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Scalar BSDEs of iterated-logarithmically sublinear generators with integrable terminal values*

Shengjun Fan^a, Ying Hu^b, Shanjian Tang^{c,*}

^a*School of Mathematics, China University of Mining and Technology, Xuzhou 221116, China*

^b*Univ. Rennes, CNRS, IRMAR-UMR6625, F-35000, Rennes, France*

^c*Department of Finance and Control Sciences, School of Mathematical Sciences, Fudan University, Shanghai 200433, China*

Abstract

We establish a general existence and uniqueness of integrable adapted solutions to scalar backward stochastic differential equations with integrable parameters, where the generator g has an iterated-logarithmic uniform continuity in the second unknown variable z . The result improves our previous one in [12].

Keywords: backward stochastic differential equation, integrable parameters, iterated-logarithmically sublinear generator, existence and uniqueness, comparison theorem.

2010 MSC: 60H10

1. Introduction

Fix an integer $d \geq 1$ and a real $T > 0$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space equipped with augmented filtration $(\mathcal{F}_t)_{t \in [0, T]}$ generated by a standard d -dimensional Brownian motion $(B_t)_{t \in [0, T]}$, and $\mathcal{F}_T = \mathcal{F}$. Consider the following scalar backward stochastic differential equation (BSDE in short):

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s) ds - \int_t^T Z_s dB_s, \quad t \in [0, T], \quad (1.1)$$

where ξ is called the terminal condition, which is an \mathcal{F}_T -measurable real-valued random variable, the random field

$$g : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^{1 \times d} \rightarrow \mathbb{R}$$

is called the generator, which is an (\mathcal{F}_t) -adapted process for each (y, z) , and the pair of (\mathcal{F}_t) -adapted processes $(Y_t, Z_t)_{t \in [0, T]}$ is called a solution of (1.1), which takes its values in $\mathbb{R} \times \mathbb{R}^{1 \times d}$ such that $\mathbb{P} - a.s.$, $t \mapsto Y_t$ is continuous, $t \mapsto |g(t, Y_t, Z_t)| + |Z_t|^2$ is integrable, and satisfies (1.1). Denote by BSDE(ξ, g) the BSDE with the terminal condition ξ and the generator g , which are called the parameters of the BSDE.

Nonlinear BSDEs were initially introduced in [19], who established an existence and uniqueness result on adapted solutions of multidimensional BSDEs with Lipschitz continuous generators and square

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*Corresponding author

Email addresses: f_s_j@126.com (Shengjun Fan), ying.hu@univ-rennes1.fr (Ying Hu), sjtang@fudan.edu.cn (Shanjian Tang)

integrable parameters. Since then, BSDEs have been extensively studied due to their deep connections with various fields such as partial differential equations, mathematical finance, stochastic control and so on. The reader is referred to among others [6, 17, 14] for more details. In particular, much efforts have been paid on the well-posedness of adapted solutions to BSDEs under various integrability on the parameters and various growth and/or continuity of the generator g in the unknown variables (y, z) . For instance, some classical results can be found in [18, 17, 2, 3, 5, 15, 16, 4, 9, 10, 11] and the references therein.

With adapted solutions of BSDEs, Peng [20] introduced the notion of conditional g -expectation of a square-integrable random variable, which is a nonlinear extension of the conventional conditional expectation. Since the conventional conditional expectation is defined in the space of integrable random variables, it is then asked that how to define the conditional g -expectation of an only integrable random variable—which entails solution of BSDEs with only integrable parameters. It has been widely recognized that it is more difficult to solve BSDEs with only integrable parameters than those with L^p -integrable parameters ($p > 1$). To the best of our knowledge, there are only few discussions on adapted solution of BSDEs with integrable parameters. In particular, an existence and uniqueness result is available in [1] in this direction for multidimensional BSDEs, where the generator g is Lipschitz continuous and grows in z in a sublinear way (see for example $(H2S)_\alpha$ with $\alpha \in (0, 1)$ in Section 2). Subsequently in [13, 7, 8, 21], new growth and continuity of the generator g in z are given for the well-posedness of the integrable adapted solutions, such as the uniform continuity and a sublinear growth, the Hölder continuity and the quasi-Hölder continuity (see $(H5S)_\alpha$ with $\alpha \in (0, 1)$ in Section 2). In the last few years, with the test function method in [10] and the localization technique in [2], we proved in [12] the existence of integrable solutions of scalar BSDEs with integrable parameters when the generator g satisfies a logarithmic sublinear growth in z (see $(H2)_n^\lambda$ with $n = 1$ and $\lambda > 1/2$ in Section 2), and the uniqueness when further g has a logarithmic uniform continuity in z (see $(H5)_n^\lambda$ with $n = 1$ and $\lambda > 1/2$ in Section 2), which extend the above-mentioned growth and continuity (see (i) of Remark 2.5 in Section 2 for details).

The objective of the present paper is to study integrable solutions of scalar BSDEs with only integrable parameters under finer assumptions on the generator g , as a continuation of our previous work [12]. The main result is formulated in Theorem 2.2 in Section 2. It gives the existence of an L^1 solution of a BSDE with integrable parameters when the generator g has an iterated-logarithmic sublinear growth in z (see $(H2)_n^\lambda$ with $n \geq 2$ and $\lambda > 1/2$ in Section 2), and the uniqueness of integrable solutions in a proper space when the generator g further has an iterated-logarithmic uniform continuity in z (see $(H5)_n^\lambda$ with $n \geq 2$ and $\lambda > 1/2$ in Section 2), which improves that of [12] (see (i) of Remark 2.3 in Section 2 for details). In fact, it gives a sequence of weaker and weaker conditions on the generator g for the existence and uniqueness of the integrable solution of $BSDE(\xi, g)$ under only integrable parameters. For the existence, we first establish a crucial inequality (see Proposition 3.2 in Section 3), and then find a proper test function (see Proposition 3.4 in Section 3) to apply Itô's formula to obtain an a priori bound on the first component of adapted solutions to the approximating BSDEs (see Proposition 3.5 in Section 3), and finally utilize the localization technique to obtain the desired solution. For the uniqueness, we establish a

general comparison theorem for the integrable solutions to the BSDEs (see Proposition 2.4 in Section 2), where the same a priori estimate technique as above and Theorem 2.1 of [7] play a key role.

Let us close the introduction by introducing some necessary notations and spaces used in this paper. For $a, b \in \mathbb{R}$, we denote $a \wedge b := \min\{a, b\}$, $a^+ := \max\{a, 0\}$ and $a^- := -\min\{a, 0\}$, and $\text{sgn}(x) := \mathbf{1}_{x>0} - \mathbf{1}_{x \leq 0}$, where $\mathbf{1}_A$ is the indicator function of set A . Let \mathbf{S} be the set of all continuous nondecreasing function $\rho(\cdot) : [0, +\infty) \rightarrow [0, +\infty)$ with $\rho(0) = 0$. For each pair of nonnegative integer $m > k \geq 0$ and each real sequence $\{a_n\}_{n=1}^{+\infty}$, we use the following convention:

$$\prod_{i=m}^k a_i := 1 \quad \text{and} \quad \sum_{i=m}^k a_i := 0.$$

Furthermore, for each integer $n \geq 1$, by induction we denote the following function

$$\ln^{(1)}(x) := \ln x, \quad x \geq e^{(1)} \quad \text{and} \quad \ln^{(n)}(x) := \ln^{(n-1)}(\ln^{(1)}(x)) = \ln^{(1)}(\ln^{(n-1)}(x)), \quad x \geq e^{(n)},$$

where

$$e^{(1)} := e \quad \text{and} \quad e^{(n)} := e^{e^{(n-1)}}.$$

For each $p > 0$, let \mathcal{S}^p be the set of (\mathcal{F}_t) -adapted continuous real-valued processes $(Y_t)_{t \in [0, T]}$ satisfying

$$\|Y\|_{\mathcal{S}^p} := \left(\mathbb{E} \left[\sup_{t \in [0, T]} |Y_t|^p \right] \right)^{\frac{1}{p} \wedge 1} < +\infty,$$

and \mathcal{M}^p the set of all (\mathcal{F}_t) -adapted $\mathbb{R}^{1 \times d}$ -valued processes $(Z_t)_{t \in [0, T]}$ satisfying

$$\|Z\|_{\mathcal{M}^p} := \left\{ \mathbb{E} \left[\left(\int_0^T |Z_t|^2 dt \right)^{p/2} \right] \right\}^{\frac{1}{p} \wedge 1} < +\infty.$$

Denote by Σ_T the set of all (\mathcal{F}_t) -stopping times τ valued in $[0, T]$. For an (\mathcal{F}_t) -adapted real-valued process $(X_t)_{t \in [0, T]}$, if the family $\{X_\tau : \tau \in \Sigma_T\}$ is uniformly integrable, then we say that it is of class (D).

The rest of this paper is organized as follows. In section 2 we state the main result and introduce several remarks and examples to illustrate our theoretical result, and in section 3 we give the proof.

2. Statement of the main result

We always suppose that $n \geq 1$ is a positive integer, $\alpha \in (0, 1)$, $\beta, \lambda \geq 0$ and $\gamma, c > 0$ are several nonnegative constants, ξ is a terminal condition satisfying $\mathbb{E}[|\xi|] < +\infty$, and $(f_t)_{t \in [0, T]}$ is an (\mathcal{F}_t) -adapted nonnegative process satisfying

$$\mathbb{E} \left[\int_0^T f_t dt \right] < +\infty.$$

For an integer $n \geq 1$ and a real number $\lambda \geq 0$, define the function

$$\mathcal{I}\mathcal{L}_n^\lambda(x) := \prod_{i=1}^{n-1} \sqrt{\ln^{(i)}(e^{(n)} + x)} \left(\ln^{(n)}(e^{(n)} + x) \right)^\lambda, \quad x \geq 0.$$

An equality or inequality between random variables are always understood in the sense of $\mathbb{P} - a.s.$. Let us introduce the following assumptions on the generator g .

(H1) $d\mathbb{P} \times dt - a.e.$, $g(\omega, t, \cdot, \cdot)$ is continuous.

(H2) $_n^\lambda$ g has a one-sided linear growth in y and an iterated-logarithmic sublinear growth in z , i.e., $d\mathbb{P} \times dt - a.e.$, for each $(y, z) \in \mathbb{R} \times \mathbb{R}^{1 \times d}$,

$$\operatorname{sgn}(y)g(\omega, t, y, z) \leq f_t(\omega) + \beta|y| + \frac{\gamma|z|}{\mathcal{IL}_n^\lambda(|z|)}.$$

(H3) g has a general growth in (y, z) , i.e., there exists a function $h(\cdot) \in \mathbf{S}$ such that $d\mathbb{P} \times dt - a.e.$,

$$|g(\omega, t, y, z)| \leq f_t(\omega) + h(|y|) + c|z|^2, \quad \forall (y, z) \in \mathbb{R} \times \mathbb{R}^{1 \times d}.$$

(H4) g satisfies an extended monotonicity condition in y , i.e., there exists a concave function $\rho(\cdot) \in \mathbf{S}$ with $\rho(u) > 0$ for $u > 0$ and $\int_{0+} \frac{du}{\rho(u)} = +\infty$ such that $d\mathbb{P} \times dt - a.e.$,

$$g(\omega, t, y_1, z) - g(\omega, t, y_2, z) \leq \rho(y_1 - y_2), \quad \forall (y_1, y_2, z) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{1 \times d} \text{ with } y_1 > y_2.$$

(H5) $_n^\lambda$ g has an iterated-logarithmic uniform continuity in z , i.e., there is a linearly growing function $\kappa(\cdot) \in \mathbf{S}$ such that $d\mathbb{P} \times dt - a.e.$,

$$|g(\omega, t, y, z_1) - g(\omega, t, y, z_2)| \leq \kappa\left(\frac{|z_1 - z_2|}{\mathcal{IL}_n^\lambda(|z_1 - z_2|)}\right), \quad \forall (y, z_1, z_2) \in \mathbb{R} \times \mathbb{R}^{1 \times d} \times \mathbb{R}^{1 \times d}.$$

Remark 2.1. *We have the following four assertions.*

(i) *Assumptions (H1), (H3) and (H4) are also required in [12]. And, assumptions (H2) $_1^\lambda$ and (H5) $_1^\lambda$ with $\lambda \in (1/2, 1]$ are the assumptions (H2) and (H5) of [12], respectively. Assumption (H2) $_1^0$ is exactly the one (H1) of [1], where g has a one-sided linear growth in y and a linear growth in z , and (H5) $_1^0$ is exactly the uniform continuity assumption of g in z of [7, 11].*

(ii) *For each $n \geq 1$ and $\lambda > 1/2$, we have*

$$\mathcal{IL}_{n+1}^\lambda(x) \leq K \mathcal{IL}_n^\lambda(x), \quad x \geq 0$$

with a constant $K > 0$ depending only on (n, λ) . Consequently, for any $\lambda > 1/2$, both conditions (H2) $_n^\lambda$ and (H5) $_n^\lambda$ become weak as the integer n increases.

(iii) *The inequality $\ln^{(n)}(e^{(n)}) \geq \ln e = 1$ is true for each $n \geq 1$. Both conditions (H2) $_n^\lambda$ and (H5) $_n^\lambda$ become weak as the parameter λ decreases in the interval $(\frac{1}{2}, \infty)$.*

(iv) *For each $n \geq 1$ and $k > e^{(n)}$, there is a constant $K > 0$ depending only on (n, k) such that*

$$1 \leq \frac{\ln^{(n)}(k+x)}{\ln^{(n)}(e^{(n)}+x)} \leq K, \quad x \geq 0.$$

Consequently, the constant $e^{(n)}$ appearing in (H2) $_n^\lambda$ and (H5) $_n^\lambda$ can be replaced with a larger number. In addition, the three assumptions (H3), (H4) and (H5) $_n^\lambda$ yield (H2) $_n^\lambda$.

Our main result is stated as the following existence and uniqueness theorem.

Theorem 2.2. *Let $n \geq 2$, $\lambda > 1/2$ and the generator g satisfy assumptions (H1), $(\text{H2})_n^\lambda$ and (H3). Then BSDE(ξ, g) admits a solution $(Y_t, Z_t)_{t \in [0, T]}$ such that $(Y, Z) \in \mathcal{S}^p \times \mathcal{M}^p$ with each $p \in (0, 1)$, and Y belongs to class (D). And, there exists a constant $C > 0$ depending only on $(\beta, \gamma, n, \lambda, T)$ such that*

$$|Y_t| \leq |Y_t| + \int_0^t f_s ds \leq C \mathbb{E} \left[|\xi| + \int_0^T f_s ds \mid \mathcal{F}_t \right] + C, \quad t \in [0, T]. \quad (2.1)$$

Moreover, if assumptions (H4) and $(\text{H5})_n^\lambda$ also hold for the generator g , then the solution (Y, Z) with Y being of class (D) is unique.

Remark 2.3. *With respect to Theorem 2.2, we make the following two remarks.*

(i) *In view of (i) and (ii) of Remark 2.1, Theorem 2.2 improves [12, Theorem 1], since it gives a sequence of weaker and weaker conditions on the generator g for the existence and uniqueness of the integrable solution of BSDE(ξ, g) under only integrable parameters. In addition, the condition of $Z \in \mathcal{M}^p$ for each $p \in (0, 1)$ is not required for the uniqueness in Theorem 2.2 as in [12, Theorem 1], while it is assumed in [1, 13, 7].*

(ii) *Both Theorem 1 of [12] and Theorem 2.2 show that under assumptions (H1) and (H3) on the generator g , the condition $(\text{H2})_n^\lambda$ with $n \geq 1$ and $\lambda > 1/2$ is sufficient to guarantee existence of the integrable solution of BSDE(ξ, g) with integrable parameters. And, [16] showed that $(\text{H2})_n^\lambda$ with $n = 1$ and $\lambda = 0$ is not enough. Assertion (iii) of Remark 2.1 states that assumption $(\text{H2})_n^\lambda$ becomes weak as the parameter λ decreases. It is still open whether the condition $(\text{H2})_n^\lambda$ with $n \geq 1$ and $\lambda \in (0, 1/2]$ is sufficient for the existence of an integrable solution.*

A general comparison result on the integrable solutions of BSDEs with integrable parameters is established in the following proposition, which generalizes Proposition 2.5 in [12] by (i) and (ii) of Remark 2.1, and naturally yields the uniqueness part in Theorem 2.2.

Proposition 2.4. *Let $n \geq 2$, $\lambda > 1/2$, ξ and ξ' be two terminal conditions, g and g' be two generators, and $(Y_t, Z_t)_{t \in [0, T]}$ and $(Y'_t, Z'_t)_{t \in [0, T]}$ be a solution of BSDE(ξ, g) and BSDE(ξ', g'), respectively. Suppose that g (resp. g') satisfies assumptions (H4) and $(\text{H5})_n^\lambda$ and $(Y - Y')^+$ is of class (D). If $\xi \leq \xi'$ and*

$$\mathbf{1}_{Y_t > Y'_t} (g(t, Y'_t, Z'_t) - g'(t, Y'_t, Z'_t)) \leq 0 \quad (\text{resp. } \mathbf{1}_{Y_t > Y'_t} (g(t, Y_t, Z_t) - g'(t, Y_t, Z_t)) \leq 0), \quad (2.2)$$

then we have $Y_t \leq Y'_t$ for each $t \in [0, T]$.

Let us further introduce the following assumptions on the generator g , which is closely related to assumptions $(\text{H2})_n^\lambda$ and $(\text{H5})_n^\lambda$. We would like to mention that some similar assumptions to the following $(\text{H2S})_\alpha$ and $(\text{H5S})_\alpha$ have been used in [1] and [21], respectively. It will be shown in the following Remark 2.5 that $(\text{H5})_n^\lambda$ and $(\text{H5S})_\alpha$ are respectively equivalent to the following $(\text{H5}')_n^\lambda$ and $(\text{H5S}')_\alpha$.

$(\text{H2S})_\alpha$ g has a one-sided linear growth in y and a sublinear growth in z , i.e., $d\mathbb{P} \times dt - a.e.$, for each

$$(y, z) \in \mathbb{R} \times \mathbb{R}^{1 \times d},$$

$$\text{sgn}(y)g(\omega, t, y, z) \leq f_t(\omega) + \beta|y| + \gamma|z|^\alpha.$$

(H5S) $_{\alpha}$ g satisfies a quasi-Hölder continuity condition in z , i.e., there exists a function $\bar{\kappa}(\cdot) \in \mathbf{S}$ with linear growth such that $d\mathbb{P} \times dt - a.e.$, for each $(y, z_1, z_2) \in \mathbb{R} \times \mathbb{R}^{1 \times d} \times \mathbb{R}^{1 \times d}$,

$$|g(\omega, t, y, z_1) - g(\omega, t, y, z_2)| \leq \bar{\kappa}(|z_1 - z_2|^{\alpha}).$$

(H5') $_{n}^{\lambda}$ There exists a constant $A > 0$ and a function $\tilde{\kappa}(\cdot) \in \mathbf{S}$ with linear growth such that $d\mathbb{P} \times dt - a.e.$, for each $(y, z_1, z_2) \in \mathbb{R} \times \mathbb{R}^{1 \times d} \times \mathbb{R}^{1 \times d}$,

$$|g(\omega, t, y, z_1) - g(\omega, t, y, z_2)| \leq \tilde{\kappa}(|z_1 - z_2|) \quad (2.3)$$

and

$$|g(\omega, t, y, z_1) - g(\omega, t, y, z_2)| \leq \frac{A|z_1 - z_2|}{\mathcal{I}\mathcal{L}_n^{\lambda}(|z_1 - z_2|)} + A. \quad (2.4)$$

(H5S') $_{\alpha}$ There exists a constant $A > 0$ and a function $\tilde{\kappa}(\cdot) \in \mathbf{S}$ with linear growth such that $d\mathbb{P} \times dt - a.e.$, for each $(y, z_1, z_2) \in \mathbb{R} \times \mathbb{R}^{1 \times d} \times \mathbb{R}^{1 \times d}$, (2.3) holds and

$$|g(\omega, t, y, z_1) - g(\omega, t, y, z_2)| \leq A|z_1 - z_2|^{\alpha} + A.$$

Remark 2.5. *We have the following several remarks.*

(i) For each $\alpha \in (0, 1)$, $n \geq 1$ and $\lambda \geq 0$, there exists a constant $K > 0$ depending only on (α, n, λ) such that

$$|x|^{\alpha} \leq \frac{K|x|}{\mathcal{I}\mathcal{L}_n^{\lambda}(|x|)}, \quad x \in \mathbb{R},$$

which means that $(\text{H2S})_{\alpha} \Rightarrow (\text{H2})_n^{\lambda}$ and $(\text{H5S})_{\alpha} \Rightarrow (\text{H5})_n^{\lambda}$. In addition, the bigger the α , the weaker the assumptions $(\text{H2S})_{\alpha}$ and $(\text{H5S})_{\alpha}$. In fact, the assertion on $(\text{H2S})_{\alpha}$ is obvious. And, it can be easily proved that for each $0 < \alpha < \bar{\alpha} < 1$, if the generator g satisfies $(\text{H5S})_{\alpha}$ with the function $\bar{\kappa}(\cdot)$, then it has to satisfy $(\text{H5S})_{\bar{\alpha}}$ with the function

$$\tilde{\kappa}(x) := \bar{\kappa}(x^{\frac{\alpha}{\bar{\alpha}}})\mathbf{1}_{0 \leq x \leq 1} + \bar{\kappa}(x)\mathbf{1}_{x > 1}.$$

(ii) It holds that $(\text{H5})_n^{\lambda} \Leftrightarrow (\text{H5}')_n^{\lambda}$ for each $n \geq 1$ and $\lambda \geq 0$. In fact, since $\ln^{(i)}(e^{(n)}) \geq 1$ for each $n \geq 1$ and $i = 1, \dots, n$ and the function $\kappa(\cdot)$ in $(\text{H5})_n^{\lambda}$ is of linear growth, the statement of $(\text{H5})_n^{\lambda} \Rightarrow (\text{H5}')_n^{\lambda}$ is obvious. Conversely, suppose that $(\text{H5}')_n^{\lambda}$ holds with a constant $A > 0$ and a function $\tilde{\kappa}(\cdot)$. Observe that for each $n \geq 1$ and $\lambda \geq 0$, there exist two constants $K_1, K_2 > 0$ depending only on (n, λ) such that

$$\mathcal{I}\mathcal{L}_n^{\lambda}(x) \leq K_1, \quad x \in [0, 1]$$

and

$$1 \leq \frac{K_2 x}{\mathcal{I}\mathcal{L}_n^{\lambda}(x)}, \quad x \in (1, +\infty).$$

It then follows from (2.3) and (2.4) that $(\text{H5})_n^{\lambda}$ holds for the generator g with the function

$$\kappa(x) := \begin{cases} \tilde{\kappa}(K_1 x)\mathbf{1}_{0 \leq x \leq 1} + \tilde{\kappa}(K_1)x\mathbf{1}_{x > 1}, & \text{if } \tilde{\kappa}(K_1) \geq (1 + K_2)A; \\ \frac{(1+K_2)A}{\tilde{\kappa}(K_1)}\tilde{\kappa}(K_1 x)\mathbf{1}_{0 \leq x \leq 1} + (1 + K_2)Ax\mathbf{1}_{x > 1}, & \text{if } \tilde{\kappa}(K_1) < (1 + K_2)A. \end{cases}$$

Similarly, it also holds that $(\text{H5S})_\alpha \Leftrightarrow (\text{H5S}')_\alpha$ for each $\alpha \in (0, 1)$. In particular, from the above assertions we can deduce that if the generator g is uniformly continuous and has a bounded growth in z , then it must satisfy $(\text{H5S})_\alpha$ and then $(\text{H5})_n^\lambda$ for each $\alpha \in (0, 1)$, $n \geq 1$ and $\lambda \geq 0$.

(iii) In view of (ii) and (iii) in Remark 2.1 and the above (i), the following assertions can be verified.

- If g_1 and g_2 satisfy respectively $(\text{H5})_n^\lambda$ and $(\text{H5})_n^{\bar{\lambda}}$ with $n \geq 1$ and $0 \leq \lambda \leq \bar{\lambda}$, then any one of their linear combinations must satisfy $(\text{H5})_n^{\bar{\lambda}}$.
- If g_1 and g_2 satisfy respectively $(\text{H5})_n^\lambda$ and $(\text{H5})_m^{\bar{\lambda}}$ with $1 \leq n < m$, $\lambda > 1/2$ and $\bar{\lambda} \geq 0$, then any one of their linear combinations must satisfy $(\text{H5})_m^{\bar{\lambda}}$.
- If the generators g_1 and g_2 satisfy respectively $(\text{H5S})_\alpha$ and $(\text{H5S})_{\bar{\alpha}}$ with $0 < \alpha \leq \bar{\alpha} < 1$, then any one of their linear combinations satisfy $(\text{H5S})_{\bar{\alpha}}$ and then $(\text{H5})_n^\lambda$ for each $n \geq 1$ and $\lambda \geq 0$.

Finally, let us give two examples to which Theorem 2.2 applies, but none of existing results could.

Example 2.6. For each $(\omega, t, y, z) \in \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^{1 \times d}$, define

$$g(\omega, t, y, z) := B_t(\omega) - e^y \sin^2 |z| + \frac{|z| \cos |z|}{\sqrt{\ln(e^8 + |z|)} (\ln \ln(e^8 + |z|))^{3/4}} - |z|^2 \sin y.$$

It is not hard to check that this generator g satisfies assumptions (H1), $(\text{H2})_n^\lambda$ and (H3) with

$$f = B + 1, \quad \beta = 0, \quad \gamma = 1, \quad n = 2, \quad \lambda = 3/4, \quad c = 1 \quad \text{and} \quad h(u) = e^u.$$

Then, by Theorem 2.2 it can be concluded that $\text{BSDE}(\xi, g)$ admits a solution $(Y_t, Z_t)_{t \in [0, T]}$ such that $(Y, Z) \in \mathcal{S}^p \times \mathcal{M}^p$ with each $p \in (0, 1)$, and Y is of class (D).

Example 2.7. For each $(\omega, t, y, z) \in \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^{1 \times d}$, define

$$\bar{g}(\omega, t, y, z) := y^4 \mathbf{1}_{y \leq 0} + l(|y|) + \sin |z| + \sqrt{|z|} + |z|^{1/3} + \frac{|z|}{\ln(e + |z|)} + \frac{|z|}{\sqrt{\ln(e^8 + |z|)} (\ln \ln(e^8 + |z|))^{3/2}},$$

where

$$l(u) := u |\ln u| \ln |\ln u| \mathbf{1}_{0 \leq u \leq \varepsilon} + l'_-(\varepsilon)(u - l(\varepsilon)) \mathbf{1}_{u > \varepsilon}, \quad u \in (0, +\infty)$$

with $\varepsilon > 0$ being sufficiently small. Note that for each $x_1, x_2 \geq 0$, we have $|F(x_1) - F(x_2)| \leq F(|x_1 - x_2|)$ for any concave function $F(\cdot) \in \mathbf{S}$. It can be verified that for each $x_1, x_2 \geq 0$,

$$\begin{aligned} |l(x_1) - l(x_2)| &\leq l(|x_1 - x_2|) \quad \text{with} \quad \int_{0^+} \frac{du}{l(u)} = +\infty, \\ |\sin x_1 - \sin x_2| &\leq |x_1 - x_2|, \quad |\sin x_1 - \sin x_2| \leq 2, \\ |\sqrt{x_1} - \sqrt{x_2}| &\leq \sqrt{|x_1 - x_2|}, \quad \left| x_1^{1/3} - x_2^{1/3} \right| \leq |x_1 - x_2|^{1/3}, \\ \left| \frac{x_1}{\ln(e + x_1)} - \frac{x_2}{\ln(e + x_2)} \right| &\leq \frac{|x_1 - x_2|}{\ln(e + |x_1 - x_2|)} \end{aligned}$$

and

$$\begin{aligned} &\left| \frac{x_1}{\sqrt{\ln(e^8 + x_1)} (\ln \ln(e^8 + x_1))^{3/2}} - \frac{x_2}{\sqrt{\ln(e^8 + x_2)} (\ln \ln(e^8 + x_2))^{3/2}} \right| \\ &\leq \frac{|x_1 - x_2|}{\sqrt{\ln(e^8 + |x_1 - x_2|)} \ln \ln(e^8 + |x_1 - x_2|)^{3/2}}. \end{aligned}$$

Based on these above observations, by virtue of Remark 2.5 we can easily prove that this generator \bar{g} satisfies assumptions (H1), (H2) $^\lambda_n$, (H3), (H4) and (H5) $^\lambda_n$ with $\rho(u) = l(u)$, $n = 2$ and $\lambda = 2/3$. Then, by Theorem 2.2 it can be concluded that BSDE(ξ, g) admits a unique solution $(Y_t, Z_t)_{t \in [0, T]}$ such that $(Y, Z) \in \mathcal{S}^p \times \mathcal{M}^p$ with each $p \in (0, 1)$ and Y is of class (D).

3. Proof of the main result

First, we have the following crucial inequality.

Proposition 3.1. *Let $p > 1$ and $\psi(\cdot) : [0, +\infty) \rightarrow [1, +\infty)$ be a twice continuously differentiable function such that for each $x \geq 0$, $\psi'(x) > 0$, $\psi''(x) < 0$,*

$$x (\ln \psi(x))' \leq \frac{1}{4} \wedge (1 - \frac{1}{\sqrt{p}}), \quad (3.1)$$

$$-x (\ln \psi'(x))' \leq \frac{3}{2}, \quad (3.2)$$

and

$$\psi(\sqrt{p}x\psi(x)) \leq \sqrt{p}\psi(x). \quad (3.3)$$

Then, we have

$$\frac{2xy}{\psi(y)} \leq \frac{px^2}{\psi^2(x)} + y^2, \quad \forall (x, y) \in [0, \infty) \times [0, \infty). \quad (3.4)$$

Proof. For $(x, y) \in [0, +\infty) \times [0, +\infty)$, define the function

$$f(x, y) := y^2 - \frac{2xy}{\psi(y)} + \frac{px^2}{\psi^2(x)} = \left(y - \frac{x}{\psi(y)}\right)^2 + \frac{px^2}{\psi^2(x)} - \frac{x^2}{\psi^2(y)}. \quad (3.5)$$

Clearly, it suffices to prove that $f(x, y) \geq 0$ for each $x, y \geq 0$. By (3.5) it is obvious for $y \geq x$. Hence, we only need to prove that $f(x, y) \geq 0$ for each $x \in (0, +\infty)$ and $y \in [0, x]$.

Now, fix arbitrary $x \in (0, +\infty)$ and let $\bar{f}(y) := f(x, y)$, $y \in [0, x]$. Then we have

$$\bar{f}'(y) = 2y - 2x \left(\frac{y}{\psi(y)}\right)' = 2y - \frac{2x(\psi(y) - y\psi'(y))}{\psi^2(y)}, \quad y \in [0, x] \quad (3.6)$$

and

$$\bar{f}''(y) = 2 - 2x \left(\frac{y}{\psi(y)}\right)'' = 2 + \frac{2x(2\psi(y)\psi'(y) - 2y(\psi'(y))^2 + y\psi(y)\psi''(y))}{\psi^3(y)}, \quad y \in [0, x]. \quad (3.7)$$

It follows from (3.1) and (3.2) that

$$2y(\psi'(y))^2 - y\psi(y)\psi''(y) \leq \frac{1}{2}\psi(y)\psi'(y) + \frac{3}{2}\psi(y)\psi'(y) = 2\psi(y)\psi'(y),$$

which together with (3.7) yields that $\bar{f}''(\cdot) > 0$, and then $\bar{f}(\cdot)$ is a strictly convex function on $[0, x]$.

Furthermore, note by (3.6) that $\bar{f}'(0) = -2x/\psi(0) < 0$ and

$$\bar{f}'(x) = 2x - \frac{2x}{\psi(x)} + \frac{2x^2\psi'(x)}{\psi^2(x)} \geq \frac{2x^2\psi'(x)}{\psi^2(x)} > 0.$$

It follows that there exists a unique $y_0 \in (0, x)$ such that $\bar{f}'(y_0) = 0$ and

$$f(x, y) = \bar{f}(y) \geq \bar{f}(y_0) = f(x, y_0), \quad y \in [0, x]. \quad (3.8)$$

In the sequel, since the function $y\psi(y)$, $y \in [0, +\infty)$ is strictly increasing with its range being $[0, +\infty)$, we can conclude that there exists a unique real $y_1 \in (0, +\infty)$ such that

$$y_1 = \frac{x}{\sqrt{p}\psi(y_1)}. \quad (3.9)$$

Then, $y_1 \in (0, x)$ and it follows from (3.6), (3.9) and (3.1) that

$$\bar{f}'(y_1) = 2y_1 - \frac{2x}{\psi(y_1)} + \frac{2xy_1\psi'(y_1)}{\psi^2(y_1)} = -\frac{2x \left[\left(1 - \frac{1}{\sqrt{p}}\right) \psi(y_1) - y_1\psi'(y_1) \right]}{\psi^2(y_1)} \leq 0.$$

Therefore, $y_1 \leq y_0$ and then by (3.8) and (3.5) we deduce that

$$f(x, y) \geq f(x, y_0) = \left(y_0 - \frac{x}{\psi(y_0)} \right)^2 + \frac{px^2}{\psi^2(x)} - \frac{x^2}{\psi^2(y_0)} \geq \frac{px^2}{\psi^2(x)} - \frac{x^2}{\psi^2(y_1)}, \quad y \in [0, x]. \quad (3.10)$$

Finally, it follows from (3.3) and (3.9) that

$$\sqrt{p}\psi(y_1) \geq \psi(\sqrt{p}y_1\psi(y_1)) = \psi(x). \quad (3.11)$$

Then, by (3.10) and (3.11) we obtain that $f(x, y) \geq 0$ for each $x \in (0, +\infty)$ and $y \in [0, x]$, which is the desired conclusion. \square

By virtue of Proposition 3.1, we can establish the following key inequality.

Proposition 3.2. *Let $n \geq 1$, $\lambda \geq 0$ and $p > 1$. Then, there exists a positive constant $k_{n,\lambda,p} \geq e^{(n)}$ depending only on (n, λ, p) such that for each $k \geq k_{n,\lambda,p}$, we have*

$$\frac{2xy}{\mathcal{I}\mathcal{L}_{n,k}^\lambda(y)} \leq \frac{px^2}{\left(\mathcal{I}\mathcal{L}_{n,k}^\lambda(x)\right)^2} + y^2, \quad x, y \geq 0, \quad (3.12)$$

where and hereafter,

$$\mathcal{I}\mathcal{L}_{n,k}^\lambda(x) := \prod_{i=1}^{n-1} \sqrt{\ln^{(i)}(k+x)} \left(\ln^{(n)}(k+x)\right)^\lambda.$$

Proof. Note that for each $m \geq 2$, the case of $n = m$ and $\lambda = 0$ is just the case of $n = m - 1$ and $\lambda = 1/2$, and the case of $n = 1$ and $\lambda = 0$ is evident from the basic inequality. It suffices to prove the case of $n \geq 1$ and $\lambda > 0$. Fix a sufficiently large $k \geq e^{(n)}$ and let

$$\psi(x) := \mathcal{I}\mathcal{L}_{n,k}^\lambda(x) \geq 1, \quad x \in [0, +\infty).$$

We prove that for a sufficient large k , the function $\psi(\cdot)$ satisfies (3.1)-(3.3). Observe that for each $n \geq 1$ and $x \geq 0$, we have

$$(\ln^{(n)}(k+x))' = \frac{(\ln^{(n-1)}(k+x))'}{\ln^{(n-1)}(k+x)} = \dots = \frac{1}{(k+x) \prod_{j=1}^{n-1} \ln^{(j)}(k+x)}.$$

For each $n \geq 1$ and $x \geq 0$, we can calculate that

$$\ln \psi(x) = \frac{1}{2} \sum_{i=1}^{n-1} \ln^{(i+1)}(k+x) + \lambda \ln^{(n+1)}(k+x)$$

and, in view of k being large enough,

$$\begin{aligned} 0 \leq x (\ln \psi(x))' &= \frac{x}{k+x} \left\{ \frac{1}{2} \sum_{i=1}^{n-1} \left(\frac{1}{\prod_{j=1}^i \ln^{(j)}(k+x)} \right) + \frac{\lambda}{\prod_{j=1}^n \ln^{(j)}(k+x)} \right\} \\ &\leq \frac{\frac{n-1}{2} + \lambda}{\ln k} \leq \frac{1}{4} \wedge \left(1 - \frac{1}{\sqrt{p}}\right), \end{aligned} \quad (3.13)$$

which means that the function $\psi(\cdot)$ satisfies (3.1).

Furthermore, it is not very difficult to verify that for each $n \geq 1$ and $x \geq 0$,

$$\psi'(x) = (\ln \psi(x))' \psi(x) = \frac{\left[\sum_{i=1}^{n-1} \left(\frac{1}{\prod_{j=i+1}^{n-1} \ln^{(j)}(k+x)} \right) + \frac{2\lambda}{\ln^{(n)}(k+x)} \right] \psi(x)}{2(k+x) \prod_{j=1}^{n-1} \ln^{(j)}(k+x)} > 0$$

and

$$\ln \psi'(x) = \ln \psi_0(x) + \ln \psi(x) - \ln 2 - \sum_{i=0}^{n-1} \ln^{(i+1)}(k+x)$$

with

$$\psi_0(x) := \sum_{i=1}^{n-1} \left(\frac{1}{\prod_{j=i+1}^{n-1} \ln^{(j)}(k+x)} \right) + \frac{2\lambda}{\ln^{(n)}(k+x)} \geq \frac{2\lambda}{\ln(k+x)} \mathbf{1}_{n=1} + \mathbf{1}_{n \geq 2}. \quad (3.14)$$

Note that for each $n \geq 1$ and $x \geq 0$, we have with $i = 1, \dots, n-1$,

$$\left(\prod_{j=i+1}^{n-1} \ln^{(j)}(k+x) \right)' = \left(\sum_{j=i+1}^{n-1} \ln^{(j+1)}(k+x) \right)' \prod_{l=i+1}^{n-1} \ln^{(l)}(k+x) = \frac{\sum_{j=i+1}^{n-1} \left(\prod_{l=j+1}^{n-1} \ln^{(l)}(k+x) \right)}{(k+x) \prod_{l=1}^i \ln^{(l)}(k+x)} \geq 0,$$

and then

$$\psi_0'(x) = \sum_{i=1}^{n-1} \left\{ \frac{\sum_{j=i+1}^{n-1} \left(\prod_{l=j+1}^{n-1} \ln^{(l)}(k+x) \right)}{(k+x) \prod_{l=1}^i \ln^{(l)}(k+x)} \right\} - \frac{2\lambda}{(k+x) \prod_{i=1}^n \ln^{(i)}(k+x) \ln^{(n)}(k+x)}.$$

From (3.14) and (3.13), we deduce that for a sufficiently large k ,

$$\begin{aligned} -(\ln \psi'(x))' &= -\frac{\psi_0'(x)}{\psi_0(x)} - (\ln \psi(x))' + \frac{1}{k+x} \sum_{i=0}^{n-1} \left(\frac{1}{\prod_{j=1}^i \ln^{(j)}(k+x)} \right) \\ &\leq 0 + \frac{1}{k+x} \left(\frac{1}{\ln(k+x)} + \frac{2\lambda}{\ln(k+x)} \right) + 0 + \frac{1}{k+x} \left(1 + \frac{n-1}{\ln(k+x)} \right) \\ &\leq \frac{1}{x} \left(1 + \frac{n+2\lambda}{\ln k} \right) \leq \frac{3}{2x}, \quad x > 0, \end{aligned}$$

which yields that the function $\psi(\cdot)$ satisfies (3.2).

In the sequel, we prove that the function $\psi(\cdot)$ also satisfies (3.3). In fact, fix $n \geq 2$ and $\lambda > 0$, and set $\delta := p^{\frac{1}{n-1+2\lambda}} > 1$. We pick k large enough such that for each $x \geq 0$ and $i = 1, \dots, n$,

$$k + \sqrt{p}x\psi(x) \leq \sqrt{p}(k+x)\psi(x) \leq (k+x)^\delta \quad \text{and} \quad \delta \ln^{(i)}(k+x) \leq (\ln^{(i)}(k+x))^\delta. \quad (3.15)$$

Then, for each $i = 1, \dots, n$, we have

$$\ln^{(i)}(k + \sqrt{px}\psi(x)) \leq \delta \ln^{(i)}(k + x), \quad x \geq 0. \quad (3.16)$$

The last inequality can be proved by induction. In fact, (3.16) is clear for $i = 1$ since it follows from (3.15) that

$$\ln^{(1)}(k + \sqrt{px}\psi(x)) \leq \ln^{(1)}[(k + x)^\delta] = \delta \ln^{(1)}(k + x), \quad x \geq 0.$$

Now, assume that (3.16) holds for some $i = l$ with $l \in \{1, \dots, n-1\}$. Then, in view of (3.16) and (3.15) we can deduce that

$$\begin{aligned} \ln^{(l+1)}(k + \sqrt{px}\psi(x)) &= \ln^{(1)}[\ln^{(l)}(k + \sqrt{px}\psi(x))] \leq \ln^{(1)}[\delta \ln^{(l)}(k + x)] \leq \ln^{(1)}[(\ln^{(l)}(k + x))^\delta] \\ &= \delta \ln^{(1)}[\ln^{(l)}(k + x)] = \delta \ln^{(l+1)}(k + x), \quad x \geq 0, \end{aligned}$$

which means that (3.16) also holds for $i = l + 1$. Hence, (3.16) is true for each $i = 1, \dots, n$, and then

$$\begin{aligned} \psi(\sqrt{px}\psi(x)) &= \prod_{i=1}^{n-1} \sqrt{\ln^{(i)}(k + \sqrt{px}\psi(x))} \left(\ln^{(n)}(k + \sqrt{px}\psi(x)) \right)^\lambda \\ &\leq \delta^{\frac{n-1}{2} + \lambda} \prod_{i=1}^{n-1} \sqrt{\ln^{(i)}(k + x)} \left(\ln^{(n)}(k + x) \right)^\lambda = \sqrt{p}\psi(x), \quad x \geq 0. \end{aligned}$$

Then, (3.3) is true for $\psi(\cdot)$. Up to now, we have proved that the function $\psi(\cdot)$ satisfies all conditions in Proposition 3.1, by which the desired inequality (3.12) follows immediately. \square

Remark 3.3. *With respect to the above Proposition 3.2, we make the following remarks.*

(i) *The case of $n = 1$ and $\lambda \in [0, 1]$ of Proposition 3.2 is Proposition 3.2 of [12].*

(ii) *Except from the case of $n = 1$ and $\lambda = 0$, the inequality (3.12) does not hold when $p \leq 1$. In fact, let $n \geq 1$, $\lambda > 0$ and $k \geq e^{(n)}$ be any constant. Assume that constants $x, y > 0$ satisfy*

$$y := \frac{x}{\mathcal{I}\mathcal{L}_{n,k}^\lambda(y)} < x.$$

Then, in view of (3.5),

$$y^2 - \frac{2xy}{\mathcal{I}\mathcal{L}_{n,k}^\lambda(y)} + \frac{x^2}{\left(\mathcal{I}\mathcal{L}_{n,k}^\lambda(x)\right)^2} = \frac{x^2}{\left(\mathcal{I}\mathcal{L}_{n,k}^\lambda(x)\right)^2} - \frac{x^2}{\left(\mathcal{I}\mathcal{L}_{n,k}^\lambda(y)\right)^2} < 0,$$

which immediately yields the desired assertion.

Now, let $n \geq 2$, $\lambda > 1/2$ and $k \geq e^{(n)}$ be sufficiently large and depends only on (n, λ) such that the inequality (3.12) with $p = 2$ in Proposition 3.2 holds. A $C^{1,2}$ function $\phi : [0, T] \times [0, +\infty) \rightarrow (0, +\infty)$ will be called a test function, if $\phi_s > 0$, $\phi_x > 0$, $\phi_{xx} > 0$, and

$$-\beta\phi_x(s, x)x - \phi_x(s, x)\frac{\gamma|z|}{\mathcal{I}\mathcal{L}_{n,k}^\lambda(|z|)} + \frac{1}{2}\phi_{xx}(s, x)|z|^2 + \phi_s(s, x) \geq 0, \quad (3.17)$$

$$(s, x, z) \in [0, T] \times [0, +\infty) \times \mathbb{R}^{1 \times d}.$$

Here and hereafter, ϕ_s is the first-order partial derivative of ϕ in the first variable, and ϕ_x and ϕ_{xx} are respectively the first- and second-order partial derivative of ϕ in the second variable. Since (in view of (3.12) with $p = 2$)

$$\begin{aligned} -\phi_x(s, x) \frac{\gamma|z|}{\mathcal{I}\mathcal{L}_{n,k}^\lambda(|z|)} + \frac{1}{2}\phi_{xx}(s, x)|z|^2 &= \phi_{xx}(s, x) \left(-\frac{\gamma\phi_x(s, x)}{\phi_{xx}(s, x)} \frac{|z|}{\mathcal{I}\mathcal{L}_{n,k}^\lambda(|z|)} + \frac{1}{2}|z|^2 \right) \\ &\geq -\frac{\gamma^2\phi_x^2(s, x)}{\phi_{xx}(s, x) \left(\mathcal{I}\mathcal{L}_{n,k}^\lambda \left(\frac{\gamma\phi_x(s, x)}{\phi_{xx}(s, x)} \right) \right)^2}, \end{aligned}$$

a $C^{1,2}$ function $\phi(\cdot, \cdot)$ will be a test function if $\phi > 0, \phi_s > 0, \phi_x > 0, \phi_{xx} > 0$, and moreover, for each $(s, x) \in [0, T] \times [0, +\infty)$,

$$-\beta\phi_x(s, x)x - \frac{\gamma^2\phi_x^2(s, x)}{\phi_{xx}(s, x) \left(\mathcal{I}\mathcal{L}_{n,k}^\lambda \left(\frac{\gamma\phi_x(s, x)}{\phi_{xx}(s, x)} \right) \right)^2} + \phi_s(s, x) \geq 0. \quad (3.18)$$

In the sequel, we choose the following function

$$\phi(s, x) := (k+x) \left[1 - \left(\ln^{(n)}(k+x) \right)^{1-2\lambda} \right] \mu_s, \quad (s, x) \in [0, T] \times [0, +\infty)$$

to explicitly solve (3.18), where $\mu_s : [0, T] \rightarrow (0, +\infty)$ is a nondecreasing and continuously differentiable function to be assigned. First of all, letting further the constant $k \geq e^{(n)}$ depending only on (n, λ, γ) be large enough, by a simple computation we can obtain that for each $(s, x) \in [0, T] \times [0, +\infty)$,

$$\phi_x(s, x) = \left[1 - \frac{1}{\left(\ln^{(n)}(k+x) \right)^{2\lambda-1}} \left(1 - \frac{2\lambda-1}{\prod_{i=1}^n \ln^{(i)}(k+x)} \right) \right] \mu_s > 0,$$

$$\phi_{xx}(s, x) = \frac{2\lambda-1}{(k+x) \left(\mathcal{I}\mathcal{L}_{n,k}^\lambda(x) \right)^2} \left(1 - \frac{2\lambda-1}{\prod_{i=1}^n \ln^{(i)}(k+x)} - \frac{1}{\left(\prod_{i=1}^n \ln^{(i)}(k+x) \right)^2} \right) \mu_s > 0$$

and

$$\phi_s(s, x) = (k+x) \left[1 - \left(\ln^{(n)}(k+x) \right)^{1-2\lambda} \right] \mu'_s > 0,$$

which yields that for $(s, x) \in [0, T] \times [0, +\infty)$,

$$\frac{1}{2}\mu_s \leq \phi_x(s, x) \leq \mu_s, \quad (3.19)$$

$$\frac{(2\lambda-1)\mu_s}{2(k+x) \left(\mathcal{I}\mathcal{L}_{n,k}^\lambda(x) \right)^2} \leq \phi_{xx}(s, x) \leq \frac{(2\lambda-1)\mu_s}{(k+x) \left(\mathcal{I}\mathcal{L}_{n,k}^\lambda(x) \right)^2}, \quad (3.20)$$

$$\phi_s(s, x) \geq \frac{1}{2}(k+x)\mu'_s, \quad (3.21)$$

and then

$$\frac{\gamma\phi_x(s, x)}{\phi_{xx}(s, x)} \geq \frac{\gamma}{2(2\lambda-1)}(k+x) \left(\mathcal{I}\mathcal{L}_{n,k}^\lambda(x) \right)^2 \geq k+x. \quad (3.22)$$

Substituting (3.19)-(3.22) into the left hand side of (3.18), we have

$$\begin{aligned}
& -\beta\phi_x(s, x)x - \frac{\gamma^2\phi_x^2(s, x)}{\phi_{xx}(s, x)\left(\mathcal{I}\mathcal{L}_{n,k}^\lambda\left(\frac{\gamma\phi_x(s, x)}{\phi_{xx}(s, x)}\right)\right)^2} + \phi_s(s, x) \\
& \geq -\beta(k+x)\mu_s - \frac{\gamma^2\mu_s^2}{\frac{(2\lambda-1)\mu_s}{2(k+x)(\mathcal{I}\mathcal{L}_{n,k}^\lambda(x))^2}\left(\mathcal{I}\mathcal{L}_{n,k}^\lambda(k+x)\right)} + \frac{1}{2}(k+x)\mu'_s \\
& \geq (k+x)\left[-\left(\beta + \frac{2\gamma^2}{2\lambda-1}\right)\mu_s + \frac{1}{2}\mu'_s\right], \quad (s, x) \in [0, T] \times [0, +\infty).
\end{aligned}$$

Thus, if we take

$$\mu_s := \exp\left[2\left(\beta + \frac{2\gamma^2}{2\lambda-1}\right)s\right], \quad s \in [0, T],$$

then (3.18) and then (3.17) holds.

Define the function for $k \geq e^{(n)}$,

$$\varphi(s, x) := (k+x)\left[1 - \left(\ln^{(n)}(k+x)\right)^{1-2\lambda}\right] \exp\left[2\left(\beta + \frac{2\gamma^2}{2\lambda-1}\right)s\right], \quad (s, x) \in [0, T] \times [0, +\infty). \quad (3.23)$$

We have established the following proposition on the test function ϕ .

Proposition 3.4. *Let $n \geq 2$, $\lambda > 1/2$ and $k \geq e^{(n)}$ be a sufficiently large constant depending only on (n, λ, γ) and such that the inequalities (3.12) (with $p = 2$) and (3.19)-(3.22) are all satisfied. Then, we have for each $(s, x, z) \in [0, T] \times [0, +\infty) \times \mathbb{R}^{1 \times d}$,*

$$-\beta\varphi_x(s, x)x - \varphi_x(s, x)\frac{\gamma|z|}{\mathcal{I}\mathcal{L}_{n,k}^\lambda(|z|)} + \frac{1}{2}\varphi_{xx}(s, x)|z|^2 + \varphi_s(s, x) \geq 0. \quad (3.24)$$

The following Proposition 3.5 gives an a priori estimate for the solution to a BSDE.

Proposition 3.5. *Assume that $n \geq 2$, $\lambda > 1/2$, the generator g satisfies assumption $(H2)_n^\lambda$, and $(Y_t, Z_t)_{t \in [0, T]}$ is a solution of BSDE (ξ, g) . If the process $(|Y_t| + \int_0^t f_s ds)_{t \in [0, T]}$ is of class (D), then there exists a constant $C > 0$ depending only on $(n, \beta, \gamma, \lambda, T)$ such that*

$$|Y_t| \leq |Y_t| + \int_0^t f_s ds \leq C\mathbb{E}\left[|\xi| + \int_0^T f_t dt \middle| \mathcal{F}_t\right] + C, \quad t \in [0, T]. \quad (3.25)$$

Proof. In view of (iv) of Remark 2.1, we suppose that the generator g satisfies assumptions $(H2)_n^\lambda$ with $\mathcal{I}\mathcal{L}_n^\lambda$ being replaced with $\mathcal{I}\mathcal{L}_{n,k}^\lambda$ for a sufficiently large constant $k \geq e^{(n)}$ (as given in Proposition 3.4).

Define

$$\bar{Y}_t := |Y_t| + \int_0^t f_s ds \quad \text{and} \quad \bar{Z}_t := \text{sgn}(Y_t)Z_t, \quad t \in [0, T].$$

Using Itô-Tanaka's formula, we have

$$\bar{Y}_t = \bar{Y}_T + \int_t^T (\text{sgn}(Y_s)g(s, Y_s, Z_s) - f_s) ds - \int_t^T \bar{Z}_s dB_s - \int_t^T dL_s, \quad t \in [0, T],$$

where L is the local time of Y at 0. Applying Itô-Tanaka's formula with the test function φ (see (3.23))

for the definition), and noting the assumption $(\text{H2})_n^\lambda$ (but with $\mathcal{I}\mathcal{L}_n^\lambda$ being replaced with $\mathcal{I}\mathcal{L}_{n,k}^\lambda$), we have

$$\begin{aligned} d\varphi(s, \bar{Y}_s) &= \varphi_x(s, \bar{Y}_s) (-\text{sgn}(Y_s)g(s, Y_s, Z_s) + f_s) ds + \varphi_x(s, \bar{Y}_s)\bar{Z}_s dB_s \\ &\quad + \varphi_x(s, \bar{Y}_s)dL_s + \frac{1}{2}\varphi_{xx}(s, \bar{Y}_s)|Z_s|^2 ds + \varphi_s(s, \bar{Y}_s)ds \\ &\geq \left[-\varphi_x(s, \bar{Y}_s) \left(\beta|Y_s| + \frac{\gamma|Z_s|}{\mathcal{I}\mathcal{L}_{n,k}^\lambda(|Z_s|)} \right) + \frac{1}{2}\varphi_{xx}(s, \bar{Y}_s)|Z_s|^2 + \varphi_s(s, \bar{Y}_s) \right] ds \\ &\quad + \varphi_x(s, \bar{Y}_s)\bar{Z}_s dB_s, \quad s \in [0, T]. \end{aligned}$$

Then, since $|Y_s| \leq \bar{Y}_s$, we have from (3.24) in Proposition 3.4,

$$d\varphi(s, \bar{Y}_s) \geq \varphi_x(s, \bar{Y}_s)\bar{Z}_s dB_s, \quad s \in [0, T].$$

In the sequel, observe from (3.23) that

$$\frac{1}{2}(k+x) \leq \varphi(s, x) \leq k_1(k+x), \quad (s, x) \in [0, T] \times [0, +\infty),$$

with

$$k_1 := \exp \left[2 \left(\beta + \frac{2\gamma^2}{2\lambda - 1} \right) T \right].$$

By a similar analysis as that in Proposition 3.4 of [12], we conclude that

$$\frac{1}{2}(k + \bar{Y}_t) \leq k_1 \mathbb{E}[(k + \bar{Y}_T) | \mathcal{F}_t], \quad t \in [0, T],$$

which immediately yields the desired inequality (3.25). \square

We now prove Theorem 2.2 and Proposition 2.4.

Proof of Theorem 2.2. Assume first that g satisfies assumptions (H1), $(\text{H2})_n^\lambda$ and (H3). With Proposition 3.5 in hand, using the localization technique put forward initially in [2] and following closely the proof of Theorem 2.1 in [12], we can construct a solution $(Y_t, Z_t)_{t \in [0, T]}$ of BSDE (ξ, g) such that $(Y, Z) \in \mathcal{S}^p \times \mathcal{M}^p$ with each $p \in (0, 1)$ and Y is of class (D). The details are omitted here. Furthermore, suppose that g also satisfies assumptions (H4) and $(\text{H5})_n^\lambda$. The uniqueness part of Theorem 2.2 is a direct consequence of Proposition 2.4, whose proof will be given below. The proof is then complete. \square

Proof of Proposition 2.4. Let A represent the linear-growth positive constant for the functions $\rho(\cdot)$ and $\kappa(\cdot)$ defined respectively in assumptions (H4) and $(\text{H5})_n^\lambda$. In view of (iv) of Remark 2.1, we can suppose that the generator g satisfies $(\text{H5})_n^\lambda$ with a sufficiently large constant $k \geq e^{(n)}$ instead of $e^{(n)}$, where the constant k depends only on (n, λ, A) and can be similarly defined as in Proposition 3.4. In view of assumptions (H4) and $(\text{H5})_n^\lambda$ with k instead of $e^{(n)}$, Proposition 3.4 and the proof of Proposition 3.5 together with Theorem 2.1 in [7], by a similar argument as that in Proposition 2.5 of [12] we have the desired assertions. The details are omitted here. \square

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