



**HAL**  
open science

## Random interlacements: the discontinuous case

Nathalie Eisenbaum

► **To cite this version:**

| Nathalie Eisenbaum. Random interlacements: the discontinuous case. 2023. hal-04160459v1

**HAL Id: hal-04160459**

**<https://cnrs.hal.science/hal-04160459v1>**

Preprint submitted on 12 Jul 2023 (v1), last revised 22 Aug 2024 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution - NonCommercial - NoDerivatives 4.0 International License

# Random interlacements: the discontinuous case

Nathalie Eisenbaum

**Abstract:** Random interlacements have been originally defined by Sznitman for continuous time simple reversible transient random walks on graphs. Among their features, they are necessary to write Dynkin type isomorphism theorems involving Gaussian free fields. Recently we have extended Sznitman's definition to continuous Markov processes in weak duality and established the connections between their random interlacements, quasi-processes and extended Markov processes. Here we relax the assumption of continuous paths and show that this connections remain valid.

**Keywords:** Markov process, local time, Gaussian free field, random interlacements, quasi-process, bivariate Revuz measure.

**MSC2020 subject classifications:** 60G15; 60J25; 60J55.

## 1 Introduction

Random interlacements associated to a Markov process  $(X_t)_{t \geq 0}$  have been defined by Sznitman [16], [17], in the case when  $X$  is a continuous (or discrete) time simple random walk on a transient graph or  $X$  is a Brownian motion on  $\mathbb{R}^d$ . This processes are continuous and symmetric (continuity for a process living on the vertex set of a graph meaning that it can jump from  $x$  to  $y$  only if  $[x, y]$  is an edge of the graph). First studied for their own interest ([15], [18], [20],...), random interlacements have been then used by Sznitman [16] to establish a Dynkin type isomorphism theorem in the case of continuous random walk on connected graphs. This identities have been then extended to Brownian motions on the graph  $\mathbb{Z}^d$  by Lupu [11]. This tools are useful to handle questions on Gaussian free fields (see e.g. [2]).

In a previous work [4] we have extended Sznitman's definition of random interlacements definition to a general continuous transient Borel process  $X$  in weak duality with respect to a  $\sigma$ -finite measure  $\nu$ . This was done in order to explore the connection between its extended Markov processes (whose definition is reminded in section 3), its Kuznetsov

process and its quasi-process (definitions reminded in section 2). It appeared then that for such a continuous  $X$ , random interlacements were linked to its quasi-process. This link was already established by Dereich and Döring [1] in the case of Brownian motion on  $\mathbb{Z}^d, d \geq 3$ .

Our aim now is to enlarge even more the framework and set a definition of random interlacements for  $X$  standard process admitting a weak dual. This is done in section 2 where the part played by the quasi-process is shown. In section 3 we clarify the connections between the random interlacements and the extended Markov processes associated to  $X$ . In section 4 one then extends Sznitman's isomorphism Theorem to symmetric transient standard processes.

## 2 Random interlacements for standard processes

We consider a transient standard process  $X$  in weak duality with another standard process  $\hat{X}$  with respect to a measure  $\nu$ . Both processes are taking values in  $(E, \mathcal{E})$  Borel metric space. We denote by  $(P_t)_{t \geq 0}$  and  $(\hat{P}_t)_{t \geq 0}$  the respective semigroups of  $X$  and  $\hat{X}$ . The weak duality translates into:

$$\int_E P_t f(x) g(x) \nu(dx) = \int_E f(x) \hat{P}_t g(x) \nu(dx). \quad (2.1)$$

for every  $\mathcal{E}$ -measurable nonnegative functions  $f$  and  $g$  and every  $t > 0$

From (2.1), one easily obtains that the measure  $\nu$  must be excessive for  $(P_t)_{t \geq 0}$  and  $(\hat{P}_t)_{t \geq 0}$  (i.e.  $\nu P_t \leq \nu$  and  $\nu \hat{P}_t \leq \nu, \forall t > 0$ ).

For a given compact subset  $B$  of  $E$ , define

$$L_B = \sup\{t \geq 0 : X_t \in B\}.$$

When replacing  $X$  by  $\hat{X}$  one defines  $\hat{L}_B$ .

Let  $\mathcal{W}$  be the space of paths  $\omega$  from  $\mathbb{R}$  to  $E \cup \{\Delta\}$  which are right continuous with left limits and  $E$  valued on some interval  $(b(\omega), d(\omega))$  and  $\omega(t) = \Delta$  outside this interval ( $b(\omega)$  and  $d(\omega)$  are called the birth and death times of the path  $\omega$ ). We denote by  $(Z_t)_{t \in \mathbb{R}}$  the coordinate process on  $\mathcal{W}$ :  $Z_t(\omega) = \omega(t)$ . We define the  $\sigma$ -fields  $\mathcal{G} = \sigma\{Z_t : t \in \mathbb{R}\}$ ,  $\mathcal{G}_t = \sigma\{Z_s : s \leq t\}$ , and the shift operators  $\sigma_t$  on  $\mathcal{W}$ :  $\sigma_t \omega(s) = \omega(t + s), s, t \in \mathbb{R}$ . The  $\sigma$ -algebra of  $(\sigma_t)$  shift invariant events in  $\mathcal{G}$  will be denoted by  $\mathcal{A}$ .

We denote by  $\mathbf{Q}_\nu$  (resp.  $\mathbf{P}_\nu$ ) the Kuznetsov measure (resp. the quasi-process) on  $\mathcal{W}$  associated to  $\{\nu, (P_t)_{t \geq 0}\}$ . The Kuznetsov measure associated to  $\{\nu, (\hat{P}_t)_{t \geq 0}\}$  is denoted by  $\hat{\mathbf{Q}}_\nu$ . For an introduction to this measures, we recommend [6]. Here is a brief reminder of their definitions.

The *Kuznetsov measure*  $\mathbf{Q}_\nu$  on  $\mathcal{W} \setminus \{\Delta\}$  is defined by:

$$\begin{aligned} \mathbf{Q}_\nu(Z_{t_1} \in A_1, Z_{t_2} \in A_2, \dots, Z_{t_n} \in A_n) &= \\ & \int_{A_1} \nu(dx_1) \int_{A_2} P_{t_2-t_1}(x_1, dx_2) \dots \int_{A_n} P_{t_n-t_{n-1}}(x_{n-1}, dx_n) \end{aligned} \quad (2.2)$$

for  $-\infty < t_1 < t_2 < \dots < t_n < +\infty$  and  $A_1, \dots, A_n \in \mathcal{E}$ .

Under  $\mathbf{Q}_\nu$  the coordinate process  $(Z_t)_{t \in \mathbb{R}}$  is hence a stationary Markov process with one dimensional distribution at time  $t$  equal to  $\nu$  and transition semigroup  $(P_t)_{t \geq 0}$ . Since  $\nu$  is excessive, the measure  $\mathbf{Q}_\nu$  is unique (see Kuznetsov [10]).

The *quasi-process*  $\mathbf{P}_\nu$  associated with  $\{\nu, (P_t)_{t \geq 0}\}$ , is the measure on  $(\mathcal{W}, \mathcal{A})$  that is determined by the conditions:

$$\mathbf{P}_\nu \left( \int_{\mathbb{R}} f(Z_t) dt \right) = \nu(f), \quad (2.3)$$

for any nonnegative measurable function  $f$  on  $E$ , and

$$\begin{aligned} &\text{for any intrinsic stopping time } S, \{Z_{S+t}, t > 0\} \text{ under} \\ &\mathbf{P}_\nu(\cdot; S \in \mathbb{R}) \text{ is Markovian with semigroup } (P_t)_{t \geq 0}, \end{aligned} \quad (2.4)$$

where by intrinsic stopping time, one means a  $(\mathcal{G}_t)$ -stopping time that satisfies

$$b \leq S < d \text{ on } \{S < +\infty\}, \text{ and } S = t + S \circ \sigma_t \text{ for all } t \in \mathbb{R}.$$

In the time continuous setting, these measures have been introduced by Weil [19].

For a path  $\omega$  in  $\mathcal{W}$ , define  $H_B$  for any compact subset  $B$  of  $E$  by

$$H_B = \inf\{t \in (b(\omega), d(\omega)) : \omega(t_-) \in B\}$$

with  $\inf \emptyset = +\infty$ .

One immediately notes that on  $\{H_B < \infty\} : H_B \in [b, d]$ .

We will also use  $\lambda_B$  given by

$$\lambda_B = \sup\{t \in (b(\omega), d(\omega)) : \omega(t) \in B\}$$

with  $\sup \emptyset = -\infty$ .

We first remind the definition of random interacements set in [5] (the addendum of section 5.3 of [4]) in the case when  $X$  and  $\hat{X}$  are continuous.

**Definition 2.1** *Let  $X$  and  $\hat{X}$  be two transient continuous Borel processes in weak duality with respect to  $\nu$ . For  $\alpha > 0$  the random interacements at level  $\alpha$  associated to  $\{\nu, ((P_t)_{t \geq 0}, (\hat{P}_t)_{t \geq 0})\}$  is a Poisson point process with intensity measure  $\alpha \mu_\nu$  where  $\mu_\nu$  is the unique measure on  $(\mathcal{W}, \mathcal{A})$  such that  $\mu_\nu(\omega \equiv \Delta) = 0$ , characterized by the two following properties:*

- for every compact subset  $B$  of  $E$ :

$$\mu_\nu[\omega(H_B) \in dx, H_B < \infty] = \hat{e}_B(dx) \quad (2.5)$$

where  $\hat{e}_B$  is the capacitary (equilibrium) measure of  $B$  associated to  $\hat{X}$  with respect to  $\nu$ ;

- for every couple of  $\mathcal{A}$  measurable functionals  $(F_1, F_2)$

$$\begin{aligned} \mu_\nu & [F_1(\omega(H_B + t), t \geq 0); F_2(\omega((H_B - t), t > 0); H_B < \infty] \quad (2.6) \\ &= \int_E \hat{e}_B(dx) \mathbb{P}_x[F_1(X_t, t \geq 0)] \hat{\mathbb{P}}_x^B[F_2(Z_t, t > 0)], \end{aligned}$$

where for  $\hat{e}_B(dx)$  a.e.  $x$ ,  $\hat{\mathbb{P}}_x^B$  is the probability measure on the set of  $E$ -valued paths indexed by  $\mathbb{R}_+$  such that

$$\hat{\mathbb{P}}_x^B[F(Z_t, t \geq 0)] = \int_E \nu(dy) \hat{\mathbb{P}}_y[F(\hat{X}_{\hat{L}_B+t}, t \geq 0) | \hat{X}_{\hat{L}_B} = x].$$

To extend the above definition to standard processes, we will use the following notion of *bivariate Revuz measure* of an homogenous random measure.

Let  $\kappa$  be an homogenous random measure for  $X$  in the sense of Gettoor and Sharpe [8]. This means that  $\kappa$  is a random measure on  $\mathbb{R}_+$  supported by  $(0, \zeta)$  such that for every  $T$  stopping time for the natural filtration of  $X$ :  $\kappa((T, t + T]) = \kappa((0, t]) \circ \theta_T$  a.s. on  $\{T < \infty\}$ , for every  $t > 0$ .

There exists an unique measure  $\epsilon$  on  $E \times E$  such that for every nonnegative bounded measurable function  $F$  on  $E \times E$ :

$$\epsilon(F) = \sup_{t>0} \frac{1}{t} \mathbb{E}_\nu \int_{(0,t]} F(X_{s-}, X_s) \kappa(ds) = \lim_{t \rightarrow 0} \frac{1}{t} \mathbb{E}_\nu \int_{(0,t]} F(X_{s-}, X_s) \kappa(ds).$$

Indeed, using the fact that for every such  $F$ ,  $F(X_{s-}, X_s) \kappa(ds)$  is still an homogenous random measure, one can make use of the arguments developed in [8] section 8, and obtain  $\epsilon$  similarly as they set the definition (8.1) in [8] of the Revuz measure of  $\kappa$ .

For any compact subset  $B$  of  $E$ , define the homogenous random measure  $\kappa_B$ :

$\kappa_B(ds) = 1_{\{0 < \hat{L}_B < \zeta\}} \delta_{\hat{L}_B}(ds)$ . There exists then an unique measure  $\hat{e}_B$  on  $E \times E$  such that for every measurable bounded nonnegative function  $F$  on  $E \times E$

$$\begin{aligned} \hat{e}_B(F) &= \sup_{t>0} \frac{1}{t} \hat{\mathbb{E}}_\nu \int_{(0,t]} F(\hat{X}_{s-}, \hat{X}_s) \kappa_B(ds) = \lim_{t \rightarrow 0} \frac{1}{t} \hat{\mathbb{E}}_\nu \int_{(0,t]} F(\hat{X}_{s-}, \hat{X}_s) \kappa_B(ds) \\ &= \lim_{t \rightarrow 0} \frac{1}{t} \int_E \nu(dx) \hat{\mathbb{P}}_x[F(\hat{X}_{\hat{L}_B-}, \hat{X}_{\hat{L}_B}), 0 < \hat{L}_B \leq t]. \end{aligned}$$

The measure  $\hat{e}_B$  is the bivariate Revuz measure associated to the homogenous random measure  $\kappa_B$  with respect to  $\nu$ .

Note that the first marginal of  $\hat{e}_B$  is  $\hat{e}_B$  the capacitary measure of  $B$  for  $\hat{X}$  with respect to  $\nu$ . We denote the second marginal of  $\hat{e}_B$  by  $\hat{e}_B^+$ .

The notion of bivariate Revuz measure has been first set by Sharpe in [14] for additive functionals under the additional assumption that  $\nu$  is a reference measure. Our framework assumes that the two semigroups  $(P_t)_{t \geq 0}$  and  $(\hat{P}_t)_{t \geq 0}$  are in weak duality. Their respective potentials are not required to have densities with respect to  $\nu$ .

**Definition 2.2** Let  $X$  be a transient standard process in weak duality with  $\hat{X}$  with respect to a  $\sigma$ -finite measure  $\nu$ . Let  $(P_t)_{t \geq 0}$  and  $(\hat{P}_t)_{t \geq 0}$  be the respective semigroups of  $X$  and  $\hat{X}$ . For  $\alpha > 0$ , the random interacements at level  $\alpha$  associated to  $\{\nu, ((P_t)_{t \geq 0}, (\hat{P}_t)_{t \geq 0})\}$  is a Poisson point process with intensity measure  $\alpha \mu_\nu$  where  $\mu_\nu$  is the unique measure on  $(\mathcal{W}, \mathcal{A})$  characterized by the three following properties:

- for any compact subset  $B$  of  $E$

$$\mu_\nu[\omega(H_B) \in dx, \omega(H_{B-}) \in dy, H_B < \infty] = \hat{\epsilon}_B(dx dy) \quad (2.7)$$

where  $\hat{\epsilon}_B$  is the bivariate Revuz measure of  $\kappa_B$  associated to  $\hat{X}$  with respect to  $\nu$ ;

- for every couple of  $\mathcal{A}$  measurable functionals  $(F_1, F_2)$

$$\begin{aligned} \mu_\nu [F_1(\omega(H_B + t), t \geq 0); F_2(\omega((H_B - t)_-, t > 0); H_B < \infty] & \quad (2.8) \\ = \int_{E \times E} \hat{\epsilon}_B(dx dy) \mathbb{P}_x[F_1(X_t, t \geq 0)] \hat{\mathbb{P}}_y^B[F_2(Z_t, t > 0)], \end{aligned}$$

where for  $\hat{\epsilon}_B^+(dy)$  a.e.  $y$ ,  $\hat{\mathbb{P}}_y^B$  is the probability measure on the set of  $E$ -valued paths indexed by  $\mathbb{R}_+$  such that

$$\hat{\mathbb{P}}_x^B[F(Z_t, t \geq 0)] = \int_E \nu(dy) \hat{\mathbb{P}}_y[F(\hat{X}_{\hat{L}_B+t}, t \geq 0) | \hat{X}_{\hat{L}_B} = x];$$

- and

$$\mu_\nu(\omega \equiv \Delta) = 0. \quad (2.9)$$

The measure  $\hat{\mathbb{P}}_x^B$  does not depend on  $\nu$ . This has been mentioned in the continuous case [5], and we will emphasize this fact in the proof of (2.10) below.

As in the continuous case, unicity of  $\mu_\nu$  can be obtained with simple arguments. Indeed let  $\mu_1$  and  $\mu_2$  be two measures on  $\mathcal{A}$  satisfying (2.7) and (2.8) for every compact subset  $B$  of  $E$ . We compute  $\mu_1(A)$  for  $A$  in  $\mathcal{A}$ . We use a sequence of compact subsets  $(B_n)_{n \geq 1}$  increasing to  $E$ . For  $i = 1, 2$ , one has:

$$\mu_i(A) = \mu_i(A; H_{B_1} = \infty) + \mu_i(A; H_{B_1} < \infty)$$

By (2.8), and monotone class argument:  $\mu_1(A; H_{B_1} < \infty) = \mu_2(A; H_{B_1} < \infty)$ . One writes then:

$$\mu_i(A; H_{B_1} = \infty) = \mu_i(A; H_{B_1} = \infty, H_{B_2} = \infty) + \mu_i(A; H_{B_1} = \infty, H_{B_2} < \infty),$$

and keep splitting the probabilities  $\mu_i(A)$ ,  $i = 1, 2$  by using the sequence  $(B_n)_{n \geq 1}$ . By (2.9):  $\mu_i(H_{B_n} = \infty, \forall n) = 0$ ,  $i = 1, 2$ . Hence we end up with two series with identical general term, which leads to  $\mu_1(A) = \mu_2(A)$ .

Note that if the measure  $\mu_\nu$  exists then, using both (2.9) and (2.8), it must satisfy for every compact  $B$

$$\mu_\nu(H_B = b, H_B < \infty) = 0.$$

The following theorem establishes the existence of  $\mu_\nu$ .

**Theorem 2.3** *Let  $X$  and  $\hat{X}$  be two transient standard processes in weak duality with respect to a  $\sigma$ -finite reference measure  $\nu$ . Then we have*

$$\mu_\nu = \mathbf{P}_\nu. \quad (2.10)$$

In case  $X$  has continuous paths, (2.10) has been already established (Theorem 5.1 in [4],[5]). In the special example when  $X$  is a Brownian motion on  $\mathbb{R}^d$ ,  $d \geq 3$ , (2.10) has been previously established by Dereich and Döring [1].

**Proof:** It suffices to show that  $\mathbf{P}_\nu$  satisfies (2.7), (2.8) and (2.9).

First note that since  $X$  is transient, the excessive measure  $\nu$  is dissipative for  $(P_t)_{t \geq 0}$  (see e.g. Remark 4.9 c in [7]). This means that for any nonnegative function  $f$  such that  $\int_E f(x)\nu(dx) < \infty$ , one has:  $\int_0^\infty P_t f dt < \infty$   $\nu$  a.e.

Then since  $\nu$  is dissipative,  $\mathbf{Q}_\nu$  must be dissipative (see [6] section 3) and hence there exists a stationary time  $S^*$  such that  $\mathbf{Q}_\nu(S^* \notin \mathbb{R}) = 0$  (Proposition 2.7 in [6]).

We remind that a  $\mathcal{G}$ -measurable random time  $S^* : \mathcal{W} \rightarrow [-\infty, \infty]$  is said to be stationary if it satisfies  $S^* = t + S^* \circ \sigma_t$  for all  $t \in \mathbb{R}$ .

Following Fitzsimmons [6], the measure  $\mathbf{P}_\nu$  can hence be defined by

$$\mathbf{P}_\nu(A) = \mathbf{Q}_\nu[A; 0 < S^* \leq 1]$$

for every  $A$  in  $\mathcal{A}$  the  $\sigma$ -field of invariants elements of  $\mathcal{G}$ .

More generally, using (2.1) and (2.3) in [6], one obtains for every  $t > 0$

$$\mathbf{P}_\nu(A) = \frac{1}{t} \mathbf{Q}_\nu[A; 0 < S^* \leq t] = \frac{1}{t} \mathbf{Q}_\nu[A; -t \leq S^* < 0]. \quad (2.11)$$

Since by definition:  $\mathbf{Q}_\nu(\omega \equiv \Delta) = 0$ , (2.11) leads to  $\mathbf{P}_\nu(\omega \equiv \Delta) = 0$ . Hence  $\mathbf{P}_\nu$  satisfies (2.9).

Fix a compact set  $B$ . We first remind that since  $\hat{X}$  is transient:  $\mathbf{Q}_\nu(H_B = b) = 0$  (see [8] (13.12)). Thanks to (2.11) one obtains:  $\mathbf{P}_\nu(H_B \notin (b, d), H_B < \infty) = 0$ .

For any  $F$  real-valued measurable bounded function on  $E \times E$ , one has for every  $t > 0$ :

$$\begin{aligned} \mathbf{P}_\nu[F(Z_{H_B}, Z_{H_B-}), H_B < \infty] &= \frac{1}{t} \mathbf{Q}_\nu[F(Z_{H_B}, Z_{H_B-}), H_B < \infty, -t \leq S^* < 0] \\ &= \frac{1}{t} \mathbf{Q}_\nu[F(Z_{H_B}, Z_{H_B-}), -t \leq H_B < 0] \end{aligned}$$

thanks to the switching property of Proposition 2.4 in [6]. Using then the duality properties of  $\mathbf{Q}_\nu$  (see [12]):

$$\hat{\mathbf{Q}}_\nu[F(Z_t, t \in \mathbb{R})] = \mathbf{Q}_\nu[F(\hat{Z}_t, t \in \mathbb{R})], \quad (2.12)$$

where  $\hat{Z}_t = Z_{(-t)-}$ ,  $t \in \mathbb{R}$ , one obtains for every  $t > 0$ :

$$\mathbf{P}_\nu[F(Z_{H_B}, Z_{H_B-}), H_B < \infty] = \frac{1}{t} \hat{\mathbf{Q}}_\nu[F(Z_{\lambda_B-}, Z_{\lambda_B}), 0 < \lambda_B \leq t]. \quad (2.13)$$

On one hand for every nonnegative function  $f$ , one has:

$$\begin{aligned} \hat{e}_B(f) &= \lim_{t \rightarrow 0} \frac{1}{t} \int_E \nu(dx) \hat{P}_x[f(\hat{X}_{\hat{L}_B-}), 0 < \hat{L}_B \leq t] \\ &= \lim_{t \rightarrow 0} \frac{1}{t} \hat{\mathbf{Q}}_\nu[f(Z_{\lambda_B-}), b < 0 < \lambda_B \leq t]. \end{aligned} \quad (2.14)$$

On the other hand, making use of (13.7) in [8], one knows that:

$$\hat{e}_B(f) = \frac{1}{t} \hat{\mathbf{Q}}_\nu[f(Z_{\lambda_B-}), 0 < \lambda_B \leq t, \lambda_B > b]. \quad (2.15)$$

Moreover on  $\{\lambda_B \in (0, t]\}$ :  $\lambda_B \in (b, d]$ , and  $\hat{X}$  being transient one has:  $\hat{\mathbf{Q}}_\nu[\lambda_B = d] = 0$  (Proposition 13.6 in [8]). Using this last remarks together with (2.14), (2.15) one finally obtains:

$$\lim_{t \rightarrow 0} \frac{1}{t} \hat{\mathbf{Q}}_\nu[0 \leq b < \lambda_B \leq t] = 0. \quad (2.16)$$

Starting from (2.13), one writes for every  $t > 0$ :

$$\begin{aligned} \mathbf{P}_\nu[F(Z_{H_B}, Z_{H_B-}), H_B < \infty] &= \frac{1}{t} \hat{\mathbf{Q}}_\nu[F(Z_{\lambda_B-}, Z_{\lambda_B}), b < 0 < \lambda_B \leq t] \\ &\quad + \frac{1}{t} \hat{\mathbf{Q}}_\nu[F(Z_{\lambda_B-}, Z_{\lambda_B}), 0 \leq b < \lambda_B \leq t], \end{aligned}$$

and let then  $t$  tend to 0 to obtain thanks to (2.16):

$$\begin{aligned} \mathbf{P}_\nu[F(Z_{H_B}, Z_{H_B-}), H_B < \infty] &= \lim_{t \rightarrow 0} \frac{1}{t} \hat{\mathbf{Q}}_\nu[F(Z_{\lambda_B-}, Z_{\lambda_B}), b < 0 < \lambda_B \leq t] \\ &= \lim_{t \rightarrow 0} \frac{1}{t} \int_E \nu(dx) \hat{P}_x[f(\hat{X}_{\hat{L}_B-}), 0 < \hat{L}_B \leq t] \\ &= \hat{e}_B(F) \end{aligned}$$

which establishes (2.7) for  $\mathbf{P}_\nu$ .

We check now that  $\mathbf{P}_\nu$  satisfies (2.8).

$$\begin{aligned} \mathbf{P}_\nu &[F_1(Z(H_B + t), t \geq 0); F_2(Z((H_B - t)-), t > 0); H_B < \infty] \quad (2.17) \\ &= \mathbf{Q}_\nu[F_1(Z(H_B + t), t \geq 0); F_2(Z((H_B - t)-), t > 0); H_B < \infty; -1 \leq S^* < 0] \\ &= \mathbf{Q}_\nu[F_1(Z(H_B + t), t \geq 0); F_2(Z((H_B - t)-), t > 0); -1 \leq H_B < 0]. \end{aligned}$$



The random time  $H_B$  is a stopping time for  $\mathbf{Q}_\nu$  (i.e. for every  $t \in \mathbb{R}$ ,  $\{H_B \leq t\} \in \mathcal{G}_t$ ). It is an intrinsic stopping time in the sense that on  $\{H_B < \infty\}$ :  $b \leq H_B < d$  and  $H_B = t + H_B \circ \sigma_t$ ,  $\forall t \in \mathbb{R}$ . This implies in particular that for every  $t$ :  $Z_{H_B+t}$  is  $\mathcal{A}$ -measurable, and hence so is  $(Z_{(H_B-t)-}, t \geq 0)$ .

Using Proposition 2.4 (2.5) in [6], one has for every  $t > 0$

$$\begin{aligned} & \frac{1}{t} \mathbf{Q}_\nu [F_1(Z(H_B + t), t \geq 0); F_2(Z((H_B - t)-), t > 0); -t \leq H_B < 0] \quad (2.18) \\ &= \mathbf{Q}_\nu [F_1(Z(H_B + t), t \geq 0); F_2(Z((H_B - t)-), t > 0); -1 \leq H_B < 0] \end{aligned}$$

Under  $\mathbf{Q}_\nu$ , one has Markov property at time  $H_B$ , under the following form (see e.g. (10.12) in [8])

$$\begin{aligned} & \mathbf{Q}_\nu [F_1(Z(H_B + t), t \geq 0); F_2(Z((H_B - t)-), t > 0); -1 \leq H_B < 0] \\ &= \mathbf{Q}_\nu [-1 \leq H_B < 0, \mathbb{P}_{Z(H_B)}[F_1(X_s, s \geq 0)] F_2(Z((H_B - t)-), t > 0)], \end{aligned}$$

which leads to

$$\begin{aligned} & \mathbf{Q}_\nu [F_1(Z(H_B + t), t \geq 0); F_2(Z((H_B - t)-), t > 0); -1 \leq H_B < 0] \\ &= \int_E \hat{e}_B(dx) \mathbb{P}_x [F_1(X_s, s \geq 0)] \\ & \quad \mathbf{Q}_\nu [-1 \leq H_B < 0, F_2(Z((H_B - t)-), t > 0) | Z(H_B) = x] \\ &= \int_E \hat{e}_B(dx) \mathbb{P}_x [F_1(X_s, s \geq 0)] \\ & \quad \hat{\mathbf{Q}}_\nu [0 < \lambda_B \leq 1, F_2(Z((\lambda_B + t)), t > 0) | Z(\lambda_B-) = x] \\ & \quad \text{(thanks to (2.12))} \\ &= \int_E \hat{e}_B(dx) \mathbb{P}_x [F_1(X_s, s \geq 0)] \int_E \hat{\mathbf{Q}}_\nu [0 < \lambda_B \leq 1, Z(\lambda_B) \in dy | Z(\lambda_B-) = x] \\ & \quad \hat{\mathbf{Q}}_\nu [0 < \lambda_B \leq 1, F_2(Z((\lambda_B + t)), t > 0) | Z(\lambda_B-) = x, Z(\lambda_B) = y] \\ &= \int_{E \times E} \hat{e}(dxdy) \mathbb{P}_x [F_1(X_s, s \geq 0)] \\ & \quad \hat{\mathbf{Q}}_\nu [0 < \lambda_B \leq 1, F_2(Z((\lambda_B + t)), t > 0) | Z(\lambda_B-) = x, Z(\lambda_B) = y], \quad (2.19) \end{aligned}$$

using (2.7) for  $\mathbf{P}_\nu$ .

Now, using both (2.18) and (2.12), we have for every  $t > 0$

$$\begin{aligned} & \hat{\mathbf{Q}}_\nu [0 < \lambda_B \leq 1, F_2(Z((\lambda_B + s)), s > 0) f_1(Z(\lambda_B-)) f_2(Z(\lambda_B))] \\ &= \frac{1}{t} \hat{\mathbf{Q}}_\nu [0 < \lambda_B \leq t, F_2(Z((\lambda_B + s)), s > 0) f_1(Z(\lambda_B-)) f_2(Z(\lambda_B))] \\ &= \frac{1}{t} \hat{\mathbf{Q}}_\nu [b < 0 < \lambda_B \leq t, F_2(Z((\lambda_B + s)), s > 0) f_1(Z(\lambda_B-)) f_2(Z(\lambda_B))] \\ & \quad + \frac{1}{t} \hat{\mathbf{Q}}_\nu [0 \leq b < \lambda_B \leq t, F_2(Z((\lambda_B + s)), s > 0) f_1(Z(\lambda_B-)) f_2(Z(\lambda_B))] \end{aligned}$$

Letting then  $t$  tend to 0 and using (2.16) one obtains:

$$\begin{aligned}
\hat{\mathbf{Q}}_\nu[0 < \lambda_B \leq 1, & \quad F_2(Z(((\lambda_B + s))), s > 0)f_1(Z(\lambda_B-))f_2(Z(\lambda_B))] & (2.20) \\
= & \quad \lim_{t \rightarrow 0} \frac{1}{t} \hat{\mathbf{Q}}_\nu[b < 0 < \lambda_B \leq t, F_2(Z(((\lambda_B + s))), s > 0)f_1(Z(\lambda_B-))f_2(Z(\lambda_B))] \\
= & \quad \lim_{t \rightarrow 0} \frac{1}{t} \int_E \hat{\mathbb{P}}_\nu[0 < \hat{L}_B \leq t, F_2(\hat{X}(((\hat{L}_B + s))), s > 0)f_1(\hat{X}(\hat{L}_B-))f_2(\hat{X}(\hat{L}_B))]
\end{aligned}$$

We use now the splitting property of the random time  $\hat{L}_B$  to claim that

$$\begin{aligned}
\hat{\mathbb{P}}_\nu [ & \quad 0 < \hat{L}_B \leq t, F_2(\hat{X}(\hat{L}_B + s)), s > 0)f_1(\hat{X}(\hat{L}_B-))f_2(\hat{X}(\hat{L}_B))] \\
= & \quad \hat{\mathbb{P}}_\nu[0 < \hat{L}_B \leq t, \Gamma(\hat{X}_{\hat{L}_B}, F_2)f_1(\hat{X}(\hat{L}_B-))f_2(\hat{X}(\hat{L}_B))],
\end{aligned}$$

where  $(\Gamma(x, A), x \in E, A \in \mathcal{F})$  is a Markov kernel (see Theorem 2.12 in [9] for a complete description of  $\Gamma$ ) independent of  $\nu$ .

From (2.20), one hence obtains:

$$\begin{aligned}
\hat{\mathbf{Q}}_\nu[0 < \lambda_B \leq 1, & \quad F_2(Z(((\lambda_B + s))), s > 0)f_1(Z(\lambda_B-))f_2(Z(\lambda_B))] \\
= & \quad \epsilon_B(g)
\end{aligned}$$

where  $g$  denotes the function on  $E \times E$  defined by  $g(x, y) = f_1(x)f_2(y)\Gamma(y, F_2)$ .

This implies that  $\hat{e}_B^+(dy)$  a.e.  $y$ :

$$\begin{aligned}
\hat{\mathbf{Q}}_\nu[0 < \lambda_B \leq 1, & \quad F_2(Z(((\lambda_B + s))), s > 0)|Z(\lambda_B-) = x, Z(\lambda_B) = y] & (2.21) \\
= & \quad \Gamma(y, F_2) \\
= & \quad \int_E \nu(dz) \hat{\mathbb{P}}_z[F_2(\hat{X}_{\hat{L}_B+s}, s > 0)|\hat{X}_{\hat{L}_B} = y],
\end{aligned}$$

which is hence independent of  $\nu$ .

Coming back to (2.19) and using (2.17) one finally obtains:

$$\begin{aligned}
\mathbf{P}_\nu [ & \quad F_1(Z(H_B + t), t \geq 0); F_2(Z((H_B - t)-), t > 0); H_B < \infty] \\
= & \quad \int_{E \times E} \hat{e}(dxdy) \mathbb{P}_x[F_1(X_t, t \geq 0)] \hat{\mathbb{P}}_y^B[F_2(Z_t, t > 0)].
\end{aligned}$$

One concludes that  $\mathbf{P}_\nu$  satisfies (2.8).  $\square$

Suppose now that the excessive measure  $\nu$  is purely excessive, which means that

$$\int_E \nu(dx) P_t 1(x) \rightarrow_{t \rightarrow \infty} 0.$$

In this case  $\mu_\nu$  can be connected with another measure on  $\mathcal{A}$  the  $\sigma$ -field generated by the shift invariant sets of  $\mathcal{W}$ . More precisely when  $\nu$  is purely excessive there exists an entrance law  $(m_t)_{t>0}$  such that

$$\nu(f) = \int_0^\infty m_t(f) dt \quad \text{and} \quad m_s P_{t-s} = m_t, \quad 0 < s < t$$

(see [3]). Define  $P_*$  by

$$P_*(Z_{t_1} \in A_1, \dots, Z_{t_n} \in A_n) = \int_{A_1} m_{t_1}(dx_1) \cdots \int_{A_n} P_{t_n - t_{n-1}}(x_{n-1}, dx_n) \quad (2.22)$$

for  $0 < t_1 \leq \dots \leq t_n$  and extend the definition to negative  $t_i$ 's by setting  $m_s = 0$  for  $s < 0$ .

We have shown in [4] ((5.10) in [4]) that for every  $A \in \mathcal{A}$ :  $P_*(A) = \mathbf{P}_\nu(A)$ . This leads to the following corollary.

**Corollary 2.4** *Let  $X$  and  $\hat{X}$  be two transient standard processes in weak duality with respect to a purely excessive measure  $\nu$ . Then we have*

$$\mu_\nu = \mathbf{P}_\nu = P_{*|\mathcal{A}}.$$

### 3 Interpretation of random interlacements

We have noticed in [4] (Remark 2.2) that a sufficient condition for  $\nu$  to be purely excessive is  $\nu$  finite excessive and the life time  $\zeta$  of  $X$  is finite  $\mathbb{P}_x$  a.s. for every  $x$  in  $E$ . The interest of Corollary 2.4 relies on a peculiar interpretation of the measure  $P_*$  under some additional assumptions. Indeed, assume that the three following properties are satisfied

- (i)  $\nu$  is finite excessive;
- (ii) the life time  $\zeta$  of  $X$  is finite  $\mathbb{P}_x$  a.s.  $\forall x \in E$ ;
- (iii)  $\zeta$  has no atom  $\mathbb{P}_x$  a.s  $\forall x \in E$

and let  $\delta$  be a point outside  $E$ . We have shown in [4] that then there exists a Markov process  $(Y_t)_{t \geq 0}$  on  $E \cup \{\delta\}$  admitting  $\delta$  as a recurrent point such that for every  $a$  in  $E$ :

$$((X_t, t < \zeta) | X_0 = a) \stackrel{(\text{law})}{=} ((Y_t, t < T_\delta) | Y_0 = a),$$

where  $T_\delta = \inf\{s \geq 0 : Y_s = \delta\}$ . Moreover the excursion process of  $Y$  with respect to  $\delta$  is a Poisson point process with intensity  $dt \times P_*$ . One says that the process  $Y$  is an “extended Markov process” of  $X$ , in short an *extended  $X$* . Thanks to Corollary 2.4 one can hence interpret the random interlacements at level  $\alpha$  as the excursion process of  $Y$  from  $\delta$  modulo time-shift up to the first time the local time at  $\delta$  exceeds  $\alpha$ .

In case the set of the three above properties (i)-(iii) is not satisfied, one can still make a connection between the random interlacements of  $X$  and the excursion process of some Markov process on  $E \cup \{\delta\}$  by assuming that  $X$  has finite 0-potential densities with respect to  $\nu$ . Indeed we have shown ([4], Corollary 2.4) that under this assumption there exists a positive function  $q$  on  $E$  such that the measure  $\nu \cdot q$  is finite (where

$\nu \cdot q(dx) = q(x)\nu(dx)$ ) and that there exists a Markov process  $(Y_t)_{t \geq 0}$  with state space  $E \cup \{\delta\}$  admitting a local time process  $(L_t^x(Y), x \in E, t \geq 0)$  with respect to  $\nu \cdot q$  and  $\delta$  as recurrent point, satisfying for every  $a$  in  $E$ :

$$((L_\infty^x(X), x \in E) | X_0 = a) \stackrel{(\text{law})}{=} ((L_{T_\delta}^x(Y), x \in E) | Y_0 = a),$$

where  $(L_\infty^x(X), x \in E)$  denotes the total accumulated local time process of  $X$  with respect to  $\nu$ .

We still say that the process  $Y$  is an extended Markov process of  $X$ , or an extended  $X$ . Note that  $Y$  depends on the choice of  $q$  but that this choice does not affect the law of  $(L_{T_\delta}^x(Y), x \in E)$ .

Following exactly the same sequence of arguments as for the proof of Proposition 5.3 in [4], one obtains the following proposition. Note that as soon as  $X$  has finite 0-potential densities with respect to  $\nu$  then necessarily  $\nu$  is a reference measure for  $(P_t)_{t \geq 0}$ .

**Proposition 3.1** *Let  $X$  be a transient standard process with semigroup  $(P_t)_{t \geq 0}$  in weak duality with respect to a  $\sigma$ -measure  $\nu$  with a standard process with semigroup  $(\hat{P}_t)_{t \geq 0}$ . Assume that  $X$  has finite 0-potential densities with respect to  $\nu$ . Then the field of occupation time of the random interlacement at level  $\alpha$  associated to  $\{\nu, ((P_t)_{t \geq 0}, (\hat{P}_t)_{t \geq 0})\}$  equals in law the local time process of an extended  $X$  at the first time its local time at  $\delta$  exceeds  $\alpha$ .*

## 4 Illustrations

Lévy processes are standard processes. Besides a Lévy process  $X$  on  $\mathbb{R}^d$  is in weak duality with  $(-X)$  with respect to the Lebesgue measure on  $\mathbb{R}^d$ . Obviously the Lebesgue measure on  $\mathbb{R}^d$  is not finite.

In case the Lévy process  $X$  is transient, according to Definition 2.2 and Theorem 2.3, the random interlacements at level  $\alpha$  associated to  $X$  is the Poisson point process with intensity measure  $\alpha \mathbf{P}_\nu$  ( $\nu$  here is the Lebesgue measure on  $\mathbb{R}^d$ ).

If moreover  $X$  has finite 0-potential densities then  $X$  admits an extended Markov process  $Y$  on  $\mathbb{R}^d \cup \{\delta\}$  such that for every  $\alpha > 0$ , the occupation field of the random interlacements of  $X$  at level  $\alpha$  equals in law the local time process of  $Y$  at the first time the local time at  $\delta$  exceeds  $\alpha$ .

In [13], Rosen defines directly the random interlacements at level  $\alpha$  as the Poisson point process with intensity measure  $\alpha \mathbf{P}_\nu$ . He considers then exclusively the case when  $X$  is a symmetric transient  $\mathbb{R}^d$ -valued Lévy process which does not admit local times. Theorem 2.3 ensures that this Poisson point process coincides with the one given by Definition 2.2 but the interpretation provided by Proposition 3.1 is not available in his framework.

For symmetric transient Lévy processes admitting local times and more generally for any transient standard process  $X$  with finite symmetric 0-potential densities

$(u(x, y), (x, y) \in E \times E)$  with respect to a  $\sigma$ -measure  $\nu$ , one has obtained (Theorem 3.1 in [4]) the following isomorphism theorem for  $Y$ , an extended  $X$ . Under  $\mathbb{P}[\cdot | Y_0 = \delta]$ , one has:

$$\left(\frac{1}{2}\eta_x^2 + L_{\tau_\alpha}^x(Y), x \in E\right) \stackrel{(\text{law})}{=} \left(\frac{1}{2}(\eta_x + \sqrt{2\alpha})^2, x \in E\right), \quad (4.1)$$

where  $(L_{\tau_\alpha}^x(Y), x \in E)$  is the local time process (with respect to  $\nu \cdot q$ ) of  $Y$ , at the first time the local time of  $Y$  at  $\delta$  exceeds the value  $\alpha$ , and  $(\eta_x, x \in E)$  is a centered Gaussian process with covariance  $(u(x, y), (x, y) \in E \times E)$ , independent of  $Y$ . According to Proposition 3.1, one can rewrite (4.1) under the following form:

$$\left(\frac{1}{2}\eta_x^2 + L_{x,\alpha}, x \in E\right) \stackrel{(\text{law})}{=} \left(\frac{1}{2}(\eta_x + \sqrt{2\alpha})^2, x \in E\right), \quad (4.2)$$

where  $(L_{x,\alpha}, x \in E)$  is the occupation time of the random interlacements at level  $\alpha$  of  $X$ .

The identity (4.2) represents an extension to the discontinuous case of Sznitman's isomorphism Theorem for random interlacements.

## References

- [1] Dereich S. and Döring L.: Random interlacements via Kuznetsov measures. arXiv:1501.00649 (2015).
- [2] Drewitz, A., Prévost, A. and Rodriguez, P.-F.: The sign clusters of the massless Gaussian free field percolate on  $\mathbb{Z}^d$ ,  $d \geq 3$ . *Comm. Math. Phys.* 362, no. 2, 513-546 (2018).
- [3] Dynkin E.B. : Minimal excessive measures and functions. *Trans. Amer. Math. Soc.* 258, 217-244 (1980).
- [4] Eisenbaum N. and Kaspi H.: Isomorphism theorem, extended Markov processes and random interlacements. *Electron. J. Probab.* 27, 1-27 (2022).
- [5] Eisenbaum N. and Kaspi H.: Addendum to "Isomorphism theorem, extended Markov processes and random interlacements". *Electron. J. Probab.* 28, 1-3 (2023).
- [6] Fitzsimmons P.J.: On a connection between Kuznetsov processes and quasi-processes. *Seminar on Stochastic Processes 1987, 123-133. Birkhauser Boston, Basel* (1988).
- [7] Fitzsimmons P.J. and Maisonneuve B. : Excessive measures and Markov processes with random birth and death. *Probab. Th. Rel. Fields* 72,319-336 (1986).
- [8] Gettoor R.K. and Sharpe M. J. : Naturality, standardness and weak duality for Markov processes. *Z. Wahrsch. Verw. Gebiete* 67, 1-62 (1984).

- [9] Gettoor R. K. : Splitting times and shift functionals. *Z. Wahrsch. Verw. Gebiete* 47, no. 1, 69-81 (1979).
- [10] Kuznetsov S.E. : Construction of Markov processes with random birth and death. *Th. Prob. Appl.* 571-574 (1974).
- [11] Lupu, T.: From loop clusters and random interacements to the free field. *Annals of Probab.* 44, 3, 2117-2146 (2016).
- [12] Mitro J.B.: Dual Markov processes: construction of a useful auxiliary process. *Z. Wahrsch. Verw. Gebiete* 47, no. 2, 139-156 (1979).
- [13] Rosen J.: Intersection local times for interacements. *Stochastic Processes and their Applications*, 124, 1849-1880 (2014).
- [14] Sharpe M.J.: Discontinuous additive functionals of dual processes. *Z. Wahrscheinlichkeitstheorie verw. Geb.* 21, 81-95 (1972).
- [15] Sznitman A.-S.: Vacant set of random interacements and percolation. *Ann. of Math. (2)* 171, no. 3, 2039-2087 (2010).
- [16] Sznitman A.-S.: An isomorphism theorem for random interacements. *Electron. Commun. Probab.* 17, 9 (2012).
- [17] Sznitman A.-S.: On scaling limits and Brownian interacements. *Bull. Braz. Math. Soc. (N.S.)* 44 (2013), no. 4, 555-592 (2013).
- [18] Teixeira, A.: On the uniqueness of the infinite cluster of the vacant set of random interacements. *Ann. Appl. Probab.* 19, no. 1, 454-466 (2009).
- [19] Weil M. : Quasi-processes. *Séminaire de Probabilités IV*, 216-239, *LNМ 124*, ed Springer-Verlag (1970).
- [20] Windisch, D.: Random walks on discrete cylinders with large bases and random interacements. *Ann. Probab.* 38, no. 2, 841-895 (2010).

Nathalie Eisenbaum  
 MAP5, CNRS - Université Paris Cité  
 nathalie.eisenbaum@parisdescartes.fr