H_{j+1}^F$ . We have to show that  $f \in \mathcal{F}$ . It is to observe that  $\beta$  is a constant by assumption. Moreover, using



the anisotropic Hsiung-Minkowski formulas (Lemma 2.3) we have

$$\begin{aligned}
\int_M f dv_g &= \int_M (\alpha F(\nu) + \beta \langle X, \nu \rangle) dv_g \\
&= \alpha \int_M F(\nu) dv_g + \int_M \left( \sum_{j=r}^s a_j b_j H_{j+1}^F \langle X, \nu \rangle \right) dv_g \\
&= \alpha \int_M F(\nu) dv_g + \sum_{j=r}^s a_j b_j \int_M H_{j+1}^F \langle X, \nu \rangle dv_g \\
&= \alpha \int_M F(\nu) dv_g - \sum_{j=r}^s a_j b_j \int_M F(\nu) H_j^F dv_g \\
&= \alpha \int_M F(\nu) dv_g - \int_M \left( \sum_{j=r}^s a_j b_j F(\nu) H_j^F \right) dv_g \\
&= \alpha \int_M F(\nu) dv_g - \alpha \int_M F(\nu) dv_g = 0.
\end{aligned}$$

Hence, the integral of  $f$  vanishes and  $f$  is eligible as a test function. We have

$$\mathcal{J}_{r,s,F}''(0)[f] = - \int_M f \mathcal{R}_{r,s,F}[f] dv_g \geq 0.$$

Now, we compute  $\mathcal{R}_{r,s,F}[f]$ . Therefore,

$$\begin{aligned}
\mathcal{R}_{r,s,F}[f] &= \sum_{j=r}^s (j+1) a_j I_{j,F}[f] = \sum_{j=r}^s (j+1) a_j I_{j,F}[\alpha F(\nu) + \beta \langle X, \nu \rangle] \\
&= \sum_{j=r}^s (j+1) a_j \left( \alpha I_{j,F}[F(\nu)] + \beta I_{j,F}[\langle X, \nu \rangle] \right).
\end{aligned}$$

From Lemma 2.7, we have

$$\begin{aligned}
\mathcal{R}_{r,s,F}[f] &= \sum_{j=r}^s (j+1) a_j \left[ \alpha \left( - \langle \text{grad } \sigma_{j+1}, (\text{grad}_{\mathbb{S}^n} F) \circ \nu \rangle + \sigma_1 \sigma_{j+1} - (j+2) \sigma_{j+2} \right) \right. \\
&\quad \left. + \beta \left( - \langle \text{grad } \sigma_{j+1}, X^T \rangle - (j+1) \sigma_{j+1} \right) \right].
\end{aligned}$$

Since  $\sum_{j=r}^s (j+1) a_j \sigma_{j+1} = \sum_{j=r}^s a_j b_j H_{j+1}^F = \text{constant}$ , we get

$$\begin{aligned}
\mathcal{R}_{r,s,F}[f] &= \sum_{j=r}^s (j+1) a_j \left[ \alpha \left( \sigma_1 \sigma_{j+1} - (j+2) \sigma_{j+2} \right) - \beta (j+1) \sigma_{j+1} \right] \\
&= \sum_{j=r}^s a_j b_j \left[ \alpha \left( n H^F H_{j+1}^F - (n-j-1) H_{j+2}^F \right) - \beta (j+1) H_{j+1}^F \right].
\end{aligned}$$

Using Lemma 2.8, we get

$$\begin{aligned} \mathcal{J}_{r,s,F}''(0)[f] &= - \int_M f \mathcal{R}_{r,s,F}[f] dv_g = - \int_M \left( \alpha F(\nu) + \beta \langle X, \nu \rangle \right) \mathcal{R}_{r,s,F}[f] dv_g \\ (11) \qquad \qquad \qquad &= - \int_M \left( \alpha F(\nu) \mathcal{R}_{r,s,F}[f] + \beta f \mathcal{R}_{r,s,F}[\langle X, \nu \rangle] \right) dv_g. \end{aligned}$$

The first term of the integral in (11) is expressed as

$$\begin{aligned} \alpha F(\nu) \mathcal{R}_{r,s,F}[f] &= \sum_{j=r}^s a_j b_j F(\nu) \left[ \alpha^2 \left( n H^F H_{j+1}^F - (n-j-1) H_{j+2}^F \right) - \alpha \beta (j+1) H_{j+1}^F \right] \\ &= \sum_{j=r}^s a_j b_j F(\nu) \left[ \alpha^2 (n-j-1) (H^F H_{j+1}^F - H_{j+2}^F) + \alpha^2 (j+1) H^F H_{j+1}^F \right. \\ (12) \qquad \qquad \qquad &\left. - \alpha \beta (j+1) H_{j+1}^F \right]. \end{aligned}$$

Similarly, using Lemma 2.7 and the fact that  $\sum_{j=r}^s a_j b_j H_{j+1}^F$  is constant, the second term in (11) is,

$$\begin{aligned} \beta f \mathcal{R}_{r,s,F}[\langle X, \nu \rangle] &= \beta f \sum_{j=r}^s (j+1) a_j I_{j,F}[\langle X, \nu \rangle] \\ &= \beta f \sum_{j=r}^s (j+1) a_j \left( - \langle \text{grad } \sigma_{j+1}, X^T \rangle - (j+1) \sigma_{j+1} \right) \\ &= -\beta f \sum_{j=r}^s a_j (j+1)^2 \sigma_{j+1} = v - \beta f \sum_{j=r}^s a_j b_j (j+1) H_{j+1}^F \\ (13) \qquad \qquad \qquad &= - \sum_{j=r}^s a_j b_j (j+1) \beta \left( \alpha F(\nu) + \beta \langle X, \nu \rangle \right) H_{j+1}^F. \end{aligned}$$

Now, plugging (11) and (12) in (13), we get

$$\begin{aligned} \mathcal{J}_{r,s,F}''(0)[f] &= - \sum_{j=r}^s a_j b_j (n-j-1) \alpha^2 \int_M F(\nu) (H^F H_{j+1}^F - H_{j+2}^F) dv_g \\ &\quad - \sum_{j=r}^s a_j b_j (j+1) \alpha^2 \int_M F(\nu) H^F H_{j+1}^F dv_g \\ &\quad + \sum_{j=r}^s 2a_j b_j (j+1) \alpha \beta \int_M F(\nu) H_{j+1}^F dv_g \\ (14) \qquad \qquad \qquad &+ \sum_{j=r}^s a_j b_j (j+1) \beta^2 \int_M H_{j+1}^F \langle X, \nu \rangle dv_g. \end{aligned}$$

Using the anisotropic Hsiung-Minkowski formulas again, we have

$$\sum_{j=r}^s 2a_j b_j (j+1) \beta^2 \int_M H_{j+1}^F \langle X, \nu \rangle dv_g = - \sum_{j=r}^s 2a_j b_j (j+1) \beta^2 \int_M F(\nu) H_j^F dv_g.$$

Therefore, (14) becomes

$$\begin{aligned}
\mathcal{J}_{r,s,F}''(0)[f] &= - \sum_{j=r}^s a_j b_j (n-j-1) \alpha^2 \int_M F(\nu) (H^F H_{j+1}^F - H_{j+2}^F) dv_g \\
(15) \quad &- \sum_{j=r}^s a_j b_j (j+1) \int_M F(\nu) (H^F H_{j+1}^F \alpha^2 - 2H_{j+1}^F \alpha \beta + H_j^F \beta^2) dv_g.
\end{aligned}$$

Now, at a point  $x$  in  $M$ , we consider the following second order polynomial

$$P_{j,F,x}(z) = F(\nu) \left( H^F H_{j+1}^F z^2 - 2H_{j+1}^F \beta z + H_j^F \beta^2 \right).$$

The discriminant of  $P_{j,F,x}$  is

$$\Delta = 4\beta^2 F(\nu)^2 \left( (H_{j+1}^F)^2 - H^F H_{j+1}^F H_j^F \right) = 4\beta^2 F(\nu)^2 H_{j+1}^F (H_{j+1}^F - H^F H_j^F).$$

Since, by assumption,  $H_{s+1}^F > 0$ , from Lemma 2.2, we have  $H_{j+1}^F > 0$  and  $H_{j+1}^F - H^F H_j^F \geq 0$ . Hence,  $\Delta$  is nonnegative, the term of degree 2 is  $F(\nu) H^F H_{j+1}^F > 0$  and then  $P_{j,F,x}(z) \geq 0$  for any  $z \in \mathbb{R}$ . In particular, for  $z = \alpha$ , we obtain

$$H^F H_{j+1}^F \alpha^2 - 2H_{j+1}^F \alpha \beta + H_j^F \beta^2 \geq 0.$$

Reporting this in (15), we get

$$\mathcal{J}_{r,s,F}''(0)[f] \leq - \sum_{j=r}^s a_j b_j (n-j-1) \alpha^2 \int_M F(\nu) (H^F H_{j+1}^F - H_{j+2}^F) dv_g.$$

Since  $a_j$  and  $b_j$  are nonnegative constants,  $H^F H_{j+1}^F - H_{j+2}^F \geq 0$  by Lemma 2.2, and  $F(\nu) > 0$ , we get

$$\mathcal{J}_{r,s,F}''(0)[f] \leq 0.$$

Moreover, by the  $\varepsilon$ -almost  $(r, s, F)$ -stability assumption, we have

$$\mathcal{J}_{r,s,F}''(0)[f] \geq -\varepsilon \int_M f^2 dv_g,$$

so we deduce that

$$0 \leq \sum_{j=r}^s a_j b_j (n-j-1) \alpha^2 \int_M F(\nu) (H^F H_{j+1}^F - H_{j+2}^F) dv_g \leq \varepsilon \int_M f^2 dv_g.$$

Since all the  $a_j$ 's,  $b_j$ 's and  $F(\nu) (H^F H_{j+1}^F - H_{j+2}^F)$  are nonnegative, we deduce that for any  $j$  between  $r$  and  $s$

$$0 \leq a_j b_j (n-j-1) \alpha^2 \int_M F(\nu) (H^F H_{j+1}^F - H_{j+2}^F) dv_g \leq \varepsilon \int_M f^2 dv_g.$$

In particular, for  $j = s-1$ , we have

$$(16) \quad 0 \leq a_{s-1} b_{s-1} (n-s) \alpha^2 \int_M F(\nu) (H^F H_s^F - H_{s+1}^F) dv_g \leq \varepsilon \int_M f^2 dv_g.$$

On the other hand, we have

$$\begin{aligned}
\int_M f^2 dv_g &= \int_M \left( \alpha F(\nu) + \beta \langle X, \nu \rangle \right)^2 dv_g \\
&= \int_M \left( \alpha^2 F(\nu)^2 + \beta^2 \langle X, \nu \rangle^2 + 2\alpha\beta F(\nu) \langle X, \nu \rangle \right) dv_g \\
(17) \quad &\leq \left( \alpha^2 \|F\|_\infty^2 + \beta^2 R^2 + 2\alpha\beta \|F\|_\infty R \right) \mathbf{V}(M).
\end{aligned}$$

Moreover, from the definition of  $\alpha$ , we have

$$(18) \quad \alpha = \frac{\int_M \left( \sum_{j=r}^s a_j b_j F(\nu) H_j^F \right) dv_g}{\int_M F(\nu)} \leq \sum_{j=r}^s a_j b_j \|S_F\|_\infty^j = C_1,$$

where  $C_1$  is a constant depending on  $n, r, s, a$  and  $\|S_F\|_\infty$ . Similarly, we have

$$(19) \quad \beta = \sum_{j=r}^s a_j b_j H_{j+1}^F \leq \sum_{j=r}^s a_j b_j \|S_F\|^{j+1} = C_2,$$

where  $C_2$  is a constant depending on  $n, r, s, a$  and  $\|S_F\|_\infty$ .

Using (18) and (19) in (17), we get

$$(20) \quad \int_M f^2 dv_g \leq \left( C_1^2 \|F\|_\infty^2 + C_2^2 R^2 + 2C_1 C_2 \|F\|_\infty R \right) \mathbf{V}(M) = C_3 \mathbf{V}(M),$$

where  $C_3$  is a constant depending on  $n, r, s, a, \|S_F\|_\infty, F$  and  $R$ .

Therefore, we have

$$\begin{aligned}
\alpha &= \frac{\int_M \left( \sum_{j=r}^s a_j b_j F(\nu) H_j^F \right) dv_g}{\int_M F(\nu)} \geq \frac{\int_M \left( \sum_{j=r}^s a_j b_j F(\nu) (H_s^F)^{\frac{j}{s}} \right) dv_g}{\int_M F(\nu)} \\
(21) \quad &\geq \sum_{j=r}^s a_j b_j \min(H_s)^{\frac{j}{s}} = C_4,
\end{aligned}$$

where  $C_4$  is a constant depending on  $n, r, s, a$  and  $\min(H_s)$ .

Hence, from (16), we deduce with (20) and (21) that

$$(22) \quad \int_M (H^F H_{j+1}^F - H_{j+2}^F) dv_g \leq \frac{\varepsilon C_3 \mathbf{V}(M)}{a_{s-1} b_{s-1} (n-s) C_4 \min(F)} = C_5 \mathbf{V}(M) \varepsilon,$$

where  $C_5$  is a constant depending on  $n, r, s, a, \min(H_s), \|S_F\|_\infty, F$  and  $R$ .

### Step 2: Almost anisotropic umbilicity.

Now, we give the lemma which is independent of the almost  $(r, s, F)$ -stability assumption.

**Lemma 3.1.** *There exists a positive constant  $K_1$  depending on  $n, s, \min_M(H_{s+1}^F), \min_M(H_s^F)$  and  $\|S_F\|_\infty$  so that  $\|\tau_F\|^2 \leq K_1 (H^F H_s^F - H_{s+1}^F)$ .*

*Proof:* From Lemma 2.2, we have for any  $k \in \{1, \dots, s\}$ ,

$$(H_k^F)^2 - H_{k+1}^F H_{k-1}^F \geq 0.$$

We have a more precise estimate of the positivity of this term. Namely,

$$(23) \quad (H_k^F)^2 - H_{k+1}^F H_{k-1}^F \geq c_n \|\tau\|^2 (H_{k+1;n,1}^F)^2,$$

where  $c_n$  is a constant depending only on  $n$  and

$$(24) \quad H_{l;i,j}^F = \frac{\partial H_l^F}{\partial \kappa_i \partial \kappa_j} = \frac{1}{\binom{n}{l}} \sum_{\substack{1 \leq i_1 < \dots < i_{l-2} \leq n \\ i_1, \dots, i_l \neq i, j}} \kappa_{i_1} \cdots \kappa_{i_{l-2}},$$

where  $\kappa_1, \dots, \kappa_n$  are the anisotropic principal curvatures. The proof for the anisotropic case is almost similar to the isotropic case discussed in [25]. Since we assume that  $H_{s+1}^F > 0$ , then all the functions  $H_k^F$  are also positive for  $k \in \{1, \dots, s\}$ . Thus, dividing by  $H_k^F H_{k-1}^F$ , (23) becomes

$$(25) \quad \frac{H_k^F}{H_{k-1}^F} - \frac{H_{k+1}^F}{H_k^F} \geq c_n \|\tau\|^2 \frac{(H_{k+1;n,1}^F)^2}{H_k^F H_{k-1}^F}.$$

By summing (25) for  $k$  from 1 to  $s$ , we get

$$(26) \quad H^F - \frac{H_{s+1}^F}{H_s^F} \geq c_n \|\tau\|^2 \sum_{k=1}^s \frac{(H_{k+1;n,1}^F)^2}{H_k^F H_{k-1}^F},$$

and so

$$(27) \quad H^F H_s^F - H_{s+1}^F \geq c_n \|\tau\|^2 \left( \sum_{k=1}^s \frac{(H_{k+1;n,1}^F)^2}{H_k^F H_{k-1}^F} \right) H_s^F.$$

Moreover, we have  $H_k^F H_{k-1}^F \leq \|S_F\|_\infty^{2(k-1)}$ . In addition, since  $H_{s+1}^F$  is positive, then all the function  $H_k^F$  are also positive, and also, as proved by Scheuer in [25], the functions  $H_{k;n,1}^F$  are positive. Here again, the proof of Scheuer of this fact is not specific to the isotropic case. It is also valid for the anisotropic mean curvatures. In addition, since they are the normalized symmetric polynomial evaluated for  $\kappa_2, \dots, \kappa_{n-1}$ , they also satisfy the Mac Laurin inequality, up to a normalization constant, that

$$(H_{k;n,1}^F)^{\frac{1}{k-2}} \geq a_{n,k} (H_{k+1;n,1}^F)^{\frac{1}{k-1}},$$

where  $a_{n,k}$  is a positive constant depending only on  $n$  and  $k$ , and so

$$(H_{k;n,1}^F)^{\frac{1}{k-2}} \geq b_{n,k,s} (H_{s+1;n,1}^F)^{\frac{1}{s-1}},$$

where  $b_{n,k,s}$  is a positive constant depending only on  $n$ ,  $k$  and  $s$ . Note that the exponents come from the fact that  $H_{k;n,1}^F$  is the symmetric polynomial of degree  $k-2$ . Thus (27) gives

$$(28) \quad H^F H^F s - H_{s+1}^F \geq c_n \|\tau\|^2 \left( \sum_{k=1}^s \frac{b_{n,k+1,s}^{2(k-1)} (H_{s+1;n,1}^F)^{\frac{2(k-1)}{s-1}}}{\|S_F\|_\infty^{2k-1}} \right) H_s^F$$

and so  $H^F H_s^F - H_{s+1}^F \geq C_6 \|\tau_F\|^2$ , with

$$C_6 = \frac{c_n \min_M(H_s^F) \min_{1 \leq k \leq s} (b_{n,k+1,s}^{2(k-1)})}{\|S_F\|_\infty} \left( \frac{\min(H_{s+1;n,1}^F)}{\|S_F\|_\infty} \right)^{2k-1}.$$

Hence, we deduce that

$$\|\tau\|^2 \leq K_1 (H^F H_s^F - H_{s+1}^F),$$

where  $K_1 = \frac{1}{C_6}$  is a constant depending on  $n$ ,  $s$ ,  $\min_M(H_{s+1;n,1}^F)$ ,  $\min_M(H_s^F)$  and  $\|S_F\|_\infty$ .  $\square$

**Step 3: Embeddedness.** By the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \|\tau_F\|_{n+1}^{2(n+1)} &= \left( \frac{1}{V(\Sigma)} \int_M \|\tau_F\|^{(n+1)} dv_g \right)^2 \\ &\leq \frac{1}{V(M)^2} \left( \int_M \|\tau_F\|^{2n} dv_g \right) \left( \int_M \|\tau_F\|^2 dv_g \right). \end{aligned}$$

From this, we deduce immediately that

$$(29) \quad \|\tau_F\|_{n+1}^{2(n+1)} \leq \frac{1}{V(M)} \|S_F\|_\infty^{2n} \left( \int_M \|\tau_F\|^2 dv_g \right),$$

and thus by Lemma 3.1

$$(30) \quad \|\tau_F\|_{n+1}^{2(n+1)} \leq \frac{K_1}{V(M)} \|S_F\|_\infty^{2n} \left( \int_M (H^F H_r^F - H_{r+1}^F) dv_g \right).$$

Moreover, using (22), we get

$$(31) \quad \|\tau_F\|_{n+1}^{2(n+1)} \leq K_1 C_5 \|S_F\|_\infty^{2n} \varepsilon = K_2 \varepsilon,$$

with  $K_2$  is a constant depending on  $n$ ,  $r$ ,  $s$ ,  $a$ ,  $\min_M(H_{s+1;n,1}^F)$ ,  $\min_M(H_s^F)$ ,  $\|S_F\|_\infty$ ,  $F$  and  $R$ .

Now, we recall the following proposition due to de Rosa and Giofrè (see [10]).

**Proposition 3.2.** *Let  $n \geq 2$ ,  $p > n$  and  $M$  a closed hypersurface of  $\mathbb{R}^{n+1}$ . We assume that there exist a constant  $c_0 > 0$  such that  $\|S_F\|_p \leq c_0$  and  $V(M) = V(W_F)$ . Then, for every  $\eta > 0$  sufficiently small, there exists a constant  $\delta > 0$  depending on  $\eta$ ,  $n$ ,  $p$ ,  $c_0$  and  $W_F$  such that if  $\|\tau_F\|_p \leq \delta$ , then  $M$  admits an anisotropic radial parametrization*

$$\psi : W_F \longrightarrow M, \quad \psi(x) = x + u(x)\nu_{\mathcal{W}}(x), \quad \text{with } \|u\|_{C^1} \leq \eta.$$

In particular, this proposition implies that for sufficiently small  $\eta \leq \frac{1}{2}$  there exists  $\delta$  so that if  $\|\tau_F\|_p \leq \delta$  then  $M$  is embedded.

Now, we take  $p = n + 1$ , fix  $\eta \leq \frac{1}{2}$  and consider  $\delta$  given by Proposition 3.2. We

choose  $\varepsilon \leq \varepsilon_0 = \frac{\delta^{\frac{1}{2(n+1)}}}{K_2}$  so that (31) gives  $\|\tau_F\|_{n+1} \leq \delta$ . Therefore, we conclude that  $M$  is embedded.

Note that  $\varepsilon_0$  is chosen small enough depending on  $n$ ,  $r$ ,  $s$ ,  $a$ ,  $\min_M(H_{s+1;n,1}^F)$ ,  $\min_M(H_s^F)$ ,  $\|S_F\|_\infty$ ,  $F$ ,  $R$  and  $V(M)$ . This dependence comes from  $K_2$  and an additional dependence on  $V(M)$  appears from Proposition 3.2. Indeed, in this

proposition, by scaling, we can remove the hypothesis  $V(M) = V(W_F)$  and in this case  $\delta$  depends also on  $V(M)$ .

**Step 4: Conclusion.** Since the embededness of the hypersurface  $M$  is proven, we will be able to conclude the final result with the help of this last proposition.

**Proposition 3.3.** *Let  $k \in \{1, \dots, n-1\}$ , and  $M$  a closed embedded hypersurface of  $\mathbb{R}^{n+1}$  with  $H_k^F > 0$ . Assume that there exists a positive constant  $a_0$  and some nonnegative constants  $b_1, \dots, b_k$  with at least one non-vanishing such that*

$$a_0 = \sum_{j=1}^k b_j H_j^F.$$

*Then, up to translations and homotheties,  $M$  is the Wulff shape  $W_F$ .*

*Proof:* We have

$$\begin{aligned} a_0 \int_M \langle X, \nu \rangle dv_g &= \int_M \sum_{j=1}^k b_j H_j^F \langle X \nu \rangle dv_g = \int_M \sum_{j=1}^k b_j H_{j-1}^F F(\nu) dv_g \\ &\leq \int_M \sum_{j=1}^k b_j \frac{H_j^F}{H} F(\nu) dv_g = a_0 \int_M \frac{F(\nu)}{H} dv_g \leq a_0 \int_M \langle X, \nu \rangle dv_g, \end{aligned}$$

where we have used successively the Weingarten relation, the Hsiung-Minkowski formulas twice, the fact that  $HH_{j-1} \leq H_j$  and finally the Heintze-Karcher inequality. Thus, all the inequalities in this sequence are equalities. In particular, we have equality in the Heintze-Karcher inequality which implies that  $M$  is the Wulff shape  $W_F$ , up to translations and homotheties.  $\square$

The proof of Theorem 1.1 is complete.

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