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# UPPER BOUNDS FOR THE BLOW-UP TIME OF THE 2-D PARABOLIC-ELLIPTIC PATLAK-KELLER-SEGEL MODEL OF CHEMOTAXIS

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Patrick Maheux. UPPER BOUNDS FOR THE BLOW-UP TIME OF THE 2-D PARABOLIC-ELLIPTIC PATLAK-KELLER-SEGEL MODEL OF CHEMOTAXIS. 2023. hal-04139369

**HAL Id: hal-04139369**

**<https://cnrs.hal.science/hal-04139369>**

Preprint submitted on 23 Jun 2023

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# UPPER BOUNDS FOR THE BLOW-UP TIME OF THE 2-D PARABOLIC-ELLIPTIC PATLAK-KELLER-SEGEL MODEL OF CHEMOTAXIS

PATRICK MAHEUX

Friday 23<sup>rd</sup> June, 2023

ABSTRACT. In this paper, we obtain upper bounds for the critical time  $T^*$  of the blow-up for the parabolic-elliptic Patlak-Keller-Segel system on the 2D-Euclidean space. Such estimates is rarely obtained and discuss in the papers on the subject No moment condition or/and entropy condition are required on the initial data; only the usual assumptions of non-negativity and finiteness of the mass is assumed. The result is expressed not only in terms of the supercritical mass  $M > 8\pi$ , but also in terms of the *shape* of the initial data.

1

## CONTENTS

1. General upper bound on $T^*$ for the (PKS) system	1
2. Estimates on the critical time bound $T_c^*(n_0)$	11
3. Lower bounds on $T_c^*(n_0)$	25
4. Examples of initial data	29
4.1. Gaussian initial data $n_0$	29
4.2. Characteristic function of a disk	34
4.3. More examples of initial data	38
References	39

## 1. GENERAL UPPER BOUND ON $T^*$ FOR THE (PKS) SYSTEM

In this paper, we consider the following Patlak-Keller-Segel (PKS) model for chemotaxis in the whole plane  $\mathbb{R}^2$  [23, 34, 22]:

$$(1.1) \quad \frac{\partial}{\partial t} n(x, t) = \Delta n(x, t) - \nabla \cdot (n \nabla c)(x, t), \quad x \in \mathbb{R}^2, \quad t > 0,$$

$$(1.2) \quad (-\Delta)c(x, t) = n(x, t), \quad x \in \mathbb{R}^2, \quad t > 0,$$

$$(1.3) \quad n(0, x) = n_0(x), \quad x \in \mathbb{R}^2, \quad 0 \leq n_0 \in L^1(\mathbb{R}^2),$$

where  $n_0$  denotes the initial data. The function  $n(x, t)$  represents the density of cells, and  $c(x, t)$  the density of the chemical secreted by themselves that attracts them and makes them to aggregate.

There is a huge bibliography on this type of systems and its variants. The geometric framework is usually  $\mathbb{R}^n$  or bounded domains on  $\mathbb{R}^n$  with different kind of results depending on the dimension  $n = 1, 2$  or  $n \geq 3$  and assumptions of the initial data. Note that there are

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2010 *Mathematics Subject Classification.* 35Q92, 35B44, 35K55, 35M31, 92C17.

*Key words and phrases.* Patlak-Keller-Segel system; chemotaxis; critical mass; blow-up time upper bound.

This work was carried out while the author benefited from a delegation at the CNRS and a CRCT during the first semester of 2021 and the first semester of 2023, respectively.

<sup>1</sup>I would like to warmly thank my colleague Vittoria Pierfelice for introducing me to the subject of Patlak-Keller-Segel Models for Chemotaxis [26].

also some recent works for specific manifolds with curvature, see [26, 32, 37] for instance. Diverse analytic settings for the existence of the solutions for the (PKS) systems have been introduced as classical solutions, strong solutions, entropy solutions, various notions of weak solutions; see [5] and references therein. Stochastic Keller-Segel particle models have been also introduced for the existence and the uniqueness of solution of (PKS) systems, see for instance [10, 40, 17] and [16] for numerical simulations.

For various versions of the (PKS) systems, blow-up time bounds have been obtained more or less explicitly. For instance, upper bounds on the maximal existence time  $T^*$  have been mentioned on the disk  $\Omega = B_R$  ( $R > 0$ ) of  $\mathbb{R}^2$  in [11, Thm 1.1, eq. (2.1)] for the parabolic-parabolic (PKS) systems (but not explicitly given). Always for the parabolic-parabolic (PKS) systems, we can also mention [8, Thm. 1.2] for a study on the whole plane under a second moment condition for  $n_0$ . In higher dimensions, we can see for instance [9, Section 4.1], also [12, Thm 1.1, Lemma 4.1]. In many papers, the bounds on  $T^*$  are not explicit and/or impose some extra-assumptions on the initial data. It is not our intention to cite all the possible articles on this subject. So, we shall limit ourselves to mention a few of the papers on (PKS) systems related to bounds on the maximal existence time  $T^*$ , especially on  $\mathbb{R}^2$  which is the setting of study in this paper. We apologize in advance for not mentioning more articles, in particular those dealing with generalizations of (PKS) systems and/or studying other types of questions for these systems. More references can be found in the recent book by P.Biler [5].

Among many papers on the subject, we can mention some of them historically at the beginning of the study of the blowup in finite time, on chemotactic collapse (convergence to Dirac measure plus possibly a  $L^1$  function at the critical time  $T^*$ ), and on the profile of solutions at  $t = T^*$  on bounded domain, or on the whole  $\mathbb{R}^2$  for (radial) solutions of (PKS) systems: see [19, 20, 21, 29, 30, 31] and, more recently, [28, 13] and references therein. A variant of the (PKS) system on a ball of  $\mathbb{R}^n$ ,  $n \geq 2$  is studied in [43] for which a novel type of critical mass phenomenon for radially symmetric initial data and linked to the concept of concentration comparison is described implying that the solutions blow up in finite time. Note that the concept of concentration comparison takes into account in some way of the shape of the initial data  $n_0$ , see Theorem 1.1 in [43]. See also [3] for a model of (PKS) system with consumption term on  $\mathbb{R}^2$ . We shall not be involved in such refinement but our estimates depend also on the shape of the initial data  $n_0$ .

It seems that no specific attempt has been made to estimate the critical time  $T^*$  over the whole space  $\mathbb{R}^2$  with minimal assumptions on the initial data. In this paper, we make some progress in this direction by providing information on upper bounds on  $T^*$ . But we shall not be concerned with lower bounds on  $T^*$  with this generality in this article. Nevertheless, we notice that in [24, p.355], and for some specific situations, some lower bounds on  $T^*$  have been obtained over the whole space  $\mathbb{R}^2$ . On bounded convex domains of  $\mathbb{R}^2$  and  $\mathbb{R}^3$ , lower bounds on the maximal existence time  $T^*$  in case of blow-up (with various notions of blow-up) have been investigated first by Payne-Song, see [35, 36]. See also [27, Th. 3.4] for a recent work dealing with more general (PKS) systems on bounded smooth domains of  $\mathbb{R}^n$  with  $n \geq 1$ . In [15], some numerical simulations are given and compared with analytic bounds of [35, 36]. The study of lower bounds on maximal existence time  $T^*$  has regained some interest more recently, in particular for some variants of the parabolic-parabolic version of (PKS) on bounded domains with smooth boundary; see [39, 33, 27] and references therein.

Let us return to the main subject of this paper. In the framework of weak solutions, it is proved in [7, Cor.2.2], under finite second moment condition and finite entropy of  $n_0$ , the following result. For  $M > 8\pi$ , the maximal existence time  $T^*$  of the solutions is bounded as follows,

$$(1.4) \quad T^* \leq \frac{2\pi I(0)}{M(M - 8\pi)},$$

where  $I(0) = \int_{\mathbb{R}^2} |x|^2 n_0(x) dx$  is the second moment of  $n_0$ . The inequality (1.4) expresses a blow-up of the solutions of the (PKS) system, i.e.  $T^* < \infty$ . Note that this result also holds true with  $I(0)$  replaced by

$$MV(0) = \int_{\mathbb{R}^2} |x - B_0|^2 n_0(x) dx,$$

where  $V(0)$  is the variance of the initial data  $n_0$  defined by

$$V(0) = V_2(n_0) = \frac{1}{M} \int_{\mathbb{R}^2} |x - B_0|^2 n_0(x) dx,$$

and  $B_0$  is given by

$$B_0 = B_0(t) = \frac{1}{M} \int_{\mathbb{R}^2} x n_t(x) dx = \frac{1}{M} \int_{\mathbb{R}^2} x n_0(x) dx \in \mathbb{R}^2, \quad t \in (0, T^*).$$

Here, the measure  $\frac{n_0(x)}{M} dx$  can be seen as a probability measure since  $n_0$  is non-negative and non identically zero. For a study of (PKS) system with Borel measures as initial data, see [1]. It can be easily seen that the barycenter  $B_0(t)$  is (formally) independent of  $t \in (0, T^*)$  for any solution  $(n_t)$  of the (PKS) system. Thus we have the following easy improvement of (1.4), namely

$$(1.5) \quad T^* \leq T_v^* := \frac{2\pi V(0)}{M - 8\pi}.$$

Let us recall the formal arguments of the proof for these inequalities (1.4) and (1.5). It is first enough to show (1.5). We have the following exact formula,

$$I'(s) := \frac{d}{ds} \left( \int_{\mathbb{R}^2} |x|^2 n_s(x) dx \right) = 4M \left( 1 - \frac{M}{8\pi} \right), \quad s \in (0, T^*),$$

for any weak solution  $(n_t)$  of (PKS) system with mass  $M = \|n_0\|_1$  such that

$$\int_{\mathbb{R}^2} |x|^2 n_0(x) dx < +\infty,$$

see [7, Lemma 2.1]. Let  $V(s) = \frac{1}{M} \int_{\mathbb{R}^2} |x - B_0|^2 n_s(x) dx = \frac{I(s)}{M} - |B_0|^2$ . The derivative just above can also be written as

$$V'(s) = \frac{1}{M} I'(s) = 4 \left( 1 - \frac{M}{8\pi} \right), \quad s \in (0, T^*).$$

By integration with respect to  $s$ , this leads to the next formula

$$V(t) = V(0) + 4 \left( 1 - \frac{M}{8\pi} \right) t, \quad 0 < t < T^*.$$

By maximal principle  $n_t \geq 0$ , so  $V(t)$  is non-negative for all  $0 < t < T^*$ . Thus, we obtain

$$V(0) - 4t \left( \frac{M - 8\pi}{8\pi} \right) \geq 0, \quad 0 < t < T^*.$$

Hence, we conclude that  $t \left( \frac{M - 8\pi}{8\pi} \right) \leq V(0)$  for all  $0 < t < T^*$ . Letting  $t$  goes to  $T^*$ , we deduce (1.5) when  $M > 8\pi$ . Now, note that (1.5) implies (1.4) because

$$MV(0) = I(0) - M|B_0|^2 \leq I(0).$$

Note that usually the barycenter is rarely mentioned in most papers due probably to the fact that the initial data is often assumed to be radially symmetric, hence  $B_0 = 0$ . But we see some advantages to consider barycenters  $B_0 \neq 0$  and the variance in the formulation of some of our results. Recall that the variance measures the spread of the data around the mean. In particular, when we mention the variance of the density  $\frac{n_0(x)}{M} dx$ . Later on, this result has been generalized in [24, Th.2] where the second equation (1.2) of the (PKS) system is replaced by

$$(-\Delta)c(x, t) + \gamma c(x, t) = n(x, t), \quad x \in \mathbb{R}^2, \quad t > 0,$$

with  $\gamma \geq 0$ . When  $\gamma > 0$ , we say that we have a *consumption* term. In that case, an additional assumption on  $I(0)$  of the form  $I(0) \leq h_1(M)$  for some function  $h_1$  of the mass  $M$  is imposed to obtain the blow-up in the supercritical case.

A result has recently been obtained for the analogue of the (PKS) system on the 2-D hyperbolic space. More precisely, with an appropriate definition of the moment  $I(0)$  for the initial data  $n_0$ , and under a similar additional condition of the form  $I(0) \leq h(M)$  for some function  $h$  of the mass  $M$ , a blow-up is proved for the solution of (PKS) system. Moreover, the maximal time  $T^*$  is bounded by an explicit function of the mass  $M$  and  $I(0)$ ; analogue to (1.5), see [26]. This last situation shares some similarity with the Euclidean case with consumption, (i.e.  $\gamma > 0$ ), certainly due to the spectral gap of the Laplacian on the hyperbolic space. Recently, local criteria have been introduced for blow-up of radial solutions in 2-dimensional chemotaxis models have been obtained by P. Biler et al., using weighted averages on disks. See [4], where the Laplacian  $\Delta$  is replaced by some powers of the Laplacian  $(-\Delta)^{\alpha/2}$ . See also [3], when the consumption term with  $\gamma > 0$  is considered.

Let us mention also the case of critical mass  $M = 8\pi$  on the plane for which the critical time is  $T^* = +\infty$ , i.e. solutions are global in time, when assumptions of finite 2-moment and entropy are made on the initial condition  $n_0$ . Moreover, the solutions blow up as the delta Dirac  $8\pi\delta_{z_0}$  at the center of mass  $z_0 = MB_0$  when  $t \rightarrow T^* = +\infty$ , see [6] [Th.1.3, p.1453], see also [21] and the recent long preprint [14]. In our study, we shall see that the upper bound  $T_c^*$  of  $T^*$  is consistent with this result when  $M \rightarrow 8\pi^+$ . See Remark 1.4 below.

The aim of the present work is to provide an upper bound on the maximal existence time  $T^*$  in the case of blow-up  $M > 8\pi$  by removing the second moment condition and/or the entropy condition on the initial data  $n_0$ . We only assume that  $n_0$  is non-negative and integrable on  $\mathbb{R}^2$ .

Throughout this paper, we shall use the definition of mild solution  $n(t)$  of the (PKS) system on  $\mathbb{R}^2$  taken from [41, p.392]. Let  $T > 0$  be fixed. We say that  $n(t) = n(\cdot, t)$ ,  $0 < t < T$ , is a mild solution of the (PKS) system if,

$$n \in C_w([0, T], L^1(\mathbb{R}^2)), \quad \sup_{t \in (0, T)} t^{\frac{1}{4}} \|n(t)\|_{L^{\frac{4}{3}}} < +\infty,$$

and  $n(t)$  satisfies the following Duhamel integration equation for all  $t \in (0, T)$ ,

$$(1.6) \quad n(t) = e^{t\Delta} n_0 - \int_0^t e^{(t-s)\Delta} \operatorname{div}(n(s)\nabla c(s)) ds,$$

with  $-\Delta c(t) = n(t)$  in the sense

$$\nabla c(x, t) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{x-y}{|x-y|^2} n(t, y) dy, \quad x \in \mathbb{R}^2, t \in (0, T).$$

Here, the space  $C_w([0, T], L^1(\mathbb{R}^2))$  is the space of (weakly) continuous functions with values in  $L^1(\mathbb{R}^2)$ . Recall that, for any initial data  $0 \leq n_0 \in L^1(\mathbb{R}^2)$ , there exists a mild solution  $(n(t))_{0 < t < T^*}$  of the (PKS) system where  $T^* \in (0, +\infty]$  denotes the maximal existence time of the solution (see [41, 5]). For the case of existence of (global) weak solutions with measures as initial data on  $\mathbb{R}^2$ , we can refer to the recent paper [18] where a simple proof of this fact is given.

Recently, Dongyi Wei has proved the following remarkable dichotomy result.

**Theorem 1.1.** [41, Th. 1.1 p.390] *Assume that  $0 \leq n_0 \in L^1(\mathbb{R}^2)$  and  $M = \|n_0\|_1$ . Let  $T^*$  be the maximal existence time of the mild solution. Then  $T^* = +\infty$  if and only if  $M \leq 8\pi$ .*

As already described above, an additional finite second moment condition on the initial data  $n_0$  is often used to imply by a simple virial argument the blow-up of the solution and to provide an explicit upper bound on the maximal existence time  $T^*$  of the solution when  $M > 8\pi$ , see for instance [7, 24]. The main novelty of the result of D. Wei is that no moment condition (nor entropy) is imposed on the initial data  $n_0$  other than the minimal

ones (1.3). In other words, his result can be rephrased as follows: the (mild) solution blows up, i.e.  $T^* < +\infty$ , if and only  $M > 8\pi$ . But it seems that there are no explicit estimates of  $T^*$  deduced from the study carried out in [41].

Before describing the main results of this paper, we make some general comments on the approach which consists in replacing the unbounded weight  $|x|^2$  appearing in the definition of the second moment by a family of bounded weights: Gaussian weights (i.e. heat kernels). We start our discussion by some simple facts related to the heat equation which is known to play an important role in the study of (PKS) systems, in particular by the Duhamel formula (1.6). This importance is due to the fact that the heat kernel is related to the linear part of the (PKS) equation (1.1), i.e. to the Laplacian  $\Delta$ . Let us denote by  $p_s$  be the heat kernel on  $\mathbb{R}^2$  defined by

$$p_s(x) = (4\pi s)^{-1} \exp\left(-\frac{|x|^2}{4s}\right)$$

for all  $x \in \mathbb{R}^2$  and all  $s > 0$ . We also denote by  $H_z(s) = H_{z,n_0}(s)$  the following function

$$H_{z,n_0}(s) := 4\pi s e^{s\Delta} n_0(z) = 4\pi s p_s \star n_0(z) = \int_{\mathbb{R}^2} \exp\left(-\frac{|x-z|^2}{4s}\right) n_0(x) dx$$

for all  $z \in \mathbb{R}^2$  and all  $s > 0$ . The family of functions  $H_{z,n_0}$  can be seen as particular weighted integrals. The general interest of studying weighted integrals of the form

$$I_w(t) := \int_{\mathbb{R}^2} w(x) n_t(x) dx, \quad 0 < t < T^*,$$

for some positive weight  $w$  is to *detect* and *measure in average* the effect of the evolution in time of the density  $(n_t)_{0 < t < T^*}$  of cells. If the chosen weight  $w$  is constant, the only fact that can be observed is that the mass of cells is preserved. Indeed, the mass of  $n(x, t)$  given by  $I_w(t)$  with  $w \equiv 1$ , i.e.,

$$M(t) = \int_{\mathbb{R}^2} n(x, t) dx = M(0) = M = \int_{\mathbb{R}^2} n_0(x) dx,$$

is preserved along the time evolution of the (PKS) system. When one considers the weight  $w(x) = |x|^2$ , such a weight imposes an additional condition on  $n_0$ , i.e., the finiteness of the second-moment  $\int |x|^2 n_0(x) dx$ , which leads automatically to a lack of generality. See (1.4) above for the corresponding bound on  $T^*$ .

Other choices of  $w$  may detect other behaviours of the solution  $(n_t)$  in the sense that the quantity  $I_w(t)$  may evolve with  $t$  in diverse ways, see for instance [5] eq. 5.3.13 p.160. This is what happens when we choose the heat kernel as weight to describe a two parameters family of weights, namely  $w_{s,z}(x) = 4\pi s p_s(x-z)$  with  $s > 0$  and  $z \in \mathbb{R}^2$  be fixed. The use of the heat kernel as weight has at least three features. The first interest is that the weight

$$w_{s,z}(x) = 4\pi s p_s(x-z) = \exp\left(-\frac{|x-z|^2}{4s}\right)$$

is nowhere constant, smooth and uniformly bounded by 1 for all  $s > 0$  and  $z \in \mathbb{R}^2$ . In particular, the function  $(s, z) \mapsto H_{z,n_0}(s)$  defined above is always finite, more precisely  $H_z(s) \leq M$  where  $M$  is the mass of  $n_0$  which is assumed to be finite. In this way, no additional condition on  $n_0$  than  $n_0 \in L^1(\mathbb{R}^2)$  is necessary to study the quantity  $H_{z,n_0}(s)$ . The second argument is that the function  $s \mapsto H_{z,n_0}(s)$  is continuous, monotonically increasing with  $s$ , and its range is  $(0, M)$  for a non-negative non-zero initial data  $n_0$ . Furthermore, choosing the point  $z \in \mathbb{R}^2$  allows us to emphasize the behavior of the initial data  $n_0$  around this point due to the rapid decay of the heat kernel at infinity. Finally, the third reason and probably the most important is that the heat kernel has a strong relationship with the (PKS) system. Indeed, the simplest linearized form of the non-linear equation (1.1), i.e. without the aggregation term  $-\nabla \cdot (n \nabla c)$ , is the heat equation, i.e.,  $\frac{\partial}{\partial t} u(x, t) = \Delta u(x, t)$ .

The aim of this paper is to provide a general upper bound on the maximal existence time  $T^*$  of mild solutions of (PKS) system in the supercritical case  $M > 8\pi$  without further conditions on the initial data  $n_0$  other than  $0 \leq n_0 \in L^1(\mathbb{R}^2)$ . This is achieved by Theorem 1.2 below, which proposes a general formula for an upper bound  $T_c^*(n_0)$  of  $T^*$ . Later on, in Theorem 1.7, we specify the form of the bound on  $T_c^*(n_0)$  for radially symmetric initial data, which allows us to use the Laplace transform. The main interest of the theorem just below is to evaluate an upper bound on the maximal existence time  $T^*$ . No additional conditions, as a moment condition and/or an entropy condition, are imposed on the initial data  $n_0$ .

These estimates could be used as time bounds for numerical simulations on the time interval  $[0, T)$  observing with certainty the blow-up phenomenon on the plane by choosing  $T \geq T_c^*(n_0) \geq T^*$ , see Theorem 1.9. But for a reasonable simulation, i.e on a bounded domain  $\Omega$  with smooth boundary  $\partial\Omega$ , it would be necessary to consider the case where the cells are located at time  $t = 0$  in a set at a very large distance from the boundary of the domain to avoid possible boundary effects. Recall that  $n_0$  represents the repartition of the cells at time  $t = 0$ . So, the condition will certainly be described by a control of the form  $\text{dist}(\text{supp } n_0, \partial\Omega) \geq C$  with  $C > 0$  large enough.

Here is the first main result of this paper.

**Theorem 1.2.** *Let  $0 \leq n_0 \in L^1(\mathbb{R}^2)$  of mass  $M = \int_{\mathbb{R}^2} n_0(x) dx > 8\pi$  and  $T^*$  be the maximal existence time of a mild solution  $(n_t)$  of the Patlak-Keller-Segel system (1.1)-(1.2)-(1.3). Then the following statements hold true.*

(1) *The maximal existence time  $T^*$  is finite and satisfies the following estimate,*

$$(1.7) \quad H_{z,n_0}(T^*) := \int_{\mathbb{R}^2} \exp\left(-\frac{|x-z|^2}{4T^*}\right) n_0(x) dx \leq \frac{2M^2}{3M-8\pi},$$

*for all  $z \in \mathbb{R}^2$ . In particular,  $T^*$  is finite.*

*Or equivalently, for all  $0 < s < T^*$  and all  $z \in \mathbb{R}^2$ ,*

$$(1.8) \quad H_{z,n_0}(s) := \int_{\mathbb{R}^2} \exp\left(-\frac{|x-z|^2}{4s}\right) n_0(x) dx \leq \frac{2M^2}{3M-8\pi}.$$

(2) *For  $z \in \mathbb{R}^2$ , let  $T_{c,z}^*(n_0) := H_{z,n_0}^{-1}\left(\frac{2M^2}{3M-8\pi}\right)$ . Then we have*

$$(1.9) \quad T^* \leq T_c^*(n_0) := \inf_{z \in \mathbb{R}^2} T_{c,z}^*(n_0) < +\infty.$$

(3) *Moreover, if there exists  $z_0 \in \mathbb{R}^2$  such that  $H_{z,n_0}(s) \leq H_{z_0,n_0}(s)$  for all  $s > 0$  and all  $z \in \mathbb{R}^2$ , then  $T_c^*(n_0) = T_{c,z_0}^*(n_0)$ . In particular, if  $n_0$  is a non-increasing radially symmetric integrable function, then*

$$T_c^*(n_0) = T_{c,z=0}^*(n_0) = H_{0,n_0}^{-1}\left(\frac{2M^2}{3M-8\pi}\right).$$

(4) *The value  $T_c^*(n_0)$  is translation-invariant in the sense that  $T_c^*(n_0) = T_c^*(m_0)$  for any  $m_0$  of the form  $m_0(x) = n_0(x+z_0)$ ,  $x \in \mathbb{R}^2$  for fixed  $z_0 \in \mathbb{R}^2$ .*

Before giving the proof of the theorem just above, we make several comments.

**Remark 1.3.** *At first sight, the inequality (1.8) comes as a surprise since the full range of  $H_z(s) := H_{z,n_0}(s)$ ,  $s > 0$ , is a priori  $(0, M)$  whenever  $n_0 \neq 0$  and  $n_0 \in L^1(\mathbb{R}^2)$ . Note that  $H_z(s)$  only depends on the Gaussian and any initial data  $n_0$ . The inequality (1.8) is a limitation of this range since we easily see that*

$$\frac{2M^2}{3M-8\pi} < M,$$



when  $M > 8\pi$ . Hence, we deduce qualitatively that  $T^*$  is finite. Indeed, if  $T^* = +\infty$  we can take the limit as  $s$  goes to infinity in (1.8), and by monotone convergence we get

$$M = \lim_{s \rightarrow +\infty} H_{z, n_0}(s) \leq \frac{2M^2}{3M - 8\pi},$$

for any  $z \in \mathbb{R}^2$ . Contradiction. This argument also implies that all the quantities  $T_{c,z}^*(n_0)$  defined in Theorem 1.2 are finite since  $s \rightarrow H_{z, n_0}(s)$  is strictly increasing, hence it is invertible on its range  $(0, M)$  which contains  $\frac{2M^2}{3M - 8\pi}$ . In particular, the inequality  $T^* \leq T_{c,z}^*(n_0)$  holds true for all  $z \in \mathbb{R}^2$  when  $M > 8\pi$  which allows us to optimize over  $z$ . Thus, the lower bound of the upper bounds  $T_{c,z}^*(n_0)$  over  $z$  will provide an upper bound on  $T^*$ .

**Remark 1.4.** The constant  $\frac{2M^2}{3M - 8\pi}$  in the inequality (1.7)-(1.8) is sharp for the supercritical case in the sense that the inequality  $R(M) := \frac{2M^2}{3M - 8\pi} < M$  holds for all  $M > 8\pi$ , and this inequality becomes an equality when  $M \rightarrow 8\pi^+$ . Note that this sharpness is essential for the proof of the dichotomy result of Theorem 1.1 in [41], and consequently also for Theorem 1.2 of this article. Note that when  $M = 8\pi$ , then  $R(M) = M = 8\pi$ , and formally

$$T^* = T_c^*(n_0) = T_{c,z}^*(n_0) = \lim_{M \rightarrow 8\pi^+} H_{z, n_0}^{-1} \left( \frac{2M^2}{3M - 8\pi} \right) = +\infty$$

for all  $z \in \mathbb{R}^2$ . Recall that when  $M = 8\pi$  then we have  $T^* = +\infty$ , see [41, Th. 1.1 p.390], recalled in Theorem 1.1 of this paper. This is consistent with a former existence result of [6] of global solutions for the critical mass  $M = 8\pi$  when 2-moment and entropy of the initial data  $n_0$  are assumed to be finite. Moreover, in [6], a more precise study of the blow-up is made by obtaining the convergence of the solutions to the  $8\pi$ -delta-Dirac measure concentrated at the center of mass of  $n_0$ . See also [2] for a study of radially symmetric solutions for  $M = 8\pi$  on  $\mathbb{R}^2$ , [21] on a ball, and on a domain [30, 31], for instance. We can also refer to [38, Th. 2 and 3, p.24-25] for the finite number of blow-up points, and for the relation between blow-up points and chemotatic collapse (Dirac measure).

**Remark 1.5.** In the course of the proof of Theorem 1.2, we shall see that the intriguing inequality (1.8) is a consequence of the fact that there exists a mild solution  $(n_t)$  of the (PKS) system with  $n_0$  as initial data. More precisely, the weak continuity at  $0^+$  of  $n_t$  "attaches" the free initial data  $n_0$  to the solution  $(n_t)$  for  $t > 0$ .

**Remark 1.6.** The left-hand side term of the inequality (1.7) is equal to  $(4\pi T^*)p_{T^*} \star n_0(z)$ . This term is clearly related to the linear part of the (PKS) system via the heat semigroup  $e^{-t\Delta}$  without mentioning any explicit connection with the non-linear non-local term of the (PKS) system. The right-hand side term of (1.7) is mainly related to the non-linearity of the (PKS) system via the rational function  $R(M)$  of the mass  $M$  defined in Remark 1.4.

**Proof of Theorem 1.2.** The proof of (1.9) is essentially a re-interpretation of the proof of the blow-up phenomenon when  $M > 8\pi$  given in [41, p.397] in order to evaluate blow-up time upper bounds. The inequality (1.8) is due to Dongyi Wei [41]. For the sake of clarity, we provide the main part of the arguments, with some comments inserted in the course of the proof. This paper is dedicated more precisely to the study of estimates of the critical time  $T_c^*(n_0)$ .

Fix  $0 \leq n_0 \in L^1$ ,  $n_0 \neq 0$ . We assume that the mass  $M = \int_{\mathbb{R}^2} n_0(x) dx$  satisfies  $M > 8\pi$  and we consider  $n_t(x) = n(x, t)$  a mild solution of the (PKS) system on  $\mathbb{R}^2$ .

**(1) (i) Proof of the inequality (1.8) and the finiteness of  $T^*$ .**

The main tool is the introduction of the following specific heat regularization function  $t \rightarrow \tilde{n}(z, t, s) = p_{s-t} \star n_t(z)$  for all  $(t, s)$  satisfying  $t < T^*$  and  $t < s < +\infty$  for fixed  $z \in \mathbb{R}^2$ . Here  $p_t(x) = (4\pi t)^{-1} \exp(-|x|^2/4t)$  denotes the heat kernel on  $\mathbb{R}^2$ . (At this stage, note that  $s$  is not necessarily bounded by  $T^*$  itself). The function  $t \rightarrow \tilde{n}(z, t, s)$  can be seen as a perturbation and a regularization of the solution  $(n_t)_{t \in (0, T^*)}$  of the (PKS) system by the



heat semigroup. It consists also to consider the (PKS) solution  $n_t$  at time  $t$  as the initial data (in  $L^1$ ) of the heat equation, and to compute the solution of the heat equation at time  $s - t$  for a priori any  $s > t$  by convolution with the heat kernel  $p_{s-t}$ . This construction is suggested by the non-linear term of the Duhamel formula (1.6). The role of reversing the time by  $-t$  in the heat kernel is that the time derivative of  $\tilde{n}$  will depend only on the non-linear (here bilinear) term of the (PKS) system. The linear part represented by the Laplacian disappears in this process. Indeed, we have

$$\partial_t \tilde{n}(z, t, s) = \frac{1}{4\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} K_1(x - z, y - z, s - t) n_t(x) n_t(y) dx dy,$$

where the convolution kernel in space variables is given by

$$K_1(x, y, t) = -\frac{[\nabla p_t(x) - \nabla p_t(y)] \cdot (x - y)}{|x - y|^2}.$$

This kernel  $K_1$  is independent of the (PKS) solution. See [41, p.392-393] for the details of computation. In [41, Prop.3.1], the following monotonicity inequality for the map  $t \mapsto \tilde{n}(z, t, s)$  has been proved

$$(1.10) \quad \frac{3M \tilde{n}(z, t, s)}{8\pi(s-t)} - \frac{M^2}{16\pi^2(s-t)^2} \leq \partial_t \tilde{n}(z, t, s),$$

for all  $z \in \mathbb{R}^2$  and all  $(t, s)$  such that  $0 < t < \min(s, T^*)$ . This result follows from the existence of a solution and the structure of the (PKS) system, and some geometric estimates on the heat kernel  $p_t$  on  $\mathbb{R}^2$ . Multiplying the inequality (1.10) by the positive term  $(s-t)^a$  with  $a := \frac{3M}{8\pi}$  leads to

$$\partial_t [(s-t)^a \tilde{n}(z, t, s)] = -a(s-t)^{a-1} \tilde{n}(z, t, s) + (s-t)^a \partial_t \tilde{n}(z, t, s) \geq \frac{-M^2}{16\pi^2} (s-t)^{a-2},$$

for all  $z \in \mathbb{R}^2$  and all  $0 < t < s < T^*$ . Now, by integrating over the interval  $(0, s)$  with respect to  $t$  we obtain the following inequality

$$\begin{aligned} \lim_{t \rightarrow s^-} (s-t)^{a-1} (s-t) p_{s-t} \star n_t(z) - \lim_{t \rightarrow 0^+} (s-t)^a p_{s-t} \star n_t(z) &= 0 - s^a p_s \star n_0(z) \\ &\geq \frac{-M^2}{16\pi^2(a-1)} s^{a-1} = \frac{-M^2}{2\pi(3M-8\pi)} s^{a-1}, \end{aligned}$$

for all  $0 < s < T^*$ . The first limit is zero since we have

$$0 \leq (s-t)^a p_{s-t} \star n_t(z) \leq \frac{(s-t)^{a-1}}{4\pi} \int_{\mathbb{R}^2} n_t(x) dx = \frac{(s-t)^{a-1}}{4\pi} M,$$

for all  $0 < t < s < T^*$ , and  $a > 1$ . The second limit is obtained by using in a crucial way the weak continuity at  $t = 0^+$  of the solution  $n \in C_w([0, T^*), L^1(\mathbb{R}^2))$  which attaches the free data  $n_0$  to the solution  $(n_t)_{t \in [0, T^*)}$  of the (PKS) system. Note also that integration on  $t$  over  $(0, s)$  imposes necessarily that  $s < T^*$  in the conclusion since  $0 < t < \min(s, T^*)$ . Finally after simplification, we get

$$H_{z, n_0}(s) := 4\pi s p_s \star n_0(z) \leq \frac{2M^2}{3M - 8\pi}.$$

Thus, the inequality (1.8) is proved for any non-negative initial data  $n_0 \in L^1(\mathbb{R}^2)$  of the (PKS) system, and for all  $0 < s < T^*$  and all  $z \in \mathbb{R}^2$ .

We now recall how we deduce from (1.8) that  $T^*$  is finite when  $M > 8\pi$ . To prove this, we suppose the contrary, i.e.  $T^* = +\infty$ . We can take the limit as  $s$  goes to infinity in (1.8) and, by monotone convergence theorem, we obtain

$$M = \lim_{s \rightarrow +\infty} H_{z, n_0}(s) \leq \frac{2M^2}{3M - 8\pi}.$$

This implies  $M \leq 8\pi$ . Contradiction.

**(1)(ii) Proof of (1.7).** We have seen in (1)(i) that  $T^*$  is finite. Now by Fatou lemma, we can write

$$\begin{aligned} H_{z,n_0}(T^*) &:= \int_{\mathbb{R}^2} \exp\left(-\frac{|x-z|^2}{4T^*}\right) n_0(x) dx = \int_{\mathbb{R}^2} \liminf_{s \rightarrow T^{*,-}} \exp\left(-\frac{|x-z|^2}{4s}\right) n_0(x) dx \\ &\leq \liminf_{s \rightarrow T^{*,-}} \int_{\mathbb{R}^2} \exp\left(-\frac{|x-z|^2}{4s}\right) n_0(x) dx \leq \frac{2M^2}{3M-8\pi}. \end{aligned}$$

This proves (1.7).

**(1)(iii) Proof of the converse:** (1.7)  $\Rightarrow$  (1.8). Conversely, the inequality (1.7) implies the inequality (1.8) because  $s \in (0, +\infty) \mapsto H_{z,n_0}(s)$  is clearly an increasing function.

**(2)** Let  $z \in \mathbb{R}^2$ ,  $n_0 \in L^1(\mathbb{R}^2)$  be fixed. We also assume that  $n_0 \neq 0$  and  $n_0 \geq 0$ . It is easy to see that the function

$$s \in (0, +\infty) \rightarrow H_{z,n_0}(s) := \int_{\mathbb{R}^2} \exp\left(-\frac{|x-z|^2}{4s}\right) n_0(x) dx \in (0, M),$$

is continuous and strictly increasing on  $(0, +\infty)$  with range  $(0, M)$ . Indeed, by monotone convergence theorem, we deduce that  $H_{z,n_0}(0^+) = 0$ ,  $H_{z,n_0}(+\infty) = M$  and  $H_{z,n_0}$  is continuous on  $(0, +\infty)$ . The function  $H_{z,n_0}$  is strictly increasing because  $n_0 \geq 0$  and  $n_0 \neq 0$ . Hence,  $H_{z,n_0}$  is a bijection from  $(0, +\infty)$  onto  $(0, M)$ . We denote by  $H_{z,n_0}^{-1} : (0, M) \rightarrow (0, +\infty)$  its inverse function. It is also a strictly increasing continuous function. For  $M > 8\pi$ , we have  $0 < \frac{2M^2}{3M-8\pi} < M$ . Hence, the value  $T_{c,n_0}^*(z) = H_{z,n_0}^{-1}\left(\frac{2M^2}{3M-8\pi}\right)$  is well-defined and finite. By (1.7), i.e.  $H_{z,n_0}(T^*) \leq \frac{2M^2}{3M-8\pi}$ , and by monotonicity of  $H_{z,n_0}^{-1}$ , this is equivalent to the following bound

$$T^* \leq H_{z,n_0}^{-1}\left(\frac{2M^2}{3M-8\pi}\right) =: T_{c,z}^*(n_0) < +\infty,$$

for all  $z \in \mathbb{R}^2$ . Thus, we obtain simultaneously a quantitative upper bound on the blow-up time  $T^*$ , and again its finiteness. Now taking the infimum over  $z$ , we deduce the upper bound (1.9) on the maximal existence time  $T^*$ .

**(3) (i)** Now assume that there exists  $z_0 \in \mathbb{R}^2$  such that we have  $H_{z,n_0}(s) \leq H_{z_0,n_0}(s)$  for all  $s > 0$  and all  $z \in \mathbb{R}^2$ . Since  $H_{z,n_0}$  and  $H_{z_0,n_0}$  are increasing and  $H_{z,n_0}(0^+) = H_{z_0,n_0}(0^+) = 0$ ,  $H_{z,n_0}(+\infty) = H_{z_0,n_0}(+\infty) = M$ , then we have  $H_{z,n_0}^{-1}(m) \leq H_{z_0,n_0}^{-1}(m)$  for all  $m \in (0, M)$  and all  $z \in \mathbb{R}^2$ . Hence, we obtain

$$\inf_{z \in \mathbb{R}^2} H_{z,n_0}^{-1}(m) = H_{z_0,n_0}^{-1}(m),$$

for all  $m \in (0, M)$ . Applying this result to  $m = \frac{2M^2}{3M-8\pi}$  with  $M > 8\pi$ , we obtain

$$T_c^*(n_0) := \inf_{z \in \mathbb{R}^2} H_{z,n_0}^{-1}\left(\frac{2M^2}{3M-8\pi}\right) = H_{z_0,n_0}^{-1}\left(\frac{2M^2}{3M-8\pi}\right) = T_{c,z_0}^*(n_0).$$

This concludes the first part of the assertion (3).

**(3) (ii)** On  $\mathbb{R}^n$ , let  $f$  and  $g$  be two non-negative non-increasing radially symmetric integrable functions, then it is known that the convolution  $f \star g$  is also a non-negative non-increasing radially symmetric integrable function. In particular, if moreover  $z \mapsto f \star g(z)$  is continuous, we deduce that  $0 \leq f \star g(z) \leq f \star g(0)$ , for all  $z \in \mathbb{R}^n$ . Here, we have assumed that  $g = n_0$  is a non-negative non-increasing radially symmetric integrable function. Since  $f = p_s$  is also a non-negative non-increasing radially symmetric integrable function, then the function  $z \in \mathbb{R}^2 \mapsto H_{z,n_0}(s) = 4\pi s p_s \star n_0(z)$  is a (continuous) non-negative non-increasing radially symmetric integrable function.

As a consequence, we have  $H_{z,n_0}(s) \leq H_{0,n_0}(s)$  for all  $s > 0$  and all  $z \in \mathbb{R}^2$ . We conclude by applying (3)(i) proved just above with  $z_0 = 0$ . This concludes the second part of the assertion (3).

(4) First note that the masses  $M := M(n_0) = \int_{\mathbb{R}^2} n_0(x) dx = M(m_0)$  are equal since the Lebesgue measure is translation-invariant. Moreover, it is easy to check that  $H_{z,m_0}(s) = H_{z+z_0,n_0}(s)$  for all  $z, z_0 \in \mathbb{R}^2$  and all  $s > 0$  with  $m_0(x) = n_0(x + z_0)$ . This yields

$$T_c^*(m_0) := \inf_{z \in \mathbb{R}^2} H_{z,m_0}^{-1}(L(M)) = \inf_{z \in \mathbb{R}^2} H_{z+z_0,n_0}^{-1}(L(M)) = \inf_{z \in \mathbb{R}^2} H_{z,n_0}^{-1}(L(M)) =: T_c^*(n_0),$$

where  $L(M) = \frac{2M^2}{3M-8\pi}$ . This leads to the stated invariance.

This concludes the proof of Theorem 1.2.  $\square$

In the second part of this section, we are interested in estimating the critical time  $T_c^* = T_c^*(n_0)$  defined in Theorem 1.2 for the particular case where  $n_0$  is a non-increasing  $z_0$ -radially symmetric initial data. Recall that a function  $n_0$  is said radially symmetric on  $\mathbb{R}^n$  if, for all  $x, y \in \mathbb{R}^n$  such that  $|x| = |y|$  then  $n_0(x) = n_0(y)$ . We say that  $n_0$  is non-increasing radially symmetric on  $\mathbb{R}^n$  if  $n_0$  satisfies for all  $x, y \in \mathbb{R}^n$  such that  $|x| \leq |y|$ , we have  $n_0(x) \geq n_0(y)$ . This is equivalent to say that  $n_0$  is radially symmetric and the function  $\tilde{n}_0$  defined by  $r \mapsto \tilde{n}_0(r) = n_0(x)$  for  $x \in \mathbb{R}^n$  with  $r = |x|$ , is non-increasing on  $[0, +\infty)$ . Fix  $z_0 \in \mathbb{R}^n$ , we say that a function  $n_0$  is a  $z_0$ -radially symmetric (resp. non-increasing  $z_0$ -radially symmetric) function if  $x \mapsto m_0(x) := n_0(x + z_0)$  is a radially symmetric (resp. non-increasing radially symmetric) function in the sense given above.

In the following results, we shall use the Laplace transform of a function  $f$  denoted by  $\mathcal{L}f(v) = \int_0^{+\infty} e^{-vu} f(u) du$  ( $v > 0$ ). This Laplace transform will be used in association with radially symmetric initial data  $n_0$ .

**Theorem 1.7.** *Under the assumptions of Theorem 1.2 and denoting by  $T_c^*(n_0)$  the critical time defined by (1.9). We have the next statements.*

(1) *If  $n_0$  is a  $z_0$ -radially symmetric non-negative integrable function then*

$$(1.11) \quad T^* \leq T_c^*(n_0) \leq \frac{1}{4(\mathcal{L}f)^{-1}\left(\frac{L(M)}{\pi}\right)},$$

*with  $f(u) = \tilde{m}_0(\sqrt{u})$ ,  $u > 0$ , where  $\tilde{m}_0(|x|) = m_0(x) = n_0(x + z_0)$ ,  $x, z_0 \in \mathbb{R}^2$ , and  $L(M) = \frac{2M^2}{3M-8\pi}$ .*

(2) *Moreover, suppose that  $n_0$  is a non-increasing  $z_0$ -radially symmetric non-negative integrable function. Then we have*

$$(1.12) \quad T^* \leq T_c^*(n_0) = \frac{1}{4(\mathcal{L}f)^{-1}\left(\frac{L(M)}{\pi}\right)},$$

*with  $f$  as in (1.11).*

Note that the upper bound on  $T_c^*(n_0)$  in (1.11) is now an equality in (1.12) when  $n_0$  is a non-increasing  $z_0$ -radially symmetric function. So, the estimate (1.11) of  $T_c^*(n_0)$  is sharp for the sub-class of non-increasing  $z_0$ -radially symmetric data  $n_0$ .

**Proof.** By applying (1.9) and the statement (4) of Theorem 1.2, we obtain

$$T^* \leq T_c^*(n_0) = T_c^*(m_0) \leq T_{c,0}^*(m_0) = H_{0,m_0}^{-1}(L(M)).$$

Now, since the function  $m_0$  is a radially symmetric, the function  $H_{0,m_0}$  can be expressed via the Laplace transform as follows,

$$H_{0,m_0}(s) = \int_{\mathbb{R}^2} \exp\left(-\frac{|x|^2}{4s}\right) m_0(x) dx = \int_{\mathbb{R}^2} \exp\left(-\frac{|x|^2}{4s}\right) \tilde{m}_0(|x|) dx.$$

By the change of variables using polar coordinates, we deduce that

$$H_{0,m_0}(s) = 2\pi \int_0^{+\infty} e^{-r^2/4s} \tilde{m}_0(r) r dr = \pi \int_0^{+\infty} e^{-r^2/4s} \tilde{m}_0(\sqrt{r^2}) d(r^2)$$

$$= \pi \int_0^{+\infty} e^{-\rho/4s} \tilde{m}_0(\sqrt{\rho}) d\rho = \pi \int_0^{+\infty} e^{-\rho/4s} f(\rho) d\rho = \pi(\mathcal{L}f)(1/4s),$$

with  $f(\rho) = \tilde{m}_0(\sqrt{\rho})$ . The last line is obtained by the change of variable  $\rho = r^2$ . The function  $v \rightarrow \mathcal{L}f(v)$  is invertible since  $H_{0,m_0}$  is invertible. Finally, we conclude that

$$H_{0,m_0}^{-1}(L(M)) = \frac{1}{4(\mathcal{L}f)^{-1}\left(\frac{L(M)}{\pi}\right)}.$$

This proves the inequality (1.11).

We now prove the inequality (1.12) as follows. Since  $n_0$  is a non-increasing  $z_0$ -radially symmetric function, then  $m_0$  is a non-increasing radially symmetric function. We apply (3) of Theorem 1.2 to obtain the next equality

$$T_c^*(n_0) = T_c^*(m_0) = \inf_{z \in \mathbb{R}^2} H_{z,m_0}^{-1}(L(M)) = H_{0,m_0}^{-1}(L(M)) = \frac{1}{4(\mathcal{L}f)^{-1}\left(\frac{L(M)}{\pi}\right)}.$$

This concludes the proof of Theorem 1.7.  $\square$

The other sections of this paper are organized as follows.

In Section 2, since generally it is difficult to compute  $T_c^*(n_0)$  explicitly, we extend our investigation of upper bounds on  $T^*$  by providing various (more or less explicit) upper bounds on the bound  $T_c^*(n_0)$  itself. We also prove lower bounds on  $T_c^*(n_0)$  in Section 3 showing (in some cases) the sharpness of the corresponding upper bounds on  $T_c^*(n_0)$ . In Section 4, we finally exhibit explicit bounds on  $T_c^*(n_0)$  for several families of examples of initial data. In particular, we can compute explicitly  $T_c^*(n_0)$  for some examples of radially symmetric initial data  $n_0$  by applying Theorem 1.7 using the Laplace transform.

## 2. ESTIMATES ON THE CRITICAL TIME BOUND $T_c^*(n_0)$

In general, the functions  $H_{z,n_0}^{-1}$  of Theorem 1.2 are not given explicitly. Nevertheless, from this theorem we are able to derive various types of explicit upper bounds on  $T^*$  from those on  $T_c^*(n_0)$ , for all  $0 \leq n_0 \in L^1(\mathbb{R}^2)$ , with or without any specific additional assumptions. Such estimates may be useful by providing an explicit interval of time  $(0, \tilde{T}) \supseteq (0, T_c^*(n_0)) \supseteq (0, T^*)$  for numerical simulations on which the blow-up will certainly be observed.

We first obtain an upper bound on  $T^*(n_0)$  as an infimum over two parameters of a function expressed in terms of convolutions with Gaussian-type weights of the initial data  $n_0$ . We denote by  $\ln_+ v = \sup(\ln v, 0)$ ,  $v > 0$ , and  $\|f\|_\infty = \sup_{z \in \mathbb{R}^2} |f(z)|$  for a bounded function  $f$ .

**Corollary 2.1.** *Let  $T^* = T^*(n_0)$  be the maximal existence time of a solution of the (PKS) system with initial data  $0 \leq n_0 \in L^1$  satisfying  $M = \|n_0\|_1 > 8\pi$ . Let  $L(M) = \frac{2M^2}{3M-8\pi}$ . Then we have*

$$(2.13) \quad T^* \leq T_c^*(n_0) \leq T_{c_1}^* := \inf_{q>1, \lambda>0} \lambda q^{-1/q} \left[ \ln_+ \left( \frac{\|\omega_{q,\lambda} \star n_0\|_\infty}{L(M)} \right) \right]^{-1/q},$$

where  $\omega_{q,\lambda}(x) = \exp(-c_{q,\lambda}|x|^{\frac{2q}{q-1}})$  with  $c_{q,\lambda} = \frac{q-1}{q}(4\lambda)^{-\frac{q}{q-1}}$ , for all  $1 < q < \infty$  and all  $\lambda > 0$ . In particular, if  $n_0$  is a non-increasing radially symmetric function, then

$$\|\omega_{q,\lambda} \star n_0\|_\infty = \omega_{q,\lambda} \star n_0(0) = \int_{\mathbb{R}^2} \exp\left(-c_{q,\lambda}|x|^{\frac{2q}{q-1}}\right) n_0(x) dx.$$

We make some comments on these results before giving the proof.

**Remark 2.2.** *Note that the weight  $\omega_{q,\lambda}$  does not depend on  $T^*$ . In fact, the parameter  $\lambda$  in the formula (2.13) plays a similar role to  $T^*$  but now it can be taken freely in the full interval  $(0, +\infty)$ .*

**Remark 2.3.** The value  $T_{c_1}^*$  is finite. We prove it as follows. Let  $p = \frac{q}{q-1}$  with  $1 < q < \infty$ . For all  $1 < q < +\infty$  and all  $z \in \mathbb{R}^2$ , we have by dominated convergence theorem,

$$\lim_{\lambda \rightarrow +\infty} \frac{\omega_{q,\lambda} \star n_0(z)}{L(M)} = \frac{1}{L(M)} \int_{\mathbb{R}^2} \lim_{\lambda \rightarrow +\infty} \exp \left[ -\frac{1}{p} \left( \frac{|x-z|^2}{4\lambda} \right)^p \right] n_0(x) dx = \frac{M}{L(M)} > 1,$$

whenever  $M > 8\pi$ . Thus, we get for all  $1 < q < +\infty$  and  $\lambda \geq \lambda_q$  (large enough),

$$0 = \ln_+ 1 < \ln_+ \left( \frac{\|\omega_{q,\lambda} \star n_0\|_\infty}{L(M)} \right).$$

This implies that  $T_{c_1}^*$  defined by (2.13) is finite.

**Proof.** The main idea of proof is to obtain a lower bound for the integral  $H_{z,n_0}(s)$  by a product of two functions, one depending on the time variable  $s$  and the other one depending on the space variable  $z$ , in order to get a lower bound of the following form,

$$g(s)(\omega \star n_0)(z) \leq \int_{\mathbb{R}^2} \exp \left( -\frac{|x-z|^2}{4s} \right) n_0(x) dx = H_{z,n_0}(s), \quad z \in \mathbb{R}^2, \quad s > 0.$$

Here, we expect that  $s \mapsto g(s)$  will be an invertible function explicitly given, and the function  $x \in \mathbb{R}^2 \mapsto \omega(x)$  will also be an explicit (bounded) weight of the space variable  $x$ .

More explicitly, we prove the corollary as follows. We apply the next well-known Young's inequality,

$$ab \leq \frac{a^p}{p\lambda^p} + \frac{\lambda^q b^q}{q},$$

for all  $a, b \geq 0$ ,  $\lambda > 0$ , where  $1 < p, q < +\infty$  satisfies  $1/p + 1/q = 1$  (i.e.,  $p = \frac{q}{q-1}$ ). We set  $a = |x-z|^2/4$  and  $b = 1/s$  for all  $x, z \in \mathbb{R}^2$  and all  $s > 0$ . Then we deduce that

$$\frac{|x-z|^2}{4s} \leq \ell |x-z|^{2p} + \frac{k}{s^q},$$

with  $\ell = p^{-1}(4\lambda)^{-p}$  and  $k = q^{-1}\lambda^q$ . This implies that

$$e^{-ks^{-q}} \int_{\mathbb{R}^2} \exp(-\ell |x-z|^{2p}) n_0(x) dx \leq \int_{\mathbb{R}^2} \exp \left( -\frac{|x-z|^2}{4s} \right) n_0(x) dx,$$

for all  $z \in \mathbb{R}^2$  and all  $s > 0$ . By setting  $\omega_{q,\lambda}(x) = \exp(-\ell |x|^{2p})$ , our result reads as a convolution inequality,

$$e^{-ks^{-q}} \omega_{q,\lambda} \star n_0(z) \leq H_{z,n_0}(s).$$

Recall that  $T_c^* = T_c^*(n_0) := \inf_{z \in \mathbb{R}^2} T_{c,z}^*(n_0)$ , and that  $T_{c,z}^* := T_{c,z}^*(n_0)$  is defined as the unique solution of  $H_{z,n_0}(T_{c,z}^*) = L(M)$  for any fixed  $z \in \mathbb{R}^2$ . By monotonicity of the functions  $s \mapsto H_{z,n_0}(s)$  and using the definition of  $T_c^*$ , we deduce from this convolution inequality that

$$e^{-ks^{-q}} \omega_{q,\lambda} \star n_0(z) \leq H_{z,n_0}(T_c^*) \leq H_{z,n_0}(T_{c,z}^*) = L(M).$$

for all  $0 < s \leq T_c^*$  and all  $z \in \mathbb{R}^2$ . Now taking the supremum over  $z \in \mathbb{R}^2$  and  $s = T_c^*$ , we obtain the next inequality,

$$e^{-k(T_c^*)^{-q}} \|\omega_{q,\lambda} \star n_0\|_\infty \leq L(M).$$

From the inequation (1.9) of Theorem 1.2, we deduce that

$$\begin{aligned} T^* &\leq T_c^*(n_0) \leq k^{1/q} \left[ \ln_+ \left( \frac{\|\omega_{q,\lambda} \star n_0\|_\infty}{L(M)} \right) \right]^{-1/q} \\ &= \lambda q^{-1/q} \left[ \ln_+ \left( \frac{\|\omega_{q,\lambda} \star n_0\|_\infty}{L(M)} \right) \right]^{-1/q}, \end{aligned}$$

for all  $1 < q < +\infty$  and all  $\lambda > 0$ .

Finally, by taking the infimum over the parameters  $q > 1$  and  $\lambda > 0$ , we obtain the inequality (2.13).

Now we prove the last statement of Corollary 2.1 as follows. By assumption,  $n_0$  is a non-increasing radially symmetric function, and since the function

$$x \mapsto \omega_{q,\lambda}(x) = \exp(-c_{q,\lambda}|x|^{\frac{2q}{q-1}})$$

has clearly the same property, then the convolution of these two functions, i.e.  $z \mapsto \omega_{q,\lambda} \star n_0(z)$ , is also a non-increasing radially symmetric (continuous) function. Thus, it attains its supremum at  $z = 0$ . This concludes the proof of Corollary 2.1.  $\square$

Unfortunately, despite its theoretical interest, it is difficult to estimate the infimum  $T_{c_1}^*$  in (2.13) of Corollary 2.1, even for the simplest case of the characteristic function of a disk, i.e.  $n_0 = 1_{B(0,R)}$ . Nevertheless, in the general situation we can choose any fixed couple of parameter  $q > 1$  and  $\lambda > 0$  to bound  $T_c^*(n_0)$ , hence  $T^*$  too. In the next corollary, we propose several approaches to obtain more explicit bounds on  $T_c^*(n_0)$ . The first one is given in terms of averages of  $n_0$  on the family of disks  $B(z, \rho)$  of radius  $\rho > 0$  centered at  $z \in \mathbb{R}^2$ , and valid for any initial data  $n_0$ . Next, in this corollary, we rewrite this result differently. The last result mentioned in this corollary deals with the case of data with compact support. All these results are applied to obtain explicit upper bounds on  $T^*$  for several families of examples, see Section 4.

**Corollary 2.4.** *Let  $T^*$  be the maximal existence time of a solution of the (PKS) system (1.1)-(1.3) with initial data  $n_0 \geq 0$  of mass  $M > 8\pi$ . Let  $L(M) = \frac{2M^2}{3M-8\pi}$ .*

(1) *We have  $T^* \leq T_c^* := T_c^*(n_0)$  with*

$$(2.14) \quad T_c^* \leq T_{c_2}^* := \inf_{\rho > 0, z \in \mathbb{R}^2} \frac{\rho^2}{4} \left[ \ln_+ \left( \frac{\mathcal{M}_z(\rho)}{L(M)} \right) \right]^{-1},$$

where  $\mathcal{M}_z(\rho) := \int_{B(z,\rho)} n_0(x) dx$  and  $B(z, \rho)$  is the Euclidean disk of radius  $\rho > 0$  centered at  $z \in \mathbb{R}^2$ . Moreover, if  $n_0$  is a non-increasing radially symmetric (integrable) function, then we have

$$(2.15) \quad T^* \leq T_c^* \leq T_{c_2}^* = \inf_{\rho > 0} \frac{\rho^2}{4} \left[ \ln_+ \left( \frac{\mathcal{M}_0(\rho)}{L(M)} \right) \right]^{-1}.$$

(2) *For each fixed  $z \in \mathbb{R}^2$  and  $\rho > 0$ , let  $g_z(\rho) = \frac{1}{M} \int_{B(z,\rho)} n_0(x) dx$ . We denote by  $g_z^{\leftarrow}$  the generalized inverse of  $g$  defined by*

$$g_z^{\leftarrow}(m) := \inf\{\rho > 0 : g_z(\rho) \geq m\}, \quad m \in (0, 1).$$

For all  $z \in \mathbb{R}^2$ , let

$$T_{c_2}^*(z) := \inf_{\rho > 0} \frac{\rho^2}{4} \left[ \ln_+ \left( \frac{\mathcal{M}_z(\rho)}{L(M)} \right) \right]^{-1}.$$

Then we have

$$(2.16) \quad T_{c_2}^*(z) = \frac{1}{4 \ln \left( 1 + \frac{M-8\pi}{2M} \right)} \cdot \inf_{\theta \in (0,1)} \left( \frac{[g_z^{\leftarrow}(a^\theta)]^2}{(1-\theta)} \right),$$

with  $a := \frac{L(M)}{M} = \frac{2M}{3M-8\pi} \in (0, 1)$ . Hence, we also have

$$(2.17) \quad T^* \leq T_c^* \leq T_{c_2}^* = \frac{1}{4 \ln \left( 1 + \frac{M-8\pi}{2M} \right)} \cdot \inf_{z \in \mathbb{R}^2, \theta \in (0,1)} \left( \frac{[g_z^{\leftarrow}(a^\theta)]^2}{(1-\theta)} \right).$$

(3) *Assume that  $n_0$  has compact support denoted by  $K$ . Let  $i_K(z) = \sup_{x \in K} |x - z|$  for all  $z \in \mathbb{R}^2$ , and  $R_0 = \inf_{z \in \mathbb{R}^2} i_K(z)$ . Then we have*

$$(2.18) \quad T^* \leq T_c^* \leq T_{c_3}^* := \frac{R_0^2}{4 \ln \left( 1 + \frac{M-8\pi}{2M} \right)}.$$



The value  $R_0$  is also the radius of the smallest closed disk containing  $K$ . In particular, we have

$$(2.19) \quad T^* \leq T_c^* \leq T_{c_3}^* \leq \frac{D^2}{12 \ln \left(1 + \frac{M-8\pi}{2M}\right)},$$

where  $D$  is the diameter of  $K$ .

Before giving the proof of this corollary, we make some comments on these results.

**Remark 2.5.** By translation invariance of  $T_c^*(n_0)$  as mentioned in Theorem 1.2 (4), the inequality (2.15) is still valid when  $n_0$  is a non-increasing  $z_0$ -radially symmetric function changing  $\mathcal{M}_0(\rho)$  by  $\mathcal{M}_{z_0}(\rho)$ .

**Remark 2.6.** The ratio  $\frac{1}{4 \ln \left(1 + \frac{M-8\pi}{2M}\right)}$  appears naturally in the proof of (2.18). The inequality (2.17) proposes a more general version where this ratio appears without any compact support condition on the initial data  $n_0$ .

**Remark 2.7.** (1) The range of the non-decreasing continuous function  $[0, +\infty) \ni \rho \mapsto g_z(\rho)$  is either the interval  $[0, 1)$  or  $[0, 1]$ . Indeed, we have  $g_z(0) = 0$ ,  $\lim_{\rho \rightarrow +\infty} g_z(\rho) = 1$  and  $g_z$  is continuous on its domain  $[0, +\infty)$  for all  $z \in \mathbb{R}^2$ . This function may not be a strictly increasing function on  $[0, +\infty)$ . In particular, it can be constant on an interval. For instance if  $g_z(\rho_1) = g_z(\rho_2)$  for some  $0 < \rho_1 < \rho_2 \leq +\infty$  and for some fixed  $z \in \mathbb{R}^2$ , then the (non-negative) initial data  $n_0$  is almost everywhere zero in the annulus  $B(z, \rho_2) \setminus B(z, \rho_1)$ . It is easy to construct such examples of initial data. For instance, we can simply consider the characteristic function of the disk  $B(z_0, \rho_1)$  and the function  $g_{z_0}$  for a fixed  $z_0$  (here,  $\rho_2 = +\infty$ ). More generally, we can consider an initial data with compact support.

- (2) If  $g_z$  is strictly increasing on  $[0, +\infty)$ , then its range is  $[0, 1)$ . Thus, the function  $g_z : [0, +\infty) \rightarrow [0, 1)$  is a bijection and  $g_z^{\leftarrow}$  is the usual inverse function  $g_z^{-1} : [0, 1) \rightarrow [0, +\infty)$  of  $g_z$ . Indeed, the value 1 cannot be in the range of  $g_z$ , if not, there exists  $0 < \rho_0 < +\infty$  such that  $g_z(\rho_0) = 1$ , then we have  $g_z(\rho) = 1$  for all  $\rho \geq \rho_0$ , since  $g_z$  is non-decreasing and bounded by 1. Contradiction:  $g_z$  is not strictly increasing.
- (3) If the value 1 is in the range of  $g_z$ , i.e. there exists  $0 < \rho_0 < +\infty$  such that  $g_z(\rho_0) = 1$ , then we deduce that  $g_z^{\leftarrow}(1)$  is defined, and

$$g_z^{\leftarrow}(1) := \inf\{\rho > 0, g_z(\rho) \geq 1\} = \inf\{\rho > 0, g_z(\rho) = 1\} \leq \rho_0.$$

So,  $g_z^{\leftarrow}(1)$  is defined and finite. In fact,  $g_z^{\leftarrow}(1)$  is defined and finite if and only if the (essential) support of  $n_0 \geq 0$  is included in a closed disk  $\overline{B}(z, \rho_0)$  for some  $0 < \rho_0 < +\infty$ . In that situation, we shall simply say that the initial data  $n_0$  has compact support. It remains to prove the converse. Assume that  $g_z^{\leftarrow}(1)$  is defined and finite. Let  $\rho_1 := g_z^{\leftarrow}(1) < +\infty$ . Then, we have  $g_z(\rho_0) = 1$  for at least one  $\rho_0 \geq \rho_1$ . Thus, we obtain

$$0 = 1 - g_z(\rho_0) = \frac{1}{M} \int_{B^c(z, \rho_0)} n_0(x) dx.$$

Since  $n_0$  is non-negative, we deduce that  $n_0 = 0$  almost everywhere on  $\mathbb{R}^2 \setminus B(z, \rho_0)$ , which proves the assertion.

- (4) If the value 1 is not in the range of  $g_z$  (i.e.  $n_0$  has no compact support), and if needed, we shall let  $g_z^{\leftarrow}(1) = +\infty$ .
- (5) The function  $g_z^{\leftarrow}$  will be called the generalized right-inverse of  $g_z$ . It can also be called the radial cumulative distribution function centered at  $z \in \mathbb{R}^2$  of the density of probability  $d\mu(x) = \frac{1}{M} n_0(x) dx$ .

**Remark 2.8.** During the proof, we shall see that both inequalities (2.15) and (2.17) are equivalent. But in the second inequality (2.17), the following quantity

$$\frac{1}{4 \ln \left( 1 + \frac{M-8\pi}{2M} \right)}$$

appears naturally in the estimate of the upper bound of  $T_c^*(n_0)$ , and also in (2.18) and (2.19). Note that this term is independent of the particular shape of the initial data  $n_0$ , and depends only on the mass  $M$ . Such a logarithmic term is mentioned here because it also occurs in Theorem 3.1 below for lower bounds to  $T_c^*(n_0)$ .

**Remark 2.9.** The quantity  $i_K(z) = \sup_{x \in K} |x - z| = |x_0 - z|$  is the distance from  $z$  to one of the farthest point  $x_0 = x_0(z, K)$  in  $K$ . Such a point  $x_0$  exists because the function  $x \rightarrow |x - z|$  is continuous on the compact set  $K$ . Hence, the supremum is attained and is, in fact, a maximum. In particular, we have for all  $z \in \mathbb{R}^2$ ,

$$K \subset B'(z, i_K(z)) := \{y \in \mathbb{R}^2 : |z - y| \leq i_K(z)\},$$

and  $i_K(z)$  is exactly the smallest radius  $R \geq 0$  such that  $K \subset B'(z, R)$  for this fixed  $z \in \mathbb{R}^2$ . Now, the quantity  $R_0 = \inf_{z \in \mathbb{R}^2} i_K(z)$  consists in choosing ideally a closed disk of minimal radius, i.e.  $R_0$ , containing  $K$ . More precisely, it can be proved that  $z \rightarrow i_K(z)$  is 1-Lipschitz (so, it is continuous), and

$$\lim_{|z| \rightarrow +\infty} i_K(z) = +\infty.$$

Since  $i_K \geq 0$  then the infimum defining  $R_0$  is attained at some point  $z_0 \in \mathbb{R}^2$ , i.e.  $R_0 = i_K(z_0) = |z_0 - x_0|$  for some  $x_0 \in K$ . Then  $K \subset B'(z_0, R_0)$  where  $R_0$  is easily seen as the minimal radius  $R$  such that  $K$  is enclosed in a closed disk  $B'(z, R)$  for some  $z \in \mathbb{R}^2$  ( $z$  is not necessarily in  $K$ ). See the examples of initial data  $n_0$  given by the characteristic function of  $K$  where  $K$  is a closed disk or an annulus treated in Section 4. In any metric space, it can be easily shown that

$$\frac{1}{2}D \leq R_0 \leq D,$$

where  $D := \text{diam}(K) = \sup_{x, y \in K} |x - y|$  is the diameter of the bounded (compact) set  $K$ . In  $\mathbb{R}^2$ , it may happen that  $\frac{1}{2}D < R_0$  ( $R_0 = D/\sqrt{3}$  for the equilateral triangle) or  $R_0 < D$  (always true in  $\mathbb{R}^n$ ). Indeed, by Jung's Theorem [42] we have on  $\mathbb{R}^n$ ,

$$R_0 \leq D \sqrt{\frac{n}{2(n+1)}}.$$

The case of equality is attained by the regular  $n$ -simplex. In particular on  $\mathbb{R}^2$ , we obtain

$$R_0 \leq D/\sqrt{3}.$$

**Remark 2.10.** The upper bound  $T_c^*(n_0)$  of  $T^*$  depends not only of the mass of the initial data  $n_0$  but also of its shape. Indeed, for instance, the radial distribution  $\mathcal{M}_z(\rho), \rho > 0, z \in \mathbb{R}^2$  appearing in (1) of Corollary 2.4 "encodes" a part of the information on the shape of  $n_0$ . In Corollary 2.4 (2), the geometric information is encoded by the generalized inverse  $g_z^\leftarrow, z \in \mathbb{R}^2$ . For explicit shape expressions, see the examples described in Section 4.

**Remark 2.11.** It can be noticed that the result (1) of Corollary 2.4 is sharp in the sense that the set of information  $\{\mathcal{M}_z(\rho), \rho > 0, z \in \mathbb{R}^2\}$  is equivalent to the knowledge of the initial data  $n_0$  since  $n_0 \in L_{loc}^1$ . Indeed, by Lebesgue differentiation theorem, we have

$$\lim_{\rho \rightarrow 0^+} \frac{1}{|B(z, \rho)|} \mathcal{M}_z(\rho) = n_0(z),$$

for almost all  $z \in \mathbb{R}^2$ .

### Proof of Corollary 2.4

(1) (a) Our proof starts from (1.7) of Theorem 1.2. To simplify the notation, we set  $T = T_c^*(n_0)$ . Since  $n_0 \geq 0$ , we have for all  $z \in \mathbb{R}^2$  and all  $\rho > 0$ ,

$$e^{-\frac{\rho^2}{4T}} \mathcal{M}_z(\rho) = \int_{B(z, \rho)} e^{-\frac{\rho^2}{4T}} n_0(x) dx \leq \int_{\mathbb{R}^2} \exp\left(-\frac{|x-z|^2}{4T}\right) n_0(x) dx = L(M),$$

with  $L(M) = \frac{2M^2}{3M-8\pi}$ . It follows easily that, for all  $z \in \mathbb{R}^2$  and all  $\rho > 0$ ,

$$T \leq \frac{\rho^2}{4 \ln_+ \left( \frac{\mathcal{M}_z(\rho)}{L(M)} \right)} \leq +\infty.$$

Now by taking the infimum over  $z \in \mathbb{R}^2$  and  $\rho > 0$ , we deduce that

$$T = T_c^*(n_0) \leq T_{c_2}^* := \inf_{\rho > 0, z \in \mathbb{R}^2} \frac{\rho^2}{4} \left[ \ln_+ \left( \frac{\mathcal{M}_z(\rho)}{L(M)} \right) \right]^{-1}.$$

It remains to prove that  $T_{c_2}^*$  is finite for any non-negative integrable function  $n_0$ . The proof is as follows. We note that for all  $z \in \mathbb{R}^2$ ,

$$\lim_{\rho \rightarrow +\infty} \ln_+ \left( \frac{\mathcal{M}_z(\rho)}{L(M)} \right) = \ln_+ \left( \frac{M}{L(M)} \right) > 0,$$

by continuity of  $\ln_+$  and monotone convergence theorem. The last inequality is due to the fact that  $L(M) < M$  when  $M > 8\pi$ . Then for one (all)  $z \in \mathbb{R}^2$ , there exists  $0 < \rho_0 < +\infty$  large enough such that,

$$\ln_+ \left( \frac{\mathcal{M}_z(\rho_0)}{L(M)} \right) > 0.$$

Hence, it yields

$$0 \leq \frac{\rho_0^2}{4} \left[ \ln_+ \left( \frac{\mathcal{M}_z(\rho_0)}{L(M)} \right) \right]^{-1} < +\infty.$$

This implies the finiteness of  $T_{c_2}^*$ . The upper bound (2.14) on  $T_c^*(n_0)$  is now proved.

(1) (b) For the proof of the second part of (1), we just need to prove for any non-negative non-increasing radially symmetric function  $n_0$  that

$$(2.20) \quad \sup_{z \in \mathbb{R}^2} \mathcal{M}_z(\rho) = \mathcal{M}_0(\rho), \quad \rho > 0.$$

This result is well-known but we provide below details for completeness. Let  $z \in \mathbb{R}^2$  be fixed. For all  $x \in B(0, \rho) \setminus U$  and all  $y \in B(z, \rho) \setminus U$  with  $U = B(0, \rho) \cap B(z, \rho)$ , we have  $|x| \leq \rho \leq |y|$ . This implies that  $n_0(x) \geq n_0(y)$  because  $n_0$  is a non-increasing radially symmetric function. We integrate this last inequality with respect to the couple of variables  $(x, y) \in (B(0, \rho) \setminus U) \times (B(z, \rho) \setminus U)$ , and get by Fubini's theorem,

$$|B(z, \rho) \setminus U| \cdot \int_{B(0, \rho) \setminus U} n_0(x) dx \geq |B(0, \rho) \setminus U| \cdot \int_{B(z, \rho) \setminus U} n_0(y) dy.$$

Here,  $|A|$  denotes the Lebesgue measure of the measurable set  $A \subset \mathbb{R}^2$ . Since  $U \subset B(z, \rho)$  and  $U \subset B(0, \rho)$ , we have

$$|B(z, \rho) \setminus U| = |B(z, \rho)| - |U| = |B(0, \rho)| - |U| = |B(0, \rho) \setminus U|.$$

It immediately yields

$$(2.21) \quad \int_{B(0, \rho) \setminus U} n_0(x) dx \geq \int_{B(z, \rho) \setminus U} n_0(y) dy$$

when  $|B(z, \rho) \setminus U| > 0$ . If  $|B(z, \rho) \setminus U| = 0$  then  $B(z, \rho) = B(0, \rho) \cap B(z, \rho) \subset B(0, \rho)$ . This implies that  $z = 0 \in \mathbb{R}^2$  and (2.21) holds trivially. Thus, we deduce that

$$\mathcal{M}_z(\rho) - \mathcal{M}_0(\rho) = \int_{B(z, \rho) \setminus U} n_0(y) dy - \int_{B(0, \rho) \setminus U} n_0(x) dx \leq 0.$$

Finally, we have for all  $z \in \mathbb{R}^2$  and all  $\rho > 0$ ,

$$\mathcal{M}_z(\rho) \leq \mathcal{M}_0(\rho).$$

Thus, the formula (2.20) follows.

(2) The proof relies on a change of variables of exponential type. We first start with some useful properties of the distribution function  $g_z$  for any  $z \in \mathbb{R}^2$ , see also Remark 2.7 above. We denote  $g_z$  by  $g$  at some places for short.

The function  $g : [0, +\infty) \rightarrow [0, 1]$  is a non-decreasing continuous function with range included in  $[0, 1]$ . Since  $g(0) = 0$ ,  $g(+\infty) = 1$  and  $g$  is continuous, then for each  $m \in (0, 1)$  there exists  $\rho_1 > 0$  such that  $g(\rho_1) = m$  by the intermediate value theorem. Let  $G(m) = \{\rho > 0 : g(\rho) = m\}$  with  $m \in (0, 1)$ . Then the set  $G_m$  is not empty and bounded from below by 0, hence the infimum of  $G_m$  exists for all  $m \in (0, 1)$ . It is also a minimum because  $G(m)$  is closed ( $g$  is continuous). We set  $g^\leftarrow(m) := \min G_m$ . The function  $g^\leftarrow$  is a right-inverse of the function  $g$  since we have  $g \circ g^\leftarrow(m) = m$  for all  $m \in (0, 1)$ . This is simply due to the fact that  $g^\leftarrow(m) \in G(m)$ . Note that  $g^\leftarrow$  is not necessarily a left-inverse of the function  $g$ , in particular if  $g_z$  is constant on some (closed) interval. Moreover, it can be shown by monotonicity of  $g$  that we also have

$$g^\leftarrow(m) = \inf G_m = \inf\{\rho > 0 : g(\rho) \geq m\}.$$

Hence, the function  $g^\leftarrow$  is non-decreasing on  $(0, 1)$  but not necessarily continuous. Its range is included in  $(0, +\infty)$ . Of course, if  $g$  is strictly increasing from  $(0, +\infty)$  onto  $(0, 1)$  then  $g^\leftarrow = g^{-1}$  is the usual inverse function to  $g$ .

Now we are in position to prove (2.18) of Corollary 2.4. Let  $z \in \mathbb{R}^2$  be fixed. We set  $a = \frac{L(M)}{M} = \frac{2M}{3M-8\pi}$ . Since  $M > 8\pi$ , we have  $a \in (0, 1)$ . By definition of  $T_{c_2}^*(z)$ , we can write for all  $\rho > 0$ ,

$$T_{c_2}^*(z) \leq \frac{\rho^2}{4} \left[ \ln_+ \left( \frac{\mathcal{M}_z(\rho)}{L(M)} \right) \right]^{-1} = \frac{\rho^2}{4} \left[ \ln_+ \left( \frac{g_z(\rho)}{a} \right) \right]^{-1},$$

because  $\mathcal{M}_z(\rho)/L(M) = g_z(\rho)/a$ . Let  $\theta \in (0, 1)$  then we have  $a^\theta \in (0, 1)$ . We set  $\rho_* = g_z^\leftarrow(a^\theta)$  then we have  $\rho_* \in (0, +\infty)$  and  $g_z(\rho_*) = a^\theta$ . The inequality just above with  $\rho = \rho_*$  implies that

$$T_{c_2}^* \leq \frac{[g_z^\leftarrow(a^\theta)]^2}{4 \ln_+(a^{\theta-1})} = \frac{[g_z^\leftarrow(a^\theta)]^2}{4(1-\theta) \ln(\frac{1}{a})}.$$

By minimization over  $\theta \in (0, 1)$ , we obtain  $T_{c_2}^* \leq \tilde{T}$ , with

$$\tilde{T} := \inf_{0 < \theta < 1} \frac{[g_z^\leftarrow(a^\theta)]^2}{4(1-\theta) \ln(\frac{1}{a})}.$$

It remains to prove the reverse inequality, namely  $\tilde{T} \leq T_{c_2}^*$ . We fix  $z \in \mathbb{R}^2$ . We discuss three cases. (The last case may not appear for some functions  $g_z$ ).

*First case.* Let  $\rho \in (0, +\infty)$  such that  $a < g_z(\rho) < 1$ . Then there exists  $\theta \in (0, 1)$  such that  $g_z(\rho) = a^\theta$ . Since  $g_z^\leftarrow(a^\theta) \leq \rho$  by definition of  $g_z^\leftarrow$ , we can say that

$$(2.22) \quad \tilde{T} \leq \frac{[g_z^\leftarrow(a^\theta)]^2}{4(1-\theta) \ln(\frac{1}{a})} \leq \frac{\rho^2}{4 \ln(a^{\theta-1})} = \frac{\rho^2}{4 \ln_+ \left( \frac{g_z(\rho)}{a} \right)} = \frac{\rho^2}{4 \ln_+ \left( \frac{\mathcal{M}_z(\rho)}{L(M)} \right)},$$

using the relations  $a^{\theta-1} = a^{-1}g_z(\rho) = \frac{\mathcal{M}_z(\rho)}{L(M)}$ .

*Second case.* Assume that  $g_z(\rho) \leq a$ , then the inequality (2.22) holds true trivially since the right-hand side is infinite. Thus, from the first and second case, we deduce that the inequality (2.22) holds true whenever  $g_z(\rho) < 1$ .

*Third case.* If it happens that  $g_z(\rho) = 1$  for some finite  $\rho > 0$ , by the same argument defining  $g^{\leftarrow}(m)$  for  $m \in (0, 1)$ , we set

$$\rho_0 = \inf\{\rho' > 0, g_z(\rho') = 1\} = \min\{\rho' > 0, g_z(\rho') = 1\}.$$

So, we have  $0 < \rho_0 \leq \rho$  for  $\rho > 0$  such that  $g_z(\rho) = 1$ . For any sequence  $(r_n)$  such that  $r_n < \rho_0$  and  $\lim_n r_n = \rho_0$ , then we have  $g_z(r_n) < g_z(\rho_0) = g_z(\rho) = 1$ . We can apply the result of the first or second case just above to such sequence  $(r_n)$ , i.e (2.22), and write for all  $n$ ,

$$\tilde{T} \leq \frac{r_n^2}{4 \ln_+ \left( \frac{\mathcal{M}_z(r_n)}{L(M)} \right)}.$$

Now, taking the limit as  $n$  goes to infinity and using the continuity of the function  $r \mapsto \mathcal{M}_z(r)$ , we obtain

$$\tilde{T} \leq \frac{\rho_0^2}{4 \ln_+ \left( \frac{\mathcal{M}_z(\rho_0)}{L(M)} \right)} \leq \frac{\rho^2}{4 \ln_+ \left( \frac{\mathcal{M}_z(\rho)}{L(M)} \right)}.$$

The last inequality is due to the equality  $g_z(\rho_0) = g_z(\rho) = 1$ , i.e.  $\mathcal{M}_z(\rho) = \mathcal{M}_z(\rho_0) = M$  and  $0 < \rho_0 \leq \rho$ .

*Conclusion.* Gathering the three cases above, we can assert that the inequality (2.22) holds true for all  $\rho > 0$ . Hence, we obtain  $\tilde{T} \leq T_{c_2}^*$  by minimization over  $\rho > 0$  of the utmost right-hand side term of the inequality (2.22). Ultimately, we get  $\tilde{T} = T_{c_2}^*$  since  $\tilde{T} \geq T_{c_2}^*$  has already been proved in the first part of the proof. Finally, the last statement of part (2) of Corollary 2.4 holds true by noting that  $\frac{1}{a} = 1 + \frac{M-8\pi}{2M}$ .

(3) The proof is similar to the proof of the first part of Corollary 2.4. We assume that  $n_0$  has compact support in  $\mathbb{R}^2$ , here denoted by  $K$ . Let  $T = T_{c,z}^*(n_0)$  as in Theorem 1.2 for fixed  $z \in \mathbb{R}^2$ . By (1.7) of Theorem 1.2, we have

$$\begin{aligned} e^{-i_K^2(z)/4T} \cdot M &= \left( e^{-\sup_{x \in K} \frac{|x-z|^2}{4T}} \right) \cdot M = \left( \inf_{x \in K} e^{-\frac{|x-z|^2}{4T}} \right) \cdot \int_K n_0(y) dy \\ &= \int_K \inf_{x \in K} e^{-\frac{|x-z|^2}{4T}} n_0(y) dy \leq \int_K e^{-\frac{|y-z|^2}{4T}} n_0(y) dy \\ &= \int_{\mathbb{R}^2} e^{-\frac{|y-z|^2}{4T}} n_0(y) dy = H_{z,n_0}(T) = L(M), \end{aligned}$$

with  $L(M) = \frac{2M^2}{3M-8\pi}$ . This implies that, for all  $z \in \mathbb{R}^2$ ,

$$T = T_{c,z}^* \leq \frac{i_K^2(z)}{4 \ln \left( \frac{M}{L(M)} \right)},$$

due to the fact that  $\ln(M/L(M)) > 0$  since  $L(M) < M$  for  $M > 8\pi$ . By minimization over  $z \in \mathbb{R}^2$ , the conclusion follows:

$$T^* \leq T_c^* \leq T_{c_3}^* := \frac{R_0^2}{4 \ln \left( \frac{M}{L(M)} \right)} = \frac{R_0^2}{4 \ln \left( 1 + \frac{M-8\pi}{2M} \right)}.$$

An alternative proof of this inequality (2.18) is as follows. In the inequality (2.17), we can consider the value  $\theta = 0$  (i.e. the limit case  $\theta$  where goes to 0). Indeed,  $g_z^{\leftarrow}(1)$  is finite

for all  $z \in \mathbb{R}^2$  (due to the assumption of compact support on  $n_0$ ) and  $g_z^\leftarrow(a^\theta) \leq g_z^\leftarrow(1)$  since  $t \mapsto g_z^\leftarrow(t)$  is non-decreasing and  $a^\theta < 1$ . So, we can write

$$T_{c_2}^* \leq \frac{1}{4 \ln \left(1 + \frac{M-8\pi}{2M}\right)} \cdot \inf_{z \in \mathbb{R}^2, \theta \in (0,1)} \left( \frac{[g_z^\leftarrow(a^\theta)]^2}{(1-\theta)} \right) \leq \frac{1}{4 \ln \left(1 + \frac{M-8\pi}{2M}\right)} \cdot \inf_{z \in \mathbb{R}^2} [g_z^\leftarrow(1)]^2,$$

Now, it can be easily shown that  $g_z^\leftarrow(1) = i_K(z)$  for all  $z \in \mathbb{R}^2$ . Hence, we get

$$R_0^2 = \inf_{z \in \mathbb{R}^2} [g_z^\leftarrow(1)]^2.$$

Thus, we conclude that  $T_{c_2}^* \leq T_{c_3}^*$ . So, the inequality (2.18) also follows from (2.17). This finishes this second proof.

The inequality (2.19) is a consequence of Jung's Theorem [42] applied to the two-dimensional case. Indeed, we have  $R_0 \leq D/\sqrt{3}$ , where  $D$  is the diameter of  $K$ , see Remark 2.9 above. This concludes the proof of Corollary 2.4.  $\square$

In the next statement, we give practical criteria for obtaining bounds on  $T_{c_2}^*(0)$  of Corollary 2.4. Under natural additional assumptions on the function  $h := (g_0^\leftarrow)^2$  appearing in the inequality (2.18) of Corollary 2.4, we prove the existence of a unique extremum at  $\theta_0 \in (0, 1)$ , or at  $\theta_0 = 0^+$  of the infimum over  $\theta \in (0, 1)$  used in (2.17). We also provide an expression of  $T_{c_2}^*(0)$  by evaluating the value of  $\theta_0$  by inverting some function, namely  $F$  canonically associated with  $g_0^\leftarrow$  when  $g_0^\leftarrow$  is smooth enough (see below). In Section 4, we apply these estimates to obtain explicit bounds on the critical time  $T_{c_2}^*(0)$  for several families of examples of initial data.

**Corollary 2.12.** *Let  $M > 8\pi$  and  $T_{c_2}^*(0)$  given in (2) of Corollary 2.4, i.e.*

$$T_{c_2}^*(0) = \frac{1}{4 \ln(1/a)} \cdot \inf_{\theta \in (0,1)} \frac{h(a^\theta)}{(1-\theta)},$$

with  $a = \frac{2M}{3M-8\pi} \in (\frac{2}{3}, 1)$  (thus  $\frac{1}{a} = 1 + \frac{M-8\pi}{2M} \in (1, \frac{3}{2})$ ). Here, the function  $h$  is defined by  $h(t) := [g_0^\leftarrow(t)]^2$ ,  $t \in (a, 1)$ . Assume that  $h$  is continuous and has derivative  $h'(t) > 0$  for all  $t \in (a, 1)$ . Let  $F$  be given by

$$(2.23) \quad F(X) = X + e^X \left( \frac{h}{h'} \right) (e^{-X}), \quad X \in (0, \ln(1/a)).$$

We have the next two results.

- (1) *If  $F$  is non-decreasing and  $F(0^+) \geq \ln(1/a)$ , i.e.  $(\frac{h}{h'}) (1^-) \geq \ln(1/a)$ , then we have*

$$(2.24) \quad T_{c_2}^*(0) = \frac{h(1^-)}{4 \ln(1/a)}.$$

- (2) *If  $F$  is continuous and strictly increasing on  $(0, \ln(1/a))$ , and satisfies the next conditions,*

- (i)  $F(0^+) < \ln(1/a)$ , i.e.  $(\frac{h}{h'}) (1^-) < \ln(1/a)$ ,  
(ii)  $F([\ln(1/a)]^-) > \ln(1/a)$ , i.e.  $(\frac{h}{h'}) (a^+) > 0$ ,  
then we have

$$(2.25) \quad T_{c_2}^*(0) = S \left( \ln \frac{1}{a} \right),$$

where

$$(2.26) \quad S(Y) := \frac{h(e^{-F^{-1}(Y)})}{4[Y - F^{-1}(Y)]} = \frac{1}{4} e^{-F^{-1}(Y)} h'(e^{-F^{-1}(Y)}), \quad Y \in \text{Dom}(F^{-1}),$$

and  $F^{-1}$  is the inverse function of  $F$ .

We make some comments on these results.



**Remark 2.13.** The functions  $S$  and  $h$  are formally independent of the mass  $M$ . These functions depends on the shape of the initial data  $n_0$  via the repartition function  $g_0^{\leftarrow}$ . Both expressions (2.24) and (2.25) are expressed respectively in terms of the functions  $h$  and  $S$  of the variable  $Y$  evaluated at  $Y_0 = Y_0(M) := \ln(\frac{1}{a}) = \ln(1 + \frac{M-8\pi}{2M})$  where  $M$  is the mass of the initial data  $n_0$  with  $M > 8\pi$ . Note that we have  $\ln(\frac{1}{a}) \in (0, \ln(3/2))$  where  $\ln(3/2) \sim 0.40$ , thus  $\ln(\frac{1}{a}) \in (0, 0.41)$  for all  $M > 8\pi$ . These comments are provided so that the reader can get an idea of the quantities involved in the computations.

**Remark 2.14.** In Section 4, we evaluate the function  $F$  for some families of examples of initial data. But unfortunately, the inverse function  $F^{-1}$  can not always be given explicitly as simple functions. Nevertheless for some examples, the function  $F^{-1}$  can be bounded above and below by explicit functions useful for estimating  $T_c^*(n_0)$ . Hence, it is also useful to provide an upper bound on  $T^*$ . Since only the evaluation of  $F^{-1}$  at  $Y_0 = \ln(\frac{1}{a})$  is involved in (2.25), the quantity  $F^{-1}(\ln \frac{1}{a})$  may certainly be estimated by numerical methods as the unique zero of the function  $Q := F + \ln a$ . But we shall not continue in that direction in this paper.

**Remark 2.15.** The map  $M \in (8\pi, +\infty) \rightarrow (\frac{2}{3}, 1)$  defined by  $a = a(M) = \frac{2M}{3M-8\pi}$  is a decreasing bijection. Hence  $h$  is defined at least on  $(a, 1)$  for all  $a \in (\frac{2}{3}, 1)$ , i.e. on  $(\frac{2}{3}, 1)$  itself. This implies that  $F$  defined in Corollary 2.12 must have a domain containing  $(0, \ln(\frac{3}{2}))$ . So, the function  $F$  does not depend on  $a$ . In fact, the function  $F$  is often defined on the whole interval  $(0, +\infty)$ .

**Remark 2.16.** The assumption  $(\frac{h}{h'}) (a^+) > 0$  in the second part of the corollary can be replaced by the simple assumption  $h'(a^+) > 0$ . Indeed, it is enough to show that we always have  $h(a^+) > 0$ . This is deduced as follows. Since we have  $h = (g_0^{\leftarrow})^2 \geq 0$ , we have  $h(a^+) \geq 0$ . It remains to show that  $h(a^+) \neq 0$ . To prove this, we suppose the contrary, i.e.  $h(a^+) = 0$ . Thus, for any sequence  $(a_n)$  such that  $a \leq a_n < 1$  and  $\lim_n a_n = a$ , then we have  $h(a^+) = \lim_n h(a_n) = 0$ . This implies that  $\lim_n g_0^{\leftarrow}(a_n) = 0$ . Thus, on one hand we get

$$\lim_n g_0(g_0^{\leftarrow}(a_n)) = g_0(0) = 0,$$

by continuity of  $g_0$ . On the other hand, we have

$$\lim_n (g_0 \circ g_0^{\leftarrow})(a_n) = \lim_n a_n = a,$$

due to the fact that  $g_0^{\leftarrow}$  is a right-inverse of  $g_0$ . So, we deduce that  $a = \frac{2M}{3M-8\pi} = 0$ , which contradicts the assumption  $M > 8\pi$ .

**Remark 2.17.** For any  $z \in \mathbb{R}^2$ , and under similar assumptions on  $h_z := (g_z^{\leftarrow})^2$  as those imposed on  $h = h_0$ , we can prove similar estimates as (2.24) and (2.25) for  $T_{c_2}^*(z)$  with the corresponding  $F_z$ .

**Proof of Corollary 2.12 .** We first need some preparation. Let  $a$  and  $h := h_0$  defined as in Corollary 2.12 ( $z = 0 \in \mathbb{R}^2$ ). We assume that  $h$  is continuous and it has a positive derivative. We set  $V(\theta) := \frac{h(a^\theta)}{(1-\theta)}$  with  $\theta \in (0, 1)$ . The derivative of the function  $V$  satisfies the following equation

$$(1-\theta)^2 V'(\theta) = (1-\theta)(\ln a)h'(a^\theta)a^\theta + h(a^\theta).$$

We make the following change of variables  $X = X(\theta) = -\theta \ln a = \theta \ln(1/a)$ . The map  $X$  is an increasing bijection from  $(0, 1)$  onto  $(0, \ln(1/a))$ . We deduce that  $a^\theta = e^{-X}$ , and

$$\begin{aligned} (1-\theta)^2 V'(\theta) &= (1-\theta)(\ln a)h'(e^{-X})e^{-X} + h(e^{-X}) \\ &= (\ln a)h'(e^{-X})e^{-X} + Xh'(e^{-X})e^{-X} + h(e^{-X}) = h'(e^{-X})e^{-X}Q(X), \end{aligned}$$

with  $Q(X) := F(X) + \ln a$ , and  $F$  given by (2.23). From the assumption  $h' > 0$ , we get  $Q(X) > 0 \iff V'(\theta) > 0$  and  $Q(X) = 0 \iff V'(\theta) = 0$ , where the relationship between  $X$  and  $\theta$  is given by  $X = -\theta \ln a$ . Now we are in position to prove our statements.

(1) Under the assumptions of (1) of Corollary 2.12, we have  $Q(0^+) = F(0^+) + \ln a \geq 0$ , and  $Q$  (also  $F$ ) is non-decreasing. This implies that  $Q(X) \geq Q(0^+) \geq 0$ , for all  $X \in (0, \ln(1/a))$ . Thus, we have  $V'(\theta) \geq 0$  for all  $\theta \in (0, 1)$ , i.e.  $V$  is non-decreasing. Hence, we deduce that

$$\inf_{0 < \theta < 1} V(\theta) = V(0^+) = h(1^-).$$

Then the formula (2.24) follows from (2.17) of Corollary 2.4 with  $z = 0$ . This proves the first statement.

(2) Under the assumptions of (2) of Corollary 2.12, the function  $Q(X) = F(X) + \ln a$  is a strictly increasing continuous function, and by (i) and (ii), we have successively  $Q(0^+) < 0$  and  $Q([\ln(1/a)]^-) > 0$ . By the intermediate value theorem, there exists a unique zero  $X_0 \in (0, \ln(1/a))$  of  $Q(X) = F(X) + \ln a$ , i.e.  $F(X_0) = \ln(1/a)$ , and finally  $X_0 = F^{-1}(\ln \frac{1}{a})$ . Indeed,  $F$  is bijective from  $(0, \ln(1/a))$  onto its range which contains  $\ln(1/a)$  since  $F(0^+) < \ln \frac{1}{a} < F([\ln(1/a)]^-)$  by assumptions (i) and (ii). We also obtain

$$Q(X) < 0 \iff 0 < X < X_0, \quad \text{and} \quad Q(X) = 0 \iff X = X_0,$$

which is equivalent to

$$V'(\theta) < 0 \iff 0 < \theta < \theta_0, \quad \text{and} \quad V'(\theta) = 0 \iff \theta = \theta_0,$$

with the relations  $X = \theta \ln(1/a)$  and  $X_0 = \theta_0 \ln(1/a)$ . This implies that the infimum of  $V(\theta)$  over  $\theta \in (0, 1)$  is attained at this point  $\theta_0 \in (0, 1)$ , i.e.

$$\inf_{0 < \theta < 1} V(\theta) = V(\theta_0),$$

with

$$\theta_0 = \frac{X_0}{\ln(1/a)} = \frac{F^{-1}(\ln \frac{1}{a})}{\ln(1/a)}.$$

Thus, we derive the next formula

$$T_{c_2}^*(0) = \frac{1}{4 \ln(1/a)} \cdot V(\theta_0) = \frac{1}{4 \ln(1/a)} \cdot \frac{h(a^{\theta_0})}{(1 - \theta_0)} = \frac{h\left(e^{-F^{-1}(\ln \frac{1}{a})}\right)}{4 \left[\ln\left(\frac{1}{a}\right) - F^{-1}\left(\ln \frac{1}{a}\right)\right]}.$$

Now let  $X = F^{-1}(Y)$  with  $Y \in \text{Dom}(F^{-1}) := \text{Im } F$  in the definition (2.23) of  $F(X)$ . We get the following expression

$$Y - F^{-1}(Y) = e^{F^{-1}(Y)} \left( \frac{h}{h'} \right) (e^{-F^{-1}(Y)}).$$

From which we deduce both formulas for the function  $S$ , namely

$$S(Y) := \frac{h(e^{-F^{-1}(Y)})}{4 [Y - F^{-1}(Y)]} = \frac{1}{4} e^{-F^{-1}(Y)} h'(e^{-F^{-1}(Y)}).$$

Finally, this yields the expected result

$$T_{c_2}^*(0) = S\left(\ln \frac{1}{a}\right)$$

where  $S(Y)$  is described just above, and  $Y = \ln(\frac{1}{a})$ . Statement (2) is proved and the proof of Corollary 2.12 is now complete.  $\square$

Now, we present another consequence of Theorem 1.2 under an additional assumption of finite centralized and normalized  $\beta$ -variance of the initial data  $n_0$  defined as follows. We assume that  $n_0$  has a moment of order  $\beta \geq 1$ , i.e.

$$\int_{\mathbb{R}^2} |x|^\beta n_0(x) dx < +\infty.$$

Then, we denote the (normalized) barycenter of  $n_0$  by  $B_0 = \frac{1}{M} \int_{\mathbb{R}^2} x n_0(x) dx$ . This is a well defined vector in  $\mathbb{R}^2$ , since by Hölder's inequality, we obtain

$$\frac{1}{M} \int_{\mathbb{R}^2} |x| n_0(x) dx \leq \left( \frac{1}{M} \int_{\mathbb{R}^2} |x|^\beta n_0(x) dx \right)^{1/\beta} < +\infty,$$

for any  $1 \leq \beta < +\infty$ . We also denote by  $V_\beta(n_0)$  the centralized and normalized  $\beta$ -variance of  $n_0$  defined by

$$V_\beta(n_0) := \left[ \frac{1}{M} \int_{\mathbb{R}^2} |x - B_0|^\beta n_0(x) dx \right]^{2/\beta}.$$

Let us mention some basic properties of the  $\beta$ -variance of  $n_0$  useful for this paper.

(a) The  $\beta$ -variance, seen as a function of  $\beta$ , i.e.  $\beta \in [2, +\infty) \mapsto V_\beta(n_0)$ , is non-decreasing. Indeed, by Hölder's inequality, we have for all  $2 \leq \beta \leq \gamma$ ,

$$V_\beta(n_0) = \left[ \frac{1}{M} \int_{\mathbb{R}^2} |x - B_0|^\beta n_0(x) dx \right]^{2/\beta} \leq \left[ \frac{1}{M} \int_{\mathbb{R}^2} |x - B_0|^{p\beta} n_0(x) dx \right]^{2/p\beta} = V_\gamma(n_0),$$

where  $p = \gamma/\beta \geq 1$ . As a particular case, we get

$$V_2(n_0) = \frac{1}{M} \int_{\mathbb{R}^2} |x - B_0|^2 n_0(x) dx \leq V_\beta(n_0) = \left[ \frac{1}{M} \int_{\mathbb{R}^2} |x - B_0|^\beta n_0(x) dx \right]^{2/\beta},$$

for all  $2 \leq \beta < +\infty$ . As a consequence, if  $V_\beta(n_0)$  is finite for some  $\beta \geq 2$  then  $V_2(n_0)$  is also finite.

(b) We have the following infimum estimate

$$(2.27) \quad \frac{1}{4} V_\beta(n_0) \leq \inf_{z \in \mathbb{R}^2} \left[ \frac{1}{M} \int_{\mathbb{R}^2} |x - z|^\beta n_0(x) dx \right]^{2/\beta} \leq V_\beta(n_0).$$

(c) In case  $\beta = 2$ , we have an equality of the infimum with the upper bound in (2.27),

$$(2.28) \quad \inf_{z \in \mathbb{R}^2} \left[ \int_{\mathbb{R}^2} |x - z|^2 \frac{n_0(x)}{M} dx \right] = V_2(n_0) = \int_{\mathbb{R}^2} |x - B_0|^2 \frac{n_0(x)}{M} dx.$$

Note that the quantity  $V_2(n_0)$  is simply denoted by  $V(0)$  in the first section of this paper.

The proof of the lower bound of (2.27) for  $\beta \geq 2$  is as follows. (The upper bound is obvious taking  $z = B_0$ ). From (discrete) Hölder's inequality, we get

$$(a + b)^\beta \leq 2^{\beta-1} (a^\beta + b^\beta), \quad a, b \geq 0.$$

Then, from this inequality and triangle inequality, we deduce that

$$|x - B_0|^\beta \leq 2^{\beta-1} (|x - z|^\beta + |z - B_0|^\beta),$$

for all  $x, z \in \mathbb{R}^2$ . We multiply this expression by the non-negative function  $n_0/M$  and integrate it with respect to  $x$  over  $\mathbb{R}^2$ , we then obtain

$$\int_{\mathbb{R}^2} |x - B_0|^\beta \frac{n_0(x)}{M} dx \leq 2^{\beta-1} \left( \int_{\mathbb{R}^2} |x - z|^\beta \frac{n_0(x)}{M} dx + |z - B_0|^\beta \right).$$

On the other hand, we have

$$|z - B_0|^\beta = \left| \int_{\mathbb{R}^2} (z - x) \frac{n_0(x)}{M} dx \right|^\beta \leq \left( \int_{\mathbb{R}^2} |z - x| \frac{n_0(x)}{M} dx \right)^\beta \leq \int_{\mathbb{R}^2} |z - x|^\beta \frac{n_0(x)}{M} dx.$$

The last inequality is obtained by applying Jensen's inequality to the probability measure  $\frac{n_0(x)}{M} dx$  (or simply by applying Hölder's inequality). It follows that

$$\int_{\mathbb{R}^2} |x - B_0|^\beta \frac{n_0(x)}{M} dx \leq 2^\beta \int_{\mathbb{R}^2} |x - z|^\beta \frac{n_0(x)}{M} dx,$$

for all  $z \in \mathbb{R}^2$ . We conclude the proof of the lower bound of (2.27) by raising the expression just above to the power of  $2/\beta$  and by taking the infimum over  $z \in \mathbb{R}^2$ .

The next result emphasizes the importance of the role of the normalized barycenter  $B_0$  of  $n_0$  in the issue of estimating  $T_c^*(n_0)$  (hence  $T^*$  also) under a  $\beta$ -moment assumption on the initial data  $n_0$ .

**Corollary 2.18.** (*Finite Variance*)

- (1) Assume that the initial data  $n_0$  has a 2-moment. Then, we have the following estimate

$$(2.29) \quad T^* \leq T_c^*(n_0) \leq T_{c_4}^* := \frac{V_2(n_0)}{4 \ln \left(1 + \frac{M-8\pi}{2M}\right)},$$

where  $V_2(n_0)$  is the normalized variance of  $n_0$ . ( $V_2(n_0)$  is also denoted by  $V(0)$  in (1.4) above).

- (2) More generally, assume that  $n_0$  has a  $\beta$ -moment with  $\beta \geq 2$ . Then we have

$$(2.30) \quad T^* \leq T_c^*(n_0) \leq T_{c_5}^* := \frac{\inf_{z \in \mathbb{R}^2} \left[ \int_{\mathbb{R}^2} |x-z|^\beta n_0(x) dx \right]^{2/\beta}}{4M^{2/\beta} \ln \left(1 + \frac{M-8\pi}{2M}\right)}.$$

In particular, we obtain

$$(2.31) \quad T^* \leq T_c^*(n_0) \leq \frac{V_\beta(n_0)}{4 \ln \left(1 + \frac{M-8\pi}{2M}\right)}, \quad \beta \geq 2.$$

We make some comments on these results.

**Remark 2.19.** The estimate (2.30) is apparently better than the estimate (2.31) (except the case  $\beta = 2$  for which we have equality). See Remarks (2.27) and (2.28).

**Remark 2.20.** Here again, the term  $[\ln(1/a)]^{-1} = \left[\ln \left(1 + \frac{M-8\pi}{2M}\right)\right]^{-1}$  appears in our estimates of  $T_c^*(n_0)$  as in Corollary 2.4 and Corollary 2.12. Note that this term tends to  $+\infty$  when  $M \rightarrow 8\pi^+$ , see comments in first section.

**Remark 2.21.** In the case of finite second-moment for the initial data  $n_0$ , we can compare two asymptotic results. From (1.5), we first recall that we can write the next upper bound

$$(2.32) \quad T^* \leq \frac{2\pi V_2(n_0)}{M-8\pi},$$

for all  $M > 8\pi$ . On the other hand, from (2.29) we have

$$(2.33) \quad T^* \leq T_c^*(n_0) \leq \frac{V_2(n_0)}{4 \ln \left(1 + \frac{M-8\pi}{2M}\right)}, \quad M > 8\pi.$$

- (i) When  $M$  is closed to  $8\pi$ . The right-hand side of (2.33) is estimated formally as follows

$$\frac{V_2(n_0)}{4 \ln \left(1 + \frac{M-8\pi}{2M}\right)} \sim \frac{4\pi V_2(n_0)}{M-8\pi}, \quad M \rightarrow 8\pi^+.$$

(i.e. when considering  $V_2(n_0)$  is fixed). So, the estimate of  $T^*$  obtained from (2.33) is twice larger than the estimate (2.32) as  $M$  tends to  $8\pi^+$ .

- (ii) The asymptotic case  $M \rightarrow +\infty$ . From (2.33), we have formally

$$\frac{V_2(n_0)}{4 \ln \left(1 + \frac{M-8\pi}{2M}\right)} \sim \frac{V_2(n_0)}{4 \ln(3/2)},$$

as  $M$  tends to infinity, which is clearly not as good as the estimate (2.32) of  $T^*$ . Indeed, the right-hand side term of (2.32) tends to zero as  $M$  tends to infinity.

The fact that the estimate (2.32) of  $T^*$  is better than (2.33) is due to a direct study of the evolution in time of the second moment of the solution ( $n_t$ ). For this

particular case, the general approach used in this paper with the heat kernel is not specific enough for obtaining an accurate result.

**Remark 2.22.** Assume that  $n_0$  has compact support. Let  $K$  denote the support of  $n_0$ . Then for any  $R_0 > 0$ , and any  $z_0 \in \mathbb{R}^2$  such that  $K \subset B'(z_0, R_0)$  (closed ball). Then, it is easy to show that

$$\inf_{z \in \mathbb{R}^2} \left[ \frac{1}{M} \int_{\mathbb{R}^2} |x - z|^\beta n_0(x) dx \right]^{2/\beta} \leq \left[ \frac{1}{M} \int_{B'(z_0, R_0)} |x - z_0|^\beta n_0(x) dx \right]^{2/\beta} \leq R_0^2,$$

for all  $\beta \geq 2$ . Thus, the inequality (2.30) immediately implies the inequality (2.18) of Corollary 2.4. Note also that the  $\beta$ -variance  $V_\beta(n_0)$  is finite since we have  $\frac{1}{4}V_\beta(n_0) \leq R_0^2$  obtained from the inequality (2.27). More precisely for the case  $\beta = 2$ , we obtain the following family of inequalities,

$$V_2(n_0) = \frac{1}{M} \int_{\mathbb{R}^2} |x|^2 n_0(x) dx - |B_0|^2 \leq R_0^2 - |B_0|^2 \leq R_0^2,$$

where  $B_0 = \frac{1}{M} \int_{\mathbb{R}^2} x \cdot n_0(x) dx \in \mathbb{R}^2$  is the (normalized) barycenter of  $n_0$ . From these last remarks, we can see that the bounds on  $T_c^*(n_0)$  obtained with the variance are stronger in general than the one obtained with the minimal radius  $R_0$  for the case where  $n_0$  has compact support. Note also that the scope of applications with the variance is wider. Indeed, the variance of  $n_0$  can be finite without compact support condition for  $n_0$ . See the examples of Section 4.

### Proof of Corollary 2.18

Assume that  $n_0 \in L^1$  with  $n_0 \neq 0$  and it has a  $\beta$ -moment with  $\beta \geq 2$ . So, the  $\beta$ -variance  $V_\beta(n_0)$  is finite. To prove the corollary, it is enough to prove the inequality (2.30). Indeed, this implies (2.29) with  $\beta = 2$  by using the relation (2.28). This also gives (2.31) by using the upper bound of (2.27) proved independently and above the statement of Corollary 2.18. The lower bound in (2.27) shows the sharpness of the result (2.31) with respect to (2.30) (up to the multiplicative constant 1/4).

Let  $M > 8\pi$ ,  $z \in \mathbb{R}^2$ , and  $\beta \geq 2$  be fixed. By definition of  $T := T_c^*(n_0)$  as defined in (1.9) of Theorem 1.2, we have  $T \leq T_{c,z}^*$  for all  $z \in \mathbb{R}^2$ . Since  $s \mapsto H_{z,n_0}(s)$  is non-decreasing, then we deduce that

$$\int_{\mathbb{R}^2} \exp\left(-\frac{|x-z|^2}{4T}\right) n_0(x) dx = H_{z,n_0}(T) \leq H_{z,n_0}(T_{c,z}^*) = \frac{2M^2}{3M-8\pi}.$$

We can rewrite this inequality in the following form,

$$\int_{\mathbb{R}^2} \Psi\left(\left[\frac{|x-z|}{2\sqrt{T}}\right]^\beta\right) d\mu(x) = \int_{\mathbb{R}^2} \exp\left(-\frac{|x-z|^2}{4T}\right) \frac{n_0(x)}{M} dx \leq \frac{2M}{3M-8\pi},$$

with  $\Psi(r) = \exp(-r^\gamma)$ ,  $\gamma = 2/\beta \leq 1$  and  $d\mu(x) = \frac{n_0(x)}{M} dx$ . Now, because the function  $\Psi$  is convex and  $\mu$  is a probability measure, we can apply Jensen's inequality and deduce that

$$\Psi\left(\int_{\mathbb{R}^2} \left[\frac{|x-z|}{2\sqrt{T}}\right]^\beta d\mu(x)\right) \leq \int_{\mathbb{R}^2} \Psi\left(\left[\frac{|x-z|}{2\sqrt{T}}\right]^\beta\right) d\mu(x).$$

Thus, we obtain

$$(2.34) \quad \Psi\left(\int_{\mathbb{R}^2} \left[\frac{|x-z|}{2\sqrt{T}}\right]^\beta d\mu(x)\right) \leq a,$$

where  $a = \frac{2M}{3M-8\pi} < 1$  (since  $M > 8\pi$ ). The function  $\Psi$  is strictly decreasing from  $(0, +\infty)$  onto  $(0, 1)$ , then it is invertible with  $\Psi^{-1}$  its decreasing inverse. By applying  $\Psi^{-1}$  on both sides of the inequality (2.34) (and reversing the inequality), this leads to

$$\frac{1}{2^\beta T^{\beta/2}} \int_{\mathbb{R}^2} |x-z|^\beta \frac{n_0(x)}{M} dx = \int_{\mathbb{R}^2} \left[\frac{|x-z|}{2\sqrt{T}}\right]^\beta d\mu(x) \geq \Psi^{-1}(a) > 0.$$

This yields,

$$T_c^*(n_0) =: T \leq \frac{1}{4[\Psi^{-1}(a)]^{2/\beta}} \left[ \int_{\mathbb{R}^2} |x-z|^\beta \frac{n_0(x)}{M} dx \right]^{2/\beta}.$$

It is easy to verify that  $[\Psi^{-1}(a)]^{2/\beta} = \ln(1/a)$ . For all  $z \in \mathbb{R}^2$ , we finally get

$$T_c^* \leq \frac{1}{4 \ln(1/a)} \left[ \int_{\mathbb{R}^2} |x-z|^\beta \frac{n_0(x)}{M} dx \right]^{2/\beta}.$$

This implies the inequality (2.30) after minimizing over  $z \in \mathbb{R}^2$ . This concludes the proof of Corollary 2.18.  $\square$

**Remark 2.23.** *With the same argument of proof as in the proof of Corollary 2.18, we can also show that*

$$T_{c,z}^* \leq \frac{1}{4 \ln(1/a)} \left[ \int_{\mathbb{R}^2} |x-z|^\beta \frac{n_0(x)}{M} dx \right]^{2/\beta},$$

for all  $z \in \mathbb{R}^2$ . Obviously, this also implies the inequality (2.30).

**Remark 2.24.** *A variant of the proof of (2.31), for any  $\beta \geq 2$ , can be given using the inequality (2.29) proved as the inequality (2.31) for the particular case  $\beta = 2$ , and using the fact that the map*

$$\beta \in (0, +\infty) \rightarrow \tilde{V}_z(\beta) := \left( \int_{\mathbb{R}^2} |x-z|^\beta \frac{n_0(x)}{M} dx \right)^{2/\beta}$$

is non-decreasing for all  $z \in \mathbb{R}^2$ . Indeed, this last fact can be shown as follows. For any  $0 < \gamma \leq \beta = p\gamma$ , we have by Jensen's inequality (or Hölder's inequality) the next inequality

$$\left( \int_{\mathbb{R}^2} |x-z|^\gamma \frac{n_0(x)}{M} dx \right)^p \leq \int_{\mathbb{R}^2} |x-z|^{\gamma p} \frac{n_0(x)}{M} dx = \int_{\mathbb{R}^2} |x-z|^\beta \frac{n_0(x)}{M} dx,$$

since  $p = \frac{\beta}{\gamma} \geq 1$ . So, we immediately deduce the inequality  $\tilde{V}_z(\gamma) \leq \tilde{V}_z(\beta)$  for all  $z \in \mathbb{R}^2$ . Thus, the inequality (2.31) follows from (2.29) for all  $\beta \geq \gamma = 2$  by taking  $z = B_0$ , since  $V_2(n_0) = \tilde{V}_{B_0}(2) \leq \tilde{V}_{B_0}(\beta) = V_\beta(n_0)$ .

### 3. LOWER BOUNDS ON $T_c^*(n_0)$

The aim of this section is to provide a lower bound on the critical time  $T_c^*(n_0)$  (but unfortunately not on  $T^*$ ) under some additional  $L^p$  properties of the initial data  $n_0$ . Note that during the proof, we have the opportunity to use the best bounds of the  $L^p - L^q$  embedding theorem of the heat semigroup. For the definition of  $T_c^*(n_0)$ , see (1.9) above.

**Theorem 3.1.** *Assume that  $n_0 \in L^1 \cap L^p$  for some  $1 < p \leq +\infty$  and  $M = \|n_0\|_1 > 8\pi$ . The following lower bounds on  $T_c^*(n_0)$  hold true.*

(1) *We have*

$$(3.35) \quad T_{c_5}^* := \frac{1}{4\pi} \sup_{1 < q \leq p} q' \left[ \frac{L(M)}{\|n_0\|_q} \right]^{q'} \leq T_c^*(n_0),$$

where  $L(M) = \frac{2M^2}{3M-8\pi}$  and  $\frac{1}{q} + \frac{1}{q'} = 1$ .

(2) *Let  $p_0 = \lceil \ln(\frac{2\epsilon}{3}) \rceil^{-1}$ , ( $p_0 \sim 1, 682$ ).*

(a) *If  $p_0 \leq p \leq +\infty$  or if  $1 < p < p_0$  and  $M \leq \frac{8\pi}{3-2e^{(1-1/p)}}$ , then we have*

$$(3.36) \quad \frac{(\pi e)^{-1}}{4 \ln \left( 1 + \frac{M-8\pi}{2M} \right)} \left[ \frac{M}{\|n_0\|_p} \right]^{p'} \leq T_c^*(n_0).$$

*In particular for  $p = +\infty$ , we have*

$$(3.37) \quad \frac{(\pi e)^{-1}}{4 \ln \left( 1 + \frac{M-8\pi}{2M} \right)} \left[ \frac{M}{\|n_0\|_\infty} \right] \leq T_c^*(n_0).$$



(b) If  $1 < p < p_0$  and  $M > \frac{8\pi}{3-2e^{(1-1/p)}}$ , then we have

$$(3.38) \quad \frac{p'}{4\pi} \left[ \frac{L(M)}{\|n_0\|_p} \right]^{p'} \leq T_c^*(n_0),$$

where  $\frac{1}{p} + \frac{1}{p'} = 1$ .

For applications of Theorem 3.1, see Section 4.

We make some comments on these results.

**Remark 3.2.** If  $n_0 \in L^1 \cap L^p$  then  $n_0 \in L^q$  for all  $q \in [1, p]$  by interpolation (generalized Hölder's inequality). Thus, the expression  $T_{c_5}^*$  of the left-hand side of (3.35) is well-defined.

**Remark 3.3.** By  $L^\infty - L^\infty$  contraction of the heat semigroup  $(e^{t\Delta})_{t \geq 0}$ , the following sharp inequality in  $L^\infty$  is well-known,

$$\|p_s \star f\|_\infty \leq \|f\|_\infty,$$

for all  $f \in L^\infty$  and all  $s > 0$ . Let  $T = T_{c,z}^*(n_0)$  with  $z \in \mathbb{R}^2$  be fixed as in Theorem 1.2 (2), and  $n_0 \in L^1 \cap L^\infty$ . As a consequence of the above inequality, we can deduce that

$$L(M) = H_{z,n_0}(T) = (4\pi T) p_T \star n_0(z) \leq 4\pi T \|n_0\|_\infty,$$

where  $L(M) = \frac{2M^2}{3M-8\pi}$  and  $M = \|n_0\|_1$ . Thus, we obtain

$$\frac{L(M)}{4\pi \|n_0\|_\infty} \leq T_{c,z}^*(n_0),$$

for all  $z \in \mathbb{R}^2$ . Hence, by minimization over  $z \in \mathbb{R}^2$  and replacing  $L(M)$  by its explicit expression, we can write

$$(3.39) \quad \frac{\pi^{-1}}{4 \left(1 + \frac{M-8\pi}{2M}\right)} \cdot \left[ \frac{M}{\|n_0\|_\infty} \right] \leq \inf_{z \in \mathbb{R}^2} T_{c,z}^*(n_0) = T_c^*(n_0),$$

for all  $n_0 \in L^1 \cap L^\infty$ .

We can prove that the inequality (3.37) is stronger than (3.39). Indeed, we can easily prove the next sharp pointwise inequality,

$$\frac{1}{1+u} \leq \kappa \frac{e^{-1}}{\ln(1+u)} \leq \frac{e^{-1}}{\ln(1+u)}, \quad 0 \leq u \leq \frac{1}{2},$$

where  $\kappa = (2e/3) \ln(3/2) \sim 0,735 < 1$ . But note that a reverse inequality doesn't hold. Indeed, there exists no  $\delta > 0$  such that

$$\frac{\delta}{\ln(1+u)} \leq \frac{1}{1+u}, \quad 0 < u \leq \frac{1}{2}.$$

(Just by taking  $u \rightarrow 0^+$ ). Then, we can apply the inequalities just above with  $u = \frac{M-8\pi}{2M}$  with  $M > 8\pi$ , since  $0 < u < 1/2$ , which shows that the inequality (3.37) is stronger than the inequality (3.39). The reason of this improvement (3.37) with respect to (3.39) is that  $n_0 \in L^1 \cap L^\infty$  implies  $n_0 \in L^p$  for any  $1 \leq p \leq \infty$  by interpolation, not only  $n_0 \in L^\infty$ . This allows us to use all the scale of  $L^p$ -spaces, and not only  $L^1$  and  $L^\infty$ . For details, see the proof of Theorem 3.1 below.

**Remark 3.4.** As a consequence of (3.37), the singularity of the ratio  $\frac{1}{4 \ln(1 + \frac{M-8\pi}{2M})}$  as  $M$  tends to  $8\pi^+$  leads to the following expected consequence,

$$\lim_{M \rightarrow 8\pi^+} T_c^*(n_0) = +\infty,$$

for all  $n_0 \in L^1 \cap L^p$ ,  $1 < p \leq +\infty$ . Note that this result can not be obtained directly from (3.39).

**Remark 3.5.** A lower bound on  $T^*$  of similar form as in (3.35) for  $T_c^*$  has been obtained in [24, Th. 1 (i)] when  $n_0 \in L^p$  with the restriction  $1 < p < 2$ . See also [24, Prop.1.1] for  $n_0 \in L^\infty$ .

**Proof of Theorem 3.1.** (1) We first recall a general result about  $L^p - L^q$  contractions of the heat semigroup  $(e^{t\Delta})_{t \geq 0}$  defined by convolution with the heat kernel  $p_t$ . On  $\mathbb{R}^n$ , we have the next inequality,

$$\|p_t \star f\|_q = \|e^{t\Delta} f\|_q \leq C(n, p, q) t^{-\frac{n}{2}(\frac{1}{p} - \frac{1}{q})} \|f\|_p, \quad t > 0, f \in L^p(\mathbb{R}^n),$$

with sharp constants

$$C(n, p, q) = \left( \frac{C_p}{C_q} \right)^n \left[ \frac{4\pi}{(\frac{1}{p} - \frac{1}{q})} \right]^{-\frac{n}{2}(\frac{1}{p} - \frac{1}{q})},$$

where  $1 \leq p \leq q \leq +\infty$ , and  $C_p^2 = p^{1/p}/p^{1/p'}$  with  $p'$  the conjugate index of  $p \in [1, +\infty]$  in the sense that  $\frac{1}{p} + \frac{1}{p'} = 1$ . This is a particular use of the sharp version of Young's inequality for which the maximizers exist and are Gaussian functions; see [25] p.98. This is explicitly written page 223 Equation (2) in [25]. In particular, this yields for  $q = +\infty$ , and for all  $p$  such that  $1 \leq p \leq +\infty$ ,

$$\|p_t \star f\|_\infty = \|e^{t\Delta} f\|_\infty \leq C(n, p) t^{-\frac{n}{2p}} \|f\|_p, \quad t > 0,$$

where

$$(3.40) \quad C(n, p) := C(n, p, +\infty) = (p')^{-\frac{n}{2p'}} (4\pi)^{-\frac{n}{2p}}.$$

We use this contraction property of the heat semigroup on  $\mathbb{R}^2$  ( $n = 2$ ) as follows. For all  $n_0 \in L^1(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$ , we have

$$\|p_s \star n_0\|_\infty \leq \frac{C(2, p)}{s^{1/p}} \|n_0\|_p, \quad s > 0,$$

where  $1 \leq p \leq +\infty$ , and the constant  $C(2, p) = (p')^{-\frac{1}{p'}} (4\pi)^{-\frac{1}{p}}$  is obtained from (3.40). Hence, we deduce that, for all  $s > 0$  and  $z \in \mathbb{R}^2$ ,

$$H_z(s) = 4\pi s p_s \star n_0(z) \leq (4\pi s)^{1-1/p} W_p \|n_0\|_p,$$

with  $W_p = (p')^{-\frac{1}{p'}}$ .

Now by definition of  $T := T_{c,z}^*(n_0)$ , we deduce that

$$H_{z,n_0}(T) = L(M) \leq (4\pi T)^{1-\frac{1}{p}} W_p \|n_0\|_p = (4\pi T)^{\frac{1}{p'}} W_p \|n_0\|_p.$$

Since we assume that  $p > 1$ , we have  $\frac{1}{p'} = 1 - \frac{1}{p} > 0$ . We easily solve this inequality for  $T$ , and get the following lower bound for  $T = T_{c,z}^*(n_0)$ ,

$$\frac{1}{4\pi} \left[ \frac{L(M)}{W_p \|n_0\|_p} \right]^{p'} \leq T_{c,z}^*(n_0).$$

By taking the infimum over  $z \in \mathbb{R}^2$ , we get

$$\frac{1}{4\pi} \left[ \frac{L(M)}{W_p \|n_0\|_p} \right]^{p'} \leq T_c^*(n_0).$$

Since  $n_0 \in L^1 \cap L^p \subset L^q$  for all  $1 < q \leq p$ , the same estimate holds with  $q$ , i.e.

$$\frac{q'}{4\pi} \left[ \frac{L(M)}{\|n_0\|_q} \right]^{q'} = \frac{1}{4\pi} \left[ \frac{L(M)}{W_q \|n_0\|_q} \right]^{q'} \leq T_c^*(n_0).$$

Finally, we conclude the proof of (3.35) by taking the supremum over  $q \in (1, p]$ .

(2) Let  $1 < p \leq +\infty$ . By the generalized Hölder's inequality, we can write for all  $1 < q \leq p$ ,

$$\|n_0\|_q \leq M^{1-p'/q'} \|n_0\|_p^{p'/q'},$$

where  $M = \|n_0\|_1$ . Here,  $p', q'$  are respectively the indices conjugate to  $p, q$ . From the inequality (3.35), we deduce that

$$\frac{1}{4\pi} \sup_{1 < q \leq p} q' \left[ \frac{L(M)}{M^{1-p'/q'} \|n_0\|_p^{p'/q'}} \right]^{q'} \leq T_c^*(n_0),$$

or equivalently,

$$\frac{1}{4\pi} \sup_{q' \in [p', +\infty)} q' \left[ \frac{L(M)}{M} \right]^{q'} \left[ \frac{M}{\|n_0\|_p} \right]^{p'} \leq T_c^*(n_0).$$

This lower bound on  $T_c^*(n_0)$  can be written as,

$$\frac{1}{4\pi} \left[ \frac{M}{\|n_0\|_p} \right]^{p'} \sup_{q' \in [p', +\infty)} q' a^{q'} \leq T_c^*(n_0).$$

where  $a := \frac{L(M)}{M} = \frac{2M}{3M-8\pi}$ .

Thus, we just need to estimate the supremum just above. Set  $h(r) = ra^r$ ,  $r > 0$ . Recall that  $a \in (0, 1)$  when  $M > 8\pi$ . It is easily shown that  $h$  is increasing on the interval  $(0, r_0)$  and decreasing on  $(r_0, +\infty)$ , where

$$r_0 = \frac{1}{\ln(1/a)} = \frac{1}{\ln\left(1 + \frac{M-8\pi}{2M}\right)}.$$

Thus,  $h$  attains its maximum at  $r_0$  on  $(0, +\infty)$ . We deduce that

$$\sup_{r>0} h(r) = \max_{r>0} h(r) = h(r_0) = \frac{e^{-1}}{\ln\left(1 + \frac{M-8\pi}{2M}\right)}.$$

This leads us to discuss two cases for evaluating  $\sup_{q' \in [p', +\infty)} q' a^{q'}$  for fixed  $p' \in [1, +\infty)$ .

(i) If  $p' \leq r_0$ , then we have

$$\sup_{q' \in [p', +\infty)} q' a^{q'} = \frac{e^{-1}}{\ln\left(1 + \frac{M-8\pi}{2M}\right)}.$$

(ii) If  $p' > r_0$ , then we have

$$\sup_{q' \in [p', +\infty)} q' a^{q'} = p' a^{p'}.$$

This implies that

(iii) If  $3 - 2e^{1-1/p} \leq \frac{8\pi}{M}$  (i.e.  $p' \leq r_0$ ), then we have

$$\frac{(\pi e)^{-1}}{4 \ln\left(1 + \frac{M-8\pi}{2M}\right)} \left[ \frac{M}{\|n_0\|_p} \right]^{p'} \leq T_c^*(n_0).$$

(iv) If  $3 - 2e^{1-1/p} > \frac{8\pi}{M}$  (i.e.  $p' > r_0$ ), then we have

$$\frac{p'}{4\pi} \left( \frac{L(M)}{\|n_0\|_p} \right)^{p'} \leq T_c^*(n_0).$$

Now we are in position to discuss both cases (a) and (b) of Theorem 3.1 (2).

(a) Let  $p_0 = \left[ \ln\left(\frac{2e}{3}\right) \right]^{-1}$  ( $\sim 1,682$ ). Assume that  $p_0 \leq p \leq +\infty$ . This condition is equivalent to  $3 - 2e^{1-1/p} \leq 0$  ( $p_0$  is the unique root of  $v(p) = 3 - 2e^{1-1/p}$ ). Thus, the condition of (iii) just above is trivial for any  $M > 8\pi$ . Consequently, the inequality (3.36) holds true. Now, let  $1 < p < p_0$  and  $M \leq \frac{8\pi}{3-2e^{1-1/p}}$ . Then this last inequality is equivalent to  $0 < 3 - 2e^{1-1/p} \leq \frac{8\pi}{M}$ , and again (3.36) holds true by virtue of (iii).

(b) Let  $1 < p < p_0$  and  $M > \frac{8\pi}{3-2e^{1-1/p}}$ . Then this second inequality is equivalent to  $3 - 2e^{1-1/p} > \frac{8\pi}{M}$ , hence the inequality (3.38) holds true by virtue of (iv).

This concludes the proof of Theorem 3.1.  $\square$

**Remark 3.6.** Let  $a = \frac{L(M)}{M} = \frac{2M}{3M-8\pi}$  with  $M > 8\pi$ . From the study of the function  $h$  defined above, it is easy to compare formally both inequalities (3.36) and (3.38). We can see that (3.36) is stronger than (3.38) (at least for  $p' \leq r_0$ ). We have the next inequality,

$$h(p') = p'a^{p'} < \sup_{r \in [p', +\infty)} h(r) = h(r_0) = \frac{e^{-1}}{\ln\left(\frac{1}{a}\right)}.$$

Now, assuming (3.36) we would have the next inequality,

$$\frac{p'}{4\pi} \left( \frac{L(M)}{\|n_0\|_p} \right)^{p'} = \frac{p'}{4\pi} a^{p'} \left( \frac{M}{\|n_0\|_p} \right)^{p'} \frac{e^{-1}}{4\pi \ln\left(\frac{1}{a}\right)} \left( \frac{M}{\|n_0\|_p} \right)^{p'} \leq T_c^*(n_0),$$

i.e. (3.38). This proves the assertion.

#### 4. EXAMPLES OF INITIAL DATA

This section is devoted to applications of Theorem 1.2, 1.7, and Corollaries 2.1, 2.4, 2.12, 2.18, providing upper estimates on the critical time  $T_c^*(n_0)$  to solutions of the (PKS) system for several families of explicit initial data  $n_0$  with supercritical mass  $M > 8\pi$ . Theorem 3.1 is applied to provide lower bounds on  $T_c^*(n_0)$ . We shall also compare the different estimates of  $T_c^*(n_0)$  obtained from these methods. As a consequence, we deduce explicit upper bounds for the maximal existence time  $T^*$  of the (PKS) system. See Sections 1, 2 and 3. Note that Corollary 2.1 is not used in this section for the examples described below due to the difficulty for obtaining explicit computations.

**4.1. Gaussian initial data  $n_0$ .** Our first family of examples of initial data  $n_0$  consists of Gaussian functions written in the following form,

$$n_0(x) = \frac{M}{4\pi\sigma} e^{-\frac{|x-z_0|^2}{4\sigma}} = Mp_\sigma(x-z_0), \quad x \in \mathbb{R}^2,$$

where  $\sigma > 0$  and  $z_0 \in \mathbb{R}^2$  are fixed. Here, the function  $p_\sigma$  denotes the Gaussian (or heat) kernel. For all these data  $n_0$ , the mass is  $\|n_0\|_1 = M$  and the normalized barycenter is  $B_0 = z_0$ . In the whole section, we shall assume that  $M > 8\pi$ . By applying part (3) and (4) of Theorem 1.2, we obtain the exact value of  $T_c^*(n_0)$ , and also the next upper bound for the maximal existence time  $T^*$  of the (PKS) system,

$$(4.41) \quad T^* \leq T_c^*(n_0) := \sigma \frac{2M}{M-8\pi}.$$

The proof is as follows. The function  $n_0$  is a non-increasing  $z_0$ -radially symmetric function. We set  $m_0(x) = n_0(x+z_0)$ ,  $x \in \mathbb{R}^2$ . So,  $m_0$  is a non-increasing radially symmetric function. By (4) of Theorem 1.2, we have  $T_c^*(n_0) = T_c^*(m_0)$ . We determine the function  $H_{z,m_0}$  evaluated at  $z = 0$  and get the next formula

$$H_{0,m_0}(s) = 4\pi s M p_s \star p_\sigma(0) = 4\pi s M p_{s+\sigma}(0) = \frac{sM}{s+\sigma}, \quad s > 0.$$

The inverse of  $H_{0,m_0}$  is easily deduced, we have

$$s = H_{0,m_0}^{-1}(u) = \frac{\sigma u}{M-u}, \quad u \in (0, M).$$

Finally, we deduce by (3) of Theorem 1.2 that

$$T_c^*(n_0) = T_c^*(m_0) = H_{0,m_0}^{-1} \left( \frac{2M^2}{3M-8\pi} \right) = \sigma \frac{2M}{M-8\pi}.$$

Alternatively, we can apply (1.12) of Theorem 1.7 (but the computation is longer) and obtain the same result for the value of  $T_c^*(n_0)$ . Note that  $T_c^*(n_0)$  depends not only of the

mass  $M$  but also of the *shape* of the Gaussian  $p_\sigma$  via the function  $H_{0,m_0}^{-1}$ , in particular through the variance parameter  $\sigma$ . Of course, the *shape* does not depend on the spacial *position* parametrized by the translation term  $z_0$  of the Gaussian function  $p_\sigma$ . In the limit case of concentration  $\sigma = 0$ , we would obtain  $T_c^*(n_0) = 0$  (the mass  $M$  remains fixed), i.e. an *immediate blow-up* of the solution of the (PKS) system as it can be expected. This is not physically surprising since the initial data should be considered at this limit case as a Dirac measure at  $z_0$  and the result is consistent with our intuition. On the other hand, for the behavior of a radially symmetric solution (i.e.  $z_0 = 0$ ) closed to the blow-up time  $T^*$ , we can refer for instance to [7, 6] (and the references therein) for a mathematical statement of the blow-up profile.

As already mentioned, whenever the second moment of  $n_0$  is finite and for which an estimate is available, we shall compare the upper bound  $T_v^*$  of  $T^*$  (see (1.5)) and the one given by  $T_c^*(n_0)$  obtained in this paper. This is done in (1) below.

- (1) *Second moment method.* We deduce from the inequality (1.5) the next estimate of  $T^*$ ,

$$T^* \leq T_v^* := \frac{2\pi V_2(n_0)}{M - 8\pi} = \sigma \frac{8\pi}{M - 8\pi},$$

for all  $\sigma > 0$  and all  $z_0 \in \mathbb{R}^2$ . Indeed, it is easy to check that  $B_0 = z_0$  and the well-known variance of the Gaussian function  $p_\sigma$  in  $\mathbb{R}^2$  is given by

$$(4.42) \quad V_2(n_0) = \frac{1}{M} \int_{\mathbb{R}^2} |x - B_0|^2 n_0(x) dx = \int_{\mathbb{R}^2} |x|^2 p_\sigma(x) dx = 4\sigma.$$

This upper bound  $T_v^*$  is obviously a better bound of  $T^*$  than the one given by (4.41) since  $M > 8\pi$ . This is due to the fact that  $T_v^*$  is deduced directly from the second moment evolution equation for a solution of the (PKS) system under finite second moment assumption of the initial data  $n_0$  (see details below (1.5)). This additional information on  $n_0$  leads to a better estimate compared to the general bound  $T_c^*(n_0)$  in (1.9) of  $T^*$  valid for any initial data  $n_0$ .

- (2) *Various estimates from the general case.* As much as possible, we now explicit various estimates of  $T_c^*(n_0) = \sigma \frac{2M}{M-8\pi}$  obtained by the corollaries described after Theorem 1.2 and by Theorem 1.7, and compare them with the exact formula (4.41) of  $T_c^*(n_0)$ . Note that this comparison concerns the upper bound  $T_c^*(n_0)$  of  $T^*$  and not  $T^*$  itself. Below, we present the results not necessarily in the order of the corollaries.

*About lower bounds given by Theorem 3.1.* The inequality (3.35) is, in fact, an equality, i.e.

$$(4.43) \quad \frac{1}{4\pi} \sup_{1 < q \leq p} q' \left[ \frac{L(M)}{\|n_0\|_q} \right]^{q'} = \sigma \frac{2M}{M - 8\pi} = T_c^*(n_0).$$

It comes as no surprise because the proof of Corollary 3.1 relies on sharp Young's inequalities for which optimizers are Gaussians (see [25] p.99). This shows that the inequality (3.35) can be optimal at least for this family of examples. Let us prove this equality. For  $1 < q < +\infty$ , the  $L^q$ -norm of  $n_0$  is given explicitly by

$$\|n_0\|_q = M \|p_\sigma\|_q = M \frac{(4\pi\sigma)^{-1+1/q}}{q^{1/q}}.$$

Thus, we can set

$$\mathcal{R}(q') := \frac{q'}{4\pi} \left[ \frac{L(M)}{\|n_0\|_q} \right]^{q'} = \sigma a^{q'} q' q^{q'/q},$$

with  $a = \frac{L(M)}{M} = \frac{2M}{3M-8\pi}$ . Now, since  $q'/q = q' - 1$  and  $q = \frac{q'}{q'-1}$ , we can write

$$\mathcal{R}(q') = \sigma a^{q'} q' q^{q'-1} = \sigma a^{q'} (q'/q) q^{q'} = \sigma a^{q'} (q' - 1) \left( \frac{q'}{q' - 1} \right)^{q'} = \sigma a^{q'} (q' - 1)^{1-q'} (q')^{q'}.$$

We set  $\rho = q' \geq p'$ . Then, for  $p = p(M)$  large enough so that  $p' \leq \frac{1}{1-a}$ , we have

$$\begin{aligned} \frac{1}{4\pi} \sup_{1 < q \leq p} q' \left[ \frac{L(M)}{\|n_0\|_q} \right]^{q'} &= \sup_{q' \geq p'} \mathcal{R}(q') = \sigma \sup_{q' \geq p'} a^{q'} (q' - 1)^{1-q'} (q')^{q'} \\ &= \sigma \sup_{\rho \geq p'} e^{H(\rho)} = \sigma \sup_{\rho \geq 1} e^{H(\rho)} = \sigma e^{H(\frac{1}{1-a})} = \sigma \frac{a}{1-a} = \sigma \frac{2M}{M-8\pi}, \end{aligned}$$

where  $H(\rho) = \rho \ln a + (1 - \rho) \ln(\rho - 1) + \rho \ln \rho$ . Indeed, from the study of the derivative

$$H'(\rho) = \ln a + \ln \left( \frac{\rho}{\rho - 1} \right),$$

we deduce that the function  $\rho \mapsto H(\rho)$  is increasing on the interval  $(1, \frac{1}{1-a})$ , and decreasing on the interval  $(\frac{1}{1-a}, +\infty)$ . Thus, the supremum of  $H$  on  $(1, +\infty)$  is achieved for  $\rho_0 = \frac{1}{1-a} > 1$ . Now, since  $n_0$  is in  $L^p$  for all  $p \geq 1$ , we can choose  $p$  large enough, so that  $p' \leq \rho_0$ . This leads to the expected equality (4.43).

The estimates of  $T_c^*(n_0)$  obtained from part 2 of Theorem 3.1 are, of course, less precise. Just below, we list them. Under the corresponding assumptions for each assertion, we can write

(a) from (3.36),

$$\sigma \frac{e^{-1} p^{p'-1}}{\ln(1 + \frac{M-8\pi}{2M})} \leq T_c^*(n_0),$$

(b) from (3.37),

$$\sigma \frac{e^{-1}}{\ln(1 + \frac{M-8\pi}{2M})} \leq T_c^*(n_0),$$

(c) from (3.38),

$$\sigma \frac{p' p^{p'-1}}{(1 + \frac{M-8\pi}{2M})^{p'}} \leq T_c^*(n_0).$$

A priori, none of these inequalities are sharp. They don't lead to the optimal estimate (3.35) of  $T_c^*(n_0)$  (case of equality for Gaussians). Nevertheless, the inequalities (3.36) and (3.37) are all of the next form,

$$\sigma \frac{k}{\ln(1 + \frac{M-8\pi}{2M})} \leq T_c^*(n_0),$$

for some constant  $k > 0$  (independent of  $M$ ). Then we can compare these estimates with the exact value of  $T_c^*(n_0)$  as  $M$  goes to  $8\pi^+$ ,

$$\sigma \frac{k}{\ln(1 + \frac{M-8\pi}{2M})} \sim k \frac{2\sigma M}{M-8\pi} = k T_c^*(n_0).$$

We now compare the exact value of  $T_c^*(n_0)$  with various upper bounds on  $T_c^*(n_0)$  obtained from the corollaries of Theorem 1.2 of Section 2 for Gaussian initial data.

From Corollary 2.4. In order to apply (2.15), we first compute

$$\mathcal{M}_{z_0}(\rho) := \int_{B(z_0, \rho)} n_0(x) dx = M(1 - e^{-\frac{\rho^2}{4\sigma}}), \quad \rho > 0.$$

Hence, we get

$$\frac{\mathcal{M}_{z_0}(\rho)}{L(M)} = \frac{1}{a} (1 - e^{-\frac{\rho^2}{4\sigma}}),$$

where  $\frac{1}{a} = 1 + \frac{M-8\pi}{2M}$ . Using the fact that  $n_0$  is non-increasing  $z_0$ -radially symmetric, we can apply (2.15) (by invariance, it is sufficient to consider the case  $z_0 = 0$ ). Thus we obtain

$$T_c^*(n_0) \leq T_{c_2}^* := \inf_{\rho>0} \frac{\rho^2}{4} \left[ \ln_+ \left( \frac{1 - e^{-\frac{\rho^2}{4\sigma}}}{a} \right) \right]^{-1}.$$

By setting  $s = \frac{\rho^2}{4\sigma}$ , this implies that

$$T_c^*(n_0) \leq T_{c_2}^* := \sigma \inf_{s>0} s \left[ \ln_+ \left( \frac{1 - e^{-s}}{a} \right) \right]^{-1} = \sigma \inf_{s>0, 1-e^{-s}>a} \frac{s}{\ln(1 - e^{-s}) + \ln(1/a)}.$$

Unfortunately, it is not clear if we can estimate this infimum in such a way that it is closed to the value of  $T_c^*(n_0)$  given above. We can deduce that we have for all  $\varepsilon \in (0, 1 - a)$ ,

$$T_c^*(n_0) \leq \sigma \frac{c_\varepsilon}{\ln\left(\frac{1-\varepsilon}{a}\right)},$$

with  $c_\varepsilon = \ln\left(\frac{1}{\varepsilon}\right) > 0$  by taking  $s = c_\varepsilon$ . We shall see below that we can obtain a more interesting upper bound on  $T_c^*(n_0)$  of the form  $\sigma \frac{1}{\ln\left(\frac{1}{a}\right)}$  using the finite variance of the initial data  $n_0$  by applying Corollary 2.18. This will improve the upper bound on  $T_c^*(n_0)$  just above obtained by Corollary 2.4.

Another bound is also obtained from the expression of  $\mathcal{M}_{z_0}(\rho)$ . Indeed, we deduce that

$$g_{z_0}(\rho) := \frac{1}{M} \int_{B(z_0, \rho)} n_0(x) dx = 1 - e^{-\frac{\rho^2}{4\sigma}}.$$

Thus, for all  $m \in (0, 1)$  we have

$$h(m) = [g_{z_0}^{\leftarrow}(m)]^2 = [g_{z_0}^{-1}(m)]^2 = \rho^2 = -4\sigma \ln(1 - m).$$

Applying (2.17), we can write

$$T_{c_2}^* \leq \frac{\sigma}{\ln\left(1 + \frac{M-8\pi}{2M}\right)} \cdot \inf_{\theta \in (0, 1)} \left( \frac{-\ln(1 - a^\theta)}{(1 - \theta)} \right).$$

Here again, it is not clear if we can estimate this infimum in such a way that the right-hand side of the preceding inequality is closed to  $T_c^*(n_0)$ .

Finally, note that the inequalities (2.18) and (2.19) of Corollary 2.4 do not apply here since  $n_0$  has no compact support.

From Corollary 2.12. We can compute explicitly  $F$  and study its properties. We have

$$\frac{h}{h'}(m) = -(1 - m) \ln(1 - m), \quad m \in (0, 1).$$

Then  $F$  is given by

$$F(X) = X + e^X \left( \frac{h}{h'} \right) (e^{-X}) = X - (e^X - 1) \ln(1 - e^{-X}), \quad X > 0.$$

and its derivative by

$$F'(X) = -e^X \ln(1 - e^{-X}), \quad X > 0.$$

The condition (2.24), namely  $\left(\frac{h}{h'}\right)(1^-) = 0 \geq \ln(1/a)$ , is not satisfied since  $a \in (0, 1)$ . Then (1) of Corollary 2.12 does not apply. But the conditions of part (2) are easily satisfied. Indeed, the function  $F$  is clearly continuous and strictly increasing ( $F'(X) > 0$  for all  $X > 0$ ). Condition (i) is satisfied, i.e.  $\left(\frac{h}{h'}\right)(1^-) = 0 < \ln(1/a)$  holds true, and (ii) is also satisfied, i.e.

$$\left(\frac{h}{h'}\right)(a^+) = -(1 - a) \ln(1 - a) > 0.$$



By applying (2.25), we get

$$T_{c_2}^*(0) = S(Y_0) = \frac{\sigma}{e^{F^{-1}(Y_0)} - 1},$$

where  $Y_0 = \ln \frac{1}{a}$ .

We now compare  $T_{c_2}^*(0)$  with  $T_c^*(n_0)$ . It is clear that for all  $X > 0$ , we have  $X < F(X)$ . So, we get  $F^{-1}(Y) < Y$  for all  $Y \in \text{Im}(F)$ . This implies that

$$\frac{\sigma}{e^Y - 1} < \frac{\sigma}{e^{F^{-1}(Y)} - 1}$$

which, evaluated at  $Y = Y_0$ , gives us

$$T_c^*(n_0) = \sigma \frac{2M}{M - 8\pi} = \frac{\sigma}{e^{Y_0} - 1} < \frac{\sigma}{e^{F^{-1}(Y_0)} - 1} = T_{c_2}^*(0).$$

This proves again that  $T_c^*(n_0) \leq T_{c_2}^*(0)$  but clearly  $T_c^*(n_0) \neq T_{c_2}^*(0)$  for the family of Gaussian initial data.

The inverse function  $F^{-1}$  of  $F$  can not be given explicitly for this particular case. Then we look for an upper bound on  $T_{c_2}^*(0)$ . A first approach should be to note that  $F(X) \leq X + G(\ln(1/a)) < X + 1$  for  $X \in (0, \ln(1/a))$  with  $G(X) = F(X) - X$  ( $G$  is an increasing function). This yields  $Y_0 - G(\ln(1/a)) \leq F^{-1}(Y_0)$ . But unfortunately  $Y_0 - G(\ln(1/a)) < 0$ . Thus we cannot deduce an upper bound for  $T_{c_2}^*(0)$  from this approach. Another approach will be to find some finite constant  $c > 0$  such that  $F(x) \leq cX$  for all  $X \in (0, \ln(1/a)) \subset (0, 0.41)$ . But we are facing to the problem that  $\lim_{X \rightarrow 0^+} F(X)/X = +\infty$  which contradicts such an estimate. Since we cannot conclude here on this point, we shall not go further in our analysis of  $F^{-1}$ .

From Corollary 2.18. We obtain an explicit upper bound on  $T_c^*(n_0)$  using  $T_{c_4}^*$  given by (2.29). This estimate is sharp when  $M$  is closed to  $8\pi$ . We also deduce interesting upper and lower bounds of  $T_c^*(n_0)$  in terms of  $T_{c_4}^*$ . First, from the value of the variance  $V_2(n_0)$  of  $n_0$  given by (4.42) and applying (2.29), it follows immediately that

$$T_c^*(n_0) = \sigma \frac{2M}{M - 8\pi} \leq T_{c_4}^* = \sigma \frac{1}{\ln(1 + \frac{M-8\pi}{2M})}.$$

This explicit result is sharp when  $M \rightarrow 8\pi^+$ , since we have the following asymptotic behaviour

$$T_{c_4}^* = \sigma \frac{1}{\ln(1 + \frac{M-8\pi}{2M})} \sim \sigma \frac{2M}{M - 8\pi} = T_c^*(n_0), \quad M \rightarrow 8\pi^+.$$

On the other hand, by a direct study of the function  $y \mapsto v(y) = \frac{\ln(1+y)}{y}$  which is a decreasing function on  $(0, 1/2)$ , we deduce that

$$c_0 \frac{1}{\ln(1+y)} \leq \frac{1}{y} \leq \frac{1}{\ln(1+y)}, \quad y \in (0, 1/2),$$

where  $c_0 := v(1/2) = 2 \ln(3/2)$ . We apply this inequality to  $y = \frac{M-8\pi}{2M} \in (0, 1/2)$  with  $M > 8\pi$ , and get

$$c_0 \frac{\sigma}{\ln(1 + \frac{M-8\pi}{2M})} \leq \sigma \frac{2M}{M - 8\pi} \leq \frac{\sigma}{\ln(1 + \frac{M-8\pi}{2M})}, \quad M > 8\pi.$$

In other words, we have obtained the following comparison estimates between  $T_c^*(n_0)$  and  $T_{c_4}^*$ ,

$$c_0 T_{c_4}^* \leq T_c^*(n_0) \leq T_{c_4}^*,$$

where  $c_0 \sim 0.8109$  and  $c_0 \geq 0.8109$ . Note that the upper bound just above is already known from Corollary 2.18. The lower bound is of interest for estimating  $T_c^*(n_0)$  by keeping in mind the fact that  $T_c^*(n_0)$  tends to infinity when  $M \rightarrow 8\pi^+$  (hence also  $T_{c_4}^*$ ).

Recall here that our main goal in this section is to compare general results obtained in Corollaries 2.4, 2.12, 2.18 and Theorem 3.1 with the known and explicit value

$$T_c^*(n_0) = \sigma \frac{2M}{M - 8\pi},$$

obtained by Theorem 1.2, or Theorem 1.7. We summarize the most interesting results of this comparison of  $T_c^*(n_0)$  with various  $T_{c_i}^*$  for the gaussian initial data as follows. From Corollary 2.18 using the variance of the initial data and Theorem 3.1, we have obtained the following estimates of  $T_c^*(n_0)$ , namely

$$T_{c_5}^* = T_c^*(n_0) = \sigma \frac{2M}{M - 8\pi} \leq T_{c_4}^* = \sigma \frac{1}{\ln(1 + \frac{M-8\pi}{2M})} \leq \frac{1}{c_0} T_c^*(n_0).$$

Moreover, Theorem 3.1 provides an optimal result (equality case). We have also seen that the estimate of  $T_c^*(n_0)$  by  $T_{c_4}^*$  of Corollary 2.18 is asymptotically sharp in the sense that  $T_{c_4}^* \sim T_c^*(n_0)$  when  $M \rightarrow 8\pi^+$ . The method of estimating  $T_c^*(n_0)$  from the variance is particularly interesting for the Gaussian initial data.

**4.2. Characteristic function of a disk.** A natural situation is when the cells are uniformly distributed in a disk  $B(z_0, R)$  with some accumulation height  $\sigma > 0$ . This leads to consider an initial data of the type  $n_0 = \sigma \mathbf{1}_{B(z_0, R)}$  for fixed  $z_0 \in \mathbb{R}^2$  and  $R > 0$ . Here  $\mathbf{1}_{B(z_0, R)}$  denotes the characteristic function of the disk  $B(z_0, R)$  of radius  $R$  centered at  $z_0$ . We assume to be in the supercritical case  $M > 8\pi$ , i.e.  $\sigma R^2 > 8$ . In that case, we can describe the bound  $T_c^*(n_0)$  as follows.

**Proposition 4.1.** *Let  $n_0 = \sigma \mathbf{1}_{B(z_0, R)}$  be the initial data for the (PKS) system. Here,  $z_0 \in \mathbb{R}^2$  and  $\sigma > 0$  are fixed. Assume that  $M = \pi \sigma R^2 > 8\pi$ .*

- (1) *Then the maximal existence time  $T^*$  of the solution of (PKS) system with initial data  $n_0$  is bounded as follows:*

$$(4.44) \quad T^* \leq T_c^*(n_0) := \frac{R^2}{4f^{-1}(\frac{2M}{3M-8\pi})} = \frac{M}{4\pi\sigma f^{-1}(\frac{2M}{3M-8\pi})},$$

where  $f^{-1}$  is the inverse of the function  $f(\lambda) = \frac{1-e^{-\lambda}}{\lambda}$ ,  $\lambda > 0$ . In particular, we have

$$(4.45) \quad 0,286.R^2 \sim \frac{R^2}{4f^{-1}(\frac{2}{3})} \leq T_c^*(n_0),$$

with  $f^{-1}(\frac{2}{3}) \sim 0.87421$ . As a consequence of (4.44), let  $\sigma \rightarrow +\infty$  for fixed  $R > 0$  (so,  $M \rightarrow +\infty$ ), we have

$$(4.46) \quad T_c^*(n_0) \sim \frac{R^2}{4f^{-1}(\frac{2}{3})} \sim 0,286.R^2.$$

- (2) *We have also the following uniform estimate for  $M > 8\pi$ ,*

$$(4.47) \quad e^{-1} \frac{R^2}{4 \ln(1 + \frac{M-8\pi}{2M})} \leq T_c^*(n_0) \leq \frac{R^2}{4 \ln(1 + \frac{M-8\pi}{2M})}.$$

- (3) *The following asymptotic estimate holds true when  $M$  is closed to the supercritical mass  $8\pi$ .*

- (a) *Let  $R > 0$  be fixed. Then we have*

$$(4.48) \quad T_c^*(n_0) \sim \frac{2\pi R^2}{M - 8\pi} = \frac{2R^2}{\sigma R^2 - 8}, \text{ as } \sigma \rightarrow (8/R^2)^+.$$

- (b) *Let  $\sigma > 0$  be fixed. Then we have*

$$(4.49) \quad T_c^*(n_0) \sim \frac{16\pi}{\sigma(M - 8\pi)} = \frac{16}{\sigma(\sigma R^2 - 8)}, \text{ as } R \rightarrow (\sqrt{8/\sigma})^+.$$

As already mentioned, the bound  $T_c^*(n_0)$  does not depend on the normalized barycenter  $B_0 = z_0$  by translation invariance. Indeed, we should expect from a physical point of view that the evolution of cells should be the same if we make a translation in the space variable by any  $z_0 \in \mathbb{R}^2$  of the initial data  $n_0$  by considering the new initial data  $x \mapsto n_0(x + z_0)$ . Of course, this due to the implicit isotropic environment of  $\mathbb{R}^2$  here (flat curvature).

As described by (4.44), the critical time  $T_c^*(n_0)$  can be alternatively expressed with the radius  $R$  (which measures essentially the size of the support of  $n_0$ ), or with the height  $\sigma$  (i.e. the sup norm of  $n_0$ ) for  $n_0$  with fixed mass  $M > 8\pi$ . As a consequence, if we fix the mass  $M > 8\pi$  and let  $R \rightarrow 0$  (i.e.,  $\sigma \rightarrow +\infty$ ), we deduce that  $T_c^*(n_0)$  tends to 0, and consequently  $T^*$  too. In particular, again with a fixed mass  $M > 8\pi$ , if we consider a sequence of initial data of the form  $n_{0,k} = \sigma_k \mathbf{1}_{B(z_0, R_k)}$  with  $R_k \rightarrow 0$  (i.e.,  $\sigma_k = \frac{M}{\pi R_k^2} \rightarrow +\infty$ ), which approximates  $M\delta_{z_0}$  in a weak sense, then we obtain from (4.44) that  $\lim_k T_c^*(n_{0,k}) = 0$ . Here,  $\delta_{z_0}$  denotes the Dirac measure at  $z_0 \in \mathbb{R}^2$ . Heuristically, it should not be surprising to observe an instantaneous explosion if we consider  $M\delta_{z_0}$  as initial data with  $M > 8\pi$ .

Now if we fix  $R > 0$  (i.e. the support of  $n_0$  is fixed) and let the accumulation height  $\sigma$  of cells goes to infinity, then we deduce from (4.46) that the critical time  $T_c^*(n_0)$ , respectively  $T^*$ , is uniformly bounded above by  $cR^2$  for some constant  $c$ .

**Proof of Proposition 4.1 .** (1) We start from the upper bound (1.7) on  $T^*$ . Since the function  $n_0(x) = \sigma \mathbf{1}_{B(z_0, R)}(x) = \sigma \mathbf{1}_{B(0, R)}(x - z_0)$  is the translated by  $z_0$  of a radial function, then we have

$$T_c^*(n_0) = H_{z_0}^{-1} \left( \frac{2M^2}{3M - 8\pi} \right),$$

where

$$\begin{aligned} H_{z_0}(T) &= \int_{\mathbb{R}^2} \exp \left( -\frac{|x - z_0|^2}{4T} \right) n_0(x) dx = \sigma \int_{B(0, R)} \exp \left( -\frac{|y|^2}{4T} \right) dy \\ &= 2\pi\sigma \int_0^R \exp \left( -\frac{r^2}{4T} \right) r dr = 2\pi\sigma \left[ -2T \exp \left( -\frac{r^2}{4T} \right) \right]_{r=0}^{r=R} = 4\pi\sigma T (1 - e^{-\frac{R^2}{4T}}). \end{aligned}$$

This result is obtained by the change of variables  $y = x - z_0$  and using polar coordinates. Now from the relation  $\pi\sigma = \frac{M}{R^2}$ , this implies

$$H_{z_0}(T) = M \left( \frac{4T}{R^2} \right) \left( 1 - e^{-\frac{R^2}{4T}} \right) = M f \left( \frac{R^2}{4T} \right),$$

where the function  $f : (0, +\infty) \rightarrow (0, 1)$  is defined by  $f(\lambda) = \frac{1 - e^{-\lambda}}{\lambda}$ .

Since the function  $f$  is a continuous (convex) strictly decreasing function with range  $(0, 1)$  then  $f$  is invertible. Then the inverse function  $f^{-1} : (0, 1) \rightarrow (0, +\infty)$  of  $f$  is also a continuous (convex) strictly decreasing function. Hence, we obtain the following expression

$$H_{z_0}^{-1}(u) = \frac{R^2}{4f^{-1}(\frac{u}{M})},$$

for all  $u \in (0, M)$ . From the relation  $M = \sigma\pi R^2$ , this yields the next upper bound on  $T^*$ ,

$$T^* \leq T_c^*(n_0) := H_{z_0}^{-1} \left( \frac{2M^2}{3M - 8\pi} \right) = \frac{R^2}{4f^{-1}(\frac{2M}{3M - 8\pi})} = \frac{M}{4\pi\sigma f^{-1}(\frac{2M}{3M - 8\pi})}.$$

This proves the inequality (4.44) which is the first part of (1) of the proposition.

We finish the proof of this part (1) as follows. First, we note that  $\frac{2M}{3M - 8\pi} \geq 2/3$  for  $M > 8\pi$  and  $f^{-1}$  is decreasing, thus the lower bound (4.45) on  $T_c^*(n_0)$  follows immediately.

The approximation  $f^{-1}(\frac{2}{3}) \sim 0.87421$  is obtained using any simple numerical calculation software.

(2) The upper bound in (4.47) follows from (4.44) and the fact that

$$\frac{1}{f^{-1}(a)} = \frac{1}{f^{-1}(\frac{2M}{3M-8\pi})} \leq \frac{1}{\ln(1 + \frac{M-8\pi}{2M})} = \frac{1}{\ln(\frac{1}{a})},$$

where  $a = \frac{2M}{3M-8\pi} = (1 + \frac{M-8\pi}{2M})^{-1} \in (0, 1)$ . Indeed, if we set  $f^{-1}(a) = \lambda > 0$ , i.e.  $f(\lambda) = a$ , then it is enough to show that

$$\ln\left(\frac{\lambda}{1 - e^{-\lambda}}\right) \leq \lambda, \quad \lambda > 0.$$

This is equivalent to the trivial inequality  $1 + \lambda \leq e^\lambda$  for  $\lambda \geq 0$ . So, the upper bound (4.47) follows.

The lower bound of (4.47) is straightforward from (3.37) of Theorem 3.1 since  $M = \pi \|n_0\|_\infty R^2$ . It can also be obtained from (3.36) with a well-chosen finite  $p$ . Indeed, the function  $n_0$  is in all  $L^p$  and its  $L^p$ -norm is explicitly computable. More precisely, we have  $\|n_0\|_p = \sigma(\pi R^2)^{1/p}$  for all  $1 \leq p \leq +\infty$ . So, we get  $M := \|n_0\|_1 = \sigma(\pi R^2)$  and

$$\left[ \frac{M}{\|n_0\|_p} \right]^{p'} = \pi R^2, \quad 1 \leq p \leq +\infty,$$

which proves (4.47) by applying (3.36) with  $p$  chosen large enough.

(3) One can easily prove that  $f^{-1}(v) \sim 2(1 - v)$  as  $v \rightarrow 1^-$ . Now since we have  $\lim_{M \rightarrow 8\pi^+} \frac{2M}{3M-8\pi} = 1^-$ , this implies that

$$f^{-1}\left(\frac{2M}{3M-8\pi}\right) \sim \frac{M-8\pi}{8\pi}, \quad \text{as } M \rightarrow 8\pi^+.$$

Finally, we conclude the estimates (4.48) and (4.49) since  $\pi\sigma R^2 = M$  and  $M \rightarrow 8\pi^+$ .

This concludes the proof of Proposition 4.1.  $\square$

Next, we compare the estimates of  $T_c^*(n_0)$  that can be obtained from the corollaries of Section 2 with the results of Proposition 4.1.

- The first alternative to obtain the upper bound of (4.47) is by applying (2.18) of Corollary 2.4 (or equivalently in this case (2.14)). This can be done because we can only consider the case  $z_0 = 0$  by translation invariance of  $T_c^*(n_0)$ , and by the fact that  $n_0$  is a non-increasing radially symmetric function when  $z_0 = 0$ . In our situation, we can explicitly compute  $T_{c_2}^*(z)$  for  $z = 0$ . For this purpose, we compute  $g_0^\leftarrow(t)$  for  $t \in (0, 1)$ . We first easily obtain

$$g(\rho) = \frac{\mathcal{M}_0(\rho)}{M} = [\inf(1, \rho/R)]^2, \quad \rho > 0.$$

Hence, we get  $g_0^\leftarrow(t) = R\sqrt{t}$  for all  $t \in (0, 1)$ . By taking  $t = a^\theta \in (0, 1)$  where  $a = \frac{2M}{3M-8\pi}$ , it follows that

$$T_{c_2}^*(0) = \frac{1}{4 \ln\left(1 + \frac{M-8\pi}{2M}\right)} \cdot \inf_{\theta \in (0, 1)} \left( \frac{[g_0^\leftarrow(a^\theta)]^2}{1 - \theta} \right) = \frac{R^2}{4 \ln\left(1 + \frac{M-8\pi}{2M}\right)} \cdot \inf_{\theta \in (0, 1)} \left( \frac{a^\theta}{1 - \theta} \right).$$

It is easily shown that  $\inf_{\theta \in (0, 1)} \left( \frac{a^\theta}{1 - \theta} \right) = \lim_{\theta \rightarrow 0^+} \left( \frac{a^\theta}{1 - \theta} \right) = 1$  using the fact that  $a \in (2/3, 1)$ , hence  $a \geq e^{-1}$ . Finally, we conclude that

$$T_c^* \leq T_{c_2}^*(0) = \frac{R^2}{4 \ln\left(1 + \frac{M-8\pi}{2M}\right)}.$$

This proves once again the upper bound as in (4.47).

• A second possibility of getting the exact value of  $T_{c_2}^*(z_0)$  is to apply part (1) of Corollary 2.12. As already said, by translation invariance we only need to consider the case  $z_0 = 0$ . From the expression of  $g_0^{\leftarrow}(t) = R\sqrt{t}$  described above, we deduce that  $h(t) := (g_0^{\leftarrow})^2(t) = R^2t$  for  $t \in (0, 1)$ . (Note that this formula is also valid for  $t \in [0, 1]$ ). It is easy to check the sufficient conditions of part (1) of Corollary 2.12. Indeed, the function  $h$  is continuous and  $h' = R^2 > 0$ . The associated function  $F$  is given by  $F(X) = X + 1$  is a non-decreasing. We check that  $F(0^+) = 1 \geq \ln(\frac{1}{a})$  since we always have  $\frac{1}{a} \leq \frac{3}{2} \leq e$ . Thus, the inequality (2.24) of Corollary 2.12 asserts that

$$T_{c_2}^*(0) = \frac{h(1^-)}{4 \ln(1/a)} = \frac{R^2}{4 \ln(1/a)},$$

which is the expected result.

• Again, a third alternative to obtain the upper bound of (4.47) is direct application of (2.18) of Corollary 2.4 because  $n_0$  has compact support. To get the value of  $T_{c_3}^*$  in (2.18), we need to compute  $R_0$  with  $K = \overline{B(0, R)}$ . This can be done as follows. We first prove that  $i_K(z) = |z| + R$  for all  $z \in \mathbb{R}^2$ , and we deduce that  $R_0 = \inf_{z \in K} i_K(z) = i_K(0) = R$ . We apply (2.18) of Corollary 2.4 to finally obtain

$$T^* \leq T_{c_3}^* := \frac{R^2}{4 \ln\left(1 + \frac{M-8\pi}{2M}\right)}.$$

A more heuristic and geometric proof to see that  $R_0 = R$  in (2.18) is to find the smallest closed disk containing  $K = \overline{B(0, R)}$ , the support of  $n_0$ , which is obviously  $K = \overline{B(0, R)}$  itself.

**Remark 4.2.** *The lower and upper bound of  $T_c^*(n_0)$  in (4.47) show that Corollary 2.4 and (1) of Corollary 2.12, and also Theorem 3.1 are sharp for estimating  $T_c^*(n_0)$  for the class of initial data  $n_0$  of characteristic functions of disks (up to universal multiplicative constants). More precisely, each result of Corollary 2.4 or Theorem 3.1 shows the sharpness of each other. A similar remark also holds with part (1) of Corollary 2.12 and Theorem 3.1.*

**Remark 4.3.** *From the upper bound of  $T^*$  by  $T_{c_3}^*$  described just above, we deduce the next asymptotic estimate*

$$T^* \leq T_{c_3}^* \sim \frac{2M^2}{4\pi\sigma(M-8\pi)} \sim \frac{2(16\pi)}{\sigma(M-8\pi)}, \quad M \rightarrow 8\pi^+,$$

for fixed  $\sigma > 0$ . This asymptotic behaviour of  $T_{c_3}^*$  can be compared with the asymptotic behaviour of  $T_c^*(n_0)$  obtained in (4.49) as  $M \rightarrow 8\pi^+$  (for fixed  $\sigma > 0$ ). In this case, it follows immediately that  $T_{c_3}^* \sim 2T_c^*(n_0)$ . Thus, the upper estimate (2.18) of  $T^*$  in Corollary 2.4 is twice greater than the upper estimate of  $T^*$  obtained from  $T_c^*(n_0)$  in (4.49). The same remark holds true with the estimate (4.48) instead of (4.49) when now  $R > 0$  is fixed and  $M \rightarrow 8\pi^+$ . This is deduced from the next asymptotic estimate

$$T_{c_3}^* \sim \frac{4\pi R^2}{M-8\pi}, \quad M \rightarrow 8\pi^+,$$

for fixed  $R > 0$ .

**Remark 4.4.** *For practical purposes, the quantity  $f^{-1}(\frac{2M}{3M-8\pi})$  can certainly be evaluated by numerical approximations for any given mass  $M > 8\pi$ . So, the critical time  $T^*$  can explicitly be bounded using the inequality (4.44).*

We conclude this section with an improvement for the upper bound on  $T_c^*(n_0)$  by applying the inequality (2.29) of Corollary 2.18. Indeed, it is easy to show that the 2-variance of the initial data  $n_0$  is finite and given by  $V_2(n_0) = R^2/2$ . Thus, we have the next result.

**Proposition 4.5.** Let  $n_0 = \sigma \mathbf{1}_{B(z_0, R)}$ , for fixed  $z_0 \in \mathbb{R}^2$  and  $\sigma, R > 0$ . Assume that  $M = \pi\sigma R^2 > 8\pi$ . Then we have

$$(4.50) \quad T^* \leq T_c^*(n_0) \leq T_{c_4}^* = \frac{2^{-1}R^2}{4 \ln \left(1 + \frac{M-8\pi}{2M}\right)}.$$

**Remark 4.6.** The inequality (4.50) clearly improves by half the upper bound of  $T_c^*(n_0)$  in the inequality (4.47) of Proposition 4.1.

**4.3. More examples of initial data.** Many other examples could be presented in this paper, but we shall limit ourselves to a few of them in this last section.

(1) *Initial data uniformly supported by an annulus*

For instance, we can consider the situation where the cells are uniformly distributed on an annulus in the plane corresponding to the following initial data,

$$n_0(x) = \sigma \left( \mathbf{1}_{B(z_0, R_2)}(x) - \mathbf{1}_{B(z_0, R_1)}(x) \right), \quad x \in \mathbb{R}^2,$$

where  $\sigma > 0$  and  $0 < R_1 < R_2$ . For this example, we can prove similarly to the case of the characteristic function of a disk the following result,

$$(4.51) \quad T^* \leq T_c^*(n_0) \leq \frac{1}{4h^{-1} \left( \frac{L(M)}{\sigma\pi} \right)},$$

where  $h^{-1}$  the inverse of the function  $h(s) = s^{-1} \left[ e^{-R_1^2 s} - e^{-R_2^2 s} \right]$ ,  $s > 0$ , and  $L(M) = \frac{2M^2}{3M-8\pi}$ . This example is an example of a  $z_0$ -radially symmetric initial data with compact support but not non-increasing. Similar results as in Proposition 4.1 obtained for the case of a characteristic function of a disk could be given. Let just mention at least two explicit results about  $T_c^*(n_0)$ :

$$e^{-1} \frac{R_2^2 - R_1^2}{4 \ln \left(1 + \frac{M-8\pi}{2M}\right)} \leq T_c^*(n_0) \leq \frac{R_2^2 + R_1^2}{8 \ln \left(1 + \frac{M-8\pi}{2M}\right)}.$$

The upper bound is obtained from an easy computation of the variance  $V_2(n_0)$ . We shall not provide the details here.

Other examples of radially symmetric (but not necessarily non-increasing) initial data  $n_0$  for which the inverse function of the Laplace transform is explicitly computable can be treated by applying Theorem 1.7. For instance, we can consider the following limited list of examples. Recall that  $a > 0$  is defined by the next formula:  $\frac{1}{a} = \frac{M}{L(M)} = 1 + \frac{M-8\pi}{2M}$ .

(2) *Initial data of the polynomial-Gaussian form.* Let  $n_0$  defined by

$$n_0(x) = \sigma |x - z_0|^{2n} e^{-\alpha |x - z_0|^2}, \quad x \in \mathbb{R}^2,$$

where  $n \in \mathbb{N}$ ,  $\sigma, \alpha > 0$  and  $z_0 \in \mathbb{R}^2$ . Let  $M = \sigma\pi n! \alpha^{-n-1} > 8\pi$ . By applying Theorem 1.7, we can prove that

$$(4.52) \quad T^* \leq T_c^*(n_0) \leq (4\alpha)^{-1} \left[ \left( \frac{1}{a} \right)^{\frac{1}{n+1}} - 1 \right]^{-1}.$$

(3) *Initial data obtained by difference of two Gaussian functions.* Let  $z_0 \in \mathbb{R}^2$ ,  $0 < d < b < +\infty$  and  $\sigma > 0$  be fixed. Let

$$n_0(x) = \frac{\sigma}{b-d} \left( e^{-d|x-z_0|^2} - e^{-b|x-z_0|^2} \right), \quad x \in \mathbb{R}^2,$$

and let  $M = \frac{\pi\sigma}{db} > 8\pi$  (i.e.  $\sigma > 8db$ ). By applying Theorem 1.7, we can prove that

$$(4.53) \quad T^* \leq T_c^*(n_0) \leq \left[ 2 \left( \sqrt{(b-d)^2 + \frac{4bd}{a}} - (b+d) \right) \right]^{-1}.$$



As already seen for the other examples treated in this paper, the computation of the 2-variance is rather easy. We have  $V_2(n_0) = \frac{b+d}{bd}$ . By applying Corollary 2.18, we get immediately

$$(4.54) \quad T^* \leq T_c^*(n_0) \leq T_{c_4}^* = \frac{\left(\frac{b+d}{bd}\right)}{4 \ln \left(1 + \frac{M-8\pi}{2M}\right)}.$$

We do not provide here the details of computations for the examples introduced in this short section, nor a detailed comparison of the results that could be deduced from Theorem 3.1 and Corollaries 2.1, 2.4, 2.12, 2.18 to avoid a lengthy paper.

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