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HIDEKI MIYACHI, KEN'ICHI OHSHIKA, AND ATHANASE PAPADOPOULOS

Abstract. We continue the study of the analogue of Thurston's metric on the Teichmüller space of Euclidean triangle which was started by Saglam-Papadopoulos in [11]. By direct calculation, we give explicit expressions of the distance function and the Finsler structure of the metric restricted to the subspace of acute triangles. We deduce from the form of the Finsler unit sphere a result on the infinitesimal rigidity of the metric. We give a description of the maximal stretching loci for a family of extreme Lipschitz maps.

Keywords: Thurston's asymmetric metric, Lipschitz metric, extreme Lipschitz maps, stretch locus, Teichmüller theory, space of Euclidean triangles, geodesics, Finsler structure

AMS codes: 30F60, 51F99, 57M50, 32G15, 57K20

1. Introduction

In his paper [15], William Thurston introduced an asymmetric metric on the Teichmüller space of complete hyperbolic metrics of finite type on a given surface, where the distance between two points is defined as the logarithm of the smallest dilatation constant of a Lipschitz homeomorphism between two marked surfaces equipped with hyperbolic structures representing these points. Thurston discovered several properties of this metric, called now *Thurston's metric*. In particular, he gave a description of a large class of geodesics, called stretch lines, he showed that any two points are connected by a geodesic, he proved that this metric is Finsler, and he gave a description of the unit sphere of the conorm of this Finsler structure at each cotangent space of Teichmüller space, as an embedding of the space of projective measured laminations on the surface.

In the last couple of decades, Thurston's metric has been adapted to many different settings. We list here some of these settings, to give an indication of some of the developments: Teichmüller spaces of surfaces with boundary, see e.g. the recent papers [5] and [1]; Teichmüller space of the torus, see [2] and [6]; Randers–Teichmüller metrics [9]; singular flat metrics underlying a fixed quadrangulation [12]; higher dimensional hyperbolic manifolds [3]; higher Teichmüller theory, see [14], and there are others. This list is necessarily very partial, and the work and the literature on this topic is growing at a fast rate. See also the list of problems in [13], and [8] and [16] for two recent surveys on Thurston's metric and its developments.

Recently, İsmail Sağlam and the third author of the present paper developed the theory of Thurston's Lipschitz metric on the Teichmüller space of Euclidean triangles, that is, the moduli space of marked Euclidean triangles (see [11]). They proved in particular that restricted to the space of acute triangles, the Lipschitz distance between two marked triangles is equivalent to another distance, defined as the logarithm of the maximum of the lengths of the three edges and the three

altitudes of the two triangles. This new formula is an analogue of Thurston's alternative description of his distance in terms of ratios of lengths of simple closed geodesics (see [15, p. 4]). In [11], the authors also proved that for $A > 0$, the metric induced on the space of acute triangles having fixed area A is Finsler. They gave a characterisation of geodesics in $\mathfrak{AT}_A$: a path is geodesic if and only if the angle at each labelled vertex of triangles in this family varies monotonically. Finally, they proved that the isometry group of $\mathfrak{AT}_A$ is isomorphic to S_3, the symmetric group on $\{1, 2, 3\}$.

The aim of the present paper is to continue studying Thurston's metric on the Teichmüller space of Euclidean triangles inaugurated in [11]. Following [11], for $A > 0$, we let $\mathfrak{T}_A$ and $\mathfrak{AT}_A$ denote respectively the Teichmüller spaces of Euclidean triangles and acute triangles of fixed area A (see §2). Unlike the metric described by Thurston on the Teichmüller space of hyperbolic surfaces, the Lipschitz metric on $\mathfrak{AT}_A$ is symmetric as shown in [11]. In the present paper, developing the theory of Thurston's metric on the Teichmüller spaces of triangles further, we obtain the following:

- Explicit formulae for the Lipschitz metric and its associated Finsler infinitesimal structure on $\mathfrak{AT}_A$, after identifying the space $\mathfrak{T}_A$ with the upper-half plane $\mathbb{H}$ in the complex plane $\mathbb{C}$.
- A description of the Finsler ball at a point in the tangent space of $\mathfrak{AT}_A$ using the new parameters, and an infinitesimal rigidity result, saying that the infinitesimal unit ball at a give point determines the triangle as a point in Teichmüller space.
- A characterisation of the stretch locus for an extremal Lipschitz map between acute triangles, and an analogue of Teichmüller's result (respectively Thurston's result) on the existence of foliations (respectively laminations) on which the Lipschitz constant for the extremal Lipschitz map is optimal.
- The perhaps surprising result that, contrary to the usual case of Thurston's metric, there are instances where there are no extremal Lipschitz homeomorphisms between two triangles.

The present setting of the Teichmüller space of the triangle is a simple case where Thurston's ideas on his Lipschitz metric have simple analogues and can be described in an elementary way, without heavy use of Thurston's theory of surfaces.

2. Teichmüller space of triangles

2.1. **labelled triangles.** We identify the Euclidean plane with the complex plane $\mathbb{C}$ in the canonical way. Consider a triangle in the Euclidean plane. We label its vertices by an ordered triple so that this labelling induces a counter-clockwise orientation on the boundary of the triangle. We call such a triangle *labelled*. Consider a 2-simplex σ_0 with vertices labelled as a, b, c, fixed once and for all. The labelled triangle is thought of as a pair (T, f) of a triangle T equipped with a vertex-preserving homeomorphism $f\colon \sigma_0 \to T$, called the *marking* of the triangle. In this setting, for $v \in \{a, b, c\}$, the vertex of T corresponding to the vertex labelled as v of σ_0 via the marking is called the *v-vertex* of T.

Henceforth, for a labelled triangle T and for $v \in \{a, b, c\}$, we use the notation $z_v(T) \in \mathbb{C}$ for the v-vertex of T. Let $\theta_v(T)$ be the angle of T at the v-vertex. We denote by $e_v(T)$ the edge of the triangle which is opposite to the v-vertex, and by $|e_v(T)|$ its Euclidean length.

2.2. **The space of triangles.** We say that two labelled triangles T and T' are *(Teichmüller)-equivalent* if there is a label-preserving isometry (in terms of the Euclidean metric) from T onto T'.

Let $\mathfrak{T}_A$ be the set of Teichmüler equivalence classes of triangles of area A. Let $\mathfrak{AT}_A$ be the subset of $\mathfrak{T}_A$ consisting of acute triangles. For $v \in \{a, b, c\}$, we denote by $\mathfrak{OT}_v$, or, more simply, $\mathfrak{O}_v$, the subset of $\mathfrak{T}_A$ consisting of obtuse triangles with obtuse angle at the v-vertex.

Convention 1. *From now on, to simplify notation, we omit the subscript A (which concerns the area of triangles) from the symbols in the case of unit area triangles (i.e. $A = 1$). For instance, we set $\mathfrak{T} = \mathfrak{T}_1$. We also simply write T to denote the Teichmüller equivalence class of a triangle T.*

2.3. **Realisation of the space of triangles.** In §6 of the paper [11], the authors suggested an identification between the space $\mathfrak{AT}_{\frac{1}{2}}$ of acute triangles of area $\frac{1}{2}$ and an ideal triangle in the hyperbolic plane. We shall use this identification.

The space $\mathfrak{T}_A$ is naturally identified with the upper-half-plane $\mathbb{H} \subset \mathbb{C}$, by taking $z \in \mathbb{H}$ to the equivalence class of a labelled triangle T_z with

$$z_a(T_z) = (2A/y)^{1/2} z, z_b(T_z) = 0, z_c(T) = (2A/y)^{1/2} \tag{2.1}$$

with area A in $\mathbb{C}$, where $z = x + iy$. We denote by z the inverse of this map, which takes each point in $\mathfrak{T}_A$ to a number in $\mathbb{H}$. For $T \in \mathfrak{T}_A$ with $z = z(T)$, we call T_z the *normalised position* of T. Under this identification, the space $\mathfrak{AT}_A$ of acute triangles is realised as the interior of the ideal (hyperbolic) triangle $\mathfrak{A}$ with vertices 0, 1, ∞. The boundary $\partial\mathfrak{A}$ consists of right triangles. For instance, the hyperbolic geodesic connecting 0 and 1 parametrises the right triangles of area A whose right angles are at the vertex labelled a. Let $\mathfrak{r}_a$ (resp. $\mathfrak{r}_b$, $\mathfrak{r}_c$) denote the complete hyperbolic geodesic connecting 0 and 1 (resp. ∞ and 0, ∞ and 1). The complement of the closure of the ideal triangle with vertices $0, 1$ and ∞ consists of obtuse triangles. For a vertex $v \in \{a, b, c\}$, we denote by $\mathfrak{O}_v$ the domain enclosed by $\mathfrak{r}_x$ and $\partial\mathbb{H}$ (see Figure 1).

For $v \in \{a, b, c\}$, we set $\mathfrak{T}_A(v) = \mathfrak{AT}_A \cup \mathfrak{r}_v \cup \mathfrak{OT}_v$.

2.4. **Lipschitz metric.** Consider two labelled triangles T and T' on the plane, and let $f\colon T \to T'$ be a label-preserving continuous map. The Lipschitz constant of f is defined as

$$L(f) = \sup_{x,y \in T, x \neq y} \frac{d_{euc}(f(x), f(y))}{d_{euc}(x, y)}$$

where d_{euc} denotes the Euclidean metric of the plane. A map f is said to be *Lipschitz* if $L(f) < \infty$. A label-preserving map $f\colon T \to T'$ is said to be *edge-preserving* if for $v \in \{a, b, c\}$, $f(e_v(T)) \subset e_v(T')$. Note that a label-preserving homeomorphism is always edge-preserving.

We define $L(T, T')$ by

$$L(T, T') := \inf\{L(f) \mid f \text{ is a label-preserving homeomorphism between } T \text{ and } T'\},$$

and set

$$d_L(T, T') = \log L(T, T').$$

A label-preserving and edge-preserving Lipschitz map $g\colon T \to T'$ is said to be *extremal* if $L(T, T') = L(g)$. Notice that an extremal Lipschitz map is not necessarily a homeomorphism.

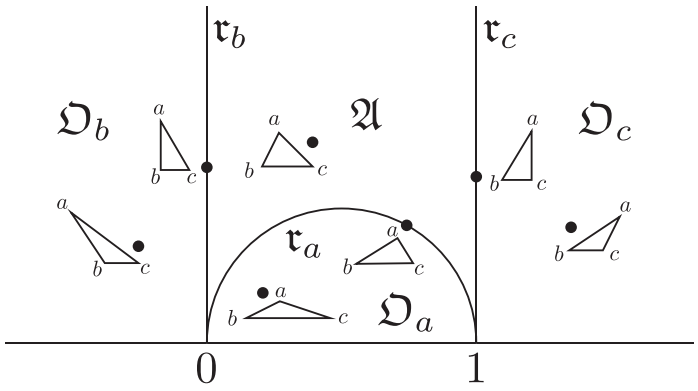

FIGURE 1. The space of triangles $\mathfrak{T}_A$. The hyperbolic ideal triangle $\mathfrak{A}$ with vertice $\{0, 1, \infty\}$ is identified with the space $\mathfrak{A}\mathfrak{T}_A$ of acute triangles. For $v \in \{a, b, c\}$, $\mathfrak{r}_v$ consists of right triangles whose angle is $\pi/2$ at the v-vertex, and $\mathfrak{O}_v$ consists of obtuse triangles which have an obtuse angle at the vertex labelled v.

One can easily see that d_L is a distance function on $\mathfrak{T}_A$, cf. [11]. We call d_L the *Lipschitz metric* on $\mathfrak{T}_A$. We give a topology the set $\mathfrak{T}_A$ (and hence $\mathfrak{A}\mathfrak{T}_A$) induced from the Lipschitz metric.

Convention 2. *A similarity of the complex plane with factor $\sqrt{A}$ gives a natural isometric identification (with respect to the Lipschitz metric) between $\mathfrak{T}$ and $\mathfrak{T}_A$. For this reason, henceforth, we shall only work with the space of of triangles of unit area.*

2.5. **Parametrisation.** For a labelled triangle T of unit area, the corresponding parameter $z = z(T)$ is expressed by

$$z(T) = \frac{|e_c(T)|}{|e_a(T)|} e^{i\theta_b(T)}, \quad \cos\theta_b(T) = \frac{|e_a(T)|^2 + |e_c(T)|^2 - |e_b(T)|^2}{2|e_a(T)||e_c(T)|}. \tag{2.2}$$

Notice that $T = (a, b, c)$ is similar to the triangle with (ordered) vertices $(z(T), 0, 1)$ by a label-preserving similarity map. A label-preserving affine map from $T_1 = T_{z_1}$ to $T_2 = T_{z_2}$ is given by

$$A_{T_1;T_2}(\zeta) = \sqrt{\frac{\mathrm{Im}(z_1)}{\mathrm{Im}(z_2)}} \left(\frac{z_2 - \overline{z_1}}{z_1 - \overline{z_1}} \zeta + \frac{z_1 - z_2}{z_1 - \overline{z_1}} \overline{\zeta} \right).$$

Its Lipschitz constant, which we denote by $L_{Af}(T_1, T_2)$, is expressed as

$$L_{Af}(T_1, T_2) = \sqrt{\frac{\mathrm{Im}(z_1)}{\mathrm{Im}(z_2)}} \left| \frac{z_2 - \overline{z_1}}{z_1 - \overline{z_1}} \right| + \left| \frac{z_1 - z_2}{z_1 - \overline{z_1}} \right| = \frac{|z_2 - \overline{z_1}| + |z_2 - z_1|}{2\mathrm{Im}(z_1)^{1/2}\mathrm{Im}(z_2)^{1/2}} \tag{2.3}$$

(cf. [10, §2]). We note the following.

Proposition 2.1. *The parametrisation $z \colon \mathfrak{T} \to \mathbb{H}$ is a homeomorphism.*

Proof. We only need to show the bi-continuity. Let T_1 and T_2 be points in $\mathfrak{T}$. By the definition of the Lipschitz metric, we have

$$L(T_1, T_2) \geq \left| \max \left\{ \frac{|e_a(T_1)|}{|e_a(T_2)|}, \frac{|e_b(T_1)|}{|e_b(T_2)|}, \frac{|e_c(T_1)|}{|e_c(T_2)|} \right\} \right|.$$

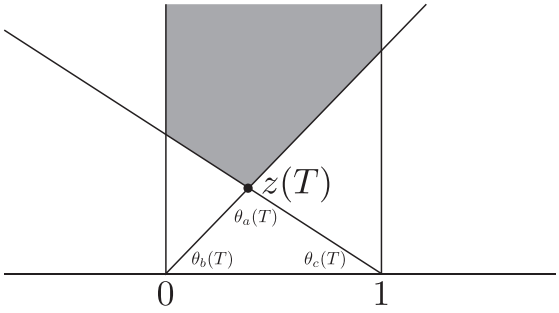

FIGURE 2. The shaded region represents tha a-pencil $P(T;a)$ for $z(T) \in \overline{\mathfrak{A}} \cup \mathfrak{O}_a$.

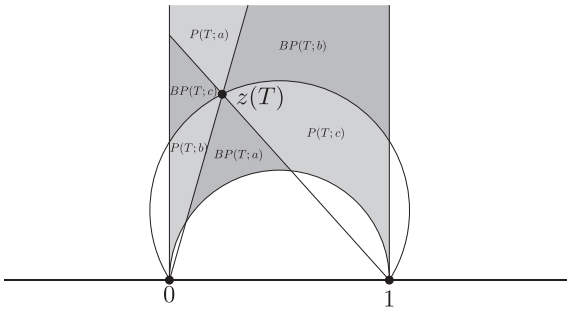

FIGURE 3. $P(T;v)$ and $BP(T;v)$ for $z(T) \in \mathfrak{A}$ and $v \in \{a,b,c\}$.

Hence, when a sequence $\{T_n\} \subset \mathfrak{T}$ converges to T in the Lipschitz metric L, the lengths of the edges also converge. By (2.2), this implies that $\{z(T_n)\}$ converges to $z(T)$ in $\mathbb{H}$. The continuity of the inverse map z^{-1} follows from (2.3) and the relation $\log L(T_1,T_2) \le \log L_{Af}(T_1,T_2)$ for any pair of triangles T_1 and T_2. □

2.6. **Pencils and Backward pencils.** For $\xi, \eta \in \partial\mathbb{H}$ with $\xi \neq \eta$, we denote by $\mathfrak{r} = \mathfrak{r}(\xi,\eta)$ the complete hyperbolic geodesic in $\mathbb{H}$ joining them. For $d > 0$, we denote by $N(\mathfrak{r},d)$ the d-neighborhood of the geodesic $\mathfrak{r}$. Let T be a labelled triangle. Suppose that $z(T)$ lies in $\mathfrak{T}(v)$ for $v \in \{a,b,c\}$. For the vertex v of T, we define the *v-pencil* $P(T;v)$ by

$$P(T;v) := \overline{N(\mathfrak{r}_{v'}, d_{v'}) \cap N(\mathfrak{r}_{v''}, d_{v''}) \cap (\mathfrak{A} \cup \mathfrak{O}_v)}$$

with $\{v,v',v''\} = \{a,b,c\}$, where $d_w = -\log\tan(\theta_w(T)/2)$ for $w \in \{v',v''\}$, (cf. Figure 2).

Given an acute triangle T, we define the *v-backward pencil* $BP(T;v)$ as

$$BP(T;v) = \overline{(\mathbb{C} - N(\mathfrak{r}_{v'}, d_{v'})) \cap (\mathbb{C} - N(\mathfrak{r}_{v''}, d_{v''})) \cap \mathfrak{A}}$$

(see Figure 3). After identifying $\mathfrak{T}$ with $\mathbb{H}$, $P(T;v)$ and $BP(T;v)$ are also regarded as subsets of $\mathfrak{T}$ for $v \in \{a,b,c\}$.

2.7. **Symmetries.** The symmetric group $\mathfrak{S}_3$ of degree 3 acts on $\mathfrak{T}$ naturally as permutations of the labelled vertices. This symmetric group is thought of as the

mapping class group of a triangle (see §6 of the paper [11]). Under the identification between $\mathfrak{T}$ and $\mathbb{H}$, the action of $\mathfrak{S}_3$ on $\mathbb{H}$ is generated by two transformations

$$\omega_{ab}(z) = \frac{\overline{z}}{\overline{z}-1}, \quad \omega_{ac}(z) = \frac{1}{\overline{z}}, \tag{2.4}$$

where the former permutes the a and b-vertices, and the latter permutes the a and c-vertices. For $z_0 \in \mathbb{H} \cong \mathfrak{T}$, the above actions are induced by the (orientation-reversing) congruences

$$R_{ab}\colon T_{z_0} \ni \zeta \mapsto -\frac{|z_0-1|}{\overline{z_0}-1}\left(\overline{\zeta} - \sqrt{\frac{2}{\mathrm{Im}(z_0)}}\right) + \frac{|z_0-1|}{\sqrt{2\mathrm{Im}(z_0)}} \in T_{\omega_{ab}(z_0)}$$

$$R_{ac}\colon T_{z_0} \ni \zeta \mapsto \frac{|z_0|}{\overline{z_0}}\overline{\zeta} \in T_{\omega_{ac}(z_0)}.$$

For the record, we note that the transformation of $\mathfrak{T}$ corresponding to the permutation between the b and c-vertices is expressed as

$$\omega_{bc}(z) = 1 - \overline{z},$$

and it is induced by the congruence

$$R_{bc}(\zeta) = \sqrt{\frac{2}{\mathrm{Im}(z_0)}} - \overline{\zeta}.$$

3. The Lipschitz distance and its Finsler structure

In this section, we discuss the Lipschitz metric on the Teichmüller space of acute triangles. The goal of this section is to prove the following theorem, which follows from Proposition 3.2 and Proposition 3.5 given later.

Theorem 3.1 (An expression of the Lipschitz distance)**.** *Let T be a point in $\mathfrak{A}$, and T a point in $\overline{\mathfrak{A}}$. Set $z = z(T) \in \mathbb{H}$, and suppose that $w = z(T')$ lies in $P(T;x) \cup BP(T;x)$ for $x \in \{a,b,c\}$. Then, we have*

$$d_L(T,T') = \begin{cases} \dfrac{1}{2}\left|\log \dfrac{\mathrm{Im}(w)}{\mathrm{Im}(z)}\right| & (\textit{if } x=a) \\ \dfrac{1}{2}\left|\log \dfrac{\mathrm{Im}(w)}{|w|^2}\dfrac{|z|^2}{\mathrm{Im}(z)}\right| & (\textit{if } x=b) \\ \dfrac{1}{2}\left|\log \dfrac{\mathrm{Im}(w)}{|w-1|^2}\dfrac{|z-1|^2}{\mathrm{Im}(z)}\right| & (\textit{if } x=c). \end{cases}$$

We shall show Theorem 3.1 by giving extremal maps concretely for each case.

Finsler structure. In [11], Sağlam and Papadopoulos gave an expression of the Finsler structure of the deformation space of acute triangles using the side-length parametrisation. In this section, we shall give a new description of the Finsler structure which arises naturally from our setting, using a direct calculation. It is often interesting to have a formula for the norm of a vector and to draw the unit ball in the tangent space, once we know that a geometrically defined distance function arises from a Finsler structure. This will imply in particular an infinitesimal rigidity result which is similar to the one obtained in [7, 4].

Let z_0 be a point in $\mathfrak{A}$. Let θ_1 and θ_2 be the arguments of z_0 and $z_0 - 1$ respectively, with $-\pi < \theta_1, \theta_2 \leq \pi$. We consider six regions in the tangent space $T_{z_0}\mathfrak{T} = \mathbb{C}$ divided by three lines passing through the origin with angles θ_1, θ_2, and

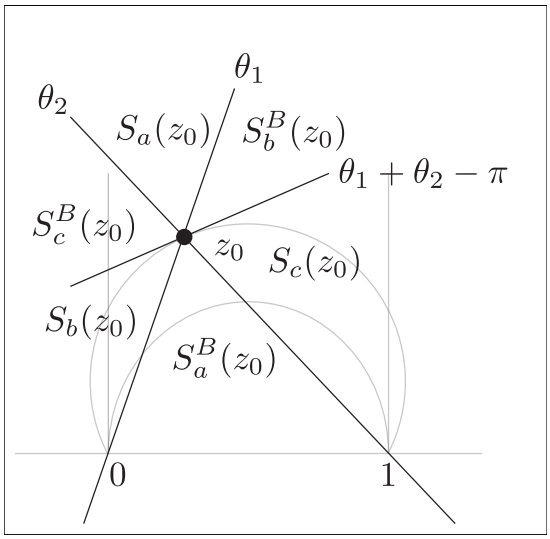

FIGURE 4. The decomposition of the tangent space at z_0 into six sectors

$\theta_1+\theta_2-\pi$. We denote these regions by $S_x(z_0)$, $S_x^B(z_0)$ $(x\in\{a,b,c\})$, as in Figure 3. We define the norm $F^{\mathfrak{A}}(z_0,v)$ of a vector $v\in T_{z_0}\mathfrak{T}=\mathbb{C}$ by

$$F^{\mathfrak{A}}(z_0,v)=\begin{cases}\dfrac{1}{2}\dfrac{|\mathrm{Im}(v)|}{\mathrm{Im}(z_0)} & (v\in S_a(z_0)\cup S_a^B(z_0))\\ \dfrac{1}{2}\left|\dfrac{\mathrm{Im}(v)}{\mathrm{Im}(z_0)}-2\mathrm{Re}\left(\dfrac{v}{z_0}\right)\right| & (v\in S_b(z_0)\cup S_b^B(z_0))\\ \dfrac{1}{2}\left|\dfrac{\mathrm{Im}(v)}{\mathrm{Im}(z_0)}-2\mathrm{Re}\left(\dfrac{v}{z_0-1}\right)\right| & (v\in S_c(z_0)\cup S_c^B(z_0)).\end{cases}$$

One can check easily that $F^{\mathfrak{A}}$ is continuous on $\mathfrak{A}\times\mathbb{C}$. The Finsler infinitesimal norm is obtained by differentiating the distance in Theorem 3.1. Thus, we we have the following.

Theorem 3.2 (Finsler structure)**.** *The Lipschitz distance d_L on $\mathfrak{A}$ is a Finsler distance with Finsler norm $F^{\mathfrak{A}}$.*

Proof. For any $v\in\mathbb{C}$, we have

$$\begin{aligned}\frac{|\mathrm{Im}(z_0+tv)|}{\mathrm{Im}(z_0)}&=1+t\frac{\mathrm{Im}(v)}{\mathrm{Im}(z_0)}\\ \frac{\mathrm{Im}(z_0+tv)}{|z_0+tv|^2}\frac{|z_0|^2}{\mathrm{Im}(z_0)}&=\left(1-2t\mathrm{Re}\left(\frac{v}{z_0}\right)+o(t)\right)\left(1+t\frac{\mathrm{Im}(v)}{\mathrm{Im}(z_0)}\right)\\ &=1+t\left(\frac{\mathrm{Im}(v)}{\mathrm{Im}(z_0)}-2\mathrm{Re}\left(\frac{v}{z_0}\right)\right)+o(t)\\ \frac{\mathrm{Im}(z_0+tv)}{|z_0+tv-1|^2}\frac{|z_0-1|^2}{\mathrm{Im}(z_0)}&=\left(1-2t\mathrm{Re}\left(\frac{v}{z_0-1}\right)+o(t)\right)\left(1+t\frac{\mathrm{Im}(v)}{\mathrm{Im}(z_0)}\right)\\ &=1+t\left(\frac{\mathrm{Im}(v)}{\mathrm{Im}(z_0)}-2\mathrm{Re}\left(\frac{v}{z_0-1}\right)\right)+o(t)\end{aligned}$$

as $t \to 0$. Suppose first that v lies in $S_a(z_0)$. Since $\mathrm{Im}(v) \geq 0$ for $v \in S_a(z_0)$, from Theorem 3.1 and the above calculation, we have

$$d_L(z_0, z_t) = \frac{1}{2}\frac{\mathrm{Im}(v)}{\mathrm{Im}(z_0)} + o(t) = \frac{1}{2}\frac{|\mathrm{Im}(v)|}{\mathrm{Im}(z_0)} + o(t) \quad (t \to 0)$$

for a differentiable path in $P(T_{z_0}; a)$ which is tangent to v at $t = 0$. We have the same conclusion for a differentiable path in $BP(T_{z_0}; a)$ which tangent to v at $t = 0$.

Suppose next that v lies in $S_b(z_0)$. Express v as $v = v_1 + iv_2 = \xi e^{i\theta}$. As depicted in Figure 3, we have $\theta_1 + \theta_2 - 2\pi \leq \theta \leq \theta_1 - \pi$ then, and hence $\theta_2 - \theta_1 - 2\pi \leq \theta - 2\theta_1 \leq -\pi - \theta_1$. Then we have

$$\mathrm{Im}(v) - 2\mathrm{Re}(\sin\theta_1 e^{-i\theta_1} v) = \xi \sin\theta - 2\xi \sin\theta_1 \cos(\theta - \theta_1) = \xi \sin(\theta - 2\theta_1) > 0,$$

where the last inequality follows from the fact that both $\sin(\theta_2 - \theta_1)$ and $\sin(\pi - \theta_1)$ are positive and fromthe bounds for θ given above. Since $z_0 = (e^{i\theta_1}/\sin\theta_1)\mathrm{Im}(z_0)$, we have

$$\frac{\mathrm{Im}(v)}{\mathrm{Im}(z_0)} - 2\mathrm{Re}\frac{v}{z_0} = \frac{\xi \sin(\theta - 2\theta_1)}{\mathrm{Im}(z_0)} > 0.$$

Hence, from the above calculation, we get, for a differentiable path in $P(T_{z_0}; b)$ tangent to v at $t = 0$,

$$\begin{aligned} d_L(z_0, z_t) &= t\frac{1}{2}\left(\frac{\mathrm{Im}(v)}{\mathrm{Im}(z_0)} - 2\mathrm{Re}\frac{v}{z_0}\right) + o(t) \\ &= t\frac{1}{2}\left|\frac{\mathrm{Im}(v)}{\mathrm{Im}(z_0)} - 2\mathrm{Re}\frac{v}{z_0}\right| + o(t). \end{aligned}$$

The remaining cases for $BP(T_{z_0}; b)$, $P(T_{z_0}; c)$, and $PB(T_{z_0}; c)$ can be dealt with by a similar argument. □

Some remarks. Regarding Theorem 3.1 and Theorem 3.2, we remark the following.

(1) Theorem 3.2 gives another way of seeing the non-uniqueness of geodesics between two points, a fact which was already noticed in [11]. Indeed, we can see that there are uncountably many geodesics between two points in $\mathfrak{A}$. For instance, take $z_0 \in \mathfrak{A}$ and $z_1 \in P(T_{z_0}; a)$. Then, a piecewise C^1-path z_t $(0 \leq 1 \leq 1)$ connecting z_0 to z_1 whose imaginary part $\mathrm{Im}(z_t)$ is increasing is a geodesic between z_0 and z_1. Similarly, when $z_1 \in BP(T_{z_0}; a)$, if the imaginary part is decreasing, z_t is a geodesic between z_0 and z_1.

(2) For $z_0 \in \mathfrak{A}$, the unit ball associated with the Finsler norm $F^{\mathfrak{A}}$ at z_0 is a hexagon with vertices $2z_0$, $2(z_0 - 1)$, $2z_0(z_0 - 1)$, $-2z_0$, $-2(z_0 - 1)$ and $-2z_0(z_0 - 1)$. The diagonals divides the unit ball into six triangles, each of which is similar to T_{z_0}. See Figure 3.

(3) From the second remark, we deduce the following *infinitesimal rigidity* result: The isometry type of the unit ball determines uniquely a triangle in the parameter space. The same kind of infinitesimal rigidity concerning cotangent spaces of Teichmüller spaces of closed surfaces with Thurston's metric was proved in [7, 4]. With the preceding notation, we can state our infinitesimal rigidity result as follows.

Corollary 3.1. *Let z_0 and z_1 be points in $\mathfrak{A}$. Suppose that there is a linear isometry from $(T_{z_0}\mathfrak{A}, F^{\mathfrak{A}}(z_0, \cdot))$ to $(T_{z_1}\mathfrak{A}, F^{\mathfrak{A}}(z_1, \cdot))$. Then the triangles T_{z_1} and T_{z_0} are congruent.*

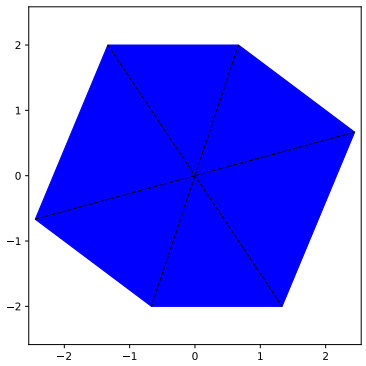

FIGURE 5. The shape of the Finsler unit ball at $z_0 = 1/3 + i$. The diagonals (dashed lines) divide the hexagon into six triangles and each triangle is similar to T_{z_0}.

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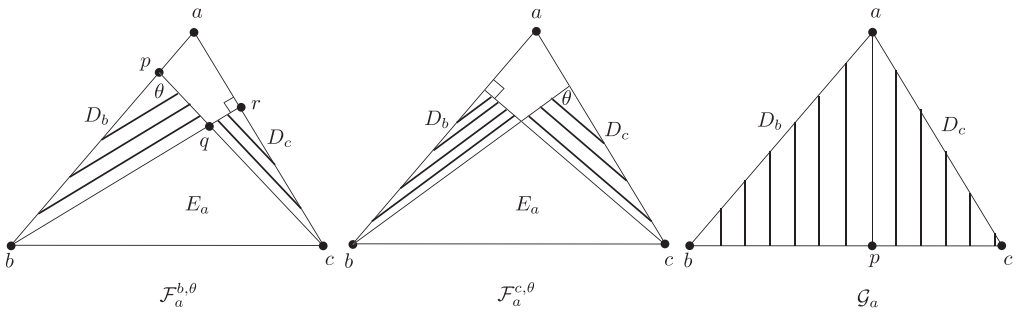

FIGURE 6. The stretching loci $\mathcal{F}_a^{b,\theta}$, $\mathcal{F}_a^{c,\theta}$, $\mathcal{G}_a$

3.1. **Stretching loci.** Let us fix an acute triangle T. With a vertex $v \in \{a, b, c\}$ and an angle $\theta \in (\theta_v(T), \pi/2)$, we associate three kinds of pictures $\mathcal{F}_v^{v',\theta} = \mathcal{F}_v^{v',\theta}(T)$ and $\mathcal{G}_v = \mathcal{G}_v(T)$ for the vertex v, where $v, v' \in \{a, b, c\}$ are distinct vertices, as in Figure 6. The figure only show the case when $v = a$, but the other cases can be drawn just by changing the symbols of vertices.

In the leftmost triangle of Figure 6, $\mathcal{F}_a^{b,\theta}$ is drawn as follows. Let a, b, and c be the a-vertex, the b-vertex, and the c-vertex of T, respectively. Draw a perpendicular from b to its opposite side with its foot r, and then a segment from c to its opposite side so that the angle $\angle bpc$ is equal to θ, where p is the foot of the segment. We denote the intersection between the segments br and cp by q. Let E_a, D_b and D_c be the triangles $\triangle bcq$, $\triangle bpq$, and $\triangle cqr$, respectively. Consider the foliations on D_b (resp. D_c) whose leaves are Euclidean segments parallel to the segment rb (resp. the segment pc). We call these foliations the *maximally stretched foliations*. The *maximal stretching locus* $\mathcal{F}_a^{b,\theta}$ consists of the foliated triangles D_b and D_c, and the triangle E_a. We call the triangle E_a the *expanding region*. The maximal stretching locus $\mathcal{F}_a^{c,\theta}$ for the second triangle in Figure 6 is defined in the same way. The names will be justified below (see Remark 3.1).

In the rightmost triangle of Figure 6, its maximal stretching locus $\mathcal{G}_a$ is drawn as follows. Draw the perpendicular from a to the opposite side, with its foot p. Divide the triangle into two right triangles $D_b = \triangle apb$ and $D_c = \triangle apc$. Consider the foliations on D_b and D_c consisting of Euclidean segments parallel to ap. The *maximal stretching locus* $\mathcal{G}_a$ consists of foliated triangles D_b and D_c.

3.2. **Extremal Lipschitz maps associated with pencils.**

Notation 3.1. *For $z \in \mathbb{H}$, we set $\tilde{T}_z$ to be the labelled triangle with vertices*

$$z_a(\tilde{T}_z) = z, z_b(\tilde{T}_z) = 0, z_c(\tilde{T}_z) = 1.$$

In this section, we discuss an extremal Lipschitz map from a normalised triangle T to a normalised triangle in the pencil $P(T; v)$ for $v \in \{a, b, c\}$. We use the notation 3.1, and shall discuss only the case when $v = a$. In the case when $v = b$ and c, we can define the Lipschitz maps in the same way, taking conjugates by congruences (cf. §2.7).

3.2.1. Let T_0 and T_1 be triangles with $z_0 = x_0 + iy_0 = z(T_0) \in \mathfrak{T}(a)$, and $z_1 = u_0 + iv_0 = z(T_1) \in P(T_0; a)$.

We define

$$f_{z_0,z_1}(\xi + i\eta) = \begin{cases} \sqrt{\dfrac{y_0}{v_0}} \left(\dfrac{u_0}{x_0}\, \xi + i \dfrac{v_0}{y_0}\, \eta \right) & (\xi + i\eta \in \overline{D_b}) \\ \sqrt{\dfrac{2}{v_0}} \left(u_0 + \sqrt{\dfrac{y_0}{2}} \left(\dfrac{1-u_0}{1-x_0} \left(\xi - \sqrt{\dfrac{2}{y_0}} x_0 \right) + i \dfrac{v_0}{y_0} \eta \right) \right) & (\xi + i\eta \in \overline{D_c}). \end{cases}$$

One can check that the map f_{z_0,z_1} is a label-preserving Lipschitz map from T_{z_0} to T_{z_1}.

3.2.2. For $0 \le k_1, k_2 \le 1$, we define a *stretch map $f_{\mathcal{G}_v,(k_1,k_2)}$ associated with $\mathcal{G}_a$ and* (k_1, k_2) on T as follows.

Let $T \in \mathfrak{T}(a)$ be in normalised position. We define

$$f_{\mathcal{G}_a,(k_1,k_2)}(\xi + i\eta) = \begin{cases} \dfrac{k_1\xi + i\eta}{(k_1x_0 + k_2(1-x_0))^{1/2}} & \left(\text{if } \xi + i\eta \in \overline{D_b}\right) \\ \dfrac{k_1\xi_0 + k_2(\xi - \xi_0) + i\eta}{(k_1x_0 + k_2(1-x_0))^{1/2}} & (\xi + i\eta \in \overline{D_c}), \end{cases} \tag{3.1}$$

where $\xi_0 = \sqrt{2/y_0}x_0$ (cf. (2.1)). Since

$$\begin{aligned} f_{\mathcal{G}_a,(k_1,k_2)}(z_a(T)) &= \left(\frac{2}{y_0}\right)^{1/2} \frac{k_1x_0 + iy_0}{(k_1x_0 + k_2(1-x_0))^{1/2}} \\ f_{\mathcal{G}_a,(k_1,k_2)}(z_b(T)) &= 0 \\ f_{\mathcal{G}_a,(k_1,k_2)}(z_c(T)) &= \left(\frac{2}{y_0}\right)^{1/2} (k_1x_0 + k_2(1-x_0)))^{1/2}, \end{aligned}$$

$f_{\mathcal{G}_a,(k_1,k_2)}$ coincides with f_{z_0,z_1}, which maps T to a triangle T_1 with

$$z_1 = z(T_1) = \frac{k_1x_0 + iy_0}{k_1x_0 + k_2(1-x_0)}. \tag{3.2}$$

We denote such a triangle T_1 by $\mathrm{St}(T, \mathcal{G}_a, (k_1, k_2))$. We note that $f_{\mathcal{G}_a,(k_1,k_2)}$ is label-preserving and edge-preserving, and that $f_{\mathcal{G}_a,(1,1)}$ is the identity map. Note also that $f_{\mathcal{G}_a,(0,k_2)}$ and $f_{\mathcal{G}_a,(k_1,0)}$ are not homeomorphisms. In these two cases, the images are right triangles and the maps are not injective.

Remark 3.1. *Let ζ_1, ζ_2 be points on T with $\mathrm{Re}(\zeta_1) = \mathrm{Re}(\zeta_2)$. Then, we have*

$$\begin{aligned} d_{euc}(f_{\mathcal{G}_a,(k_1,k_2)}(\zeta_1), f_{\mathcal{G}_a,(k_1,k_2)}(\zeta_2)) &= \frac{1}{(k_1x_0 + k_2(1-x_0)))^{1/2}} d_{euc}(\zeta_1, \zeta_2) \\ &> d_{euc}(\zeta_1, \zeta_2) \end{aligned}$$

when $k_1k_2 < 1$ since $0 < x_0,\ 1 - x_0 < 1$. Therefore, $f_{\mathcal{G}_a,(k_1,k_2)}$ stretches T along the foliations in D_b and D_c. This is the reason why $\mathcal{G}_a$ (and hence $\mathcal{G}_v$ for $v = b,\ c$) is called the maximal stretching locus, and the map $f_{\mathcal{G}_a,(k_1,k_2)}$ (and hence $f_{\mathcal{G}_v,(k_1,k_2)}$ for $v = b,\ c$) is named a stretch map.

We have the following.

Proposition 3.1 (Parameterisations of Pencils)**.** *For $v \in \{a, b, c\}$ and $T \in \mathfrak{T}(v)$, $z(\mathrm{St}(T, \mathcal{G}_v, (k_1, k_2))))$ is contained in $P(T; v)$ for any $0 \le k_1, k_2 \le 1$ with $|k_1| + |k_2| > 0$. Conversely, for any $z \in P(T; v)$, there is a unique pair (k_1, k_2) with $0 \le k_1, k_2 \le 1$ with $|k_1| + |k_2| > 0$ such that $z = z(\mathrm{St}(T, \mathcal{G}_v, (k_1, k_2)))$. Therefore, the map*

$$[0, 1] \times [0, 1] \setminus \{(0, 0)\} \ni (k_1, k_2) \mapsto \mathrm{St}(T, \mathcal{G}_v, (k_1, k_2)) \in P(T; x)$$

is a homeomorphism.

Proof. We shall prove the proposition only in the case when $v = a$. Set $z(T) = x_0 + iy_0$ and $T' = \mathrm{St}(T, \mathcal{G}_v, (k_1, k_2))$ as above. From (3.2), we have $0 \le \mathrm{Re}(z(T')) \le 1$. Since $0 \le k_1,\ k_2 \le 1$ and $0 \le x_0 \le 1$, we have

$$\begin{aligned} \frac{\mathrm{Im}(z(T'))}{\mathrm{Re}(z(T'))} = \frac{y_0}{k_1x_0} \ge \frac{y_0}{x_0} = \frac{\mathrm{Im}(z(T))}{\mathrm{Re}(z(T))} \text{ and} \\ \frac{\mathrm{Im}(z(T'))}{\mathrm{Re}(z(T')) - 1} = -\frac{y_0}{k_2(1-x_0)} \le -\frac{y_0}{1-x_0} = \frac{\mathrm{Im}(z(T))}{\mathrm{Re}(z(T)) - 1}, \end{aligned}$$

hence $z(T') \in P(T; a)$.

Let $z = x+iy$ be a point in $P(T; a)$. By solving the equation $z = z(\mathrm{St}(T, \mathcal{G}_a, (k_1, k_2)))$, we obtain

$$k_1 = \frac{xy_0}{x_0y}, \quad k_2 = \frac{(1-x)y_0}{(1-x_0)y}.$$

The condition $z = x + iy \in P(T; a)$ implies that $0 \le k_1, k_2 \le 1$. Furthermore, $k_1 = 0$ implies that $z(\mathrm{St}(T, \mathcal{G}_a, (k_1, k_2))) \in \mathfrak{r}_b$, and $k_2 = 0$ implies that $z(\mathrm{St}(T, \mathcal{G}_a, (k_1, k_2))) \in \mathfrak{r}_c$. Therefore, k_1 and k_2 cannot vanish simultaneously. □

Proposition 3.2 (Stretch maps are extremal)**.** *For $v \in \{a, b, c\}$ and $0 \le k_1,\ k_2 \le 1$ with $|k_1| + |k_2| > 0$, the stretch map $f_{\mathcal{G}_v,(k_1,k_2)}$ is an extremal Lipschitz map.*

Proof. As in the proof of Proposition 3.1, we only consider the case when $v = a$. By definition, $f_{\mathcal{G}_a,(k_1,k_2)}$ is the composition of a contraction of the horizontal direction with Lipschitz constant 1 and an expansion with factor $1/(x_0k_1 + (1 - x_0)k_2))^{1/2}$. Furthermore, the contraction preserves distances in the imaginary direction. Hence we have

$$L(f_{\mathcal{G}_a,(k_1,k_2)}) = \frac{1}{(x_0k_1 + (1-x_0)k_2))^{1/2}}. \tag{3.3}$$

On the other hand, the altitudes of T and $T' = \mathrm{St}(T, \mathcal{G}_a, (k_1, k_2))$ are $(2Ay_0)^{1/2}$ and $(2Ay_0)^{1/2}/(x_0k_1 + (1 - x_0)k_2))^{1/2}$. Since each altitude is equal to the length of the perpendicular from the a-vertex to its opposite side, the Lipschitz constant

of any label-preserving and edge-preserving Lipschitz map from T to T' is at least $1/(x_0k_1+(1-x_0)k_2))^{1/2}$. □

Corollary 3.2 (Lipschitz distance for pencils)**.** *For $v\in\{a,b,c\}$ and $0\le k_1,\ k_2\le 1$ with $|k_1|+|k_2|>0$, we have*

$$L(T,\mathrm{St}(T,\mathcal{G}_v,(k_1,k_2)))=\begin{cases}\dfrac{1}{2}\log\dfrac{1}{x_0k_1+(1-x_0)k_2} & (\textit{if } v=a)\\ \dfrac{1}{2}\log\dfrac{|z_0|^2}{x_0k_1+(|z_0|^2-x_0)k_2} & (\textit{if } v=b)\\ \dfrac{1}{2}\log\dfrac{|z_0-1|^2}{(|z_0|^2-x_0)k_1+(1-x_0)k_2} & (\textit{if } v=c)\end{cases}\tag{3.4}$$

where we set $z(T)=z_0=x_0+iy_0$. In particular, if we set $z_1=z(\mathrm{St}(T,\mathcal{G}_v,(k_1,k_2)))$, we obtain

$$L(T_{z_0},T_{z_1})=\begin{cases}\dfrac{1}{2}\log\dfrac{\mathrm{Im}(z_1)}{\mathrm{Im}(z_0)} & (\textit{if } v=a)\\ \dfrac{1}{2}\log\dfrac{1}{2}\left|\log\dfrac{\mathrm{Im}(z_1)}{|z_1-1|^2}\dfrac{|z_0-1|^2}{\mathrm{Im}(z_0)}\right| & (\textit{if } v=b)\\ \dfrac{1}{2}\log\dfrac{\mathrm{Im}(z_1)}{|z_1|^2}\dfrac{|z_0|^2}{\mathrm{Im}(z_0)} & (\textit{if } v=c)\end{cases}\tag{3.5}$$

Proof. We shall only give a proof in the case when $v=a$ (cf. (2.4)). If $k_1k_2\neq 0$, the stretch map $f_{\mathcal{G}_a,(k_1,k_2)}$ is a homeomorphism. Therefore, (3.4) and (3.5) follow from (3.3).

Assume now that $k_1=0$. For $\epsilon\ge 0$, we set $T_\epsilon=\mathrm{St}(T,\mathcal{G}_a,(\epsilon,k_2))$. We take a label-preserving affine map $A_\epsilon\colon T_\epsilon\to T_0$ and set $g_\epsilon=A_\epsilon\circ f_{\mathcal{G}_a,(\epsilon,k_2)}$. Then, g_ϵ is a label-preserving Lipschitz homeomorphism from T to T_0. From (2.3) and Proposition 3.2, we have

$$L(f_{\mathcal{G}_a,(0,k_2)})\le L(T,T_0)\le L(g_\epsilon)\le L(A_\epsilon)L(f_{\mathcal{G}_a,(\epsilon,k_2)})\to L(f_{\mathcal{G}_a,(0,k_2)})$$

as $\epsilon\to 0$, which implies (3.4) and (3.5) for the case when $k_1=0$. The case when $k_2=0$ can be dealt with in the same way. □

3.2.3. *Extremal Lipschitz maps are not always homeomorphisms.* Let v, v' and v'' be vertices such that $\{v,v',v''\}=\{a,b,c\}$. Let T be a triangle in $\mathfrak{AT}$. For $T'\in P(T;v)\cap(\mathfrak{r}_{v'}\cup\mathfrak{r}_{v''})$, our stretch map from T to T' is not a homeomorphism. In fact, we have the following general statement.

Proposition 3.3. *In the above setting, there is no extremal label-preserving Lipschitz homeomorphism between T and T'.*

Proof. Let $g\colon T\to T'$ be a label-preserving Lipschitz homeomorphism. We may assume that $v=a$ and $T'\in\mathfrak{r}_b$. As was shown in the proof of Proposition 3.2, the Lipschitz constant of the stretch map is the ratio of the altitudes of T and T'.

Let $p\in e_a(T)$ be the foot of the perpendicular in T from the a-vertex $z_a(T)$ to the opposite edge $e_a(T)$. Since T is an acute triangle, p is different from the b-vertex $z_b(T)$ of T. Since g is a homeomorphism, $g(p)\neq z_b(T')$ but $g(p)\in e_a(T')$. Therefore,

$$L(T,T')=\frac{d_{euc}(z_a(T'),z_b(T'))}{d_{euc}(z_a(T),p)}<\frac{d_{euc}(z_a(T'),g(p))}{d_{euc}(z_a(T),p)}\le L(g).$$

Hence g is not extremal. □

3.3. **Extremal Lipschitz maps associated with backward pencils.**

3.3.1. *Contractions.* Let $z_0 \in \mathbb{H}$ be a point in the ideal triangle with vertices 0, 1 and ∞. Let $z_1 = u_0 + iv_0$ be a point in $BP(T; a)$. We divide $\tilde{T}_{z_0}$ into two right angled triangles by the perpendicular from the b-vertex 0 of $\tilde{T}_{z_0}$ and its ac-side. For $x \in \{a, c\}$, we denote by $D_b^x = D_b^x(\tilde{T}_{z_0})$ the component which contains the x-vertex in its boundary. We define the *contraction* $C^b_{z_0,z_1}$ *on* $\tilde{T}_{z_0}$ *associated with the b-vertex* by

$$C^b_{z_0,z_1}(\zeta) = \begin{cases} \zeta & (\zeta \in D_b^c) \\ \dfrac{1+k}{2}\zeta + \dfrac{1-k}{2}\dfrac{\zeta_0}{\overline{\zeta_0}}\,\overline{\zeta} & (\zeta \in D_b^a), \end{cases}$$

where $\zeta_0 = i\dfrac{\mathrm{Im}(z_0)}{1-\overline{z_0}}$ and $k = \dfrac{y_0}{|z_0|^2 - x_0}\dfrac{\mathrm{Re}(z_1(\overline{z_0}-1))}{\mathrm{Im}(z_1(\overline{z_0}-1))}$. Notice that ζ_0 is the foot of the perpendicular from the b-vertex 0 of $\tilde{T}_{z_0}$ to the ac-side. We can check easily that $0 < k \leq 1$. Indeed, by assumption, $z_1 \in D_b^a$. By a calculation, we can see that a Euclidean ray emanating from the b-vertex 0 passing by z_1 intersects the ac-side of $\tilde{T}_{z_0}$ at

$$\zeta_1 = \frac{y_0}{\mathrm{Im}(z_1(\overline{z_0}-1))} w_0. \tag{3.6}$$

We can also see that $k = |\zeta_1 - \zeta_0|/|z_0 - \zeta_0|$, and hence that $0 < k \leq 1$. By definition, $C^b_{z_0,z_1}$ is a contraction from the triangle with vertices 0, z_0 and ζ_0 to the one with vertices 0, ζ_1 and ζ_0, and takes $\tilde{T}_{z_0}$ to $\tilde{T}_{\zeta_1}$. See the map represented by the right-lower arrow in Figure 7.

We define the *contraction* $C^c_{z_0,z_1}$ *associated with the c-vertex* by

$$C^c_{z_0,z_1}(\zeta) = 1 - \overline{C^b_{1-\overline{z_0},1-\overline{z_1}}(1-\overline{\zeta})},$$

obtained by conjugating the contraction $C^b_{\omega_{bc}(z_0),\omega_{bc}(z_0)}$ on the vertical line in $\mathbb{C}$ passing through the midpoint $1/2$ between the b-vertex 0 and the c-vertex 1 of $\tilde{T}_{z_0}$ (see §2.7).

We define the *contraction* $G^b_{z_0,z_1}$ *from* $\tilde{T}_{z_0}$ *and* $\tilde{T}_{z_1}$ *associated with the b-vertex* by

$$G^b_{z_0,z_1}(\zeta) = C^c_{\zeta_1,z_1} \circ C^b_{z_0,z_1}(\zeta)$$

for ζ_1 defined in (3.6).

We also define the *contraction* $G^c_{z_0,z_1}$ *from* $\tilde{T}_{z_0}$ *and* $\tilde{T}_{z_1}$ *associated with the b-vertex* and also satisfying the properties of Proposition 3.4 just by interchanging the roles of the b and c-vertices.

The following proposition is an immediate consequence of the definition.

Proposition 3.4 (Lipschitz constants of contractions)**.** *Let z_0 be a point in $\mathfrak{A}$ and z_1 a point in $BP(T_{z_0}; a)$. Let x be either b or c.*

1. *The Lipschitz constant of the contraction $G^x_{z_0,z_1}$ is 1.*
2. *After identifying T_{z_0} with $\tilde{T}_{z_0}$ by a similarity with factor $\sqrt{\mathrm{Im}(z_0)/2}$, the Lipschitz constant of the contraction is attained at two points both of which lie in either the expanding region E_a or on any leaf of the foliation $\mathcal{F}_a^{x,\theta}$, where θ is the angle at ζ_1 of $\tilde{T}_{\zeta_1}$.*

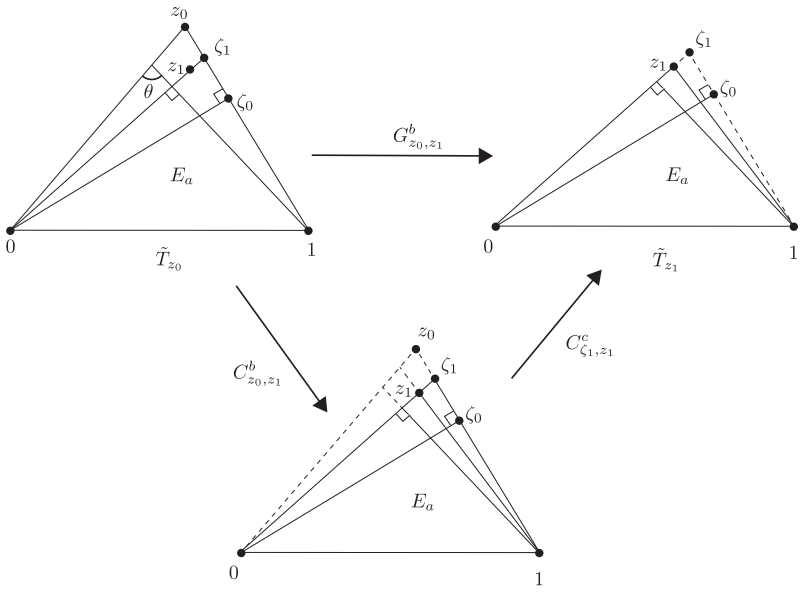

FIGURE 7. Contractions $C^b_{z_0,z_1}$, $C^c_{z_0,z_1}$ and $G^b_{z_0,z_1}$.

3.3.2. *Extremal maps.* Let T be a (marked) acute triangle and $z_0 = x_0 + iy_0 = z(T) \in \mathfrak{A}$. Let $z_1 = u_0 + iv_0 \in BP(T; a)$. We define the piecewise linear maps $g^b_{z_0,z_1}$ and $g^c_{z_0,z_1}$ from T_{z_0} to T_{z_1} by

$$g^x(\zeta) = \sqrt{\frac{2}{v_0}} G^b_{z_0,z_1}\left(\sqrt{\frac{y_0}{2}}\zeta\right)$$

for $x = b$ or c.

Proposition 3.5 (Extremal Lipschitz maps for backward pencils)**.** *Let $z_0 \in \mathfrak{A}$ and $z_1 \in BP(T_{z_0}; a)$. For $x \in \{b, c\}$, we have the following:*

1. *The contraction $g^x_{z_0,z_1}$ is an extremal Lipschitz map from T_{z_0} to T_{z_1} with Lipschitz constant $\sqrt{\mathrm{Im}(z_0)/\mathrm{Im}(z_1)}$.*
2. *The Lipschitz constant is attained by two points both of which lie either in the expanding region E_a or on each leaf of the foliation $\mathcal{F}^{x,\theta}_a$, where θ is defined in the same manner as in Proposition 3.4.*

Proof. From Proposition 3.4, we need only verify that $g^x_{z_0,z_1}$ is extremal. Let $g\colon T_{z_0} \to T_{z_1}$ be a label-preserving and edge-preserving Lipschitz map. Since the length of the edge $e_a(T_{z_i})$ is $\sqrt{2/\mathrm{Im}(z_i)}$ for $i = 0, 1$, the Lipschitz constant of $L(g)$ satisfies

$$L(g) \geq \frac{\sqrt{2/\mathrm{Im}(z_1)}}{\sqrt{2/\mathrm{Im}(z_0)}} = \sqrt{\frac{\mathrm{Im}(z_1)}{\mathrm{Im}(z_0)}} = L(g^x_{z_0,z_1}),$$

which implies what we wanted. □

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