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Random interlacements: the discontinuous case

Nathalie Eisenbaum

Abstract: Random interlacements have been originally defined by Sznitman to study the torus disconnection problem of the simple random walk on \mathbb{Z}^d ($d \geq 3$). Later, Sznitman wrote Dynkin type isomorphism theorems connecting random interlacements to Gaussian free fields. These theorems have been then used to handle questions related to Gaussian free fields.

The notion of random interlacements has not been used to study Markov processes other than simple random walks or Brownian motions. The first obstacle is the lack of a general appropriate definition.

Recently with Kaspi, we have extended Sznitman's definition to continuous Markov processes in weak duality. We exploited this definition to extend Sznitman's isomorphism Theorem and to relate random interlacements to quasi-processes. The aim of this note is to relax the assumption of continuous paths and set a proper definition of random interlacements for standard processes. Once this obstacle suppressed, one can in particular enunciate Sznitman's isomorphism theorem in this general framework.

Keywords: Markov process, standard process, local time, Gaussian free field, random interlacements, quasi-process, bivariate Revuz measure.

MSC2020 subject classifications: 60G15; 60J25; 60J55.

1 Introduction

Random interlacements associated to a Markov process $(X_t)_{t \geq 0}$ have been defined by Sznitman [16], [17], in the case when X is a continuous time simple random walk on a transient graph or X is a Brownian motion on \mathbb{R}^d . These processes are continuous and symmetric (continuity for a process living on the vertex set of a graph meaning that it can jump from x to y only if $[x, y]$ is an edge of the graph). First studied for their own interest ([15], [18], [20],..), random interlacements have been then used by Sznitman [16] to establish a Dynkin type isomorphism theorem in the case of continuous random

walk on connected graphs. These identities have been then extended to Brownian motions on the graph \mathbb{Z}^d by Lupu [11]. These tools are useful to handle questions on Gaussian free fields (see e.g. [2]).

Before introducing the contribution of this note, we mention that a preliminary section, section 2, reminds the definition of all the needed classical notions and notation which are used here.

In a previous work ([4],[5]) we have extended Sznitman's definition of random interlacements to a general continuous transient strong Markov process X in weak duality. This was done in order to explore the connection between its extended Markov processes, its Kuznetsov process and its quasi-process. It appeared then that for such a continuous X , random interlacements were linked to its quasi-process. This link was already established by Dereich and Döring [1] in the case of Brownian motion on $\mathbb{Z}^d, d \geq 3$.

Our aim now is to generalize even more the framework and set a definition of random interlacements for X standard process admitting a weak dual. This is done in section 3 where the part played by the quasi-process is shown. In section 4 we clarify the connections between the random interlacements and the extended Markov processes associated to X . In section 5 one presents an extension of Sznitman's isomorphism Theorem to symmetric transient standard processes.

2 Preliminaries

We will make use of the following classical notions. In the sequel (E, \mathcal{E}) will denote a Borel metric space.

Borel right process A Borel right process $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, \mathbb{P}_x; x \in E)$ is such that $(X_t)_{t \geq 0}$ is a stochastic process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with right continuous paths; θ_t denotes the usual shift operator ($X_s \circ \theta_t = X_{t+s}; \forall t, s \geq 0$) and for every x in E , $(X_t)_{t \geq 0}$ satisfies the strong Markov property with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$ (which is augmented and right continuous) under the probability \mathbb{P}_x such that $\mathbb{P}_x(X_0 = x) = 1$.

Denote by $(P_t)_{t \geq 0}$ the semigroup of X , Borel right process. We do not assume that $P_t 1 = 1$. For x in E , one has: $P_t 1(x) = P_t(x, E) = \mathbb{P}_x(X_t \in E)$. Denote by Δ a point outside E and extend P_t to $E \cup \{\Delta\}$ by setting: $P_t(x, \Delta) = 1 - P_t(x, E), \forall x \in E$ and $P_t(\Delta, \Delta) = 1$. That way one obtains a Borel right process on $E \cup \{\Delta\}$ that one still denotes by X .

Life time One defines the life time ζ of X as the nonnegative random variable:

$$\zeta = \inf\{t \geq 0 : X_t = \Delta\}.$$

Its law is given by: $\mathbb{P}_x(\zeta > t) = P_t 1(x), \forall t > 0, \forall x \in E$.

Standard process A standard process $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, \mathbb{P}_x; x \in E)$ is a Borel right process such that for every stopping time T , every x in E , for every increasing

sequence of stopping times $(T_n)_{n \geq 0}$ converging \mathbb{P}_x a.s. to T , then X_{T_n} tends to X_T on $\{T < \zeta\}$ \mathbb{P}_x a.s.

We mention that as a consequence of the definition, the paths of the standard processes are right continuous with left limits.

We also mention that Lévy processes are standard processes.

Weak duality A standard process X is in weak duality with another standard process \hat{X} with respect to a σ -measure ν if both are taking values in (E, \mathcal{E}) and for every \mathcal{E} -measurable nonnegative functions f and g and every $t > 0$:

$$\int_E P_t f(x) g(x) \nu(dx) = \int_E f(x) \hat{P}_t g(x) \nu(dx), \quad (2.1)$$

where $(P_t)_{t \geq 0}$ and $(\hat{P}_t)_{t \geq 0}$ denote the respective semigroups of X and \hat{X} .

Excessive measure A measure m on \mathcal{E} is excessive with respect to a semi-group $(P_t)_{t \geq 0}$ if m is a σ -finite measure on \mathcal{E} such that $mP_t \leq m, \forall t > 0$.

From (2.1), one easily obtains that the measure ν must be excessive for $(P_t)_{t \geq 0}$ and $(\hat{P}_t)_{t \geq 0}$ (i.e. $\nu P_t \leq \nu$ and $\nu \hat{P}_t \leq \nu, \forall t > 0$).

Last visit For a given compact subset B of E , the last visit of X to B is

$$L_B = \sup\{t \geq 0 : X_t \in B\}.$$

When replacing X by \hat{X} one writes \hat{L}_B .

Transient Markov process A Markov process is transient if for any compact subset B of E , one has: $\mathbb{P}_x[L_B < \infty] = 1, \forall x \in E$.

Let \mathcal{W} be the space of paths ω from \mathbb{R} to $E \cup \{\Delta\}$ (with $\Delta \notin E$) which are right continuous with left limits and E valued on some interval $(b(\omega), d(\omega))$ and $\omega(t) = \Delta$ outside this interval ($b(\omega)$ and $d(\omega)$ are called the birth and death times of the path ω). We denote by $(Z_t)_{t \in \mathbb{R}}$ the coordinate process on \mathcal{W} : $Z_t(\omega) = \omega(t)$. We define the σ -fields $\mathcal{G} = \sigma\{Z_t : t \in \mathbb{R}\}$, $\mathcal{G}_t = \sigma\{Z_s : s \leq t\}$, and the shift operators σ_t on \mathcal{W} : $\sigma_t \omega(s) = \omega(t + s), s, t \in \mathbb{R}$. The σ -algebra of (σ_t) shift invariant events in \mathcal{G} will be denoted by \mathcal{A} .

For a path ω in \mathcal{W} , define H_B for any compact subset B of E by

$$H_B = \inf\{t \in (b(\omega), d(\omega)) : \omega(t_-) \in B\}$$

with $\inf \emptyset = +\infty$.

One immediately notes that on $\{H_B < \infty\} : H_B \in [b, d)$.

We will also use λ_B given by

$$\lambda_B = \sup\{t \in (b(\omega), d(\omega)) : \omega(t) \in B\}$$

with $\sup \emptyset = -\infty$.

Given a Borel right process X and an excessive measure ν with respect to its semi-group $(P_t)_{t \geq 0}$, we denote by \mathbf{Q}_ν (resp. \mathbf{P}_ν) the Kuznetsov measure (resp. the quasi-process) on \mathcal{W} associated to $\{\nu, (P_t)_{t \geq 0}\}$. For an introduction to these measures, we recommend [6]. Here is a brief reminder of their definitions.

Kuznetsov measure The Kuznetsov measure \mathbf{Q}_ν on $\mathcal{W} \setminus \{\Delta\}$ is defined by:

$$\begin{aligned} \mathbf{Q}_\nu(Z_{t_1} \in A_1, Z_{t_2} \in A_2, \dots, Z_{t_n} \in A_n) &= \\ & \int_{A_1} \nu(dx_1) \int_{A_2} P_{t_2-t_1}(x_1, dx_2) \dots \int_{A_n} P_{t_n-t_{n-1}}(x_{n-1}, dx_n) \end{aligned} \quad (2.2)$$

for $-\infty < t_1 < t_2 < \dots < t_n < +\infty$ and $A_1, \dots, A_n \in \mathcal{E}$.

Under \mathbf{Q}_ν the coordinate process $(Z_t)_{t \in \mathbb{R}}$ is hence a stationary Markov process with one dimensional distribution at time t equal to ν and transition semigroup $(P_t)_{t \geq 0}$. Since ν is excessive, the measure \mathbf{Q}_ν is unique (see Kuznetsov [10]).

For X in weak duality with \hat{X} with respect to ν , the Kuznetsov measure associated to $\{\nu, (\hat{P}_t)_{t \geq 0}\}$ will be denoted by $\hat{\mathbf{Q}}_\nu$.

Quasi-process The quasi-process \mathbf{P}_ν associated with $\{\nu, (P_t)_{t \geq 0}\}$, is the measure on $(\mathcal{W}, \mathcal{A})$ that is determined by the conditions:

$$\mathbf{P}_\nu \left(\int_{\mathbb{R}} f(Z_t) dt \right) = \nu(f), \quad (2.3)$$

for any nonnegative measurable function f on E , and

$$\begin{aligned} & \text{for any intrinsic stopping time } S, \{Z_{S+t}, t > 0\} \text{ under} \\ & \mathbf{P}_\nu(\cdot; S \in \mathbb{R}) \text{ is Markovian with semigroup } (P_t)_{t \geq 0}, \end{aligned} \quad (2.4)$$

where by intrinsic stopping time, one means a (\mathcal{G}_t) -stopping time that satisfies

$$b \leq S < d \text{ on } \{S < +\infty\}, \text{ and } S = t + S \circ \sigma_t \text{ for all } t \in \mathbb{R}.$$

In the time continuous setting, these measures have been introduced by Weil [19].

The notion of **extended process** associated to a Markov process is not a classical notion and requires a full introduction which will be done in section 4. This notion will be then used in section 4 and section 5.

3 Random interlacements for standard processes

We consider now a transient standard process X in weak duality with another standard process \hat{X} with respect to a σ -measure ν , both taking values in (E, \mathcal{E}) . We denote by $(P_t)_{t \geq 0}$ and $(\hat{P}_t)_{t \geq 0}$ the respective semigroups of X and \hat{X} .

The aim of this section is to set a definition of random interlacements for transient standard processes in weak duality. This definition should be compatible with the existing one for continuous strong Markov processes in weak duality, have a clear meaning and of course be not empty. We first remind the definition of random interlacements set in [5] (the addendum of section 5.3 of [4]) in the case when X and \hat{X} are continuous.

Definition 3.1 *Let X and \hat{X} be two transient continuous Borel processes in weak duality with respect to ν . For $\alpha > 0$ the random interlacements at level α associated to $\{\nu, ((P_t)_{t \geq 0}, (\hat{P}_t)_{t \geq 0})\}$ is a Poisson point process with intensity measure $\alpha \mu_\nu$ where μ_ν is the unique measure on $(\mathcal{W}, \mathcal{A})$ such that $\mu_\nu(\omega \equiv \Delta) = 0$, characterized by the two following properties:*

- for every compact subset B of E :

$$\mu_\nu[\omega(H_B) \in dx, H_B < \infty] = \hat{e}_B(dx) \quad (3.1)$$

where \hat{e}_B is the capacitary (equilibrium) measure of B associated to \hat{X} with respect to ν ;

- for every couple of \mathcal{A} measurable functionals (F_1, F_2)

$$\begin{aligned} \mu_\nu [F_1(\omega(H_B + t), t \geq 0); F_2(\omega((H_B - t), t > 0)); H_B < \infty] \\ = \int_E \hat{e}_B(dx) \mathbb{P}_x[F_1(X_t, t \geq 0)] \hat{\mathbb{P}}_x^B[F_2(Z_t, t > 0)], \end{aligned} \quad (3.2)$$

where for $\hat{e}_B(dx)$ a.e. x , $\hat{\mathbb{P}}_x^B$ is the probability measure on the set of E -valued paths indexed by \mathbb{R}_+ such that

$$\hat{\mathbb{P}}_x^B[F(Z_t, t \geq 0)] = \int_E \nu(dy) \hat{\mathbb{P}}_y[F(\hat{X}_{\hat{L}_B+t}, t \geq 0) | \hat{X}_{\hat{L}_B} = x].$$

To extend the above definition to standard processes, we will use the following notion of **bivariate Revuz measure** of a homogenous random measure.

Let κ be a homogenous random measure for X in the sense of Gettoor and Sharpe [8]. This means that κ is a random measure on \mathbb{R}_+ supported by $(0, \zeta)$ such that for every T stopping time for the natural filtration of X : $\kappa((T, t + T]) = \kappa((0, t]) \circ \theta_T$ a.s. on $\{T < \infty\}$, for every $t > 0$.

There exists a unique measure ϵ on $E \times E$ such that for every nonnegative bounded measurable function F on $E \times E$:

$$\epsilon(F) = \sup_{t>0} \frac{1}{t} \mathbb{E}_\nu \int_{(0,t]} F(X_{s-}, X_s) \kappa(ds) = \lim_{t \rightarrow 0} \frac{1}{t} \mathbb{E}_\nu \int_{(0,t]} F(X_{s-}, X_s) \kappa(ds).$$

Indeed, using the fact that for every such F , $F(X_{s-}, X_s) \kappa(ds)$ is still an homogenous random measure, one can make use of the arguments developed in [8] section 8, and obtain ϵ similarly as they set the definition (8.1) in [8] of the Revuz measure of κ has been set.

One calls ϵ the bivariate Revuz measure of κ with respect to ν .

For any compact subset B of E , define the homogenous random measure κ_B :

$$\kappa_B(ds) = 1_{\{0 < \hat{L}_B < \hat{\zeta}\}} \delta_{\hat{L}_B}(ds),$$

where $\hat{\zeta}$ denotes the life time of \hat{X} .

There exists then a unique measure $\hat{\epsilon}_B$ on $E \times E$ such that for every measurable bounded nonnegative function F on $E \times E$

$$\begin{aligned} \hat{\epsilon}_B(F) &= \sup_{t>0} \frac{1}{t} \hat{\mathbb{E}}_\nu \int_{(0,t]} F(\hat{X}_{s-}, \hat{X}_s) \kappa_B(ds) = \lim_{t \rightarrow 0} \frac{1}{t} \hat{\mathbb{E}}_\nu \int_{(0,t]} F(\hat{X}_{s-}, \hat{X}_s) \kappa_B(ds) \\ &= \lim_{t \rightarrow 0} \frac{1}{t} \int_E \nu(dx) \hat{\mathbb{P}}_x[F(\hat{X}_{\hat{L}_B-}, \hat{X}_{\hat{L}_B}), 0 < \hat{L}_B \leq t]. \end{aligned}$$

The measure $\hat{\epsilon}_B$ is the bivariate Revuz measure of the homogenous random measure κ_B with respect to ν .

Note that the first marginal of $\hat{\epsilon}_B$ is $\hat{\epsilon}_B$ the capacitary measure of B for \hat{X} with respect to ν . We denote the second marginal of $\hat{\epsilon}_B$ by $\hat{\epsilon}_B^+$.

The notion of bivariate Revuz measure has been first set by Sharpe in [14] for additive functionals under the additional assumption that ν is a reference measure. Our framework assumes that the two semigroups $(P_t)_{t \geq 0}$ and $(\hat{P}_t)_{t \geq 0}$ are in weak duality. Their respective potentials are not required to have densities with respect to ν .

Definition 3.2 *Let X be a transient standard process in weak duality with \hat{X} with respect to a σ -finite measure ν . Let $(P_t)_{t \geq 0}$ and $(\hat{P}_t)_{t \geq 0}$ be the respective semigroups of X and \hat{X} . For $\alpha > 0$, the random interlacements at level α associated to $\{\nu, ((P_t)_{t \geq 0}, (\hat{P}_t)_{t \geq 0})\}$ is a Poisson point process with intensity measure $\alpha \mu_\nu$, where μ_ν is the unique measure on $(\mathcal{W}, \mathcal{A})$ characterized by the three following properties:*

- for any compact subset B of E

$$\mu_\nu[\omega(H_B) \in dx, \omega(H_{B-}) \in dy, H_B < \infty] = \hat{\epsilon}_B(dxdy) \quad (3.3)$$

where $\hat{\epsilon}_B$ is the bivariate Revuz measure of κ_B associated to \hat{X} with respect to ν ;

- for every couple of \mathcal{A} measurable functionals (F_1, F_2)

$$\begin{aligned} \mu_\nu & [F_1(\omega(H_B + t), t \geq 0); F_2(\omega((H_B - t)_-, t > 0); H_B < \infty] \quad (3.4) \\ & = \int_{E \times E} \hat{\epsilon}_B(dx dy) \mathbb{P}_x[F_1(X_t, t \geq 0)] \hat{\mathbb{P}}_y^B[F_2(Z_t, t > 0)], \end{aligned}$$

where for $\hat{\epsilon}_B^+(dy)$ a.e. y , $\hat{\mathbb{P}}_y^B$ is the probability measure on the set of E -valued paths indexed by \mathbb{R}_+ such that

$$\hat{\mathbb{P}}_y^B[F(Z_t, t \geq 0)] = \int_E \nu(dz) \hat{\mathbb{P}}_z[F(\hat{X}_{\hat{L}_B+t}, t \geq 0) | \hat{X}_{\hat{L}_B} = y];$$

- and

$$\mu_\nu(\omega \equiv \Delta) = 0. \quad (3.5)$$

The measure $\hat{\mathbb{P}}_y^B$ does not depend on ν . This has been mentioned in the continuous case [5], and we will emphasize this fact in the proof of (3.6) below.

As in the continuous case, uniqueness of μ_ν can be obtained with simple arguments. Indeed let μ_1 and μ_2 be two measures on \mathcal{A} satisfying (3.3) and (3.4) for every compact subset B of E . We compute $\mu_1(A)$ for A in \mathcal{A} . We use a sequence of compact subsets $(B_n)_{n \geq 1}$ increasing to E . For $i = 1, 2$, one has:

$$\mu_i(A) = \mu_i(A; H_{B_1} = \infty) + \mu_i(A; H_{B_1} < \infty)$$

By (3.4), and monotone class argument: $\mu_1(A; H_{B_1} < \infty) = \mu_2(A; H_{B_1} < \infty)$. One writes then:

$$\mu_i(A; H_{B_1} = \infty) = \mu_i(A; H_{B_1} = \infty, H_{B_2} = \infty) + \mu_i(A; H_{B_1} = \infty, H_{B_2} < \infty),$$

and keep splitting the probabilities $\mu_i(A), i = 1, 2$ by using the sequence $(B_n)_{n \geq 1}$. By (3.5): $\mu_i(H_{B_n} = \infty, \forall n) = 0, i = 1, 2$. Hence we end up with two series with identical general term, which leads to $\mu_1(A) = \mu_2(A)$.

Note that if the measure μ_ν exists then, using both (3.5) and (3.4), it must satisfy for every compact B

$$\mu_\nu(H_B = b, H_B < \infty) = 0.$$

The following theorem establishes the existence of μ_ν .

Theorem 3.3 *Let X and \hat{X} be two transient standard processes in weak duality with respect to a σ -finite reference measure ν . Then we have*

$$\mu_\nu = \mathbf{P}_\nu, \quad (3.6)$$

where \mathbf{P}_ν is the quasi-process associated with $\{\nu, (P_t)_{t \geq 0}\}$.

In case X has continuous paths, (3.6) has been already established (Theorem 5.1 in [4],[5]). In the special example when X is a Brownian motion on \mathbb{R}^d , $d \geq 3$, (3.6) has been previously established by Dereich and Döring [1].

Proof: It suffices to show that \mathbf{P}_ν satisfies (3.3), (3.4) and (3.5).

First note that since X is transient, the excessive measure ν is dissipative for $(P_t)_{t \geq 0}$ (see e.g. Remark 4.9 c in [7]). This means that for any nonnegative function f such that $\int_E f(x)\nu(dx) < \infty$, one has: $\int_0^\infty P_t f dt < \infty$ ν a.e.

As noticed in [6] section 3, since ν is dissipative then \mathbf{Q}_ν , the Kuznetsov measure associated to $\{\nu, (P_t)_{t \geq 0}\}$, must be dissipative. This means that for any nonnegative \mathcal{G} -measurable F such that $\mathbf{Q}_\nu(F) < \infty$, one has: $\mathbf{Q}_\nu(\int_{\mathbb{R}} F \circ \sigma_t dt = \infty) = 0$. Using (Proposition 2.7 in [6]), one hence obtains that there exists a stationary time S^* such that $\mathbf{Q}_\nu(S^* \notin \mathbb{R}) = 0$.

We remind that a \mathcal{G} -measurable random time $S^* : \mathcal{W} \rightarrow [-\infty, \infty]$ is said to be stationary if it satisfies $S^* = t + S^* \circ \sigma_t$ for all $t \in \mathbb{R}$.

Following Fitzsimmons [6], the measure \mathbf{P}_ν can hence be defined by

$$\mathbf{P}_\nu(A) = \mathbf{Q}_\nu[A; 0 < S^* \leq 1]$$

for every A in \mathcal{A} the σ -field of invariants elements of \mathcal{G} .

More generally, using (2.1) and (2.3) in [6], one obtains for every $t > 0$

$$\mathbf{P}_\nu(A) = \frac{1}{t} \mathbf{Q}_\nu[A; 0 < S^* \leq t] = \frac{1}{t} \mathbf{Q}_\nu[A; -t \leq S^* < 0]. \quad (3.7)$$

Since by definition: $\mathbf{Q}_\nu(\omega \equiv \Delta) = 0$, (3.7) leads to $\mathbf{P}_\nu(\omega \equiv \Delta) = 0$. Hence \mathbf{P}_ν satisfies (3.5).

Fix a compact set B . We first remind that since \hat{X} is transient: $\mathbf{Q}_\nu(H_B = b) = 0$ (see [8] (13.12)). Thanks to (3.7) one obtains: $\mathbf{P}_\nu(H_B \notin (b, d), H_B < \infty) = 0$.

For any F real-valued measurable bounded function on $E \times E$, one has for every $t > 0$:

$$\begin{aligned} \mathbf{P}_\nu[F(Z_{H_B}, Z_{H_B-}), H_B < \infty] &= \frac{1}{t} \mathbf{Q}_\nu[F(Z_{H_B}, Z_{H_B-}), H_B < \infty, -t \leq S^* < 0] \\ &= \frac{1}{t} \mathbf{Q}_\nu[F(Z_{H_B}, Z_{H_B-}), -t \leq H_B < 0] \end{aligned}$$

thanks to the switching property of Proposition 2.4 in [6]. Using then the duality properties of \mathbf{Q}_ν (see [12]):

$$\hat{\mathbf{Q}}_\nu[F(Z_t, t \in \mathbb{R})] = \mathbf{Q}_\nu[F(\hat{Z}_t, t \in \mathbb{R})], \quad (3.8)$$

where $\hat{Z}_t = Z_{(-t)-}$, $t \in \mathbb{R}$, one obtains for every $t > 0$:

$$\mathbf{P}_\nu[F(Z_{H_B}, Z_{H_B-}), H_B < \infty] = \frac{1}{t} \hat{\mathbf{Q}}_\nu[F(Z_{\lambda_B-}, Z_{\lambda_B}), 0 < \lambda_B \leq t]. \quad (3.9)$$

On one hand for every nonnegative function f , one has:

$$\begin{aligned}\hat{e}_B(f) &= \lim_{t \rightarrow 0} \frac{1}{t} \int_E \nu(dx) \hat{\mathbb{P}}_x[f(\hat{X}_{\hat{L}_B-}), 0 < \hat{L}_B \leq t] \\ &= \lim_{t \rightarrow 0} \frac{1}{t} \hat{\mathbf{Q}}_\nu[f(Z_{\lambda_B-}), b < 0 < \lambda_B \leq t].\end{aligned}\quad (3.10)$$

On the other hand, making use of (13.7) in [8], one knows that:

$$\hat{e}_B(f) = \frac{1}{t} \hat{\mathbf{Q}}_\nu[f(Z_{\lambda_B-}), 0 < \lambda_B \leq t, \lambda_B > b].\quad (3.11)$$

Moreover on $\{\lambda_B \in (0, t]\}$: $\lambda_B \in (b, d]$, and \hat{X} being transient one has: $\hat{\mathbf{Q}}_\nu[\lambda_B = d] = 0$ (Proposition 13.6 in [8]). Using these last remarks together with (3.10), (3.11) one finally obtains:

$$\lim_{t \rightarrow 0} \frac{1}{t} \hat{\mathbf{Q}}_\nu[0 \leq b < \lambda_B \leq t] = 0.\quad (3.12)$$

Starting from (3.9), one writes for every $t > 0$:

$$\begin{aligned}\mathbf{P}_\nu[F(Z_{H_B}, Z_{H_B-}), H_B < \infty] &= \frac{1}{t} \hat{\mathbf{Q}}_\nu[F(Z_{\lambda_B-}, Z_{\lambda_B}), b < 0 < \lambda_B \leq t] \\ &\quad + \frac{1}{t} \hat{\mathbf{Q}}_\nu[F(Z_{\lambda_B-}, Z_{\lambda_B}), 0 \leq b < \lambda_B \leq t],\end{aligned}$$

and let then t tend to 0 to obtain thanks to (3.12):

$$\begin{aligned}\mathbf{P}_\nu[F(Z_{H_B}, Z_{H_B-}), H_B < \infty] &= \lim_{t \rightarrow 0} \frac{1}{t} \hat{\mathbf{Q}}_\nu[F(Z_{\lambda_B-}, Z_{\lambda_B}), b < 0 < \lambda_B \leq t] \\ &= \lim_{t \rightarrow 0} \frac{1}{t} \int_E \nu(dx) \hat{\mathbb{P}}_x[f(\hat{X}_{\hat{L}_B-}), 0 < \hat{L}_B \leq t] \\ &= \hat{e}_B(F)\end{aligned}$$

which establishes (3.3) for \mathbf{P}_ν .

We check now that \mathbf{P}_ν satisfies (3.4).

$$\begin{aligned}\mathbf{P}_\nu [F_1(Z(H_B + t), t \geq 0); F_2(Z((H_B - t)_-), t > 0); H_B < \infty] &\quad (3.13) \\ &= \mathbf{Q}_\nu[F_1(Z(H_B + t), t \geq 0); F_2(Z((H_B - t)_-), t > 0); H_B < \infty; -1 \leq S^* < 0] \\ &= \mathbf{Q}_\nu[F_1(Z(H_B + t), t \geq 0); F_2(Z((H_B - t)_-), t > 0); -1 \leq H_B < 0].\end{aligned}$$

The random time H_B is a stopping time for \mathbf{Q}_ν (i.e. for every $t \in \mathbb{R}$, $\{H_B \leq t\} \in \mathcal{G}_t$). It is an intrinsic stopping time in the sense that on $\{H_B < \infty\}$: $b \leq H_B < d$ and $H_B = t + H_B \circ \sigma_t$, $\forall t \in \mathbb{R}$. This implies in particular that for every t : Z_{H_B+t} is \mathcal{A} -measurable, and hence so is $(Z_{(H_B-t)_-}, t \geq 0)$.

Using Proposition 2.4 (2.5) in [6], one has for every $t > 0$

$$\begin{aligned} & \frac{1}{t} \mathbf{Q}_\nu [F_1(Z(H_B + t), t \geq 0); F_2(Z((H_B - t)_-), t > 0); -t \leq H_B < 0] \quad (3.14) \\ & = \mathbf{Q}_\nu [F_1(Z(H_B + t), t \geq 0); F_2(Z((H_B - t)_-), t > 0); -1 \leq H_B < 0] \end{aligned}$$

Under \mathbf{Q}_ν , one has Markov property at time H_B , under the following form (see e.g. (10.12) in [8])

$$\begin{aligned} & \mathbf{Q}_\nu [F_1(Z(H_B + t), t \geq 0); F_2(Z((H_B - t)_-), t > 0); -1 \leq H_B < 0] \\ & = \mathbf{Q}_\nu [-1 \leq H_B < 0, \mathbb{P}_{Z(H_B)} [F_1(X_s, s \geq 0)] F_2(Z((H_B - t)_-), t > 0)], \end{aligned}$$

which leads to

$$\begin{aligned} & \mathbf{Q}_\nu [F_1(Z(H_B + t), t \geq 0); F_2(Z((H_B - t)_-), t > 0); -1 \leq H_B < 0] \\ & = \int_E \hat{e}_B(dx) \mathbb{P}_x [F_1(X_s, s \geq 0)] \\ & \quad \mathbf{Q}_\nu [-1 \leq H_B < 0, F_2(Z((H_B - t)_-), t > 0) | Z(H_B) = x] \\ & = \int_E \hat{e}_B(dx) \mathbb{P}_x [F_1(X_s, s \geq 0)] \\ & \quad \hat{\mathbf{Q}}_\nu [0 < \lambda_B \leq 1, F_2(Z((\lambda_B + t)), t > 0) | Z(\lambda_B -) = x] \\ & \quad \text{(thanks to (3.8))} \\ & = \int_E \hat{e}_B(dx) \mathbb{P}_x [F_1(X_s, s \geq 0)] \int_E \hat{\mathbf{Q}}_\nu [0 < \lambda_B \leq 1, Z(\lambda_B) \in dy | Z(\lambda_B -) = x] \\ & \quad \hat{\mathbf{Q}}_\nu [0 < \lambda_B \leq 1, F_2(Z((\lambda_B + t)), t > 0) | Z(\lambda_B -) = x, Z(\lambda_B) = y] \\ & = \int_{E \times E} \hat{e}(dxdy) \mathbb{P}_x [F_1(X_s, s \geq 0)] \\ & \quad \hat{\mathbf{Q}}_\nu [0 < \lambda_B \leq 1, F_2(Z((\lambda_B + t)), t > 0) | Z(\lambda_B -) = x, Z(\lambda_B) = y], \quad (3.15) \end{aligned}$$

using (3.3) for \mathbf{P}_ν .

Now, using both (3.14) and (3.8), we have for every $t > 0$

$$\begin{aligned} & \hat{\mathbf{Q}}_\nu [0 < \lambda_B \leq 1, F_2(Z((\lambda_B + s)), s > 0) f_1(Z(\lambda_B -)) f_2(Z(\lambda_B))] \\ & = \frac{1}{t} \hat{\mathbf{Q}}_\nu [0 < \lambda_B \leq t, F_2(Z((\lambda_B + s)), s > 0) f_1(Z(\lambda_B -)) f_2(Z(\lambda_B))] \\ & = \frac{1}{t} \hat{\mathbf{Q}}_\nu [b < 0 < \lambda_B \leq t, F_2(Z((\lambda_B + s)), s > 0) f_1(Z(\lambda_B -)) f_2(Z(\lambda_B))] \\ & \quad + \frac{1}{t} \hat{\mathbf{Q}}_\nu [0 \leq b < \lambda_B \leq t, F_2(Z((\lambda_B + s)), s > 0) f_1(Z(\lambda_B -)) f_2(Z(\lambda_B))] \end{aligned}$$

Letting then t tend to 0 and using (3.12) one obtains:

$$\begin{aligned} & \hat{\mathbf{Q}}_\nu [0 < \lambda_B \leq 1, F_2(Z((\lambda_B + s)), s > 0) f_1(Z(\lambda_B -)) f_2(Z(\lambda_B))] \quad (3.16) \\ & = \lim_{t \rightarrow 0} \frac{1}{t} \hat{\mathbf{Q}}_\nu [b < 0 < \lambda_B \leq t, F_2(Z((\lambda_B + s)), s > 0) f_1(Z(\lambda_B -)) f_2(Z(\lambda_B))] \\ & = \lim_{t \rightarrow 0} \frac{1}{t} \int_E \hat{\mathbb{P}}_\nu [0 < \hat{L}_B \leq t, F_2(\hat{X}((\hat{L}_B + s)), s > 0) f_1(\hat{X}(\hat{L}_B -)) f_2(\hat{X}(\hat{L}_B))] \end{aligned}$$

We use now the splitting property of the random time \hat{L}_B to claim that

$$\begin{aligned} \hat{\mathbb{P}}_\nu [0 < \hat{L}_B \leq t, F_2(\hat{X}(\hat{L}_B + s)), s > 0) f_1(\hat{X}(\hat{L}_B -)) f_2(\hat{X}(\hat{L}_B))] \\ = \hat{\mathbb{P}}_\nu[0 < \hat{L}_B \leq t, \Gamma(\hat{X}_{\hat{L}_B}, F_2) f_1(\hat{X}(\hat{L}_B -)) f_2(\hat{X}(\hat{L}_B))], \end{aligned}$$

where $(\Gamma(x, A), x \in E, A \in \mathcal{F})$ is a Markov kernel (see Theorem 2.12 in [9] for a complete description of Γ) independent of ν .

From (3.16), one hence obtains:

$$\begin{aligned} \hat{\mathbf{Q}}_\nu[0 < \lambda_B \leq 1, F_2(Z((\lambda_B + s)), s > 0) f_1(Z(\lambda_B -)) f_2(Z(\lambda_B))] \\ = \epsilon_B(g) \end{aligned}$$

where g denotes the function on $E \times E$ defined by $g(x, y) = f_1(x) f_2(y) \Gamma(y, F_2)$.

This implies that $\hat{e}_B^+(dy)$ a.e. y :

$$\begin{aligned} \hat{\mathbf{Q}}_\nu[0 < \lambda_B \leq 1, F_2(Z((\lambda_B + s)), s > 0) | Z(\lambda_B -) = x, Z(\lambda_B) = y] \quad (3.17) \\ = \Gamma(y, F_2) \\ = \int_E \nu(dz) \hat{\mathbb{P}}_z[F_2(\hat{X}_{\hat{L}_B + s}, s > 0) | \hat{X}_{\hat{L}_B} = y], \end{aligned}$$

which is hence independent of ν .

Coming back to (3.15) and using (3.13) one finally obtains:

$$\begin{aligned} \mathbf{P}_\nu [F_1(Z(H_B + t), t \geq 0); F_2(Z((H_B - t) -), t > 0); H_B < \infty] \\ = \int_{E \times E} \hat{e}(dx dy) \mathbb{P}_x[F_1(X_t, t \geq 0)] \hat{\mathbb{P}}_y^B[F_2(Z_t, t > 0)]. \end{aligned}$$

One concludes that \mathbf{P}_ν satisfies (3.4). \square

Suppose now that the excessive measure ν is purely excessive, which means that

$$\int_E \nu(dx) P_t 1(x) \rightarrow_{t \rightarrow \infty} 0. \quad (3.18)$$

In this case μ_ν can be connected with another measure on \mathcal{A} the σ -field generated by the shift invariant sets of \mathcal{W} . More precisely when ν is purely excessive there exists a family of measures on \mathcal{E} , $(m_t)_{t > 0}$, such that

$$\nu(f) = \int_0^\infty m_t(f) dt \quad \text{and} \quad m_s P_{t-s} = m_t, \quad 0 < s < t.$$

One says that $(m_t)_{t > 0}$ is an entrance law associated to ν (see [3]). Define P_* by

$$P_*(Z_{t_1} \in A_1, \dots, Z_{t_n} \in A_n) = \int_{A_1} m_{t_1}(dx_1) \cdots \int_{A_n} P_{t_n - t_{n-1}}(x_{n-1}, dx_n) \quad (3.19)$$

for $0 < t_1 \leq \dots \leq t_n$ and extend the definition to negative t_i 's by setting $m_s = 0$ for $s < 0$.

We have shown in [4] ((5.10) in [4]) that for every $A \in \mathcal{A}$: $P_*(A) = \mathbf{P}_\nu(A)$. This leads to the following corollary.

Corollary 3.4 *Let X and \hat{X} be two transient standard processes in weak duality with respect to a purely excessive measure ν . Then we have*

$$\mu_\nu = \mathbf{P}_\nu = P_{*|\mathcal{A}}.$$

The interest of Corollary 3.4 relies on an interpretation of the measure P_* which can be done under some additional assumptions. This is the object of the next section.

4 Interpretation of random interlacements

We have noticed in [4] (Remark 2.2) that a sufficient condition for ν to be purely excessive (3.18) is ν finite excessive and the life time ζ of X is finite \mathbb{P}_x a.s. for every x in E . The interest of Corollary 3.4 relies on a peculiar interpretation of the measure P_* under some additional assumptions. Indeed, assume that the three following properties are satisfied

- (i) ν is finite and excessive;
- (ii) the life time ζ of X is finite \mathbb{P}_x a.s. $\forall x \in E$;
- (iii) ζ has no atom \mathbb{P}_x a.s. $\forall x \in E$

and let δ be a point outside E . We have shown in [4] that then there exists a Markov process $(Y_t)_{t \geq 0}$ on $E \cup \{\delta\}$ admitting δ as a recurrent point such that for every a in E :

$$((X_t, t < \zeta) | X_0 = a) \stackrel{(\text{law})}{=} ((Y_t, t < T_\delta) | Y_0 = a),$$

where $T_\delta = \inf\{s \geq 0 : Y_s = \delta\}$. Moreover the excursion process of Y with respect to δ is a Poisson point process with intensity $dt \times P_*$. One says that the process Y is an **extended Markov process** of X , in short an *extended X* . Thanks to Corollary 3.4 one can hence interpret the random interlacements at level α as the excursion process of Y from δ modulo time-shift up to the first time the local time at δ exceeds α .

In case the set of the three above properties (i)-(iii) is not satisfied, one can still make a connection between the random interlacements of X and the excursion process of some Markov process on $E \cup \{\delta\}$ by assuming that X has finite 0-potential densities with respect to ν . Indeed we have shown ([4], Corollary 2.4) that under this assumption there exists a positive function q on E such that the measure $\nu \cdot q$ is finite (where $\nu \cdot q(dx) = q(x)\nu(dx)$) and that there exists a Markov process $(Y_t)_{t \geq 0}$ with state space $E \cup \{\delta\}$ admitting a local time process $(L_t^x(Y), x \in E, t \geq 0)$ with respect to $\nu \cdot q$ and δ as recurrent point, satisfying for every a in E :

$$((L_\infty^x(X), x \in E) | X_0 = a) \stackrel{(\text{law})}{=} ((L_{T_\delta}^x(Y), x \in E) | Y_0 = a),$$

where $(L_\infty^x(X), x \in E)$ denotes the total accumulated local time process of X with respect to ν .

We still say that the process Y is an extended Markov process of X , or an extended X . Note that Y depends on the choice of q but that this choice does not affect the law of $(L_{T_\delta}^x(Y), x \in E)$.

Following exactly the same sequence of arguments as for the proof of Proposition 5.3 in [4], one obtains the following proposition. Note that as soon as X has finite 0-potential densities with respect to ν then necessarily ν is a reference measure for $(P_t)_{t \geq 0}$.

Proposition 4.1 *Let X be a transient standard process with semigroup $(P_t)_{t \geq 0}$ in weak duality with respect to a σ -measure ν with a standard process with semigroup $(\hat{P}_t)_{t \geq 0}$. Assume that X has finite 0-potential densities with respect to ν . Then the field of occupation times of the random interlacement at level α associated to $\{\nu, ((P_t)_{t \geq 0}, (\hat{P}_t)_{t \geq 0})\}$ equals in law the local time process of an extended X at the first time its local time at δ exceeds α .*

5 Illustrations

5.1 Lévy processes

Lévy processes are standard processes. Besides a Lévy process X on \mathbb{R}^d is in weak duality with $(-X)$ with respect to the Lebesgue measure on \mathbb{R}^d . Obviously the Lebesgue measure on \mathbb{R}^d is not finite.

In case the Lévy process X is transient, according to Definition 3.2 and Theorem 3.3, the random interlacements at level α associated to X is the Poisson point process with intensity measure $\alpha \mathbf{P}_\nu$ (ν here is the Lebesgue measure on \mathbb{R}^d).

If moreover X has finite 0-potential densities then X admits an extended Markov process Y on $\mathbb{R}^d \cup \{\delta\}$ such that for every $\alpha > 0$, the occupation field of the random interlacements of X at level α equals in law the local time process of Y at the first time the local time at δ exceeds α .

In [13], Rosen defines directly the random interlacements at level α as the Poisson point process with intensity measure $\alpha \mathbf{P}_\nu$. He considers then exclusively the case when X is a symmetric transient \mathbb{R}^d -valued Lévy process and does not admit local times. Theorem 3.3 ensures that this Poisson point process coincides with the one given by Definition 3.2 but the interpretation provided by Proposition 4.1 is not available in his framework.

5.2 Extension of Sznitman's isomorphism Theorem

For any transient standard process X with finite symmetric 0-potential densities $(u(x, y), (x, y) \in E \times E)$ with respect to a σ -measure ν , one has obtained (Theorem 3.1 in [4]) the following isomorphism theorem for Y , an extended X . Under $\mathbb{P}[\cdot | Y_0 = \delta]$,

one has:

$$\left(\frac{1}{2}\eta_x^2 + L_{\tau_\alpha}^x(Y), x \in E\right) \stackrel{(\text{law})}{=} \left(\frac{1}{2}(\eta_x + \sqrt{2\alpha})^2, x \in E\right), \quad (5.1)$$

where $(L_{\tau_\alpha}^x(Y), x \in E)$ is the local time process (with respect to $\nu \cdot q$) of Y , at the first time the local time of Y at δ exceeds the value α , and $(\eta_x, x \in E)$ is a centered Gaussian process with covariance $(u(x, y), (x, y) \in E \times E)$, independent of Y . According to Proposition 4.1, one can rewrite (5.1) under the following form:

$$\left(\frac{1}{2}\eta_x^2 + L_{x,\alpha}, x \in E\right) \stackrel{(\text{law})}{=} \left(\frac{1}{2}(\eta_x + \sqrt{2\alpha})^2, x \in E\right), \quad (5.2)$$

where $(L_{x,\alpha}, x \in E)$ is the occupation time of the random interlacements at level α of X .

The identity (5.2) represents an extension to the discontinuous case of Sznitman's isomorphism Theorem for random interlacements.

It holds in particular for any symmetric transient Lévy process admitting local times.

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