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Abstract

We build on our previous work to derive more accurate analytical approximations for the phase-plane trajectories of the standard susceptible-infectious-removed (SIR) epidemic model, including host births and deaths. From our refined analysis, we obtain closed-form analytical expressions for the maximum and minimum prevalence following an initial outbreak. As in our previous work, our analysis involves matching asymptotic expansions across branch cuts of the Lambert W function, but we carry the approximations to higher asymptotic orders.

1 Introduction

In 1927, Kermack and McKendrick [5] (hereafter KM) published the system of ordinary differential equations (ODEs) that are now known as the standard susceptible-infectious-removed (SIR) model,

$$\frac{\mathrm{d}X}{\mathrm{d}t} = \mu(1-X) - \beta XY\,, \tag{1a}$$

$$\frac{\mathrm{d}Y}{\mathrm{d}t} = (\beta X - \gamma - \mu)Y\,, \tag{1b}$$

$$\frac{\mathrm{d}Z}{\mathrm{d}t} = \gamma Y - \mu Z\,. \tag{1c}$$

Here, the state variables are the proportions of the population that are susceptible (X), infective (Y), and removed (Z), and the parameters are the *per capita* rate of birth and death (μ), the transmission rate (β), and the recovery (or removal) rate (γ). Since $X + Y + Z = 1$, the dynamical system (1) is two-dimensional. KM found the exact solution in the phase plane in the absence of vital dynamics ($\mu = 0$), but no exact solution is known for $\mu > 0$.

In a previous paper [10], we used multiple scale and singular perturbation methods [8, 6] to obtain closed-form analytical approximations to the phase-plane trajectories of Equation (1) in the presence of host births and deaths ($\mu > 0$). Here, we extend our analyses to higher order and obtain more accurate trajectory approximations.

As in our lower order analysis [10], the small parameter[1] we use for perturbation expansions is

$$\epsilon = \frac{\varepsilon}{\mathcal{R}_0}\,, \tag{2}$$

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[1]See [10, Table 1] for parameter estimates for common infectious diseases. Typically $\epsilon < 0.001$.

where ε is the expected infectious period $[1/(\gamma+\mu)]$ in units of the expected host lifetime $(1/\mu)$, and $\mathcal{R}_0$ is the basic reproduction number $[\beta/(\gamma+\mu)]$. The endemic equilibrium (EE) of the SIR ODEs (1) is

$$x_\star = \frac{1}{\mathcal{R}_0}, \qquad y_\star = \varepsilon(1-x_\star). \tag{3}$$

If we express time in units of the expected infectious period $[\tau = t(\gamma+\mu)]$, then Equation (1) can be conveniently written

$$x_\star \frac{\mathrm{d}X}{\mathrm{d}\tau} = \epsilon(1-X) - XY\,, \tag{4a}$$

$$x_\star \frac{\mathrm{d}Y}{\mathrm{d}\tau} = (X-x_\star)Y\,, \tag{4b}$$

so the phase plane equation can be written

$$\frac{\mathrm{d}Y}{\mathrm{d}x} = \frac{(x-x_\star)Y}{\epsilon(1-x)-xY} \qquad \text{or} \qquad \frac{\mathrm{d}X}{\mathrm{d}y} = \frac{\epsilon(1-X)-Xy}{(X-x_\star)y}. \tag{5}$$

As in [10], we use the probabilist's convention that lower case letters indicate independent variables and capitals indicate dependent variables.

Our previous approximation [10] is "zeroth order" in the sense that it is based on matching asymptotic expansions to KM's phase plane solution to Equation (5) for $\epsilon = 0$ (which is a good approximation away from the coordinate axes if $0 < \epsilon \ll 1$). The approximation we present here is "first order" in the sense that it is based on matching to an $\mathcal{O}(\epsilon)$ correction to KM's exact solution for $\epsilon = 0$. We put these labels in quotes because they do not reflect the ultimate asymptotic orders of the approximations we have derived.

Our "zeroth order" approximation yields an analytical formula for the proportion susceptible near the end of a major outbreak, which is an important component of an analysis of epidemic burnout that we have conducted based on the stochastic SIR model [9]. Our "first order" approximation allows us to obtain an estimate of the burnout probability that does not require an *a priori* choice of boundary layer, which in turn allows us to show that the prior estimate of the burnout probability presented by van Herwaarden [11] can be derived as an approximation to our result in [9].

Like the approximations of van Herwaarden [11], our "first order" results depend on the numerical evaluation of integrals. However, we do obtain "integral free" inner solutions—*i.e.,* we obtain closed-form analytical results for the dynamics inside all boundary and corner layers—and are consequently able to derive convenient, closed-form "first order" expressions for the peak prevalence and the minimum prevalence after the first major epidemic following disease invasion.

As in our "zeroth order" analysis, we succeed in obtaining a "first order" approximation that is valid to the same order in ϵ throughout the phase plane. Critical ingredients in our approach are the use of Lambert's W function [3] to invert implicit relations, and matching across branch cuts of the W function.

A much more detailed introduction to the SIR model and the method of asymptotic matching is given in our previous paper [10]. Our new results are summarized in Table 3.

2 Prior Work

In Table 1, we summarize the approximate solutions to Equation (4) that we constructed in [10]; each of these ***local solutions*** is valid only in a subset of the phase-plane. The ***outer solution***, obtained in [10] by assuming an asymptotic series in powers of ϵ,

$$\overset{\text{out}}{Y}(x;\epsilon) = \sum_{j=0}^{\infty} \overset{\text{out}}{Y_j}(x)\epsilon^j\,, \tag{6}$$

and neglecting terms of $\mathcal{O}(\epsilon)$, captured the behaviour of the exact solution in the majority of phase space.

The x-axis ***boundary layer*** solution, obtained in [10] via a WKB ansatz [1, Chapter 10], is valid where y

is ***transcendentally small***, *i.e.,* where y vanishes more rapidly than any power ϵ^j as $\epsilon \to 0$ [6, p. 4]. Finally, we derived two ***corner layer*** solutions by assuming that $y = \mathcal{O}(\epsilon)$, which enabled us to smoothly match the outer and boundary layer solutions. We combined these local solutions into a single ***matched asymptotic solution*** approximating the trajectory of a novel pathogen invading at the ***disease-free equilibrium*** (DFE, $(x_{\rm i}, y_{\rm i}) = (1, 0)$). This single solution was expressed as a multi-function in y, valid everywhere in the phase-plane, with different branches giving the trajectories in the left and right half-planes ($x \le x_\star$ and $x \ge x_\star$).

Solution	**Notation**	**Expression**	**Scaling**
Outer	$\overset{\text{out}}{Y}_0(x)$	$\overset{\text{out}}{C}_0 - x + x_\star \ln x$	$x = \mathcal{O}(1)$, $y = \mathcal{O}(1)$
Corner	$\overset{\text{cor}}{X}(y;\epsilon)$	$\overset{\text{cor}}{C}_0 + \left(\frac{\overset{\text{cor}}{C}_0}{x_\star - \overset{\text{cor}}{C}_0}\right) y + \epsilon \ln \epsilon^{-1} \left(\overset{\text{cor}}{C}_{\ln} + \frac{1 - \overset{\text{cor}}{C}_0}{x_\star - \overset{\text{cor}}{C}_0} \right) - \epsilon \left[\left(\frac{1 - \overset{\text{cor}}{C}_0}{x_\star - \overset{\text{cor}}{C}_0} \right) \ln y - \overset{\text{cor}}{C}_1 \right]$	$x = \mathcal{O}(1)$, $y = \mathcal{O}(\epsilon)$
x-axis bdry	$\overset{x\text{b}}{Y}(x;\epsilon)$	$(1-x)^{-\frac{1}{\epsilon}(1-x_\star)}\, e^{\frac{1}{\epsilon}(C_0^\phi - x) + C_1^\phi}$	$x = \mathcal{O}(1)$, $y =$ TS

Table 1: Local solutions constructed in [10]. TS indicates a quantity that is transcendentally small. Throughout, we adopt the convention that C, with any combination of sub- and/or superscripts, indicates a constant of integration.

3 Refined Asymptotic Solutions

As the small parameter ϵ is increased, the outer solution $\overset{\text{out}}{Y}_0(x)$ [Table 1] becomes a progressively poorer approximation to the first epidemic. As a consequence our subsequent corner and boundary layer approximations – and thus the matched solution – also perform less well. To resolve this, here we seek a second matching that includes the correction term $\overset{\text{out}}{Y}_1(x)$ (6), again focusing on the solution along the unstable manifold, $x_{\rm i} = 1$. Unlike our previous matching, where we expressed all local solutions as functions of y, we have not found a way to invert the relation $y = \overset{\text{out}}{Y}_1(x)$ that we obtain, so in this section we will match functions of x.

These higher order corrections come at the cost of (considerably) complicating our analytical expressions. In particular, we are obliged to introduce integrals that do not appear to have closed form solutions, and which are numerically sensitive due to singular terms in the integrand. In Appendix A we obtain some analytical approximations to these integrals by other means (the integrals are independent of ϵ, so we cannot use our matched asymptotic approach). Furthermore, for any given value of $\mathcal{R}_0$, fully matched solutions are available only for sufficiently small ϵ (see Appendix B for the $\mathcal{R}_0$-dependent condition on ϵ; we remark that even when this condition is violated, the outer and boundary layer solutions continue to provide excellent local approximations).

3.1 The KM Solution

Given an initial condition $(x_{\rm i}, y_{\rm i})$, the constant of integration in $\overset{\text{out}}{Y}_0(x)$ is readily found to be

$$\overset{\text{out}}{C}_0 = y_{\rm i} + x_{\rm i} - x_\star \ln x_{\rm i}\,, \tag{7}$$

so that

$$\overset{\text{out}}{Y}_0(x, x_\text{i}, y_\text{i}) = y_\text{i} + x_\text{i} - x + x_\star \ln\Big(\frac{x}{x_\text{i}}\Big), \tag{8}$$

which is the phase plane solution first discovered by KM for the SIR ODEs without vital dynamics, *i.e.,* for $\epsilon = 0$.

Provided $x_\text{i} \geq x_\star$ and $y_\text{i} \geq 0$, the solution (8) is non-negative and concave, with two positive roots,

$$x_\text{f}(x_\text{i}, y_\text{i}) = -x_\star W_0\left(\mathscr{E}(-x_\text{i}/x_\star)e^{-\frac{y_\text{i}}{x_\star}}\right). \tag{9}$$

in $(0, x_\star)$, and another in $(x_\star, 1)$, and a unique maximum $\overline{y}_0$ at $x_\star$,

$$\overline{y}_0(x_\text{i}, y_\text{i}) = \overset{\text{out}}{Y}_0(x_\star, x_\text{i}, y_\text{i}) = y_\text{i} + x_\text{i} - x_\star\big(1 + \ln(x_\text{i}/x_\star)\big). \tag{10}$$

Here, W_0 denotes the principal branch of the Lambert W-function[2]

$\overline{y}_0(x_\text{i}, y_\text{i})$ and $x_\text{f}(x_\text{i}, y_\text{i})$ are the true peak prevalence and final size for the SIR model without vital dynamics ($\epsilon = 0$) started from (x_i, y_i). $\overline{y}_0$ only approximates the peak prevalence for the model with demography, and there is no "final" size if there is a continuous inflow of new susceptible individuals. Nevertheless, we informally refer to x_f as the "final size" for convenience. Moreover, when there is no risk of confusion, for ease of notation we will write $x_\text{f}(x_\text{i}, 0) = x_\text{f}(x_\text{i})$ and $x_\text{f}(1, 0) = x_\text{f}$ and similarly for $\overline{y}_0$.

As in [10], our ultimate goal is to approximate the solution corresponding to invasion at the DFE, $(x_\text{i}, y_\text{i}) = (1, 0)$, but in order to obtain matched asymptotic approximations that continue into the second and subsequent epidemic waves, we need to consider solutions for more general initial conditions. In [10] we derived ***effective initial conditions*** $(x_{\text{i},2}, 0)$ that allow us to approximate these waves. Given an initial condition (x_i, y_i), we set

$$x_{\text{i},2}(x_\text{i}, y_\text{i}) = 1 + (1 - x_\star)W_0\bigg(\mathscr{E}\Big(-\frac{1 - x_\text{f}(x_\text{i}, y_\text{i})}{1 - x_\star}\Big)\bigg). \tag{12}$$

As we explained in [10], although $(x_{\text{i},2}, 0)$ is not a point on the actual trajectory, it is an "effective" initial condition for the second wave in the following sense: if $(x_{\text{i},2}, 0)$ were used as the initial state in the KM ($\epsilon = 0$) solution, the resulting trajectory would meet the second rise of the actual solution as it curves up from the left in the phase plane.

3.2 The First Order Outer Solution

If we now include the second term in the asymptotic expansion (6),

$$\overset{\text{out}}{Y}(x; \epsilon) = \overset{\text{out}}{Y}_0(x) + \epsilon\overset{\text{out}}{Y}_1(x) + \mathcal{O}(\epsilon^2), \tag{13}$$

we must now solve for $\overset{\text{out}}{Y}_1(x)$, which, substituting (6) into Equation (4) and collecting $\mathcal{O}(\epsilon)$ terms, satisfies

$$\frac{\mathrm{d}\overset{\text{out}}{Y}_1}{\mathrm{d}x} = \Big(\frac{1}{x} - 1\Big)\Big(\frac{x_\star}{x} - 1\Big)\frac{1}{\overset{\text{out}}{Y}_0(x)}. \tag{14}$$

[2] If $\mathscr{E}(z) = ze^z$, Lambert's W-function $W(z)$ [3] solves the "left-sided" inverse relation $\mathscr{E}(W(z)) = z$. This equation has countably many solutions, each corresponding to branches W_i of the W-function; we will need the two real branches, W_0, which maps $[-\frac{1}{e}, \infty)$ to $[-1, \infty)$, and W_{-1}, which maps $[-\frac{1}{e}, 0)$ to $(-\infty, -1]$. For these two branches, W_i is a *partial* "right-sided" inverse function for $\mathscr{E}(z)$:

$$\begin{aligned} W_{-1}(\mathscr{E}(z)) &= z \quad \text{if } z \leq -1 \\ W_0(\mathscr{E}(z)) &= z \quad \text{if } z \geq -1. \end{aligned} \tag{11}$$

This equation can be formally solved by (indefinite) integration. If we impose the initial condition $(x_{\mathrm{i}}, y_{\mathrm{i}})$, the first order term, $\overset{\text{out}}{Y}_1(x)$, in the asymptotic series for the outer solution (6), takes the form

$$\overset{\text{out}}{Y}_1(x, x_{\mathrm{i}}, y_{\mathrm{i}}) = \int \Big(\frac{1}{x} - 1\Big)\Big(\frac{x_\star}{x} - 1\Big) \frac{1}{\overset{\text{out}}{Y}_0(x, x_{\mathrm{i}}, y_{\mathrm{i}})} \, \mathrm{d}x. \tag{15}$$

This integral has no closed analytical form so we will need to evaluate it numerically. To do so, we must fix an endpoint to make it a definite integral, which requires some care: $\overset{\text{out}}{Y}_0(x, x_{\mathrm{i}}, y_{\mathrm{i}})$ (8), which appears in the denominator of the integrand in Equation (15), vanishes at $x_{\mathrm{f}}(x_{\mathrm{i}}, y_{\mathrm{i}})$, and if $y_{\mathrm{i}} = 0$ then $\overset{\text{out}}{Y}_0(x, x_{\mathrm{i}}, y_{\mathrm{i}})$ vanishes at x_{i} as well. Consequently, there is a pair of possible singularities in the integrand. If $y_{\mathrm{i}} = 0$ and $x_{\mathrm{i}} = 1$ then the integrand approaches $x_\star$ in the limit as $x \to 1$ and the singularity is removable. However, if $y_{\mathrm{i}} = 0$ then the integrand is singular for all $x_{\mathrm{i}} \neq 1$. Moreover, the integrand is always singular at $x_{\mathrm{f}}(x_{\mathrm{i}}, y_{\mathrm{i}})$. Since we need to be able to compute $\overset{\text{out}}{Y}_1(x, x_{\mathrm{i}}, y_{\mathrm{i}})$ at values of $x \in (x_{\mathrm{f}}(x_{\mathrm{i}}, y_{\mathrm{i}}), x_{\mathrm{i}})$, we must choose an endpoint that lies in that interval as well. The natural choice is $x_\star$, which always satisfies $x_{\mathrm{f}}(x_{\mathrm{i}}, y_{\mathrm{i}}) < x_\star < x_{\mathrm{i}}$. For any $x \in (x_{\mathrm{f}}, x_{\mathrm{i}})$, the integrand is then non-singular on $[x, x_\star]$ ($[x_\star, x]$) if $x < x_\star$ (resp. $x > x_\star$).

Since the specificiation of an arbitrary endpoint is just a device to be able to compute the integral (15) numerically, we must still include an arbitrary constant of integration, which will provide a degree of freedom that we will need when imposing initial conditions. Thus, we write

$$\overset{\text{out}}{Y}_1(x, x_{\mathrm{i}}, y_{\mathrm{i}}) = \int_{x_\star}^{x} \Big(\frac{1}{u} - 1\Big)\Big(\frac{x_\star}{u} - 1\Big) \frac{1}{\overset{\text{out}}{Y}_0(u, x_{\mathrm{i}}, y_{\mathrm{i}})} \, \mathrm{d}u + \overset{\text{out}}{C}_1 . \tag{16}$$

Henceforth, we will limit our attention to solutions with $y_{\mathrm{i}} = 0$, which we denote by $\overset{\text{out}}{Y}_1(x, x_{\mathrm{i}})$. When focussing more specifically on the solution with $(x_{\mathrm{i}}, y_{\mathrm{i}}) = (1, 0)$, we will write simply $\overset{\text{out}}{Y}_1(x)$, but we do need the extra freedom of an arbitrary x_{i} to consider subsequent epidemic waves. For all such solutions $\overset{\text{out}}{Y}_1(x, x_{\mathrm{i}})$, the integrand in (16) has a simple pole at $x = x_{\mathrm{f}}(x_{\mathrm{i}})$ (9) with residue

$$\lim_{x \to x_{\mathrm{f}}(x_{\mathrm{i}})} \big(x - x_{\mathrm{f}}(x_{\mathrm{i}})\big)\Big(\frac{1}{x} - 1\Big)\Big(\frac{x_\star}{x} - 1\Big) \frac{1}{\overset{\text{out}}{Y}_0(x, x_{\mathrm{i}})} = \frac{1}{x_{\mathrm{f}}(x_{\mathrm{i}})} - 1 . \tag{17}$$

If $x_{\mathrm{i}} \neq 1$, it also has a second simple pole at $x = x_{\mathrm{i}}$ with residue $\frac{1}{x_{\mathrm{i}}} - 1$. To be able to evaluate the integral (16), we shall need to remove these poles, separating the integral into singular and non-singular parts; doing so will also prove to be an essential step in matching the solutions.

Presentation of calculations will be greatly simplified by introducing the notation

$$\mathcal{Y}_a^b(x_{\mathrm{i}}) = \int_a^b \left[\Big(\frac{1}{u} - 1\Big)\Big(\frac{x_\star}{u} - 1\Big) \frac{1}{\overset{\text{out}}{Y}_0(u, x_{\mathrm{i}})} - \Big(\frac{1}{x_{\mathrm{i}}} - 1\Big) \frac{1}{u - x_{\mathrm{i}}} - \Big(\frac{1}{x_{\mathrm{f}}(x_{\mathrm{i}})} - 1\Big) \frac{1}{u - x_{\mathrm{f}}(x_{\mathrm{i}})} \right] \mathrm{d}u . \tag{18}$$

Here, the sub- and superscripts a and b are used to emphasize their role as endpoints of an integral, and in particular that

$$\mathcal{Y}_a^b(x_{\mathrm{i}}) = \mathcal{Y}_a^c(x_{\mathrm{i}}) + \mathcal{Y}_c^b(x_{\mathrm{i}}) , \qquad \text{for any } c \in \mathbb{R} . \tag{19}$$

We can then re-write (16) as

$$\begin{aligned} \overset{\text{out}}{Y}_1(x, x_{\mathrm{i}}) &= \mathcal{Y}_{x_\star}^{x}(x_{\mathrm{i}}) + \int_{x_\star}^{x} \Big(\frac{1}{x_{\mathrm{i}}} - 1\Big) \frac{1}{u - x_{\mathrm{i}}} \, \mathrm{d}u + \int_{x_\star}^{x} \Big(\frac{1}{x_{\mathrm{f}}(x_{\mathrm{i}})} - 1\Big) \frac{1}{u - x_{\mathrm{f}}(x_{\mathrm{i}})} \, \mathrm{d}u + \overset{\text{out}}{C}_1 \\ &= \mathcal{Y}_{x_\star}^{x}(x_{\mathrm{i}}) + \Big(\frac{1}{x_{\mathrm{i}}} - 1\Big) \ln\left(\frac{x_{\mathrm{i}} - x}{x_{\mathrm{i}} - x_\star}\right) + \Big(\frac{1}{x_{\mathrm{f}}(x_{\mathrm{i}})} - 1\Big) \ln\left(\frac{x - x_{\mathrm{f}}(x_{\mathrm{i}})}{x_\star - x_{\mathrm{f}}(x_{\mathrm{i}})}\right) + \overset{\text{out}}{C}_1 . \end{aligned} \tag{20}$$

The first order approximation to the outer solution is thus

$$\overset{\text{out}}{Y}(x, x_{\mathrm{i}}; \epsilon) = \overset{\text{out}}{Y}_0(x, x_{\mathrm{i}}) + \epsilon \overset{\text{out}}{Y}_1(x, x_{\mathrm{i}}) . \tag{21}$$

Note that when $x_{\mathrm{i}} = 1$, which is our focal solution, all terms in Equations (18) and (20) involving the residue $\frac{1}{x_{\mathrm{i}}} - 1$ vanish; in what follows we will write $\overset{\text{out}}{Y}(x, 1; \epsilon)$ as $\overset{\text{out}}{Y}(x; \epsilon)$.

Analytical matching In general, $\mathcal{Y}^{x}_{x_\star}(x_{\mathrm{i}})$ can only be computed numerically, so our first order solution is only semi-analytical. In the limit where $\mathcal{R}_0$ is near 1, however, we show in Appendix A that a satisfactory analytical approximation to $\mathcal{Y}^{x}_{x_\star}(x_{\mathrm{i}})$ can be found, which enables us to give a fully-analytical first order matched solution.

3.2.1 Outer solution from the DFE

Specializing now to the case $(x_{\mathrm{i}}, y_{\mathrm{i}}) = (1, 0)$, we must have $Y(1) = 0$. Since $\overset{\text{out}}{Y}_0(1) = 0$ [Equation (8)], to retain the correct initial condition we must have $\overset{\text{out}}{Y}_1(1) = 0$ also; hence from Equation (20) we infer that

$$\overset{\text{out}}{C}_1 = -\mathcal{Y}^{1}_{x_\star}(1) - \left(\frac{1}{x_{\mathrm{f}}} - 1\right)\ln\left(\frac{1 - x_{\mathrm{f}}}{x_\star - x_{\mathrm{f}}}\right), \tag{22}$$

whence

$$\overset{\text{out}}{Y}(x; \epsilon) = \overset{\text{out}}{Y}_0(x) + \epsilon\left(\left(\frac{1}{x_{\mathrm{f}}} - 1\right)\ln\left(\frac{x - x_{\mathrm{f}}}{1 - x_{\mathrm{f}}}\right) - \mathcal{Y}^{1}_{x}(1)\right). \tag{23}$$

Peak prevalence We obtain an approximation to $\overline{y}$ by evaluating Equation (23) at $x = x_\star$, which yields

$$\overline{y}_1 = \overline{y}_0 - \epsilon\left(\mathcal{Y}^{1}_{x_\star}(1) + \left(\frac{1}{x_{\mathrm{f}}} - 1\right)\ln\left(\frac{x_\star - x_{\mathrm{f}}}{1 - x_{\mathrm{f}}}\right)\right) = \overline{y}_0 - \epsilon\int_{x_\star}^{1}\left(\frac{x_\star}{u} - 1\right)\left(\frac{1}{u} - 1\right)\frac{1}{\overset{\text{out}}{Y}_0(u)}\,\mathrm{d}u\,. \tag{24}$$

In Appendix A.2, we derive a closed-form analytical approximation to the integral in Equation (24), which then yields

$$\overline{y}_1 \approx 1 - x_\star(1 - \ln x_\star) - \epsilon\frac{(1 - x_\star)(1 - x_\star + \ln x_\star)}{1 - x_\star(1 - \ln x_\star)}\,. \tag{25}$$

3.3 Outer to corner solution matching

We start by introducing an intermediate scale $\eta = \eta(\epsilon)$ that we will use to capture the overlap between the outer and corner solutions:

$$\epsilon\ln\epsilon^{-1} \ll \eta \ll \epsilon^{1/2}\,, \tag{26}$$

which implies

$$\eta^2 \ll \epsilon \ll \epsilon\ln\eta^{-1} \ll \epsilon\ln\epsilon^{-1} \ll \eta\,. \tag{27}$$

In addition, $\epsilon\ln\epsilon^{-1} \ll \eta$ implies that $\frac{1}{\epsilon}e^{-\frac{\eta x_\eta}{\epsilon}}C \ll 1$ for any $C > 0$. Expanding the outer solution $\overset{\text{out}}{Y}(x; \epsilon)$ (21) about $x = x_{\mathrm{f}} + \eta x_\eta$ in an asymptotic series with these orders gives us

$$\begin{aligned}\overset{\text{out}}{Y}(x_{\mathrm{f}} + \eta x_\eta; \epsilon) = \eta\left(\frac{x_\star}{x_{\mathrm{f}}} - 1\right)x_\eta - \epsilon\ln\eta^{-1}\left(\frac{1}{x_{\mathrm{f}}} - 1\right)\\ + \epsilon\left(\left(\frac{1}{x_{\mathrm{f}}} - 1\right)\ln x_\eta - \left(\frac{1}{x_{\mathrm{f}}} - 1\right)\ln(1 - x_{\mathrm{f}}) - \mathcal{Y}^{1}_{x_{\mathrm{f}}}(1)\right) + \mathcal{O}(\eta^2).\end{aligned} \tag{28}$$

To match $\overset{\text{out}}{Y}$ with the corner layer solution $\overset{\text{cor}}{X}(y; \epsilon)$ [Table 1], we must first re-express it as a function $\overset{\text{cor}}{Y}{}^{i}(x; \epsilon)$ of x. Rearranging $x = \overset{\text{cor}}{X}(y; \epsilon)$, we get

$$-\frac{1}{\epsilon}\left(\frac{1}{\overset{\text{cor}}{C}_0} - 1\right)^{-1} e^{-\frac{1}{\epsilon}\frac{x_\star - \overset{\text{cor}}{C}_0}{1 - \overset{\text{cor}}{C}_0}\left(x - \overset{\text{cor}}{C}_0 - \epsilon\ln\epsilon^{-1}\left(\overset{\text{cor}}{C}_{\ln} + \frac{1 - \overset{\text{cor}}{C}_0}{x_\star - \overset{\text{cor}}{C}_0}\right) - \epsilon\overset{\text{cor}}{C}_1\right)} = -\frac{1}{\epsilon}\left(\frac{1}{\overset{\text{cor}}{C}_0} - 1\right)^{-1} y e^{-\frac{1}{\epsilon}\left(\frac{1}{\overset{\text{cor}}{C}_0} - 1\right)^{-1} y} \tag{29}$$

$$= \mathscr{C}\left(-\frac{1}{\epsilon}\left(\frac{1}{\overset{\text{cor}}{C}_0}-1\right)^{-1} y\right), \tag{30}$$

which we can invert using the Lambert W function to get $y = \overset{\text{cor}}{Y}{}^{i}(x;\epsilon)$:

$$\overset{\text{cor}}{Y}{}^{i}(x;\epsilon) = -\epsilon\left(\frac{1}{\overset{\text{cor}}{C}_0}-1\right) W_i\left(-\frac{1}{\epsilon}\left(\frac{1}{\overset{\text{cor}}{C}_0}-1\right)^{-1} e^{-\frac{1}{\epsilon}\frac{x_\star-\overset{\text{cor}}{C}_0}{1-\overset{\text{cor}}{C}_0}\left(x-\overset{\text{cor}}{C}_0-\epsilon\ln\epsilon^{-1}\left(\overset{\text{cor}}{C}_{\ln}+\frac{1-\overset{\text{cor}}{C}_0}{x_\star-\overset{\text{cor}}{C}_0}\right)-\epsilon\overset{\text{cor}}{C}_1\right)}\right), \tag{31}$$

where $i = -1, 0$ correspond to values of $y \geq \epsilon\big((1/\overset{\text{cor}}{C}_0)-1\big)$ and $y \leq \epsilon\big((1/\overset{\text{cor}}{C}_0)-1\big)$, respectively. Note that the point $\big(\overset{\text{cor}}{C}_0, \epsilon\big((1/\overset{\text{cor}}{C}_0)-1\big)\big)$ is on the x nullcline, $\frac{\mathrm{d}X}{\mathrm{d}\tau} = 0$, *i.e.,* $y = \epsilon\left(\frac{1}{x}-1\right)$, which separates the upper and lower branches of the phase-plane trajectories. The two branches $i = -1, 0$ give the solutions above and below the nullcline, which we match to the outer solution $\overset{\text{out}}{Y}(x;\epsilon)$ (21) and the x-axis boundary layer solution, $\overset{\text{xb}}{Y}(x;\epsilon)$ [Table 1], respectively.

If we set $x = \overset{\text{cor}}{C}_0 + \eta x_\eta$ in Equation (31), and note that by Equation (26),

$$1 \ll \ln\epsilon^{-1} \ll \frac{\eta}{\epsilon}, \tag{32}$$

we can infer that the argument of the Lambert W function in Equation (31) is proportional to

$$e^{-\frac{\eta}{\epsilon}\frac{x_\star-\overset{\text{cor}}{C}_0}{1-\overset{\text{cor}}{C}_0}x_\eta-\ln\epsilon^{-1}} \sim e^{-\frac{\eta}{\epsilon}\frac{x_\star-\overset{\text{cor}}{C}_0}{1-\overset{\text{cor}}{C}_0}x_\eta}. \tag{33}$$

If $\overset{\text{cor}}{C}_0 < x_\star$ and $x_\eta > 0$, the argument vanishes as $\eta \to 0$ and we can use the expansion of W_{-1} for small arguments [3],

$$W_{-1}(z) = \ln(-z) - \ln(-\ln(-z)) + o(1), \tag{34}$$

to expand $\overset{\text{cor}}{Y}{}^{-1}(x;\epsilon)$ about $x_\eta = 0$ and obtain

$$\begin{aligned}\overset{\text{cor}}{Y}{}^{-1}(\overset{\text{cor}}{C}_0+\eta x_\eta;\epsilon) = {}& \eta\left(\frac{x_\star}{\overset{\text{cor}}{C}_0}-1\right)x_\eta - \epsilon\ln\eta^{-1}\left(\frac{1}{\overset{\text{cor}}{C}_0}-1\right) - \epsilon\ln\epsilon^{-1}\left(\frac{x_\star}{\overset{\text{cor}}{C}_0}-1\right)\left(\overset{\text{cor}}{C}_{\ln}+\frac{1-\overset{\text{cor}}{C}_0}{x_\star-\overset{\text{cor}}{C}_0}\right)\\ &+\epsilon\left[\left(\frac{1}{\overset{\text{cor}}{C}_0}-1\right)\ln x_\eta + \left(\frac{1}{\overset{\text{cor}}{C}_0}-1\right)\ln\left(\frac{x_\star}{\overset{\text{cor}}{C}_0}-1\right) - \left(\frac{x_\star}{\overset{\text{cor}}{C}_0}-1\right)\overset{\text{cor}}{C}_1\right] + \mathcal{O}\left(\frac{\epsilon^2\ln\epsilon^{-1}}{\eta}\right).\end{aligned} \tag{35}$$

Note that $\epsilon\ln\epsilon^{-1} \ll \eta$ implies $\frac{\epsilon^2\ln\epsilon^{-1}}{\eta} \ll \epsilon$ and the two solutions, $\overset{\text{out}}{Y}(x_{\text{f}}+\eta x_\eta;\epsilon)$ and $\overset{\text{cor}}{Y}{}^{-1}(x_{\text{f}}+\eta x_\eta;\epsilon)$, agree up to $\mathcal{O}(\epsilon)$ provided we set

$$\overset{\text{cor}}{C}_0 = x_{\text{f}}, \tag{36}$$

$$\overset{\text{cor}}{C}_{\ln} = -\frac{1-\overset{\text{cor}}{C}_0}{x_\star-\overset{\text{cor}}{C}_0} = -\frac{1-x_{\text{f}}}{x_\star-x_{\text{f}}}, \tag{37}$$

and

$$\overset{\text{cor}}{C}_1 = \frac{1-x_{\text{f}}}{x_\star-x_{\text{f}}}\left[\ln\left(\left(\frac{1}{x_{\text{f}}}-1\right)(x_\star-x_{\text{f}})\right) + \left(\frac{1}{x_{\text{f}}}-1\right)^{-1}\mathcal{Y}^1_{x_{\text{f}}}(1)\right]. \tag{38}$$

Substituting these choices into Equation (31) yields a matched ***left corner solution*** at $x_{\rm f}$,

$$\overset{\rm lc}{Y}{}^{i}(x;\epsilon) = -\epsilon\left(\frac{1}{x_{\rm f}} - 1\right) W_i\left(-\frac{x_\star - x_{\rm f}}{\epsilon} e^{-\frac{x-x_{\rm f}}{\epsilon}\frac{x_\star - x_{\rm f}}{1-x_{\rm f}} + \left(\frac{1}{x_{\rm f}}-1\right)^{-1}\mathcal{Y}^1_{x_{\rm f}}(1)}\right). \tag{39}$$

Summing the $i = -1$ branch, $\overset{\rm lc}{Y}{}^{-1}(x;\epsilon)$, and the outer solution, $\overset{\rm out}{Y}(x;\epsilon)$, and then subtracting the overlap, yields a matched solution for the region above the x nullcline,

$$Y^{\uparrow}(x;\epsilon) = \overset{\rm out}{Y}_0(x) + \overset{\rm lc}{Y}{}^{-1}(x;\epsilon) - \left(\frac{x_\star}{x_{\rm f}} - 1\right)(x - x_{\rm f}) + \epsilon\mathcal{Y}^x_{x_{\rm f}}(1). \tag{40}$$

Applicability of the first order solution Some care is required in applying the results above; $|\mathcal{Y}^x_{x_{\rm f}}(1)|$ grows very rapidly with increasing $\mathcal{R}_0$, so $\epsilon|\mathcal{Y}^x_{x_{\rm f}}(1)|$ need not be small (*e.g.,* $\epsilon|\mathcal{Y}^x_{x_{\rm f}}(1)| \approx 4.4$ for $\mathcal{R}_0 = 6$ and $\varepsilon = 0.01$), so there are realistic parameter regimes where higher order terms would be necessary to obtain a good approximation. In Appendix B we derive upper bounds on the admissible values of ϵ as a function of $\mathcal{R}_0$.

3.4 Corner to Inner Solution Matching

We can similarly match the $i = 0$ branch, $\overset{\rm lc}{Y}{}^{0}(x;\epsilon)$, to the x-axis boundary layer solution $\overset{\rm xb}{Y}(x;\epsilon)$ [Table 1], As we observed in the previous subsection, if we set $x = x_{\rm f} + \eta x_\eta$, the argument of the W function in $\overset{\rm lc}{Y}{}^{0}(x_{\rm f} + \eta x_\eta;\epsilon)$ is small [see Equation (39)]. Because we are now using the $i = 0$ branch instead of $i = -1$, we use the expansion $W_0(z) = z + \mathcal{O}(z^2)$ [3] to get

$$\overset{\rm lc}{Y}{}^{0}(x_{\rm f} + \eta x_\eta;\epsilon) = \left(\frac{1}{x_{\rm f}} - 1\right)(x_\star - x_{\rm f})\, e^{-\frac{\eta x_\eta}{\epsilon}\frac{x_\star - x_{\rm f}}{1-x_{\rm f}} + \left(\frac{1}{x_{\rm f}}-1\right)^{-1}\mathcal{Y}^1_{x_{\rm f}}(1)} + \mathcal{O}\left(e^{-2\frac{\eta x_\eta}{\epsilon}\frac{x_\star - x_{\rm f}}{1-x_{\rm f}}}\right). \tag{41}$$

On the other hand, substituting $C_0^\phi = c_0^\phi - (1 - x_\star)\ln(1 - c_0^\phi)$ into the x-axis boundary layer solution [Table 1] gives

$$\overset{\rm xb}{Y}(x;\epsilon) = \left(\frac{1-x}{1-c_0^\phi}\right)^{-\frac{1}{\epsilon}(1-x_\star)} e^{\frac{1}{\epsilon}(c_0^\phi - x) + C_1^\phi}. \tag{42}$$

Evaluating at $x = c_0^\phi + \eta x_\eta$, we have

$$\begin{aligned}
\overset{\rm xb}{Y}(c_0^\phi + \eta x_\eta;\epsilon) &= \left(1 - \frac{\eta x_\eta}{1-c_0^\phi}\right)^{-\frac{1-x_\star}{\epsilon}} e^{-\frac{\eta x_\eta}{\epsilon} + C_1^\phi} \\
&= e^{-\frac{\eta x_\eta}{\epsilon} - \frac{1-x_\star}{\epsilon}\ln\left(1 - \frac{\eta x_\eta}{1-c_0^\phi}\right) + C_1^\phi} \\
&= e^{-\frac{\eta x_\eta}{\epsilon}\frac{x_\star - c_0^\phi}{1-c_0^\phi} + C_1^\phi + \mathcal{O}\left(\frac{\eta^2}{\epsilon}\right)} \\
&= e^{-\frac{\eta x_\eta}{\epsilon}\frac{x_\star - c_0^\phi}{1-c_0^\phi} + C_1^\phi}\left(1 + \mathcal{O}\left(\frac{\eta^2}{\epsilon}\right)\right),
\end{aligned} \tag{43}$$

and the solutions (41, 43) coincide if we take

$$c_0^\phi = x_{\rm f} \tag{44}$$

$$C_0^\phi = c_0^\phi - (1 - x_\star)\ln(1 - c_0^\phi) = x_{\rm f} - (1 - x_\star)\ln(1 - x_{\rm f}) \tag{45}$$

and

$$C_1^{\phi} = \ln\left[\left(\frac{1}{x_{\rm f}} - 1\right)(x_\star - x_{\rm f})\right] + \left(\frac{1}{x_{\rm f}} - 1\right)^{-1} \mathcal{Y}^1_{x_{\rm f}}(1). \tag{46}$$

Substituting these choices into $\overset{\rm xb}{Y}(x;\epsilon)$ [Table 1] gives

$$\overset{\rm xb}{Y}(x;\epsilon) = \left(\frac{1}{x_{\rm f}} - 1\right)(x_\star - x_{\rm f})\left(\frac{1-x_{\rm f}}{1-x}\right)^{\frac{1-x_\star}{\epsilon}} e^{-\frac{x-x_{\rm f}}{\epsilon} + \left(\frac{1}{x_{\rm f}}-1\right)^{-1}\mathcal{Y}^1_{x_{\rm f}}(1)}. \tag{47}$$

As we observed in [10], we can write $\overset{\rm xb}{Y}(x;\epsilon) = (1/x_{\rm f} - 1)(x_\star - x_{\rm f}) e^{\left(\frac{1}{x_{\rm f}}-1\right)^{-1}\mathcal{Y}^1_{x_{\rm f}}(1)} e^{-\frac{\phi_0(x)}{\epsilon}}$, for

$$\phi_0(x) = x - x_{\rm f} + (1 - x_\star)\ln\left(\frac{1-x}{1-x_{\rm f}}\right). \tag{48}$$

$\phi_0(x)$ has two zeros, at $x_{\rm f}$ and $x_{\rm i,2}$ [Equations (9) and (12)]. Between these two zeros, $\phi_0(x) > 0$ and—excluding small neighbourhoods [of radius $\mathcal{O}(\epsilon \ln \epsilon^{-1})$] of the two zeroes—$\overset{\rm xb}{Y}(x;\epsilon)$ will be transcendentally small in the interval $(x_{\rm f}, x_{\rm i,2})$. The small neighbourhoods of the zeros of $\phi_0(x)$, where $\overset{\rm xb}{Y}(x;\epsilon) = \mathcal{O}(\epsilon)$, provide a "transition-zone" where we can match the boundary layer solution with the corner solutions (also $\mathcal{O}(\varepsilon)$) and via them, with the outer solution (which is $\mathcal{O}(1)$).

Expanding in series exactly as we did above with the left corner solution $\overset{\rm lc}{Y}{}^0(x;\epsilon)$ (39), we match $\overset{\rm xb}{Y}(x;\epsilon)$ (47) to $\overset{\rm cor}{Y}{}^i(x;\epsilon)$ (31) to get a ***right corner solution*** at $x_{\rm i,2}$:

$$\overset{\rm rc}{Y}{}^i(x;\epsilon) = -\epsilon\left(\frac{1}{x_{\rm i,2}} - 1\right) W_i\left(-\frac{\left(\frac{1}{x_{\rm f}} - 1\right)(x_\star - x_{\rm f})}{\epsilon\left(\frac{1}{x_{\rm i,2}} - 1\right)} e^{-\frac{x_{\rm i,2}-x}{\epsilon}\frac{x_{\rm i,2}-x_\star}{1-x_{\rm i,2}} + \left(\frac{1}{x_{\rm f}}-1\right)^{-1}\mathcal{Y}^1_{x_{\rm f}}(1)}\right) \tag{49}$$

(for the sake of completeness, we list the matching constants in Table 4).

Summing the two corner solutions [Equations (39) and (49), taking the $i = 0$ branches for the solutions below the x nullcline] and the x-axis boundary layer solution (47), and subtracting the resulting overlaps, yields a matched solution below the x nullcline,

$$\begin{aligned} Y_\downarrow(x;\epsilon) = {} & \overset{\rm xb}{Y}(x;\epsilon) + \overset{\rm lc}{Y}{}^0(x;\epsilon) + \overset{\rm rc}{Y}{}^0(x;\epsilon) \\ & - \left(\frac{1}{x_{\rm f}} - 1\right)(x_\star - x_{\rm f})\, e^{\left(\frac{1}{x_{\rm f}}-1\right)^{-1}\mathcal{Y}^1_{x_{\rm f}}(1)} \left(e^{-\frac{x-x_{\rm f}}{\epsilon}\frac{x_\star - x_{\rm f}}{1-x_{\rm f}}} + e^{-\frac{x_{\rm i,2}-x}{\epsilon}\frac{x_{\rm i,2}-x_\star}{1-x_{\rm i,2}}}\right). \end{aligned} \tag{50}$$

3.4.1 Prevalence trough

Evaluating $\overset{\rm xb}{Y}(x;\epsilon)$ (47) at $x = x_\star$ gives us an estimate of the minimum fraction infected ($\underline{y}$),

$$\underline{y}_{\_1} = \left(\frac{1}{x_{\rm f}} - 1\right)(x_\star - x_{\rm f})\left(\frac{1-x_{\rm f}}{1-x_\star}\right)^{\frac{1-x_\star}{\epsilon}} e^{-\frac{x_\star - x_{\rm f}}{\epsilon} + \left(\frac{1}{x_{\rm f}}-1\right)^{-1}\mathcal{Y}^1_{x_{\rm f}}(1)}. \tag{51}$$

Figure 3 shows the accuracy of this approximation to $\underline{y}$.

3.4.2 Closed-form approximations

Using our approximation to $\mathcal{Y}^1_{x_{\rm f}}(1)$ [Equation (A.1.1) in Appendix A], we obtain

$$\left(\frac{1}{x_{\rm f}} - 1\right)^{-1}\mathcal{Y}^1_{x_{\rm f}}(1) \approx \frac{x_\star - x_{\rm f} + x_\star x_{\rm f}}{(1-x_{\rm f})(1-x_\star)}\ln x_{\rm f} - \frac{x_\star}{x_\star - x_{\rm f}}. \tag{52}$$

Substituting Equation (52) into Equations (39), (47) and (49) to (51) gives us closed-form analytical expressions for all but the outer solution, $\overset{\text{out}}{Y}(x;\epsilon)$, which depends on $\mathcal{Y}^1_x(1)$ (we do not have a good approximation for the integral $\mathcal{Y}^1_x(1)$ for arbitrary x). We present these closed-form solutions in Tables 2 and 3.

3.5 Beyond the first epidemic wave

In principle, we could continue our phase plane trajectory approximation through any number of epidemic waves, but unlike the simple iterative expression we obtained in [10], the higher order expressions rapidly increase in complexity, and we have not identified a pattern that allows us to simplify them.

To illustrate the growth in analytical complexity as the trajectory proceeds, we will briefly sketch the derivation of the solution $Y^{\uparrow}(x;\epsilon)$ for the *second* epidemic wave. The $i=-1$ branch of the right corner solution (49) extends the solution above the x nullcline.

To match into the outer solution $\overset{\text{out}}{Y}(x, x_{\text{i},2};\epsilon)$ [Equations (20) and (21)], we expand $\overset{\text{rc}}{Y}{}^{-1}(x;\epsilon)$ as usual in an asymptotic series. Since the trajectory always lies in the set $x < x_{\text{i},2}$, we make the change of variables $x = x_{\text{i},2} - \eta x_\eta$ in $\overset{\text{rc}}{Y}{}^{-1}(x;\epsilon)$ and expand using Equation (34) to get

$$\begin{aligned}
\overset{\text{rc}}{Y}{}^{-1}(x_{\text{i},2} - \eta x_\eta;\epsilon) = {} & -\epsilon\left(\frac{1}{x_{\text{i},2}} - 1\right) W_{-1}\left(-\frac{1}{\epsilon}\frac{\left(\frac{1}{x_{\text{f}}} - 1\right)(x_\star - x_{\text{f}})}{\left(\frac{1}{x_{\text{i},2}} - 1\right)} e^{-\frac{\eta x_\eta}{\epsilon}\frac{x_{\text{i},2}-x_\star}{1-x_{\text{i},2}} - \left(\frac{1}{x_{\text{f}}}-1\right)^{-1}\mathcal{Y}^1_{x_{\text{f}}}(1)}\right) \\
= {} & \eta\left(1 - \frac{x_\star}{x_{\text{i},2}}\right)x_\eta + \epsilon \ln \eta^{-1}\left(\frac{1}{x_{\text{i},2}} - 1\right) \\
& - \epsilon\left(\frac{1}{x_{\text{i},2}} - 1\right)\left[\ln x_\eta + \ln\left(\frac{\left(\frac{1}{x_{\text{f}}} - 1\right)(x_\star - x_{\text{f}})(1 - x_{\text{i},2})}{\left(\frac{1}{x_{\text{i},2}} - 1\right)(x_{\text{i},2} - x_\star)}\right) - \left(\frac{1}{x_{\text{f}}} - 1\right)^{-1}\mathcal{Y}^1_{x_{\text{f}}}(1)\right] \\
& + \mathcal{O}\left(\frac{\epsilon^2 \ln \epsilon^{-1}}{\eta}\right),
\end{aligned} \tag{53}$$

which has a logarithmic singularity at $x_{\text{i},2}$. While this singularity may initially appear to be a defect in the approximation, it is necessary to match to the outer solution for the second epidemic wave (21): once we impose the effective initial condition $x_{\text{i},2} < 1$ (12), the second wave outer solution also has a logarithmic singularity at $x_{\text{i},2}$.

Expanding $\overset{\text{out}}{Y}(x, x_{\text{i},2};\epsilon)$ [Equations (20) and (21)] about $x = x_{\text{i},2}$, we have

$$\begin{aligned}
\overset{\text{out}}{Y}(x_{\text{i},2} - \eta x_\eta, x_{\text{i},2};\epsilon) = {} & \eta\left(1 - \frac{x_\star}{x_{\text{i},2}}\right)x_\eta - \epsilon \ln \eta^{-1}\left(\frac{1}{x_{\text{i},2}} - 1\right) \\
& + \epsilon\left(\left(\frac{1}{x_{\text{i},2}} - 1\right)\ln x_\eta + \mathcal{Y}^{x_{\text{i},2}}_{x_\star}(x_{\text{i},2}) - \left(\frac{1}{x_{\text{i},2}} - 1\right)\ln\left(x_{\text{i},2} - x_\star\right) + \left(\frac{1}{x_{\text{f},2}} - 1\right)\ln\left(\frac{x_{\text{i},2} - x_{\text{f},2}}{x_\star - x_{\text{f},2}}\right) + \overset{\text{out}}{C}_1\right) \\
& + \mathcal{O}(\eta^2),
\end{aligned} \tag{54}$$

where

$$x_{\text{f},2} = x_{\text{f}}(x_{\text{i},2}) \tag{55}$$

is the "final size" of the second epidemic wave.

Note that the offending logarithmic term from Equation (53) appears with opposite sign in Equation (54): when we sum the two solutions, the singularities will cancel. Choosing

$$\overset{\text{out}}{C}_1 = -\left(\frac{1}{x_{\text{f},2}} - 1\right)\ln\left(\frac{x_{\text{i},2} - x_{\text{f},2}}{x_\star - x_{\text{f},2}}\right) - \left(\frac{1}{x_{\text{i},2}} - 1\right)\ln\left(\frac{\left(\frac{1}{x_{\text{f}}} - 1\right)(x_\star - x_{\text{f}})(1 - x_{\text{i},2})}{\left(\frac{1}{x_{\text{i},2}} - 1\right)}\right) + \frac{\frac{1}{x_{\text{i},2}} - 1}{\frac{1}{x_{\text{f}}} - 1}\mathcal{Y}^1_{x_{\text{f}}}(1) - \mathcal{Y}^{x_{\text{i},2}}_{x_\star}(x_{\text{i},2}) \tag{56}$$

so that the expansions (53) and (54) agree to $\mathcal{O}(\epsilon)$ one obtains

$$\overset{\text{out}}{Y}_0(x,x_{\text{i},2}) + \epsilon\left[-\mathcal{Y}_x^{x_{\text{i},2}}(x_{\text{i},2}) + \left(\frac{1}{x_{\text{i},2}}-1\right)\ln\left(\frac{x_{\text{i},2}-x}{1-x_{\text{i},2}}\right) + \left(\frac{1}{x_{\text{f},2}}-1\right)\ln\left(\frac{x-x_{\text{f},2}}{x_{\text{i},2}-x_{\text{f},2}}\right)\right.$$
$$\left. + \frac{\frac{1}{x_{\text{i},2}}-1}{\frac{1}{x_{\text{f}}}-1}\mathcal{Y}^1_{x_{\text{f}}}(1) - \left(\frac{1}{x_{\text{i},2}}-1\right)\ln\left(\frac{\left(\frac{1}{x_{\text{f}}}-1\right)(x_\star - x_{\text{f}})}{\left(\frac{1}{x_{\text{i},2}}-1\right)(x_{\text{i},2}-x_\star)}\right)\right] \tag{57}$$

Finally, replacing x_{f} by $x_{\text{f},2}$ in the left corner solution (39) and its expansion (35) gives us the second left-corner solution and its expansion (up to the unknown matching constant $\overset{\text{cor}}{C}_1$). Comparing (35) (suitably modified with $x_{\text{f},2}$ replacing x_{f}) with (54) (for the appropriate choice of $\overset{\text{out}}{C}_1$), we again determine $\overset{\text{cor}}{C}_1$ by requiring the series agree to $\mathcal{O}(\epsilon)$. This yields

$$\overset{\text{lc}}{Y}{}^i(x;\epsilon) = -\epsilon\left(\frac{1}{x_{\text{f},2}}-1\right)W_i\left(-\frac{x_{\text{i},2}-x_{\text{f},2}}{\epsilon}\left(\frac{x_\star - x_{\text{f},2}}{1-x_{\text{f},2}}\right)\left(\frac{\left(\frac{1}{x_{\text{f}}}-1\right)(x_\star - x_{\text{f}})(1-x_{\text{i},2})}{\left(\frac{1}{x_{\text{i},2}}-1\right)(x_{\text{i},2}-x_\star)(x_{\text{i},2}-x_{\text{f},2})}\right)^{\frac{\frac{1}{x_{\text{i},2}}-1}{\frac{1}{x_{\text{f},2}}-1}}\right.$$
$$\left.\times\exp\left(-\frac{x-x_{\text{f},2}}{\epsilon}\frac{x_\star - x_{\text{f},2}}{1-x_{\text{f},2}} + \left(\frac{1}{x_{\text{f},2}}-1\right)^{-1}\mathcal{Y}^{x_{\text{i},2}}_{x_{\text{f},2}}(x_{\text{i},2}) + \frac{\left(\frac{1}{x_{\text{i},2}}-1\right)}{\left(\frac{1}{x_{\text{f}}}-1\right)\left(\frac{1}{x_{\text{f},2}}-1\right)}\mathcal{Y}^1_{x_{\text{f}}}(1)\right)\right), \tag{58}$$

whereas the matched solution above the x nullcline is

$$\overset{\text{out}}{Y}_0(x,x_{\text{i},2}) + \overset{\text{rc}}{Y}{}^{-1}(x;\epsilon) + \overset{\text{lc}}{Y}{}^{-1}(x;\epsilon) - \left(1-\frac{x_\star}{x_{\text{i},2}}\right)(x_{\text{i},2}-x) - \left(\frac{x_\star}{x_{\text{f},2}}-1\right)(x-x_{\text{f},2})$$
$$+\epsilon\left[\mathcal{Y}^x_{x_{\text{f},2}}(x_{\text{i},2}) + \frac{\left(\frac{1}{x_{\text{i},2}}-1\right)}{\left(\frac{1}{x_{\text{f}}}-1\right)}\mathcal{Y}^1_{x_{\text{f}}}(1) + \left(\frac{1}{x_{\text{i},2}}-1\right)\ln\left(\frac{\left(\frac{1}{x_{\text{f}}}-1\right)(x_\star - x_{\text{f}})(1-x_{\text{i},2})}{\left(\frac{1}{x_{\text{i},2}}-1\right)(x_{\text{i},2}-x_\star)(x_{\text{i},2}-x_{\text{f},2})}\right)\right] \tag{59}$$

While one might hope to recognize a pattern that facilitates continuing the approximation through subsequent waves, we have not succeeded and do not pursue the question further. We leave it as an exercise for the motivated reader.

4 Discussion

Building on our previous work [10], we have developed a higher order approximation for the phase plane trajectories of the standard SIR model with vital dynamics Equation (4). Throughout the phase plane, the "first order" solution we have presented here is more accurate than the "zeroth order" solution we derived previously [10], but the higher order results are much more cumbersome and, in practice, useful only until the peak of the second epidemic wave (whereas our "zeroth order" approximation is useful for an arbitrary number of epidemic waves). Much of the "first order" gain in accuracy is in the part of the phase plane where the frequency of infected hosts is implausibly low (reminiscent of Mollison's "atto-fox" [7]).

Our higher order trajectory approximation ($Y^\uparrow$ and $Y_\downarrow$ in Table 3) depends on an integral that must be evaluated numerically [$\mathcal{Y}^b_a(x_{\text{i}})$, Equation (18)]. However, as noted in §3.4.2, our closed-form approximation (52) to $\mathcal{Y}^1_{x_{\text{f}}}(1)$ allows us to obtain closed-form analytical expressions for all components of our matched asymptotic expansion *except* the outer solution. Our most useful closed-form results are our expressions for

the maximum and minimum prevalence following disease invasion ($\overline{y}$ in Equation (25) and $\underline{y}$ in Equation (51) using Equation (52)). We emphasize that our peak prevalence formula (25) includes the first order effect of demography, whereas in [10] our peak prevalence formula was the same as KM's.

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A Analytical Approximations for $\mathcal{Y}_a^b(x_\mathrm{i})$

A.1 Approximating $\mathcal{Y}^{x_\mathrm{i}}_{x_\mathrm{f}(x_\mathrm{i})}(x_\mathrm{i})$

While simple to formulate, in addition to having no analytical closed form, the integral (18) proves inconvenient to compute numerically. Here we consider analytical approximations obtained via polynomial approximations to $\overset{\text{out}}{Y}_0(x, x_\mathrm{i})$. In order to be consistent with our matchings, we require that the approximation preserve the poles and residues of the integrand, or equivalently that we approximate $\overset{\text{out}}{Y}_0(x, x_\mathrm{i})$ by a polynomial with the same zeros x_i and x_f, and the same derivatives at these points, $\frac{x_\star}{x_\mathrm{i}} - 1$ and $\frac{x_\star}{x_\mathrm{f}} - 1$.

When $x_\mathrm{i} = 1$, the singularity at x_i is removable, and we can approximate $\overset{\text{out}}{Y}_0(x) = \overset{\text{out}}{Y}_0(x, 1)$ by its linear approximation at x_f, $L(x,1) = \left(\frac{x_\star}{x_\mathrm{f}} - 1\right)(x - x_\mathrm{f})$. This gives us a particularly simple analytical expression,

$$\begin{aligned}
\mathcal{Y}^1_x(1) &= \int_x^1 \left[\left(\frac{x_\star}{u} - 1\right)\left(\frac{1}{u} - 1\right)\frac{1}{\overset{\text{out}}{Y}_0(u)} - \left(\frac{1}{x_\mathrm{f}} - 1\right)\frac{1}{u - x_\mathrm{f}}\right] \mathrm{d}u \\
&\approx \int_x^1 \left[\left(\frac{x_\star}{u} - 1\right)\left(\frac{1}{u} - 1\right)\frac{1}{L(u,1)} - \left(\frac{1}{x_\mathrm{f}} - 1\right)\frac{1}{u - x_\mathrm{f}}\right] \mathrm{d}u \\
&= \left(\frac{1}{x_\mathrm{f}} - 1 - \frac{x_\mathrm{f}}{x_\star - x_\mathrm{f}}\right)\ln x - \frac{x_\star}{x_\star - x_\mathrm{f}}\left(\frac{1}{x} - 1\right).
\end{aligned} \tag{A.1.1}$$

For $x = x_\mathrm{f}$, the value of the integral is largely determined in a neighbourhood of x_f and Equation (A.1.1) agrees very well with the numerically computed value of $\mathcal{Y}^1_{x_\mathrm{f}}(1)$ (Figure 1). Unfortunately, Equation (A.1.1) fails to provide a good uniform approximation for generic x.

For a general initial condition x_i, we can approximate $\overset{\text{out}}{Y}_0(x, x_\mathrm{i})$ by the cubic Hermite spline [2] determined by its zeroes and the derivatives at the zeroes:

$$\begin{aligned}
H(x, x_\mathrm{i}) &= \left(\frac{x_\star}{x_\mathrm{i}} - 1\right)(x - x_\mathrm{i}) + \frac{\frac{x_\star}{x_\mathrm{i}} - 1}{x_\mathrm{i} - x_\mathrm{f}(x_\mathrm{i})}(x - x_\mathrm{i})^2 + \frac{\left(\frac{x_\star}{x_\mathrm{i}} - 1\right) + \left(\frac{x_\star}{x_\mathrm{f}(x_\mathrm{i})} - 1\right)}{(x_\mathrm{i} - x_\mathrm{f}(x_\mathrm{i}))^2}(x - x_\mathrm{i})^2(x - x_\mathrm{f}(x_\mathrm{i})) \\
&= \frac{\left(\frac{x_\star}{x_\mathrm{i}} - 1\right) + \left(\frac{x_\star}{x_\mathrm{f}(x_\mathrm{i})} - 1\right)}{(x_\mathrm{i} - x_\mathrm{f}(x_\mathrm{i}))^2}(x - x_\mathrm{i})(x - x_\mathrm{f}(x_\mathrm{i}))(x - x_\mathrm{H}(x_\mathrm{i}))
\end{aligned} \tag{A.1.2}$$

where

$$x_\mathrm{H}(x_\mathrm{i}) = \frac{\left(\frac{x_\star}{x_\mathrm{i}} - 1\right)x_\mathrm{f}(x_\mathrm{i}) + \left(\frac{x_\star}{x_\mathrm{f}(x_\mathrm{i})} - 1\right)x_\mathrm{i}}{\left(\frac{x_\star}{x_\mathrm{i}} - 1\right) + \left(\frac{x_\star}{x_\mathrm{f}(x_\mathrm{i})} - 1\right)} \tag{A.1.3}$$

is the third root of the Hermite spline $H(x, x_\mathrm{i})$.

We then have

$$\begin{aligned}
\mathcal{Y}^{x_\mathrm{i}}_x(x_\mathrm{i}) &= \int_x^{x_\mathrm{i}} \left[\left(\frac{1}{u} - 1\right)\left(\frac{x_\star}{u} - 1\right)\frac{1}{\overset{\text{out}}{Y}_0(u, x_\mathrm{i})} - \left(\frac{1}{x_\mathrm{i}} - 1\right)\frac{1}{u - x_\mathrm{i}} - \left(\frac{1}{x_\mathrm{f}(x_\mathrm{i})} - 1\right)\frac{1}{u - x_\mathrm{f}(x_\mathrm{i})}\right] \mathrm{d}u \\
&\approx \int_x^{x_\mathrm{i}} \left[\left(\frac{1}{u} - 1\right)\left(\frac{x_\star}{u} - 1\right)\frac{1}{H(u, x_\mathrm{i})} - \left(\frac{1}{x_\mathrm{i}} - 1\right)\frac{1}{u - x_\mathrm{i}} - \left(\frac{1}{x_\mathrm{f}(x_\mathrm{i})} - 1\right)\frac{1}{u - x_\mathrm{f}(x_\mathrm{i})}\right] \mathrm{d}u \\
&= -\frac{(x_\mathrm{i} - x_\mathrm{f}(x_\mathrm{i}))^2}{\left(\frac{x_\star}{x_\mathrm{i}} - 1\right) + \left(\frac{x_\star}{x_\mathrm{f}(x_\mathrm{i})} - 1\right)}\left[\frac{x_\star}{x_\mathrm{i}x_\mathrm{f}(x_\mathrm{i})x_\mathrm{H}(x_\mathrm{i})}\left(\frac{1}{x} - \frac{1}{x_\mathrm{i}} + \left(1 + \frac{1}{x_\star} - \frac{1}{x_\mathrm{i}} - \frac{1}{x_\mathrm{f}(x_\mathrm{i})} - \frac{1}{x_\mathrm{H}(x_\mathrm{i})}\right)\ln\left(\frac{x}{x_\mathrm{i}}\right)\right)\right. \\
&\qquad \left. + \frac{\left(\frac{x_\star}{x_\mathrm{H}(x_\mathrm{i})} - 1\right)\left(\frac{1}{x_\mathrm{H}(x_\mathrm{i})} - 1\right)}{(x_\mathrm{H}(x_\mathrm{i}) - x_\mathrm{i})(x_\mathrm{H}(x_\mathrm{i}) - x_\mathrm{f}(x_\mathrm{i}))}\ln\left(\frac{x - x_\mathrm{H}(x_\mathrm{i})}{x_\mathrm{i} - x_\mathrm{H}(x_\mathrm{i})}\right)\right].
\end{aligned} \tag{A.1.4}$$

Equation (A.1.4) provides a good approximation that is uniform in x, *provided* $\mathcal{R}_0$ is close to 1. Away from this limit, Equation (A.1.4) performs similarly to Equation (A.1.1), *i.e.,* it does not provide a good uniform approximation in x, but for $x = x_{\mathrm{f}}(x_{\mathrm{i}})$ (for a general x_{i}) it does approximate $\mathcal{Y}^{x_{\mathrm{i}}}_{x_{\mathrm{f}}(x_{\mathrm{i}})}(x_{\mathrm{i}})$ very well (*cf.* Equations (58) and (59) and Figure 1). The failure for general $\mathcal{R}_0$ is due to the Hermite cubic spline substantially overestimating $\overline{y}$ for larger values of $\mathcal{R}_0$, which results in a poor approximation to $\mathcal{Y}^{x_{\mathrm{i}}}_{x}(x_{\mathrm{i}})$ for x near to $x_\star$. We attempted to obtain an improved estimate to $\overset{\text{out}}{Y}_0(x, x_{\mathrm{i}})$ by imposing a maximum at $\big(x_\star, \overline{y}_0(x_{\mathrm{i}})\big)$, either by using a pair of cubic Hermite splines on the intervals $[x_{\mathrm{f}}(x_{\mathrm{i}}), x_\star]$ and $[x_\star, x_{\mathrm{i}}]$ or by using a quintic Hermite spline; unfortunately, the resulting approximations are overwhelemed by Runge oscillations [4].

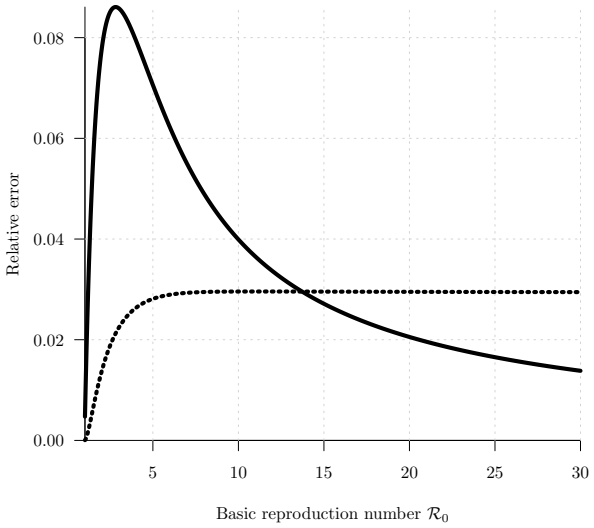


Figure 1: Relative error in our numerical approximations of $\mathcal{Y}^{x_{\mathrm{i},j}}_{x_{\mathrm{f},j}}(x_{\mathrm{i},j})$ for $j = 1$ (solid) and $j = 2$ (dotted). The exact expression as an integral is given in Equation (18). The closed-form approximations are given in Equations (A.1.1) and (A.1.4).

A.2 Approximating $\overline{y}$

We can obtain an excellent explicit approximation to

$$\mathcal{Y}_{\max} = \int_{x_\star}^{1} \left(\frac{x_\star}{u} - 1\right)\left(\frac{1}{u} - 1\right) \frac{1}{\overset{\text{out}}{Y}_0(u)}\, \mathrm{d}u \tag{A.2.1}$$

(see Equation (24)) by approximating $\overset{\text{out}}{Y}_0(u)$ with a rational function on the interval $[x, 1]$. The simplest rational approximation is the line joining $\big(x, \overset{\text{out}}{Y}_0(x)\big)$ to $\big(1, \overset{\text{out}}{Y}_0(1)\big)$, *i.e.,*

$$\overset{\text{out}}{Y}_0(u) = 1 - u + x_\star \ln u \tag{A.2.2a}$$

$$\simeq \overset{\text{out}}{Y}_0(x) + \frac{\overset{\text{out}}{Y}_0(1) - \overset{\text{out}}{Y}_0(x)}{1 - x}(u - x) \tag{A.2.2b}$$

$$= \overset{\text{out}}{Y}_0(x) - \frac{\overset{\text{out}}{Y}_0(x)}{1-x}(u-x) \tag{A.2.2c}$$

We need to evaluate $\overset{\text{out}}{Y}_0(x)$ at its maximum point, namely $x = x_\star$, hence the relevant approximation is [*cf.* Equation (10)]

$$\overset{\text{out}}{Y}_0(u) \simeq \overline{y}_0 - \frac{\overline{y}_0}{1-x_\star}\big(u - x_\star\big) = \overline{y}_0\,\frac{1-u}{1-x_\star} = \frac{1-x_\star\big(1-\ln x_\star\big)}{1-x_\star}\big(1-u\big)\,. \tag{A.2.3}$$

Inserting Equation (A.2.3) in Equation (A.2.1), where factors of $(1-u)$ in the numerator and denominator of the integrand now cancel out, we obtain

$$\mathcal{Y}_{\max} \simeq \left(\frac{1-x_\star}{1-x_\star\big(1-\ln x_\star\big)}\right)\int_{x_\star}^{1}\Big(\frac{x_\star}{u^2} - \frac{1}{u}\Big)\,\mathrm{d}u = \left(\frac{1-x_\star}{1-x_\star\big(1-\ln x_\star\big)}\right)\Big(1 - x_\star + \ln x_\star\Big)\,, \tag{A.2.4}$$

which yields Equation (25).

B Domain of Applicability of the First Order Approximation

We observed in [10] that for sufficiently large values of $\mathcal{R}_0$, x_{f} is $\mathcal{O}(\epsilon)$. In this limit, the higher-order corrections in the first order left corner layer approximation (39) cause it to extend into an $\mathcal{O}(\epsilon)$-strip in the biologically meaningless half-plane $x < 0$; consequently, for sufficiently large $\mathcal{R}_0$, $Y^{\uparrow}(x;\epsilon)$ (40) fails to approximate the true trajectory $Y(x)$. Here, we compute an upper bound $\bar{\epsilon} = \bar{\epsilon}(\mathcal{R}_0)$ such that the corner solution remains in the biological right half plane for $\epsilon < \bar{\epsilon}$, and thus $(0, \bar{\epsilon})$ provides the domain of applicability of $Y^{\uparrow}(x;\epsilon)$.

Substituting $\overset{\text{cor}}{C}_0$ (36), $\overset{\text{cor}}{C}_{\ln}$ (37), and $\overset{\text{cor}}{C}_1$ (38) into $\overset{\text{cor}}{X}(y;\epsilon)$ [Table 1] gives us the matched first order left corner layer solution as a function of y:

$$\overset{\text{lc}}{X}(y;\epsilon) = x_{\mathrm{f}} + \left(\frac{x_{\mathrm{f}}}{x_\star - x_{\mathrm{f}}}\right)y - \epsilon\left(\frac{1-x_{\mathrm{f}}}{x_\star - x_{\mathrm{f}}}\right)\left[\ln y - \ln\left(\left(\frac{1}{x_{\mathrm{f}}} - 1\right)(x_\star - x_{\mathrm{f}})\right) - \left(\frac{1}{x_{\mathrm{f}}} - 1\right)^{-1}\mathcal{Y}^1_{x_{\mathrm{f}}}(1)\right]. \tag{B.0.1}$$

Solving $\frac{\mathrm{d}}{\mathrm{d}y}\overset{\text{lc}}{X}(y;\epsilon) = 0$, we find that the minimum value of $\overset{\text{lc}}{X}(y;\epsilon)$, $\underline{x}_1$, occurs at $y = \epsilon\left(\frac{1}{x_{\mathrm{f}}} - 1\right)$:

$$\underline{x}_1 = \overset{\text{lc}}{X}\left(\epsilon\left(\tfrac{1}{x_{\mathrm{f}}} - 1\right);\epsilon\right) = x_{\mathrm{f}} + \epsilon\ln\epsilon^{-1}\left(\frac{1-x_{\mathrm{f}}}{x_\star - x_{\mathrm{f}}}\right) + \epsilon\left(\frac{1-x_{\mathrm{f}}}{x_\star - x_{\mathrm{f}}}\right)\left[1 + \ln\left(x_\star - x_{\mathrm{f}}\right) + \left(\frac{1}{x_{\mathrm{f}}} - 1\right)^{-1}\mathcal{Y}^1_{x_{\mathrm{f}}}(1)\right]. \tag{B.0.2}$$

(Similarly, we can obtain the right corner solution as a function of y, $\overset{\text{lc}}{X}(y;\epsilon)$; evaluating the latter at $y = \epsilon\left(\frac{1}{x_{\mathrm{i,2}}} - 1\right)$ gives an approximation to the maximum fraction susceptible, $\overline{x}_1$. See Table 2). Proceeding as in §3.3, we may use the Lambert W function to convert the inequality $\underline{x}_1 > 0$ into an inequality for ϵ. We find $\underline{x}_1 > 0$ if and only if

$$\epsilon < \bar{\epsilon} = \frac{x_\star - x_{\mathrm{f}}}{\left(\frac{1}{x_{\mathrm{f}}} - 1\right)W_0\left(\left(\frac{1}{x_{\mathrm{f}}} - 1\right)^{-1}e^{-\left(\frac{1}{x_{\mathrm{f}}}-1\right)^{-1}\mathcal{Y}^1_{x_{\mathrm{f}}}(1)-1}\right)}. \tag{B.0.3}$$

Quantity		**Expression**	**Equation**
Equilibrium susceptible density	$x_\star$	$\frac{1}{\mathcal{R}_0}$	(3)
Final size (KM)	x_f	$-x_\star W_0\left(\mathscr{E}\left(-1/x_\star\right)\right)$	(9)
Peak prevalence (KM)	$\overline{y}_0$	$1-x_\star\left(1-\ln\left(x_\star\right)\right)$	(10)
Effective initial condition	$x_{\mathrm{i},2}$	$1+\left(1-x_\star\right)W_0\left(\mathscr{E}\left(-\frac{1-x_\mathrm{f}}{1-x_\star}\right)\right)$	(12)
Peak prevalence ($\epsilon>0$)	$\overline{y}_1$	$\overline{y}_0-\epsilon\frac{(1-x_\star)(1-x_\star+\ln x_\star)}{1-x_\star\left(1-\ln x_\star\right)}$	(25)
Minimum prevalence	$\underline{y}_1$	$\left(\frac{1}{x_\mathrm{f}}-1\right)\left(x_\star-x_\mathrm{f}\right)x_\mathrm{f}^{\frac{x_\star-x_\mathrm{f}+x_\star x_\mathrm{f}}{(1-x_\mathrm{f})(1-x_\star)}}\left(\frac{1-x_\mathrm{f}}{1-x_\star}\right)^{\frac{1-x_\star}{\epsilon}}e^{-\frac{x_\star-x_\mathrm{f}}{\epsilon}-\frac{x_\star}{x_\star-x_\mathrm{f}}}$	(51)
Minimum susceptible density	$\underline{x}_1$	$x_\mathrm{f}+\epsilon\left(\frac{1-x_\mathrm{f}}{x_\star-x_\mathrm{f}}\right)\left[\ln\epsilon^{-1}+1+\ln\left(x_\star-x_\mathrm{f}\right)+\frac{x_\star-x_\mathrm{f}+x_\star x_\mathrm{f}}{(1-x_\mathrm{f})(1-x_\star)}\ln x_\mathrm{f}-\frac{x_\star}{x_\star-x_\mathrm{f}}\right]$	(B.0.2)
Maximum susceptible density	$\overline{x}_1$	$x_{\mathrm{i},2}+\epsilon\frac{1-x_{\mathrm{i},2}}{x_{\mathrm{i},2}-x_\star}\left[\ln\left(\frac{\epsilon\left(\frac{1}{x_{\mathrm{i},2}}-1\right)}{\left(\frac{1}{x_\mathrm{f}}-1\right)\left(x_\star-x_\mathrm{f}\right)}\right)-\frac{x_\star-x_\mathrm{f}+x_\star x_\mathrm{f}}{(1-x_\mathrm{f})(1-x_\star)}\ln x_\mathrm{f}-\frac{x_\star}{x_\star-x_\mathrm{f}}\right]$	–

Table 2: Approximations of quantities of epidemiological interest for disease invasions, *i.e.,* on the trajectory that emanates from the disease-free equilibrium (DFE), $(x_\mathrm{i},y_\mathrm{i})=(1,0)$. Each entry may depend upon entries above it in the table (but never on entries below). These quantities are used in our approximations to the full trajectories in Table 3. We use "(KM)" to indicate quantities that are exact for the Kermack-McKendrick SIR model *without* vital dynamics ($\epsilon=0$). With vital dynamics ($\epsilon>0$), the peak prevalence $\overline{y}_0$ is an approximation, and there is no "final" size, but the quantity x_f appears in the approximation to the minimum fraction susceptible ($\underline{x}_1$). The expression for $\underline{x}_1$ is valid provided $\epsilon<\bar{\epsilon}$ Equation (B.0.3). We write the formulae for $\underline{x}_1$ and $\overline{x}_1$ as compactly as possible here; see Equation (B.0.2) for $\underline{x}_1$ written out with separate terms for each asymptotic order.

Solution	Notation	Expression	Branch	Domain	Equation
KM	$\overset{\text{out}}{Y}_0(x, x_{\text{i}})$	$x_{\text{i}} - x + x_\star \ln\left(\frac{x}{x_{\text{i}}}\right)$	–	$[x_{\text{f}}, x_{\text{i}}]$	(8)
Outer	$\overset{\text{out}}{Y}(x;\epsilon)$	$\overset{\text{out}}{Y}_0(x, x_{\text{i}}) - \epsilon \mathcal{Y}^1_x(1) + \epsilon\left(\frac{1}{x_{\text{f}}} - 1\right) \ln \frac{x - x_{\text{f}}}{1 - x_{\text{f}}}$	–	$[x_{\text{f}}, x_{\text{i}}]$	(23)
x-axis bdry	$\overset{\text{xb}}{Y}(x;\epsilon)$	$\left(\frac{1}{x_{\text{f}}} - 1\right)(x_\star - x_{\text{f}}) \left(\frac{1-x_{\text{f}}}{1-x}\right)^{\frac{1-x_\star}{\epsilon}} x_{\text{f}}^{\frac{x_\star - x_{\text{f}} + x_\star x_{\text{f}}}{(1-x_{\text{f}})(1-x_\star)}} e^{-\frac{x - x_{\text{f}}}{\epsilon} - \frac{x_\star}{x_\star - x_{\text{f}}}}$	–	$[x_{\text{f}}, x_\star]$	(47)
Left corner	$\overset{\text{lc}}{Y}{}^i(x;\epsilon)$	$-\epsilon\left(\frac{1}{x_{\text{f}}} - 1\right) W_i\left(-\frac{x_\star - x_{\text{f}}}{\epsilon} x_{\text{f}}^{\frac{x_\star - x_{\text{f}} + x_\star x_{\text{f}}}{(1-x_{\text{f}})(1-x_\star)}} e^{-\frac{x - x_{\text{f}}}{\epsilon} \frac{x_\star - x_{\text{f}}}{1 - x_{\text{f}}} - \frac{x_\star}{x_\star - x_{\text{f}}}}\right)$	$i = \begin{cases} -1 & \text{above} \\ 0 & \text{below} \end{cases}$	$[x_{\text{f}}, x_\star]$	(39)
Right corner	$\overset{\text{rc}}{Y}{}^i(x;\epsilon)$	$-\epsilon\left(\frac{1}{x_{\text{i},2}} - 1\right) W_i\left(-\frac{\left(\frac{1}{x_{\text{f}}} - 1\right)(x_\star - x_{\text{f}})}{\epsilon\left(\frac{1}{x_{\text{i},2}} - 1\right)} x_{\text{f}}^{\frac{x_\star - x_{\text{f}} + x_\star x_{\text{f}}}{(1-x_{\text{f}})(1-x_\star)}} e^{-\frac{x_{\text{i},2} - x}{\epsilon} \frac{x_{\text{i},2} - x_\star}{1 - x_{\text{i},2}} - \frac{x_\star}{x_\star - x_{\text{f}}}}\right)$	$i = \begin{cases} -1 & \text{above} \\ 0 & \text{below} \end{cases}$	$[x_\star, x_{\text{i},2}]$	(49)
Matched, above	$Y^{\uparrow}(x;\epsilon)$	$\overset{\text{out}}{Y}_0(x, x_{\text{i}}) + \overset{\text{lc}}{Y}{}^{-1}(x;\epsilon) - \left(\frac{x_\star}{x_{\text{f}}} - 1\right)(x - x_{\text{f}}) + \epsilon \mathcal{Y}^x_{x_{\text{f}}}(1)$	–	$[x_{\text{f}}, x_{\text{i}}]$	(40)
Matched, below	$Y_{\downarrow}(x;\epsilon)$	$\overset{\text{xb}}{Y}(x;\epsilon) + \overset{\text{lc}}{Y}{}^0(x;\epsilon) + \overset{\text{rc}}{Y}{}^0(x;\epsilon)$ $-\left(\frac{1}{x_{\text{f}}} - 1\right)(x_\star - x_{\text{f}}) x_{\text{f}}^{\frac{x_\star - x_{\text{f}} + x_\star x_{\text{f}}}{(1-x_{\text{f}})(1-x_\star)}} e^{-\frac{x_\star}{x_\star - x_{\text{f}}}} \left(e^{-\frac{x - x_{\text{f}}}{\epsilon} \frac{x_\star - x_{\text{f}}}{1 - x_{\text{f}}}} + e^{-\frac{x_{\text{i},2} - x}{\epsilon} \frac{x_{\text{i},2} - x_\star}{1 - x_{\text{i},2}}}\right)$	–	$[x_{\text{f}}, x_{\text{i},2}]$	(50)

Table 3: First order matched approximations to the initial epidemic following disease invasion $[(x_{\text{i}}, y_{\text{i}}) = (1, 0)]$. See Table 2 for $x_\star$, x_{f}, and $x_{\text{i},2}$ expressed in terms of $\mathcal{R}_0$ and ϵ. Equation (18) defines $\mathcal{Y}^b_a(1) = \int_a^b \left[\left(\frac{1}{u} - 1\right)\left(\frac{x_\star}{u} - 1\right) \Big/ \overset{\text{out}}{Y}_0(u, x_{\text{i}}) - \left(\frac{1}{x_{\text{f}}} - 1\right)\frac{1}{u - x_{\text{f}}}\right] \mathrm{d}u$. We use (52) to approximate $\mathcal{Y}^1_{x_{\text{f}}}(1)$ analytically. Above and below indicate solutions that are valid above and below the x nullcline, $y = \epsilon\left(\frac{1}{x} - 1\right)$. The expression for $\overset{\text{rc}}{Y}{}^i(x;\epsilon)$, and thus the matched solutions $Y^{\uparrow}(x;\epsilon)$ and $Y_{\downarrow}(x;\epsilon)$ are valid provided $\epsilon < \bar{\epsilon}$ Equation (B.0.3).

Constant	Expression	Equation
$\overset{\text{out}}{C}_0$	$y_{\text{i}} + x_{\text{i}} - x_\star \ln x_{\text{i}}$	(7)
$\overset{\text{out}}{C}_1$	$-\mathcal{Y}^1_{x_\star}(1) - \left(\frac{1}{x_{\text{f}}} - 1\right) \ln\left(\frac{1-x_{\text{f}}}{x_\star - x_{\text{f}}}\right)$	(22)
$\overset{\text{cor}}{C}_0$ (left)	x_{f}	(36)
$\overset{\text{cor}}{C}_{\ln}$ (left)	$-\frac{1-x_{\text{f}}}{x_\star - x_{\text{f}}}$	(37)
$\overset{\text{cor}}{C}_1$ (left)	$\frac{1-x_{\text{f}}}{x_\star - x_{\text{f}}}\left[\ln\left(\left(\frac{1}{x_{\text{f}}} - 1\right)(x_\star - x_{\text{f}})\right) + \left(\frac{1}{x_{\text{f}}} - 1\right)^{-1} \mathcal{Y}^1_{x_{\text{f}}}(1)\right]$	(38)
c_0^ϕ	x_{f}	(44)
C_0^ϕ	$x_{\text{f}} - (1 - x_\star)\ln(1 - x_{\text{f}})$	(45)
C_1^ϕ	$\ln\left[\left(\frac{1}{x_{\text{f}}} - 1\right)(x_\star - x_{\text{f}})\right] + \left(\frac{1}{x_{\text{f}}} - 1\right)^{-1} \mathcal{Y}^1_{x_{\text{f}}}(1)$	(46)
$\overset{\text{cor}}{C}_0$ (right)	$x_{\text{i},2}$	–
$\overset{\text{cor}}{C}_{\ln}$ (right)	$\frac{1-x_{\text{i},2}}{x_{\text{i},2} - x_\star}$	–
$\overset{\text{cor}}{C}_1$ (right)	$-\frac{1-x_{\text{i},2}}{x_{\text{i},2} - x_\star}\left[\ln\left[\left(\frac{1}{x_{\text{f}}} - 1\right)(x_\star - x_{\text{f}})\right] + \left(\frac{1}{x_{\text{f}}} - 1\right)^{-1} \mathcal{Y}^1_{x_{\text{f}}}(1)\right]$	–
$\overset{\text{out}}{C}_0$ (2^{nd})	$x_{\text{i},2} - x_\star \ln x_{\text{i},2}$	–
$\overset{\text{out}}{C}_1$ (2^{nd})	$-\left(\frac{1}{x_{\text{f},2}} - 1\right)\ln\left(\frac{\left(\frac{1}{x_{\text{f}}}-1\right)(x_\star - x_{\text{f}})(1-x_{\text{i},2})}{\left(\frac{1}{x_{\text{i},2}}-1\right)(x_{\text{i},2} - x_\star)(x_{\text{i},2} - x_{\text{f},2})}\right) + \frac{\frac{1}{x_{\text{f},2}}-1}{\frac{1}{x_{\text{f}}}-1}\mathcal{Y}^1_{x_{\text{f}}}(1) - \mathcal{Y}^{x_{\text{i},2}}_{x_\star}(x_{\text{i},2})$	(56)
$\overset{\text{cor}}{C}_0$ (left, 2^{nd})	$x_{\text{f},2}$	–
$\overset{\text{cor}}{C}_{\ln}$ (left, 2^{nd})	$-\frac{1-x_{\text{f},2}}{x_\star - x_{\text{f},2}}$	–
$\overset{\text{cor}}{C}_1$ (left, 2^{nd})	$\frac{1-x_{\text{f},2}}{x_\star - x_{\text{f},2}}\left[\left(\frac{1}{x_{\text{f},2}} - 1\right)^{-1} \mathcal{Y}^{x_{\text{i},2}}_{x_{\text{f},2}}(x_{\text{i},2}) - \frac{\frac{1}{x_{\text{i},2}} - 1}{\left(\frac{1}{x_{\text{f}}} - 1\right)\left(\frac{1}{x_{\text{f},2}} - 1\right)}\mathcal{Y}^1_{x_{\text{f}}}(1) + \frac{\frac{1}{x_{\text{i},2}} - 1}{\frac{1}{x_{\text{f},2}} - 1}\ln\left(\frac{\left(\frac{1}{x_{\text{f}}} - 1\right)(x_\star - x_{\text{f}})(1 - x_{\text{i},2})}{\left(\frac{1}{x_{\text{i},2}} - 1\right)(x_{\text{i},2} - x_\star)(x_{\text{i},2} - x_{\text{f},2})}\right) + \ln\left(\frac{\left(\frac{1}{x_{\text{f},2}} - 1\right)(x_\star - x_{\text{f},2})(x_{\text{i},2} - x_{\text{f},2})}{1 - x_{\text{f},2}}\right)\right]$	–

Table 4: First order matching constants for disease invasions [$(x_{\text{i}}, y_{\text{i}}) = (1, 0)$]. Left and right indicate constants appearing in the left and right corner solutions, while 2^{nd} indicates constants for the second epidemic wave. See Table 2 for $x_\star$, x_{f}, and $x_{\text{i},2}$ expressed in terms of $\mathcal{R}_0$ and ϵ. The final expressions with these values for the matching constants are listed in Table 3.

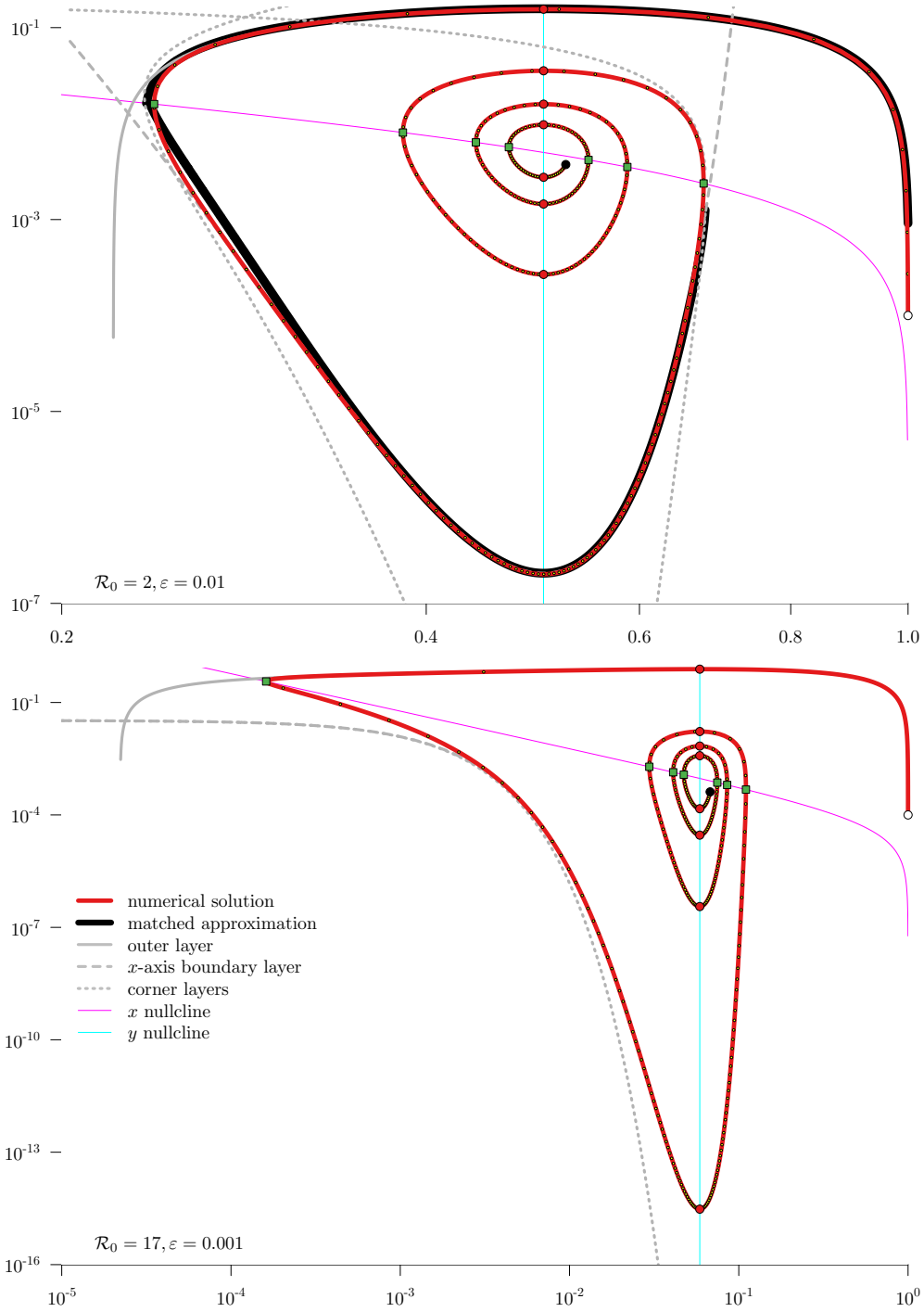


Figure 2: Solutions of the SIR ODEs Equation (4) and first order approximations (Table 3). This figure should be compared with the zeroth order approximations shown in [10, Figures 4 and 5]. *Top panel:* $\mathcal{R}_0 = 2$, $\varepsilon = 0.01$. *Bottom panel:* $\mathcal{R}_0 = 17$, $\varepsilon = 0.001$; similar to measles and whooping cough [10, Table 1]. Various outer and inner approximations are shown in grey, and the matched approximation is black. Numerically computed solutions to Equation (4) are red. In the bottom panel, the matched solution is not shown because matching fails for this parameter combination ($5.9 \times 10^{-5} = \epsilon > \bar{\epsilon} \approx 5.7 \times 10^{-9}$; *cf.* Equation (B.0.3)). Nevertheless, the local solutions that we show still provide accurate approximations, with the exception of the corner layer solution that fails and prevents successful matching.

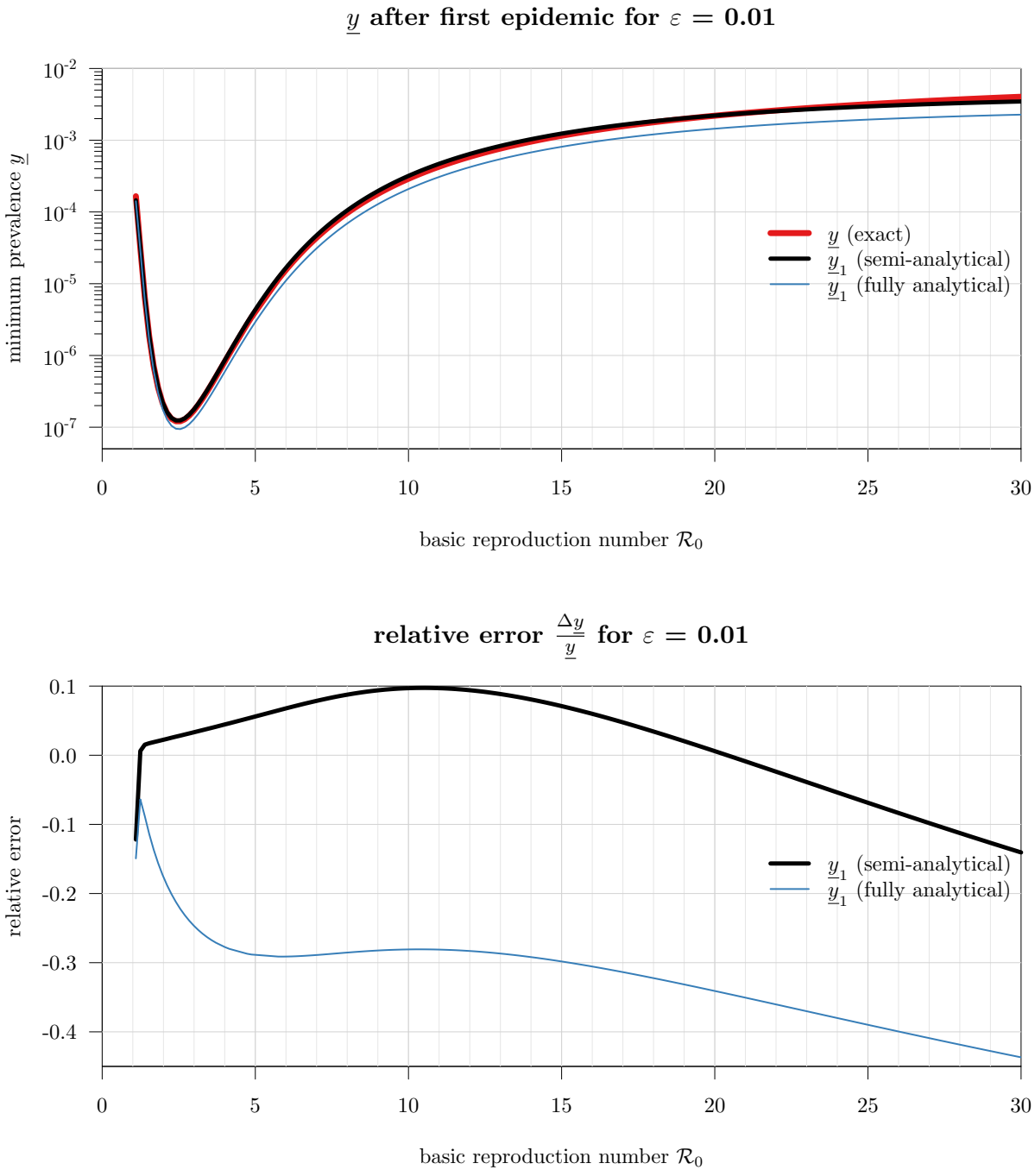


Figure 3: Minimum prevalence following an initial epidemic, as a function of $\mathcal{R}_0$ for $\varepsilon = 0.01$. *Top panel:* minimum prevalence $\underline{y}$ from "exact" numerical solution of Equation (4); from our semi-analytical approximation $\underline{y}_1$ (51, which depends on computing the integral $\mathcal{Y}^1_{x_f}(1)$ numerically); and from our fully analytical closed-form approximation to $\underline{y}_1$ (obtained using Equation (52)). *Bottom panel:* Relative errors when approximating $\underline{y}$ using the semi-analytical $\underline{y}_1$ Equation (51), and the fully analytical approximate $\underline{y}_1$.

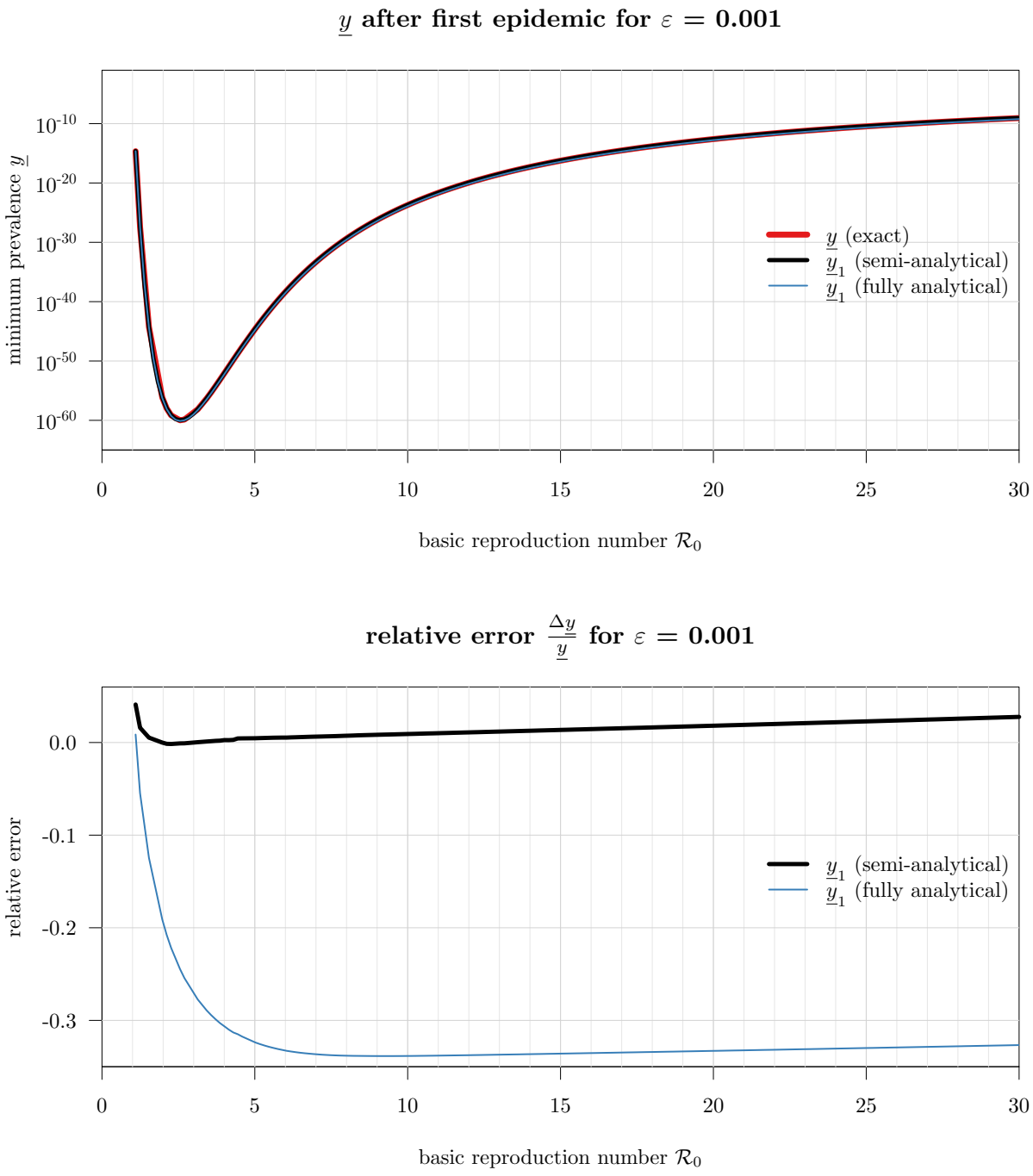


Figure 4: Minimum prevalence after a first epidemic. Like Figure 3 but for $\varepsilon = 0.001$.

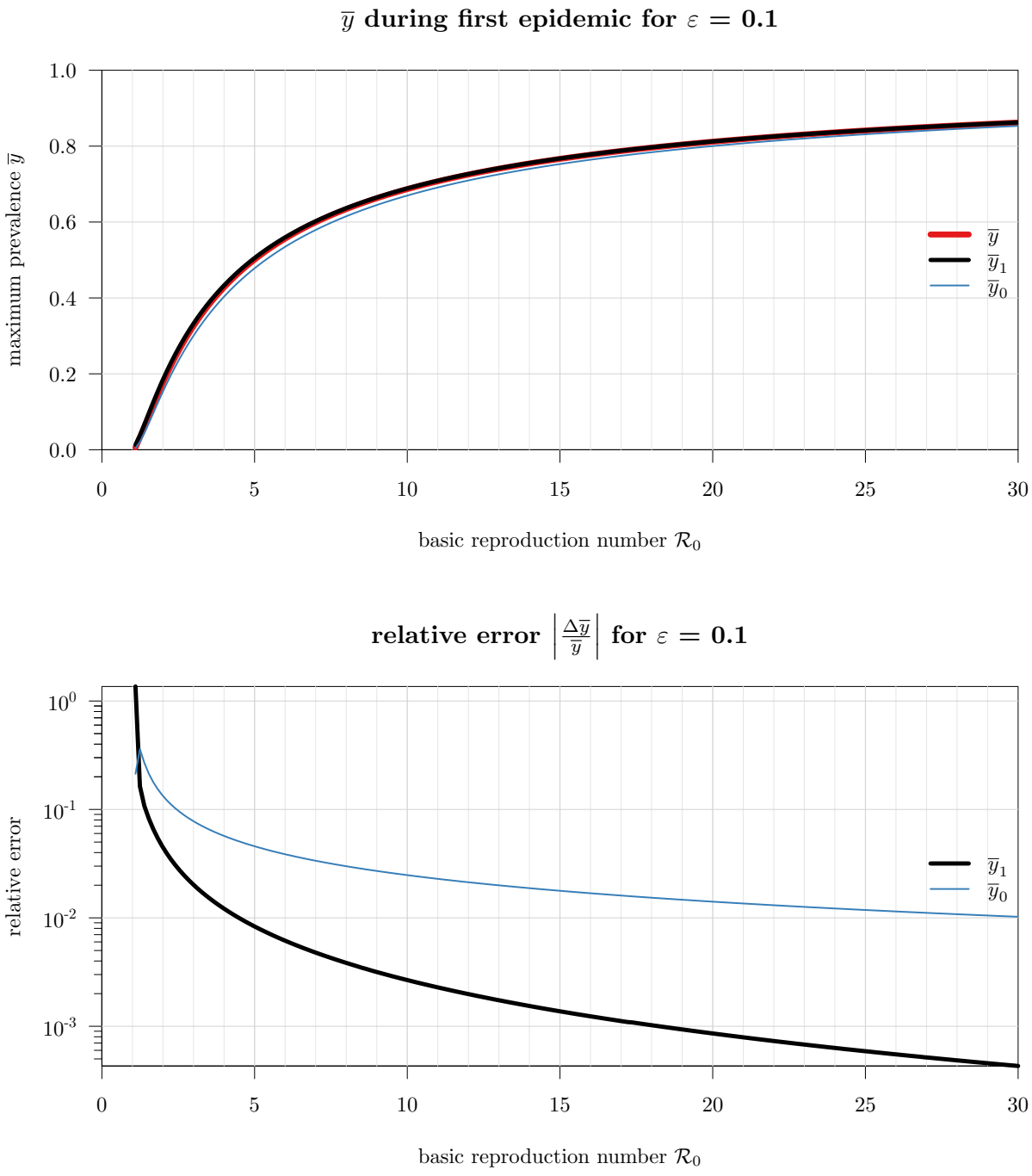
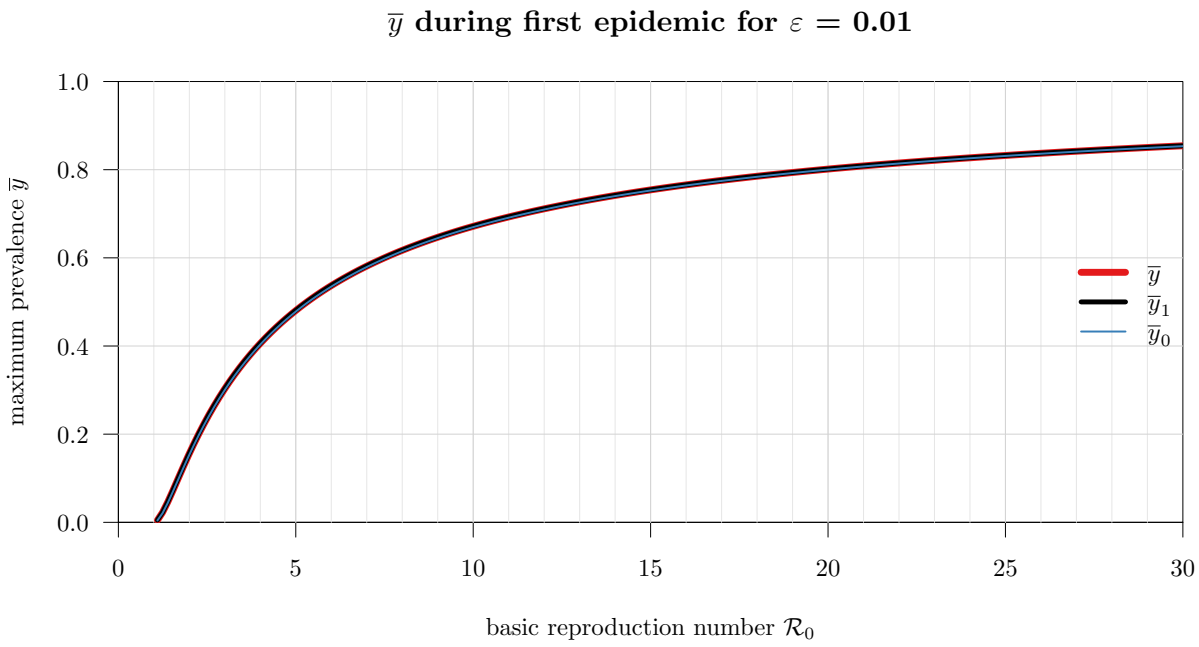


Figure 5: Maximum prevalence. Like Figure 3 but for $\overline{y}$ with $\varepsilon = 0.1$, and using a logarithmic scale in the bottom panel.



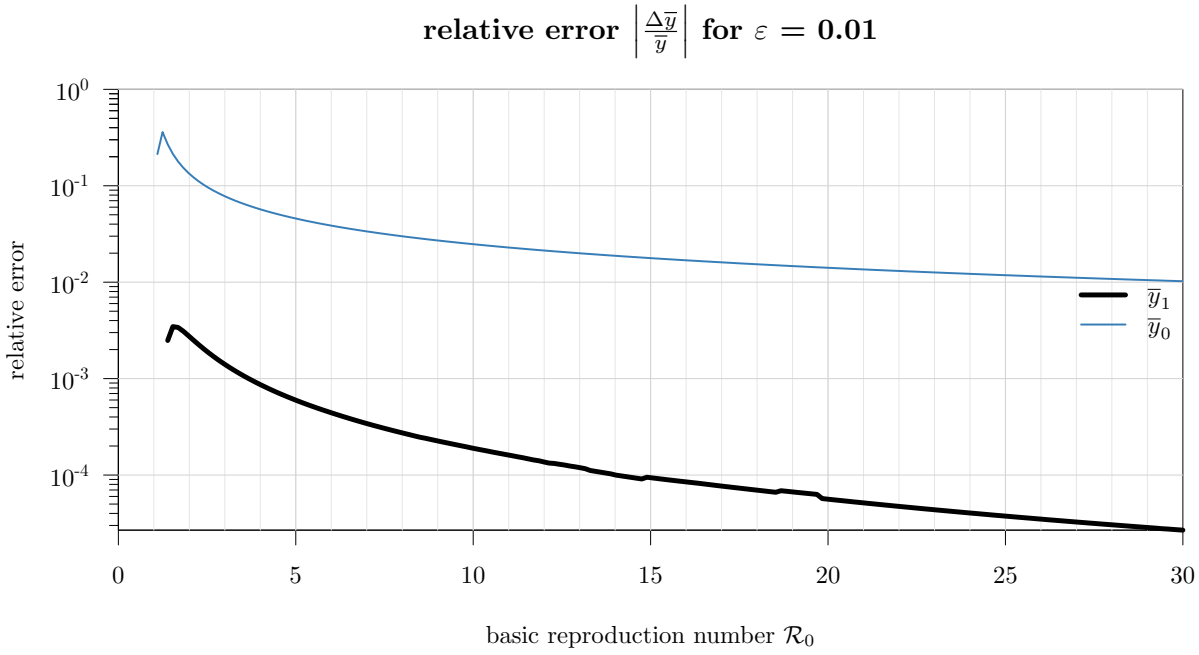


Figure 6: Maximum prevalence. Like Figure 5 but for $\varepsilon = 0.01$.