Curvature properties and Shafarevich conjecture for toroidal compactifications of ball quotients

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Abstract

We study toroidal compactifications of finite volume complex hyperbolic manifolds. We obtain results on the existence or nonexistence of Kähler metrics satisfying certain nonpositive curvature properties on these compactifications. Starting from quotients of complex hyperbolic space by deep enough non-uniform arithmetic lattices, we also verify the Shafarevich conjecture for their compactifications, by showing that their universal covers are Stein.

1 Introduction

This article is a contribution to the study of closed Kähler manifolds obtained by *toroidal compactification* of spaces of the form $\mathbb{H}^n_{\mathbb{C}}/\Gamma$, where $\mathbb{H}^n_{\mathbb{C}}$ denotes the complex hyperbolic space of dimension n and Γ a "deep enough" torsion-free non-uniform lattice of $\mathrm{PU}(n,1)$. These compactifications, also called Mumford compactification, were first described in [Mum75; Ash+10]. The metric study of these manifolds was done by Hummel and Schroeder [HS96]. They showed:

Theorem ([HS96])**.** *Let Γ_0 be a non-uniform torsion-free lattice of $\mathrm{PU}(n,1)$ for which the quotient space $\mathbb{H}^n_{\mathbb{C}}/\Gamma_0$ admits a toroidal compactification X_{Γ_0}. Then X_{Γ_0} is a Kähler manifold. Moreover, there exists a finite index subgroup $\Gamma' < \Gamma_0$ such that for any finite index subgroup $\Gamma < \Gamma'$ the toroidal compactification X_Γ of $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ admits a Riemannian metric of nonpositive sectional curvature.*

The toroidal compactification of the quotient space $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ will be denoted by X_Γ, whenever it exists: precise conditions ensuring its existence are given in Section 2, where we recall the construction of X_Γ. In fact, toroidal compactifications are projective manifolds [Mok12; DD15], and several authors have studied their properties, from the point of view of algebraic geometry [DD17; BT18; Cad21; Mem23], or from other points of view [Py17; DS18].

In this article, we first show that these manifolds X_Γ do not admit a Kähler metric with nonpositive sectional curvature (Theorem 1), confirming a claim from [HS96], but that they admit a Kähler metric with nonpositive holomorphic bisectional curvature (Theorem 2). As a corollary, we answer in the affirmative a question raised by Diverio in [Div22] on the existence of a closed complex manifold admitting a Kähler metric with quasi-negative holomorphic (bi)sectional curvature but admitting no Kähler metric with negative holomorphic sectional curvature (Corollary 3). In another direction, we also show that the universal cover $\widetilde{X_\Gamma}$ of X_Γ is Stein when the lattice Γ is arithmetic and deep enough. To do so, we study the properties of the Albanese map of the manifolds obtained by compactifying finite covers of $\mathbb{H}^n_{\mathbb{C}}/\Gamma$, and we obtain results concerning the Albanese image of the tori added during the compactification (Theorem 4), extending earlier

work of Eyssidieux [Eys18]. This improvement allows us to prove that for deep enough finite index subgroups Γ' of Γ, the universal cover of the toroidal compactification of $\mathbb{H}^n_{\mathbb{C}}/\Gamma'$ is a Stein manifold (Theorem 5). When $n = 2$ this result had been obtained earlier by Eyssidieux [Eys18].

Let us now describe more precisely and state these results. In [HS96], Hummel and Schroeder assert that toroidal compactifications do not admit Kähler metrics with nonpositive sectional curvature. A proof of this assertion has been given in complex dimension 2 for "deep enough" lattices in [Di 12]. Generalizing it to any dimension, we prove:

Theorem 1. *Let Γ be a non-uniform torsion-free lattice of $\mathrm{PU}(n,1)$ for which the quotient space $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ admits a toroidal compactification X_Γ. Then there is no Kähler metric with nonpositive sectional curvature on X_Γ.*

In particular, the techniques for constructing Kähler metrics on X_Γ developed in [HS96] cannot provide a metric with nonpositive sectional curvature. However, a computation shows that they are sufficient to construct a Kähler metric whose curvature verifies a weaker condition of nonpositivity, namely the nonpositivity of the holomorphic bisectional curvature:

Theorem 2. *Let Γ_0 be a non-uniform torsion-free lattice of $\mathrm{PU}(n,1)$. There exists a finite index subgroup $\Gamma' < \Gamma_0$ such that for any finite index subgroup $\Gamma < \Gamma'$, the toroidal compactification X_Γ of $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ admits a Kähler metric with nonpositive holomorphic bisectional curvature.*

The metric constructed in the proof of this theorem has negative holomorphic bisectional curvature on the open set $\mathbb{H}^n_{\mathbb{C}}/\Gamma \subset X_\Gamma$. In particular, it is quasi-negative in the sense of [DT19], i.e. the holomorphic bisectional curvature is nonpositive everywhere and negative at a point. In their article, Diverio and Trapani showed that every closed Kähler manifold with quasi-negative holomorphic sectional curvature has ample canonical bundle, a result which was already known under the assumption of negativity of the holomorphic sectional curvature [WY16; TY17]. This answered a question of Yau. This situation naturally lead the first author to ask in [Div22] whether there exist closed Kähler manifolds of quasi-negative holomorphic sectional curvature but which do not admit a Kähler metric of negative holomorphic sectional curvature. Toroidal compactifications contain tori, which are not Kobayashi hyperbolic, so they allow to answer this question in the affirmative:

Corollary 3. *For Γ a deep enough non-uniform lattice as in Theorem 2, X_Γ is an example of a closed complex manifold which admits a Kähler metric of quasi-negative holomorphic bisectional curvature, but which does not admit a Kähler metric of negative holomorphic sectional curvature.*

Along the way we observe that Theorem 2 combined with the main result of [DT19] provides an alternative proof of [DD17, Theorem 1.3]: for Γ a deep enough non-uniform lattice as in Theorem 2, X_Γ has ample canonical bundle. For a more direct proof, we remark in Section 4 that the metric constructed in the proof of Theorem 2 has negative Ricci curvature. In addition, applying [Gue22, Theorem 3.1], we get that any irreducible subvariety of X_Γ which is not included in the complement of $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is of general type.

We now state our results related to the Shafarevich conjecture for the compactifications X_Γ. This conjecture, stated in 1972 for projective manifolds, predicts that the universal cover of any closed Kähler manifold is holomorphically convex. It is known in the case where the fundamental group of the manifold is linear [Eys+12; CCE15]. It is also known for specific examples [EM15]. See also [Kat97; KR98; Eys04] for earlier results, and [Eys11] for a survey article on this conjecture.

For toroidal compactifications, if the lattice Γ of $\mathrm{PU}(n,1)$ is deep enough, then by the theorem of Hummel and Schroeder recalled above, the universal cover $\widetilde{X_\Gamma}$ of X_Γ is diffeomorphic to $\mathbb{R}^{2n}$

and can be endowed with a Kähler metric, thus it does not contain any compact analytic variety of positive dimension. Therefore, $\widetilde{X_\Gamma}$ is holomorphically convex if and only if it is Stein. When the lattice is arithmetic, Eyssidieux proved that, up to taking a finite index subgroup of the lattice, the Albanese map of X_Γ, denoted by $A_\Gamma : X_\Gamma \longrightarrow \mathrm{Alb}(X_\Gamma)$, is an immersion on the complement of the tori added during the compactification [Eys18]. He then concluded in dimension 2 that the universal cover of X_Γ is holomorphically convex, using a result of [Nap90] valid only in this dimension. To study the higher dimensional case, we make Eyssidieux's result about Albanese mappings more precise. To do so, we analyse the behavior of the Albanese map on the tori added during the compactification, which we will call in the following "boundary tori of X_Γ". More precisely, a boundary torus of X_Γ is a connected component of the complement of $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ in X_Γ. With this terminology, we prove:

Theorem 4. *Let Γ_0 be a non-uniform torsion-free arithmetic lattice of $\mathrm{PU}(n,1)$. There exists a finite index subgroup $\Gamma' < \Gamma_0$ such that for any finite index subgroup $\Gamma < \Gamma'$, the toroidal compactification X_Γ of $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ satisfies the following properties:*

- *Its Albanese map A_Γ is an immersion on the open set $\mathbb{H}^n_{\mathbb{C}}/\Gamma \subset X_\Gamma$ and:*
- *For any boundary torus T of X_Γ, the map $A_{\Gamma\,|T}$ is either an immersion or a constant.*

The arithmeticity of Γ_0 is used crucially in the proof of Theorem 4, as in [Eys18]. Deciding whether for every boundary torus T of X_{Γ_0}, there exists a finite index subgroup $\Gamma < \Gamma_0$ for which the Albanese map A_Γ of X_Γ is an immersion when restricted to the tori above T seems much more difficult. However the weaker result of Theorem 4 is enough for our purpose. We combine it with the arguments of [Nap90], valid in any dimension, about holomorphic convexity with respect to a sufficiently positive line bundle, and we adapt the arguments given there in dimension 2 in order to generalize Eyssidieux's result to all dimension:

Theorem 5. *Let Γ_0 be a non-uniform torsion-free arithmetic lattice of $\mathrm{PU}(n,1)$ There exists a finite index subgroup $\Gamma' < \Gamma_0$ such that for any finite index subgroup $\Gamma < \Gamma'$, the universal cover $\widetilde{X_\Gamma}$ of the toroidal compactification X_Γ of $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is a Stein manifold.*

This theorem is deduced from Proposition 13 stated in Section 6, which is a more general result on projective manifolds which are the source of a generically finite holomorphic map with values in a compact manifold with Stein universal cover.

The article is organized as follows. In Section 2, we recall how toroidal compactifications of spaces of the form $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ are constructed. Then, Sections 3, 4, 5, 6 contain respectively the proofs of Theorems 1, 2, 4, 5. They use Section 2 and are independent of each other, except for Section 6 which relies on Section 5.

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Contents

2 Preliminaries

The complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$ is the unique complete simply connected Kähler manifold of dimension n with constant holomorphic sectional curvature equal to -4. Its Riemannian sectional curvature is pinched between -4 and -1. To construct it, we first consider a Hermitian form $\langle\cdot,\cdot\rangle$ of signature $(n,1)$ on $\mathbb{C}^{n+1}$. As a complex manifold, $\mathbb{H}^n_{\mathbb{C}}$ is the projectivisation of the set of negative vectors for this form; it is an open set of $\mathbb{CP}^n$. Its boundary $\partial\mathbb{H}^n_{\mathbb{C}}$ can be defined as its topological boundary in $\mathbb{CP}^n$. We do not recall the construction of the hyperbolic metric on this space, and refer for example to [Gol99]. Its group of holomorphic isometries is the simple Lie group $\mathrm{PU}(n,1)$, and it also acts on $\partial\mathbb{H}^n_{\mathbb{C}}$. We will use the classical trichotomy of isometries into elliptic, parabolic or hyperbolic type, for which we refer for instance to [BH99, section II.6].

We first introduce some notations which are analogous to the ones used in [HS96; Mok12]. To every point $\xi \in \partial\mathbb{H}^n_{\mathbb{C}}$ can be associated a nilpotent subgroup N_ξ of $\mathrm{PU}(n,1)$, which is the unipotent radical of the stabiliser of ξ in $\mathrm{PU}(n,1)$. This Lie group is isomorphic to the Heisenberg group of dimension $2n-1$. The following calculations will be used in the proof of Proposition 6, they aim at expliciting global coordinates on the quotient of a horoball centered at ξ by the action of a non-trivial element Z in the center of N_ξ.

Let us fix a geodesic γ parametrized with unit speed such that $\lim_{t\to+\infty}\gamma(t)=\xi$. Denote also $o := \gamma(0)$ and $\xi' := \lim_{t\to-\infty}\gamma(t)$. Then ξ and ξ' lift to vectors $f_1, f_2 \in \mathbb{C}^{n+1}$ which are isotropic for the Hermitian form defined above, non-collinear, and can be normalized so that $\langle f_1, f_2\rangle = 1$ and $o = [f_1 - f_2]$. Here and in the sequel, $[\cdot]$ denotes the projection map $\mathbb{C}^{n+1}\setminus\{0\} \to \mathbb{CP}^n$. Let $(f_3,\dots,f_{n+1})$ be a basis of $P := (\mathbb{C}f_1 \oplus \mathbb{C}f_2)^\perp$, orthonormal for the Hermitian product $\langle\cdot,\cdot\rangle_{|P\times P}$. We idenfity $(P, \langle\cdot,\cdot\rangle_{|P\times P})$ with $\mathbb{C}^{n-1}$ endowed with the standard Hermitian product whose associated norm is denoted by $\|\cdot\|$. In the basis $(f_1,\dots,f_{n+1})$ of $\mathbb{C}^{n+1}$, the form $\langle\cdot,\cdot\rangle$ has the following expression:

$$\langle af_1 + bf_2 + v, af_1 + bf_2 + v\rangle = 2\Re(a\bar{b}) + \|v\|^2.$$

The group N_ξ is made of the endomorphisms of $\mathbb{C}^{n+1}$ whose matrices in the basis $(f_1,\dots,f_{n+1})$ have the form:

$$\begin{pmatrix} 1 & -\frac{\|v\|^2}{2} - is & -\langle\cdot, v\rangle \\ 0 & 1 & 0 \\ 0 & v & \mathrm{Id}_{n-2} \end{pmatrix},$$

with $(s,v) \in \mathbb{R}\times\mathbb{C}^{n-1}$. We call $n_{(s,v)}$ the element of $\mathrm{PU}(n,1)$ defined by the matrix above. A global chart of $\mathbb{H}^n_{\mathbb{C}}$, called *Siegel model* of $\mathbb{H}^n_{\mathbb{C}}$, is given by:

$$\begin{aligned}\widetilde{\Omega} := \{(a,v) \in \mathbb{C}\times\mathbb{C}^{n-1} \mid 2\Re(a) + \|v\|^2 < 0\} &\longrightarrow \mathbb{H}^n_{\mathbb{C}} \\ (a,v) &\longmapsto [af_1 + f_2 + v].\end{aligned}$$

Denote by $\widetilde{\mathcal{L}}$ the coordinates defined by the inverse of this map. For any real number t, let a_t be the hyperbolic isometry of axis γ defined in the basis $(f_1,\dots,f_{n+1})$ by the matrix $\mathrm{Diag}(e^{-t}, e^t, 1, \dots, 1)$. A calculation shows that the point $n_{(s,v)}\cdot a_t \cdot o$ has coordinates:

$$\widetilde{\mathcal{L}}\left(n_{(s,v)}\cdot a_t\cdot o\right) = \left(-\frac{\|v\|^2}{2} - is - e^{-2t}, v\right). \tag{1}$$

This shows in particular that the map $N_\xi \times \mathbb{R} \to \mathbb{H}^n_{\mathbb{C}}$ which associates to each element $(n_{(s,v)}, t)$ the point $n_{(s,v)} \cdot a_t \cdot o$ of the hyperbolic space is a diffeomorphism.

If Z is an element of N_ξ of the form $Z := n_{(l,0)}$ with $l \neq 0$, the group $\langle Z \rangle$ which it generates acts freely and properly discontinuously on $\mathbb{H}^n_{\mathbb{C}}$. The action of Z is expressed in the coordinates $\widetilde{\mathcal{L}}$ by the translation $(a, u) \mapsto (a, u) - (il, 0)$, thus the diagram of holomorphic maps:

$$\begin{array}{ccc} \mathbb{H}^n_{\mathbb{C}} & \xrightarrow{\widetilde{\mathcal{L}}} & \widetilde{\Omega} \\ \downarrow & & \downarrow{\scriptstyle (\exp(\frac{2\pi \bullet}{l}),\mathrm{Id})} \\ \mathbb{H}^n_{\mathbb{C}}/\langle Z \rangle & \xrightarrow{\mathcal{L}} & \Omega. \end{array}$$

commutes, where the map $\mathcal{L}$ defined by the diagram is a biholomorphism on its image:

$$\Omega := \{(a, v) \in \mathbb{C} \times \mathbb{C}^{n-1} \mid 0 < |a| < e^{\frac{-\pi \|v\|^2}{l}}\}.$$

A horoball centered at ξ, denoted by H_ξ or simply H in what follows, is a subset of $\mathbb{H}^n_{\mathbb{C}} \simeq N_\xi \times \mathbb{R}$ of the form $N_\xi \times (-\infty, t_0)$ for some real number t_0. This definition does not depend on the choice of the geodesic γ, for example because horoballs can also be defined as the sublevel sets of the Busemann function [Gol99]. The action of Z on the hyperbolic space is identified with its action by left multiplication on the first factor of $N_\xi \times \mathbb{R}$. In particular, it preserves the horoballs. In coordinates $\widetilde{\mathcal{L}}$ and $\mathcal{L}$ respectively, a horoball $N_\xi \times (-\infty, t_0)$ and its quotient by the action of Z identify with the open sets:

$$\widetilde{\Omega}_{t_0} := \{(a, v) \in \mathbb{C} \times \mathbb{C}^{n-1} \mid \Re(a) < -\frac{\|v\|^2}{2} - e^{-2t_0}\} \quad \text{and}$$
$$\Omega_{t_0} := \{(a, v) \in \mathbb{C} \times \mathbb{C}^{n-1} \mid 0 < |a| < \lambda(t_0) e^{\frac{-\pi \|v\|^2}{l}}\},$$

with $\lambda(t_0) := \exp(\frac{-2\pi e^{-2t_0}}{l})$.

A non-uniform lattice Γ_0 of $\mathrm{PU}(n, 1)$ is a discrete subgroup of $\mathrm{PU}(n, 1)$ for which the quotient space $\mathbb{H}^n_{\mathbb{C}}/\Gamma_0$ has finite Riemannian volume, without being compact. Let Γ_0 be a non-uniform torsion-free lattice of $\mathrm{PU}(n, 1)$, and Γ a finite index subgroup of Γ_0. Then the quotient space $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is a complex manifold naturally endowed with a Kähler metric of constant holomorphic sectional curvature -4. By Margulis' Lemma [BGS85] it admits a *thick-thin decomposition* with a finite number of cusps, which means that we can write:

$$\mathbb{H}^n_{\mathbb{C}}/\Gamma = Q \cup \coprod_{i=1}^{k} C_i,$$

where Q is a compact subset of $\mathbb{H}^n_{\mathbb{C}}/\Gamma$, $k \in \mathbb{N}^*$, and for any $1 \leq i \leq k$, C_i is a cusp, i.e., there exists a point ξ_i of the boundary of $\mathbb{H}^n_{\mathbb{C}}$ and a sufficiently deep horoball H_i centered at ξ_i, globally fixed by the stabiliser $\mathrm{Stab}_\Gamma(\xi_i)$ of ξ_i in Γ, such that C_i is biholomorphically isometric to $H_i/\mathrm{Stab}_\Gamma(\xi_i)$ [SY82]. The group $\mathrm{Stab}_\Gamma(\xi_i)$ contains no hyperbolic elements, thus it is a subgroup of $N_{\xi_i} \rtimes K_i$, where K_i is the pointwise stabiliser in $\mathrm{PU}(n, 1)$ of a geodesic ray ending at ξ_i. In particular, $\mathrm{Stab}_\Gamma(\xi_i)$ fixes each horosphere $N_{\xi_i} \times \{t\}$. Using the Auslander–Bieberbach's theorem, one can show that there exists a finite subset $S \subset \Gamma_0$ such that for every finite index normal subgroup Γ of Γ_0 which does not intersect S, the parabolic elements of Γ are purely unipotent [Hum98]. In this case, $\mathrm{Stab}_\Gamma(\xi_i)$ is a subgroup of N_{ξ_i} for all i, and finally the cusp C_i is isomorphic to the product of the nilmanifold $N_{\xi_i}/(\Gamma \cap N_{\xi_i})$ by the interval $(-\infty, t_0)$ for some real number

t_0 [Hum98]. Furthermore, since $H_i/(\Gamma \cap N_i)$ has finite volume, Fubini's theorem implies that $N_{\xi_i}/(\Gamma \cap N_{\xi_i})$ also has finite volume, and thus that $\Gamma \cap N_{\xi_i}$ is a lattice of N_{ξ_i}.

We will say that Γ is a lattice for which the quotient space $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ admits a toroidal compactification if Γ is non-uniform, torsion-free and all parabolic subgroups of Γ are purely unipotent, as explained in the previous paragraph. This is justified by the following proposition, which, for such lattices Γ allows to compactify the manifold $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ by identifying each cusp to a holomorphic punctured disk bundle over a complex torus of dimension $n-1$. The compactification can then be done by adding the zero section of this bundle. This identification is classical [Mok12], and we include it for the reader's convenience.

Proposition 6. *Let N be the $(2n-1)$-dimensional Heisenberg group, identified with N_ξ for some point $\xi \in \partial\mathbb{H}^n_{\mathbb{C}}$, Λ a lattice of N, which acts on $\mathbb{H}^n_{\mathbb{C}} \simeq N \times \mathbb{R}$ by left multiplication on the N-factor, and let $H := N \times (-\infty, t_0)$ be a horoball of the complex hyperbolic space. Let $\rho : N \to \mathbb{C}^{n-1}$ be the surjective group morphism $n_{(s,v)} \mapsto v$. Then the map $\pi : H/\Lambda \to \mathbb{C}^{n-1}/\rho(\Lambda)$ is a holomorphic punctured disk bundle over the torus $T := \mathbb{C}^{n-1}/\rho(\Lambda)$, which has negative curvature. More precisely, there exists a holomorphic line bundle $\widetilde{\pi} : L \to T$ on the torus T, endowed with a Hermitian metric h with negative curvature $\Theta := -i\partial\bar{\partial}\log h$, as well as a holomorphic embedding $i : H/\Lambda \to L$ whose image is the open set:*

$$\{v \in L \mid 0 < \|v\|_h < 1\},$$

and such that $\pi = \widetilde{\pi} \circ i$.

Proof. Since Λ is a lattice of N, $\rho(\Lambda)$ is a lattice of $\mathbb{C}^{n-1}$ and in particular T is a torus. The discrete subgroup $\Lambda \cap [N, N]$ is non-trivial, generated by an element $Z := n_{(l,0)}$ for some $l > 0$. The map $\mathcal{L}$ defined above identifies $H/\langle Z \rangle$ to an open set Ω_{t_0} of $\mathbb{C} \times \mathbb{C}^{n-1}$. Through this identification, Λ acts on Ω_{t_0} by:

$$n_{(s,v)} \cdot (t, w) = \left(e^{\frac{2\pi}{l}\left(-\frac{\|v\|^2}{2} - is - \langle w, v\rangle\right)} t, v + w \right).$$

The quotient space H/Λ can thus be identified with the quotient of Ω_{t_0} by this action. Notice that this action naturally extends to an action of Λ on $\mathbb{C} \times \mathbb{C}^{n-1}$. Let us endow the trivial line bundle $\mathrm{pr}_2 : \mathbb{C} \times \mathbb{C}^{n-1} \to \mathbb{C}^{n-1}$ with the Hermitian metric:

$$h((a, v), (a', v)) := \lambda(t_0)^{-2} \exp\left(\frac{2\pi\|v\|^2}{l}\right) a\bar{a'},$$

so that the inclusion map $j : \Omega_{t_0} \to \mathbb{C} \times \mathbb{C}^{n-1}$ defines a holomorphic embedding whose image is the open set:

$$\{(a, v) \in \mathbb{C} \times \mathbb{C}^{n-1} \mid 0 < \|(a, v)\|_h < 1\}.$$

Since Ω_{t_0} is Λ-invariant, both the inclusion j and the bundle pr_2 pass to the quotient, i.e. we have the following commutative diagram:

$$\begin{array}{ccc}
\Omega_{t_0} & \longrightarrow & \Omega_{t_0}/\Lambda \\
\downarrow{\scriptstyle j} & & \downarrow{\scriptstyle i} \\
\mathbb{C} \times \mathbb{C}^{n-1} & \longrightarrow & L \\
\downarrow{\scriptstyle \mathrm{pr}_2} & & \downarrow{\scriptstyle \widetilde{\pi}} \\
\mathbb{C}^{n-1} & \longrightarrow & T.
\end{array}$$

In this diagram, $L := (\mathbb{C} \times \mathbb{C}^{n-1})/\Lambda$, horizontal arrows are natural quotient maps and $\widetilde{\pi}$ is the only map making the bottom square of the diagram commute. Using the above expression for the action of Λ on $\mathbb{C} \times \mathbb{C}^{n-1}$, we see that $\widetilde{\pi} : L \to T$ is a line bundle and L can be endowed with the quotient Hermitian metric of h, still denoted by h, so that the image of i is the open set:

$$\{v \in L \mid 0 < \|v\|_h < 1\}.$$

The curvature of (L, h), locally given by the form $-\frac{2\pi i}{l}\partial\bar{\partial}\|\cdot\|^2$, is negative. □

Let us denote by X_Γ the toroidal compactification of $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ obtained by the previous construction. Recall that a line bundle $L \to X$ on a complex manifold X is negative if it admits a Hermitian metric h whose curvature form is negative, i.e. if the Chern class $c_1(L)$ of the bundle L can be represented by a negative $(1,1)$-form. For each boundary torus T added during the compactification, the normal bundle of T in X_Γ is identified with the normal bundle $(TL/TO)_{|O}$ of the zero section O in the total space of the bundle L, which is isomorphic to L. We deduce that:

Corollary 7. *For each boundary torus $T \subset X_\Gamma$ added during the compactification, the normal bundle of T in X_Γ is negative.*

3 Nonpositively curved Kähler manifolds containing hypersurfaces with $c_1 = 0$

In this section, we prove that for any non-uniform torsion-free lattice Γ of $\mathrm{PU}(n,1)$, the toroidal compactification X_Γ, when defined, does not admit any Kähler metric with nonpositive sectional curvature (Theorem 1). We will use that X_Γ contains a torus of complex codimension 1 whose normal bundle is negative (Corollary 7). The following proposition immediately implies the theorem. It uses the fact that the holomorphic sectional curvature decreases when passing to submanifolds, as well as relations between different notions of curvature, some of which are valid only for Kähler manifolds, and which are explained in [Div22].

Proposition 8. *Let M be a complex manifold with a Kähler metric of nonpositive sectional curvature. Suppose that M contains a codimension 1 compact submanifold V whose first Chern class $c_1(TV)$ is zero. Then V is totally geodesic and the Ricci curvature of M is zero in restriction to V. Moreover the first Chern class $c_1(N_V)$ of the normal bundle of V vanishes.*

Before proving the Proposition, we state our curvature conventions. On a Riemannian manifold (M, g), we define:

$$\begin{aligned} &R(X,Y) := \nabla_{[X,Y]} - [\nabla_X, \nabla_Y] \qquad \text{and} \\ &R(X,Y,Z,T) := g(R(X,Y)Z,T). \end{aligned} \tag{2}$$

Thus (M, g) has nonpositive sectional curvature if and only if $R(X,Y,X,Y) \leq 0$ for all vectors fields X, Y of M. Accordingly, if $(f_1, \ldots, f_n)$ is an orthonormal basis of T_xM, the Ricci curvature of M at x is defined by:

$$\mathrm{Ricci}_M(X,Y) := \sum_{j=1}^{n} R(X, f_j, Y, f_j).$$

With these notations in mind, let us now turn to the proof of the Proposition:

Proof of Proposition 8. The holomorphic sectional curvature of V is less than or equal to the one of M restricted to the tangent bundle of V, therefore, it is nonpositive. By [Div22, remark 2.8], in the case of a compact Kähler manifold, the non-positivity of the holomorphic sectional curvature implies the non-positivity of the scalar curvature, and the total scalar curvature (i.e. the integral of the scalar curvature) is zero if the Chern class $c_1(TV)$ of V is zero. Therefore, the scalar curvature of V is identically zero. From [Div22, proposition 2.9], we deduce that the holomorphic sectional curvature of V is identically zero. Because the latter determines the sectional curvature of V [KN96, chapter IX, proposition 7.1], we deduce that the metric g restricted to V is flat. The holomorphic sectional curvature of V also bounds from below the holomorphic sectional curvature of M restricted to the tangent bundle of V, so the latter is identically zero. From [KN96, chapter IX, proposition 9.2], we deduce that the second fundamental form of V is zero, thus that V is totally geodesic.

Let us show that the Ricci curvature of M restricted to TV is identically zero. Let $(z^1, \dots, z^n)$ be complex coordinates in the neighborhood of a point x of V, such that $(\frac{\partial}{\partial z^1}, \dots, \frac{\partial}{\partial z^n})$ forms an orthonormal basis of $T_x^{(1,0)}M$ and $(\frac{\partial}{\partial z^1}, \dots, \frac{\partial}{\partial z^{n-1}})$ an orthonormal basis of $T_x^{(1,0)}V$. Then, writing $z^i =: x^i + \sqrt{-1}y^i$ and $e_i := \frac{\partial}{\partial x^i}$, we have that:

- $(e_1, Je_1, \dots, e_n, Je_n)$ is an orthonormal basis of the tangent space T_xM ;
- $(e_1, Je_1, \dots, e_{n-1}, Je_{n-1})$ is an orthonormal basis of the tangent space T_xV.

It is sufficient to show that $\mathrm{Ricci}_M(e_i, e_i) = 0$ for all $i < n$. Recall that the first Bianchi identity gives for all vectors $X, Y \in T_xM$:

$$R(X, JX, Y, JY) = R(X, Y, X, Y) + R(X, JY, X, JY).$$

In particular, if the sectional curvature is nonpositive, so is the holomorphic bisectional curvature. This formula also allows to express the Ricci curvature at x:

$$\begin{aligned}
\mathrm{Ricci}_M(e_i, e_i) &= \sum_{j=1}^{n} R(e_i, e_j, e_i, e_j) + R(e_i, Je_j, e_i, Je_j) \\
&= \sum_{j=1}^{n} R(e_i, Je_i, e_j, Je_j) \\
&= R(e_i, Je_i, e_n, Je_n) + \mathrm{Ricci}_V(e_i, e_i) \\
&= R(e_i, Je_i, e_n, Je_n).
\end{aligned}$$

We assert that for all vectors v, w tangent to M at the same point, we have:

$$R(v, Jv, w, Jw)^2 \leq R(v, Jv, v, Jv)R(w, Jw, w, Jw). \tag{3}$$

This inequality holds for all Kähler manifolds with nonpositive Riemannian sectional curvature. Provided that it is true, we apply it to $v = e_i$ and $w = e_n$: since $R(e_i, Je_i, e_i, Je_i) = 0$ due to the fact that the holomorphic sectional curvature of M restricted to TV is zero, this shows that $\mathrm{Ricci}_{M\,|TV} = 0$.

The proof of Inequality (3), coming from the articles [MS80; Di 12], consists in considering a one-parameter family of tangent planes at the considered point, and in expressing the curvature of these planes as a nonpositive function of the parameter. We use the fact that the metric is Kählerian to simplify the expression of the curvature. More precisely, for any real number a, we set:

$$\begin{cases} p_a := av + w \\ q_a := J(av - w). \end{cases}$$

Since $p_{-a} = Jq_a$ and $q_{-a} = -Jp_a$, the map $a \mapsto R(p_a, q_a, p_a, q_a)$ is even. It can thus be written as a polynomial $P(a^2)$ of degree at most 2 in a^2:

$$P(a^2) = R(p_a, q_a, p_a, q_a) = R(v, Jv, v, Jv)a^4 + Aa^2 + R(w, Jw, w, Jw),$$

with:

$$\begin{aligned} A :=& R(v, Jv, w, -Jw) + R(v, -Jw, v, -Jw) + R(v, -Jw, w, Jv) + \\ & R(w, Jv, v, -Jw) + R(w, Jv, w, Jv) + R(w, -Jw, v, Jv) \\ =& -R(v, Jv, w, Jw) + R(v, -Jw, v, -Jw) - R(v, -Jw, v, -Jw) + \\ & R(w, Jv, v, -Jw) - R(w, Jv, v, -Jw) - R(v, Jv, w, Jw) \\ =& -2R(v, Jv, w, Jw). \end{aligned}$$

Thus:

$$P(a^2) = R(v, Jv, v, Jv)a^4 - 2R(v, Jv, w, Jw)a^2 + R(w, Jw, w, Jw).$$

Since the sectional curvature is nonpositive, the polynomial P is nonpositive on $\mathbb{R}_+$. Recall also that $R(X, JX, Y, JY) \leq 0$ for all vectors $X, Y \in T_xM$. We distinguish two cases:

- If the coefficient $R(v, Jv, v, Jv)$ vanishes, then when a becomes increasingly large, the inequality $R(p_a, q_a, p_a, q_a) \leq 0$ forces $R(v, Jv, w, Jw)$ to vanish as well.
- Otherwise, we want to show that the discriminant of P is nonpositive. If it were positive, then P would have two simple roots whose sum would be $2\frac{R(v,Jv,w,Jw)}{R(v,Jv,v,Jv)} \geq 0$, so at least one of them would be positive. But on the neighbourhood of a simple root, the sign of P would change, which would contradict the inequality $R(p_a, q_a, p_a, q_a) \leq 0$. Hence the discriminant of P is nonpositive.

In all cases, we get the desired inequality:

$$R(v, Jv, w, Jw)^2 \leq R(v, Jv, v, Jv)R(w, Jw, w, Jw).$$

Let us now show that the normal bundle of V has vanishing first Chern class. If $\rho := \mathrm{Ricci}_M(\cdot, J\cdot)$ denotes the Ricci form of M, the class $c_1(K_M)$ is represented by $\frac{\sqrt{-1}}{2\pi}\rho$, which vanishes in restriction to V. Since $c_1(K_V) = -c_1(TV) = 0$ and $N_V = K_V \otimes K_{M|V}^{-1}$, we deduce that $c_1(N_V) = 0$. □

4 A metric on X_Γ with nonpositive holomorphic bisectional curvature

In this section, we prove that for any non-uniform torsion-free lattice Γ_0 of $\mathrm{PU}(n, 1)$, there exists a finite index subgroup $\Gamma' < \Gamma_0$ such that for any finite index subgroup $\Gamma < \Gamma'$, the toroidal compactification X_Γ of $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ admits a Kähler metric with nonpositive holomorphic bisectional curvature (Theorem 2). We will need the following lemma, whose proof we omit:

Lemma 9. *There exists a constant $C_0 > 0$ such that for every $C \geq C_0$, there is a smooth function $f : [0, C] \to \mathbb{R}+$ which coincides with* cosh *on some neighborhood of 0, coincides with* exp *on some neighborhood of C and such that the functions f, f', f'', f''' are positive on $(0, C]$.*

Proof of Theorem 2. The proof takes up Hummel and Schroeder's construction of a Kähler metric on toroidal compactifications [HS96], adding the assumption of finite index subgroups in order to get space for the construction of a metric with nonpositive holomorphic bisectional curvature. In a nutshell, the idea is to work separately in each cusp, and, provided that the cusp is "big enough", to patch there some compact manifold, endowed with a Kähler metric which glues to the complex hyperbolic one, and whose bisectional curvature is nonpositive. For the reader's convenience, we first provide a detailed description of the construction of the metric, as carried out by Hummel and Schroeder, and then we come to the original part of the proof, which is the computation of the holomorphic bisectional curvature.

First we fix some notations. Using the model $(s,v) \mapsto n_{(s,v)}$ of the Heisenberg group N described in Section 2, we endow N with the left-invariant metric μ expressed at the identity point by:

$$\mu_{\mathrm{Id}} := ds^2 \oplus dv^2.$$

Let $\mathfrak{n}$ be the Lie algebra of N, Z the generator of $[\mathfrak{n},\mathfrak{n}] \simeq \mathbb{R}$ such that $\exp(Z) = n_{(1,0)}$ and $\mathfrak{r}$ the μ-orthogonal of $[\mathfrak{n},\mathfrak{n}]$. Assuming that Γ_0 is a torsion-free non-uniform lattice of $\mathrm{PU}(n,1)$ with purely unipotent parabolic elements as explained in Section 2, any cusp in $\mathbb{H}^n_{\mathbb{C}}/\Gamma_0$ is biholomorphically isometric to:

$$((-\infty, A_0] \times N/\Lambda_0, \mu_0, J_0),$$

where A_0 is some real number and Λ_0 a lattice in N which depend on the cusp; μ_0 is the N-invariant metric expressed at a point $(t, \mathrm{Id} \bmod \Lambda_0)$ by:

$$\mu_{0(t,\mathrm{Id} \bmod \Lambda_0)} := dt^2 \oplus e^{2t}\mu_{|\mathfrak{r}\times\mathfrak{r}} \oplus e^{4t}\mu_{|\mathbb{R}Z\times\mathbb{R}Z};$$

and J_0 is the N-invariant complex structure defined at a point $(t, \mathrm{Id} \bmod \Lambda_0)$ by:

$$\begin{cases} J_0 Z = e^{2t}\frac{\partial}{\partial t}, \\ \forall v \in \mathbb{C}^{n-1}, \exp \circ J_0 \circ \exp^{-1}(n_{(0,v)}) = n_{(0,iv)}. \end{cases}$$

In fact this biholomorphism is, up to translating the parameter t by $\ln(\sqrt{2})$, the quotient of the map $\widetilde{\mathcal{L}}$ defined in Section 2. Avoiding for each cusp a finite number of elements of Γ, there is by Malcev's Theorem a finite index subgroup $\Gamma' < \Gamma_0$ such that for any finite index subgroup $\Gamma < \Gamma'$, any cusp of $\mathbb{H}^n_{\mathbb{C}}/\Gamma$, which lies above some cusp $((-\infty, A_0] \times N/\Lambda_0, \mu_0, J_0)$ of $\mathbb{H}^n_{\mathbb{C}}/\Gamma_0$, is biholomorphically isometric to $((-\infty, A_0] \times N/\Lambda, \mu_0, J_0)$ for some finite index subgroup $\Lambda < \Lambda_0$ such that the group $\Lambda \cap [N,N]$ is generated by $n_{(l,0)}$ with $l > 2\pi \exp^{2(C_0 - A_0)}$, where the constant C_0 is given by Lemma 9. Let Γ be a finite index subgroup of Γ'. We will show that the toroidal compactification X_Γ of $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ has a metric with nonpositive bisectional holomorphic curvature.

Let $((-\infty, A_0] \times N/\Lambda, \mu_0, J_0)$ be a cusp of $\mathbb{H}^n_{\mathbb{C}}/\Gamma$. To construct the desired metric on this cusp, we first change slightly the model. Let φ be the automorphism of N given by:

$$\varphi(n_{(s,v)}) = n_{\frac{2\pi}{l}s, \sqrt{\frac{2\pi}{l}}v},$$

and set $C := \ln\sqrt{\frac{l}{2\pi}}$ and $A := A_0 + C$ in order to define the diffeomorphism:

$$\Phi : \begin{cases} (-\infty, A_0] \times N/\Lambda & \longrightarrow & (-\infty, A] \times N/\varphi(\Lambda) \\ (t, n \bmod \Lambda) & \longmapsto & (t + C, \varphi(n) \bmod \varphi(\Lambda)). \end{cases}$$

Easy computations show that $\Phi_*\mu_0 = \mu_0$ and $\Phi_* J_0 = J_0$. In this new model, we have by construction $\varphi(n_{(l,0)}) = n_{(2\pi,0)}$ and $A > C_0$.

We now make a second change of model. First let $f : [0, A] \to \mathbb{R}_+$ be the function given by Lemma 9, $g := ff'$ and $\psi : (0, A] \to (-\infty, A]$ be the solution of the following differential equation:

$$\begin{cases} \psi'(t) = & \frac{e^{2\psi(t)}}{g(t)} \\ \psi(A) = & A \end{cases}$$

It is easily seen that ψ is a well defined diffeomorphism, and that ψ is the identity in a neighborhood of A. Then let:

$$\Psi := \psi \times \mathrm{Id} : (0, A] \times N/\varphi(\Lambda) \longrightarrow (-\infty, A] \times N/\varphi(\Lambda).$$

We compute that $J := \Psi^* J_0$ is the N-invariant complex structure defined at a point $(t, \mathrm{Id} \bmod \varphi(\Lambda))$ by:

$$\begin{cases} JZ = g(t)\frac{\partial}{\partial t}, \\ \forall v \in \mathbb{C}^{n-1}, \exp \circ J \circ \exp^{-1}(n_{(0,v)}) = n_{(0,iv)}. \end{cases}$$

Then we endow the complex manifold $((0, A] \times N/\varphi(\Lambda), J)$ with the N-invariant hermitian metric $\mu_{f,g}$ expressed at a point $(t, \mathrm{Id} \mod \varphi(\Lambda))$ by:

$$\mu_{f,g\ (t,\mathrm{Id}\ \bmod\ \varphi(\Lambda))} := dt^2 \oplus f(t)^2 \mu_{|\mathfrak{r}\times\mathfrak{r}} \oplus g(t)^2 \mu_{|\mathbb{R}Z\times\mathbb{R}Z}.$$

The metric $\mu_{f,g}$ coincides with $\Psi^*\mu_0$ on a neighborhood of the boundary of the manifold, hence it glues back to the hyperbolic metric on the thick part of $\mathbb{H}^n_{\mathbb{C}}/\Gamma$. As explained in [HS96], the toroidal compactification of the cusp can be realized in this model in the following way: using polar coordinates, we first identify $(0, A) \times N/\langle n_{(2\pi,0)}\rangle$ with $\mathbb{D}^*(0, A) \times \mathbb{C}^{n-1}$, where $\mathbb{D}^*(0, A)$ is the punctured disk in $\mathbb{C}$ with radius A; then we take the quotient by the action of $\varphi(\Lambda)$.

From [HS96], the equality $g = ff'$ implies that the metric $\mu_{f,g}$ is Kählerian. The fact that $f = \cosh$ in some neighborhood of 0 implies that f extends to an even function with $f(0) \neq 0$ and that $g = ff'$ extends to an odd function with $g'(0) = 1$. From [HS96, Lemma 3.8], this imply that $\mu_{f,g}$ extends to the compactification of the cusp.

We now express the holomorphic bisectional curvature for the metric $\mu_{f,g}$, and show that the positivity conditions of Lemma 9 ensure that $\mu_{f,g}$ has negative holomorphic bisectional curvature on the cusp, and thus nonpositive holomorphic bisectional curvature on the compactified cusp. As N acts by isometries on $(0, A) \times N/\varphi(\Lambda)$, we will work at a point of the form $(t, \mathrm{Id} \bmod \varphi(\Lambda))$ with $t \in (0, A)$. Let Y, Ξ be two tangent vectors at this point. Since $\mathfrak{n} = \mathfrak{r} \oplus \mathbb{R}Z$ and $JZ = g(t)\frac{\partial}{\partial t}$, we can write $Y =: a\underline{X} + bZ + cJZ$ and $\Xi =: \alpha\widetilde{X} + \beta Z + \gamma JZ$ with $a, b, c, \alpha, \beta, \gamma \in \mathbb{R}$, and $X, \widetilde{X} \in \mathfrak{r}$ of norm $\mu(X, X) = \mu(\widetilde{X}, \widetilde{X}) = 1$. As in [HS96], we use the curvature convention given by Equation (2), and the holomorphic bisectional curvature of Y and Ξ is $R(Y, JY, \Xi, J\Xi) =: R(Y \wedge JY, \Xi \wedge J\Xi)$. We compute that:

$$\begin{aligned} Y \wedge JY = & B_1 + B_2 + B_3 \qquad \text{with:} \\ & B_1 = a^2 X \wedge JX + (b^2 + c^2) Z \wedge JZ, \\ & B_2 = -ac(X \wedge Z + JX \wedge JZ), \\ & B_3 = -ab(JX \wedge Z + (-X \wedge JZ)). \end{aligned}$$

In the same way, $\Xi \wedge J\Xi = \beta_1 + \beta_2 + \beta_3$ with β_i defined as B_i by replacing a, b, c by α, β, γ and X by $\widetilde{X}$. Following [HS96], we define:

$$(E_1, E_2, E_3, E_4, E_5, E_6) := \left(\frac{X \wedge JX}{\|X \wedge JX\|}, \frac{Z \wedge JZ}{\|Z \wedge JZ\|}, \frac{X \wedge Z}{\|X \wedge Z\|}, \frac{JX \wedge JZ}{\|JX \wedge JZ\|}, \frac{JX \wedge Z}{\|JX \wedge Z\|}, \frac{X \wedge JZ}{\|X \wedge JZ\|} \right).$$

When $g = ff'$, Hummel and Schroeder's computations show that:

$$(R(E_i, E_j))_{1\leq i,j\leq 6} =: \begin{pmatrix} G & 0 & 0 \\ 0 & F & 0 \\ 0 & 0 & F \end{pmatrix},$$

where:

$$G = \begin{pmatrix} -4\left(\frac{f'}{f}\right)^2 & -2\frac{f''}{f} \\ -2\frac{f''}{f} & -3\frac{f''}{f} - \frac{f'''}{f'} \end{pmatrix} =: \begin{pmatrix} -h_1^2 & -2k \\ -2k & -h_2^2 \end{pmatrix}$$
$$F = \begin{pmatrix} -\frac{f''}{f} & -\frac{f''}{f} \\ -\frac{f''}{f} & -\frac{f''}{f} \end{pmatrix} =: \begin{pmatrix} -h_3^2 & -k \\ -k & -h_4^2 \end{pmatrix}$$

The functions $h_i : (0, A) \to (0, +\infty)$ defined through G and F are well defined because of the positivity conditions for f and its derivatives. Here and in the sequel, the functions f, g, k, h_i shall implicitly be evaluated at t. Now we compute that:

$$\begin{aligned} R(B_1, \beta_1) &= a^2\alpha^2 R(X \wedge JX, \widetilde{X} \wedge J\widetilde{X}) - 2ka^2(\beta^2 + \gamma^2)\|X \wedge JX\|\|Z \wedge JZ\| - \\ &\qquad 2k\alpha^2(b^2 + c^2)\|\widetilde{X} \wedge J\widetilde{X}\|\|Z \wedge JZ\| - h_2^2(b^2 + c^2)(\beta^2 + \gamma^2)\|Z \wedge JZ\|^2 \\ &= a^2\alpha^2 R(X \wedge JX, \widetilde{X} \wedge J\widetilde{X}) - 2h_3^2 f^2 g^2 \left(a^2(\beta^2 + \gamma^2) + \alpha^2(b^2 + c^2)\right) - h_2^2 g^4 (b^2 + c^2)(\beta^2 + \gamma^2). \\ R(B_1, \beta_2) &= -2a^2\alpha\gamma R(X \wedge JX, \widetilde{X} \wedge Z). \\ R(B_1, \beta_3) &= -2a^2\alpha\beta R(X \wedge JX, J\widetilde{X} \wedge Z). \end{aligned}$$

According to the computations of the appendix of [HS96], $R(X, JX)Z$ is a multiple of $JZ = g\frac{\partial}{\partial t}$, which is orthogonal both to $\widetilde{X}$ and $J\widetilde{X}$. Therefore the two terms $R(B_1, \beta_2)$ and $R(B_1, \beta_3)$ vanish. By symmetry, we also have $R(B_2, \beta_1) = R(B_3, \beta_1) = 0$. Then, using that the metric is Kählerian, we find that:

$$\begin{aligned} R(B_2, \beta_2) &= ac\alpha\gamma R(X \wedge Z + JX \wedge JZ, \widetilde{X} \wedge Z + J\widetilde{X} \wedge JZ) \\ &= 4ac\alpha\gamma R(X \wedge Z, \widetilde{X} \wedge Z). \\ R(B_3, \beta_3) &= 4ab\alpha\beta R(JX \wedge Z, J\widetilde{X} \wedge Z). \\ R(B_2, \beta_3) &= 4ac\alpha\beta R(X \wedge Z, J\widetilde{X} \wedge Z). \\ R(B_3, \beta_2) &= 4ab\alpha\gamma R(JX \wedge Z, \widetilde{X} \wedge Z). \end{aligned}$$

Let us compute these four terms. The application $(U, \widetilde{U}) \mapsto R(U \wedge Z, \widetilde{U} \wedge Z)$ is bilinear symmetric, of quadratic form:

$$R(U \wedge Z, U \wedge Z) = -f^2 g^2 h_3^2 \mu(U, U).$$

By splitting, we deduce that for all $U, \widetilde{U}$, we have $R(U \wedge Z, \widetilde{U} \wedge Z) = -f^2 g^2 h_3^2 \mu(U, \widetilde{U})$. Hence:

$$\begin{aligned} R(JX \wedge Z, J\widetilde{X} \wedge Z) &= -f^2 g^2 h_3^2 \mu(JX, J\widetilde{X}) = -f^2 g^2 h_3^2 \mu(X, \widetilde{X}), \\ R(X \wedge Z, J\widetilde{X} \wedge Z) &= -f^2 g^2 h_3^2 \mu(X, J\widetilde{X}), \\ R(JX \wedge Z, \widetilde{X} \wedge Z) &= -f^2 g^2 h_3^2 \mu(JX, \widetilde{X}) = f^2 g^2 h_3^2 \mu(X, J\widetilde{X}). \end{aligned}$$

Therefore:

$$\begin{aligned}-R(Y\wedge JY,\Xi\wedge J\Xi) = &-a^2\alpha^2 R(X\wedge JX,\widetilde{X}\wedge J\widetilde{X})+\\ &h_2^2g^4(b^2+c^2)(\beta^2+\gamma^2)+\\ &2h_3^2f^2g^2\left(a^2(\beta^2+\gamma^2)+\alpha^2(b^2+c^2)\right)+\\ &f^2g^2h_3^2\mu(X,\widetilde{X})4a\alpha(b\beta+c\gamma)+\\ &f^2g^2h_3^2\mu(X,J\widetilde{X})4a\alpha(c\beta-b\gamma).\end{aligned}$$

It remains to compute $R(X\wedge JX,\widetilde{X}\wedge J\widetilde{X})$. Let us write $\widetilde{X}=dX+eJX+W$ with $d,e\in\mathbb{R}$ and W a vector μ-orthogonal to X and JX (in fact $d=\mu(X,\widetilde{X})$ and $e=\mu(JX,\widetilde{X})$). Then:

$$\begin{aligned}R(X\wedge JX,\widetilde{X}\wedge J\widetilde{X}) &= R(X,JX,dX+eJX+W,-eX+dJX+JW)\\ &= (d^2+e^2)R(X,JX,X,JX)+d(R(X,JX,X,JW)+R(X,JX,W,JX))+\\ &\quad e(R(X,JX,JX,JW)-R(X,JX,W,X))+R(X,JX,W,JW)\\ &= (d^2+e^2)R(X,JX,X,JX)+2R(X,JX,X,eW+dJW)+R(X,JX,W,JW)\\ &= -(d^2+e^2)f^4h_1^2-2(\frac{3g^2}{f^2}+f'^2)\langle JX,eW+dJW\rangle+R(X,JX,W,JW)\\ &= -(d^2+e^2)f^4h_1^2+0+R(X,JX,W,JW).\end{aligned}$$

Also:

$$\begin{aligned}R(X,JX,W,JW) &= \langle\nabla_{[X,JX]}W-\nabla_X\nabla_{JX}W+\nabla_{JX}\nabla_XW,JW\rangle\\ &= 2\langle\nabla_ZW,JW\rangle \qquad \text{because } \nabla_XW=\nabla_{JX}W=0 \text{ and } [X,JX]=2Z\\ &= \frac{-2g^2}{f^2}\|W\|^2 \qquad \text{from [HS96, (3.2)]}\end{aligned}$$

Putting together the previous calculations, we obtain:

$$\begin{aligned}-R(Y\wedge JY,\Xi\wedge J\Xi) = a^2\alpha^2\left((d^2+e^2)f^4h_1^2+\frac{2g^2}{f^2}\|W\|^2\right)+h_2^2g^4(b^2+c^2)(\beta^2+\gamma^2)+\\ 2f^2g^2h_3^2\left(a^2(\beta^2+\gamma^2)+\alpha^2(b^2+c^2)+2a\alpha(b\beta+c\gamma)\mu(X,\widetilde{X})+2a\alpha(c\beta-b\gamma)\mu(X,J\widetilde{X})\right).\end{aligned}$$

We now show that:

$$\left|2a\alpha\left((b\beta+c\gamma)\mu(X,\widetilde{X})+(c\beta-b\gamma)\mu(X,J\widetilde{X})\right)\right|\leq a^2(\beta^2+\gamma^2)+\alpha^2(b^2+c^2).$$

The two real numbers $x:=\mu(X,\widetilde{X})$ and $y:=\mu(X,J\widetilde{X})$ verify the inequality:

$$x^2+y^2=|x+iy|^2=|\mu_{\mathbb{C}}(X,\widetilde{X})|^2\leq\mu(X,X)\mu(\widetilde{X},\widetilde{X})\leq 1,$$

where $\mu_{\mathbb{C}}$ is the hermitian product associated to $\mu_{|\mathfrak{r}\times\mathfrak{r}}$, defined by:

$$\mu_{\mathbb{C}}(U,\widetilde{U}):=\mu(U,\widetilde{U})+i\mu(U,J\widetilde{U}).$$

Using the triangle inequality and the inequality $2XY\leq X^2+Y^2$, we get that:

$$\begin{aligned}|2a\alpha\left((b\beta+c\gamma)x+(c\beta-b\gamma)y\right)| &\leq 2|a\alpha||b\beta x+c\beta y|+2|a\alpha||c\gamma x-b\gamma y|\\ &\leq a^2\beta^2+\alpha^2(bx+cy)^2+a^2\gamma^2+\alpha^2(cx-by)^2\\ &\leq a^2(\beta^2+\gamma^2)+\alpha^2(b^2+c^2).\end{aligned}$$

This allows us to conclude that $R(Y \wedge JY, \Xi \wedge J\Xi)$ is nonpositive. Easy case by case examinations show that it vanishes only if $Y = 0$ or $\Xi = 0$. □

Remark. In fact the metric constructed in the proof has negative Ricci curvature. The Ricci curvature, being a sum of holomorphic bisectional curvatures, is negative on the open set $\mathbb{H}^n_{\mathbb{C}}/\Gamma \subset X_\Gamma$, and we will prove that it is negatively pinched near the boundary tori. More precisely, using the notations of the proof of Theorem 2, we claim that Ricci $\leq -2\mu_{f,g}$ on $(0,\epsilon) \times N/\phi(\Lambda)$ for some $\epsilon > 0$. Indeed, let $(X_1, \ldots, X_{n-1})$ be an orthogonal $\mathbb{C}$-basis of $(\mathfrak{r}, \mu_{|\mathfrak{r}\times\mathfrak{r}})$, so that, implictely evaluating f and $g = ff'$ at some real number $t \in (0, A]$, the family:

$$(\frac{1}{f}X_1, \frac{1}{f}JX_1, \ldots, \frac{1}{f}X_{n-1}, \frac{1}{f}JX_{n-1}, \frac{1}{g}Z, \frac{1}{g}JZ)$$

forms an orthonormal basis of the tangent space of $(0, A] \times N/\phi(\Lambda)$ at $(t, \mathrm{Id} \mod \varphi(\Lambda))$. Let:

$$\Xi = \sum_{i=1}^{n-1} (\alpha_i X_i + \alpha_{\bar{i}} JX_i) + \beta Z + \gamma JZ$$

be a tangent vector at this point, with $\alpha_i, \alpha_{\bar{i}}, \beta, \gamma \in \mathbb{R}$, and denote by α the real number:

$$\alpha := \sqrt{\sum_{i=1}^{n-1} (\alpha_i^2 + \alpha_{\bar{i}}^2)}.$$

Using the above formula for the bisectional curvature, a computation shows that the Ricci curvature of the metric has the following expression:

$$\mathrm{Ricci}(\Xi, \Xi) = -\left(2ff'' + 4f'^2 + 2(n-2)f'^2\right)\alpha^2 - \left((2n+1)ff'^2f'' + f^2f'f'''\right)(\beta^2 + \gamma^2).$$

The function f coincides with cosh on $(0, \epsilon)$, for some $\epsilon > 0$ and when $t < \epsilon$, we get:

$$\mathrm{Ricci}(\Xi, \Xi) \leq -2f^2\alpha^2 - 2g^2(\beta^2 + \gamma^2) = -2\|\Xi\|^2_{\mu_{f,g}}.$$

By continuity, this inequality remains valid on the boundary tori, hence the result.

5 Commensurators, parabolic subgroups and Albanese maps

We begin this section by stating a proposition which will be used during the proof of Theorem 4. If Γ is a discrete subgroup of $\mathrm{PU}(n,1)$, its commensurator $\mathrm{Comm}(\Gamma)$ is the set of elements $g \in \mathrm{PU}(n,1)$ such that $g\Gamma g^{-1} \cap \Gamma$ has finite index both in Γ and in $g\Gamma g^{-1}$; it is a subgroup of $\mathrm{PU}(n,1)$. Moreover a parabolic point of Γ is a point $\xi \in \partial\mathbb{H}^n_{\mathbb{C}}$ whose stabiliser $\mathrm{Stab}_\Gamma(\xi)$ in Γ contains a parabolic element. The following proposition is probably classical, we include it for the sake of being self-contained.

Proposition 10. *Let Γ_0 be a non-uniform arithmetic lattice of $\mathrm{PU}(n,1)$. For any pair $(\xi, x) \in \partial\mathbb{H}^n_{\mathbb{C}} \times \mathbb{H}^n_{\mathbb{C}}$, let $K(\xi, x)$ be the subgroup $\mathrm{Stab}_\Gamma(\xi) \cap \mathrm{Stab}_\Gamma(x)$ of $\mathrm{PU}(n,1)$. Then there exists a dense subset $S \subset \mathbb{H}^n_{\mathbb{C}}$ such that for any parabolic point ξ of Γ_0 and for any point $x \in S$, $\mathrm{Comm}(\Gamma_0) \cap K(\xi, x)$ is dense in $K(\xi, x)$.*

We defer the proof to the end of the section, and continue with Theorem 4. For every closed Kähler manifold X, one can construct its Albanese torus $\mathrm{Alb}(X)$ and a holomorphic map from X to $\mathrm{Alb}(X)$, called the Albanese map of X [Voi02, chapter 12]. We denote by $\mathrm{Alb}(X_\Gamma)$ the Albanese torus of X_Γ and $A_\Gamma : X_\Gamma \to \mathrm{Alb}(X_\Gamma)$ its Albanese map. With these notations, let us restate Theorem 4:

Theorem 4. *Let Γ_0 be a non-uniform torsion-free arithmetic lattice of $\mathrm{PU}(n,1)$. There exists a finite index subgroup $\Gamma' < \Gamma_0$ such that for any finite index subgroup $\Gamma < \Gamma'$, the toroidal compactification X_Γ of $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ satisfies the following properties:*

- *Its Albanese map A_Γ is an immersion on the open set $\mathbb{H}^n_{\mathbb{C}}/\Gamma \subset X_\Gamma$ and:*
- *For any boundary torus T of X_Γ, the map $A_{\Gamma\,|T}$ is either an immersion or a constant.*

We recall that by "boundary torus of X_Γ", we mean one of the tori added during the compactification, i.e. a connected component of the complement of $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ in X_Γ.

Proof. For the first point, we refer to [Eys18], which uses that there are non-trivial holomorphic one forms on X_Γ by results of Shimura [Shi79]. See also [LP22] for another exposition of this proof in the cocompact case. For the proof of the second point, we can and do assume that $\mathbb{H}^n_{\mathbb{C}}/\Gamma_0$ admits a toroidal compactification, and we introduce the following terminology: a boundary torus τ of X_{Γ_0} will be said to be virtually Albanese non-constant if there exists a finite index subgroup $\Gamma < \Gamma_0$ and a boundary torus T of X_Γ above τ, in restriction to which A_Γ is not constant. Let τ be a virtually Albanese non-constant boundary torus of X_{Γ_0}, and Γ and T as above. Holomorphic maps between tori have constant rank, thus for every point $x \in T$, there exists a holomorphic one form $\alpha \in \Omega^1(X_\Gamma)$ such that $i^*\alpha(x) \neq 0$, where $i : T \hookrightarrow X_\Gamma$ is the inclusion.

Let $\xi \in \partial\mathbb{H}^n_{\mathbb{C}}$ be a point corresponding to the cusp of $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ compactified by the torus T. From here on, we use the notations of Section 2. According to the results recalled there, $\mathrm{Stab}_\Gamma(\xi)$ is a lattice of the nilpotent group N_ξ. Thus it intersects non-trivially $[N_\xi, N_\xi] \simeq \mathbb{R}$. Let Z_ξ be a generator of $\Gamma \cap [N_\xi, N_\xi]$. We have the following diagram:

$$\begin{array}{ccccccc}
\mathbb{H}^n_{\mathbb{C}}/\langle Z_\xi\rangle & \xrightarrow{\mathcal{L}} & \Omega & \hookrightarrow & \Omega_0 & \xleftarrow{\widetilde{i}} & \mathbb{C}^{n-1} \\
\downarrow & & & & \downarrow & & \downarrow \\
\mathbb{H}^n_{\mathbb{C}}/\Gamma & & \hookrightarrow & & X_\Gamma & \xleftarrow{i} & T,
\end{array}$$

where $\Omega_0 := \Omega \cup (\{0\} \times \mathbb{C}^{n-1})$, hooked arrows are inclusions, $\widetilde{i}$ is defined by $\widetilde{i}(x) := (0,x)$, and the map $\Omega_0 \to X_\Gamma$ is a covering above $T \cup \mathbb{H}^n_{\mathbb{C}}/\Gamma$.

Recall that $\mathcal{L}$ depends on the choice of a geodesic γ such that $\gamma(t) \underset{t\to+\infty}{\longrightarrow} \xi$. Here we choose γ such that $o := \gamma(0)$ belongs to the set S of Proposition 10. Recall also that $K(\xi, o)$ is the intersection of the stabilisers of $\xi \in \partial\mathbb{H}^n_{\mathbb{C}}$ and $o \in \mathbb{H}^n_{\mathbb{C}}$ in $\mathrm{PU}(n,1)$, and that it is isomorphic to $\mathrm{U}(n-1)$. Now the action of $K(\xi, o)$ on $\mathbb{H}^n_{\mathbb{C}}$ commutes with $\langle Z_\xi\rangle$ so $K(\xi, o)$ also acts on Ω. Identifying $K(\xi, o)$ with $\mathrm{U}(n-1)$, this action can simply be written $A \cdot (\lambda, v) = (\lambda, Av)$ for all $(\lambda, v) \in \Omega \subset \mathbb{C}^* \times \mathbb{C}^{n-1}$. This follows from formula 1 and diagram page 4, and from the fact that $K(\xi, o)$ acts only on the "v-terms" of the coordinates $\widetilde{\mathcal{L}}$ and preserves its norm. In particular, the action of $K(\xi, o)$ extends by the same formula to Ω_0.

Let $x \in X_\Gamma$ be the point below $(0,0) \in \Omega_0$, and $\alpha \in \Omega^1(X_\Gamma)$ a holomorphic one form such that $i^*\alpha(x) \neq 0$. Denoting by β the pullback of α by the map $\Omega_0 \to X_\Gamma$, we have $\widetilde{i}^*\beta(0) \neq 0$. We deduce that there exists $g_1, \ldots, g_{n-1} \in K(\xi, o)$ such that

$$e := (\widetilde{i}^*(g_1^{-1})^*\beta(0), \ldots, \widetilde{i}^*(g_{n-1}^{-1})^*\beta(0))$$

is a basis of $T_0^{(1,0)*}\mathbb{C}^{n-1}$. All the g_i's can be chosen in $\mathrm{Comm}(\Gamma) \cap K(\xi, o)$, which is dense in $K(\xi, o)$ by Proposition 10.

Then let Γ' be some finite index subgroup of:

$$\bigcap_{i=1}^{n-1} g_i \Gamma g_i^{-1} \cap \Gamma,$$

and which is normal in Γ. We want to show that for every boundary torus T' of $X_{\Gamma'}$ above T and for every point $x' \in T'$ above x, the differential $dA_{\Gamma'|T'}(x')$ is injective. Since Γ' is normal in Γ, the automorphism group of $X_{\Gamma'}$ acts transitively on the set of tori above T, thus it is enough to show this property for the torus T' above T which compactify the cusp associated to the Γ'-orbit of ξ.

The group $\Gamma' \cap [N_\xi, N_\xi]$ is a non-trivial subgroup of $\Gamma \cap [N_\xi, N_\xi]$, it is thus generated by dZ_ξ for some $d \in \mathbb{N}^*$. We then have the following diagram:

$$\begin{array}{ccccccc}
\mathbb{H}^n_{\mathbb{C}}/\langle dZ_\xi \rangle & \xrightarrow{\mathcal{L}^{(d)}} & \Omega^{(d)} & \hookrightarrow & \Omega_0^{(d)} & \xleftarrow{\widetilde{i}^{(d)}} & \mathbb{C}^{n-1} \\
\downarrow & & & & \downarrow \Psi & & \downarrow \mathrm{Id} \\
\mathbb{H}^n_{\mathbb{C}}/\langle Z_\xi \rangle & \xrightarrow{\mathcal{L}} & \Omega & \hookrightarrow & \Omega_0 & \xleftarrow{\widetilde{i}} & \mathbb{C}^{n-1} \\
\downarrow & & & & \downarrow & & \downarrow \\
\mathbb{H}^n_{\mathbb{C}}/\Gamma & & \hookrightarrow & & X_\Gamma & \xleftarrow{i} & T,
\end{array} \tag{4}$$

where:

$$\begin{aligned}
&\Omega^{(d)} := \{(a, v) \in \mathbb{C} \times \mathbb{C}^{n-1} \mid 0 < |a| < e^{\frac{-\pi \|v\|^2}{d l}}\}, \\
&\Omega_0^{(d)} := \Omega^{(d)} \cup (\{0\} \times \mathbb{C}^{n-1}) \qquad \text{and} \\
&\Psi : (a, v) \mapsto (a^d, v).
\end{aligned}$$

Let $\delta := \Psi^* \beta$, and define $\delta_i := (g_i^{-1})^* \delta$ for $i \in \{1, \dots, n-1\}$. We claim that $\delta_1, \dots, \delta_{n-1}$ are the pullback of holomorphic forms $\epsilon_1, \dots, \epsilon_{n-1}$ on $X_{\Gamma'}$, and that denoting $T' \overset{i'}{\hookrightarrow} X_{\Gamma'}$ the inclusion of T' in $X_{\Gamma'}$, the family $e'' := (i'^* \epsilon_1(x'), \dots, i'^* \epsilon_{n-1}(x'))$ is a basis of $T^{(1,0)*}_{x'} T'$. This will imply that the differential $dA_{\Gamma'|T'}(x')$ is injective.

For all i, the form δ_i is the pullback of a holomorphic form ϵ_i because the element $g_i \in \mathrm{PU}(n,1)$ induces a biholomorphism $\mathbb{H}^n_{\mathbb{C}}/\Gamma \to \mathbb{H}^n_{\mathbb{C}}/g_i \Gamma g_i^{-1}$ which extends to a biholomorphism $\theta_i : X_\Gamma \to X_{g_i \Gamma g_i^{-1}}$. Noticing that g_i acts on $\Omega_0^{(d)}$ and that there is a ramified covering $\pi_i : X_{\Gamma'} \to X_{g_i \Gamma g_i^{-1}}$, the commutativity of the following diagram entails that δ_i is the pullback of the form $\epsilon_i := \pi_i^* (\theta_i)_* \alpha$:

$$\begin{array}{ccccc}
\Omega_0^{(d)} & \xrightarrow{g_i \cdot} & \Omega_0^{(d)} & \longrightarrow & X_{\Gamma'} \\
\downarrow \Psi & & & & \\
\Omega_0 & & & & \downarrow \pi_i \\
\downarrow & & & & \\
X_\Gamma & & \xrightarrow{\theta_i} & & X_{g_i \Gamma g_i^{-1}}.
\end{array}$$

The family e'' is a basis because, up to the cover $\mathbb{C}^{n-1} \to T'$ it can be identified with the family $e' := (\widetilde{i}^{(d)*} \delta_1(0), \dots, \widetilde{i}^{(d)*} \delta_{n-1}(0))$ which, up to the identity map $\mathbb{C}^{n-1} \to \mathbb{C}^{n-1}$ of the commutative diagram (4), is (identified with) e. The latter was constructed to be a basis of $T_0^{(1,0)*} \mathbb{C}^{n-1}$.

We deduce that $dA_{\Gamma'|T'}(x')$ is injective. Since $A_{\Gamma'|T'}$ is a holomorphic application between tori, it is a fibration, and in particular, its differential is injective at a point if and only if it is injective at any point. Hence $A_{\Gamma'|T'}$ is an immersion.

For each virtually Albanese non-constant boundary torus τ_i, there is a finite index subgroup $\Gamma_i' < \Gamma_0$ constructed as above. Let Γ' be the intersection of the Γ_i'. Let Γ'' be a finite index subgroup of Γ', and T a boundary torus of $X_{\Gamma''}$. If T covers a boundary torus of X_{Γ_0} which is not virtually Albanese non-constant, then by definition $A_{\Gamma''|T}$ is constant. Otherwise, there exists an intermediate ramified covering $X_{\Gamma''} \to X_{\Gamma_i'} \to X_{\Gamma_0}$ such that the Albanese map of $X_{\Gamma_i'}$ in restriction to a torus below T is an immersion. *A fortiori* the Albanese map of $X_{\Gamma''}$ in restriction to T is an immersion. □

Proof of Proposition 10. We use there the explicit description of non-uniform arithmetic lattices in $\mathrm{PU}(n,1)$, see for instance section 2 of [ES14] or [PY09]. We denote $\pi : \mathrm{U}(n,1) \to \mathrm{PU}(n,1)$ and $[\cdot] : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ the projection maps. Let Γ_0 be a non-uniform arithmetic lattice of $\mathrm{PU}(n,1)$. Then there exists:

- A purely imaginary quadratic extension $l := \mathbb{Q}(\sqrt{-d})$ of $\mathbb{Q}$ whose ring of integers is denoted $\mathcal{O}_l$,
- A Hermitian form $H : l^{n+1} \times l^{n+1} \to l$ on l^{n+1} whose extension $H_{\mathbb{C}}$ to $\mathbb{C}^{n+1}$ is of signature $(n,1)$,
- An isomorphism $\Phi : (\mathbb{C}^{n+1}, H_{\mathbb{C}}) \to (\mathbb{C}^{n+1}, \langle\cdot,\cdot\rangle)$ where $\langle\cdot,\cdot\rangle$ denotes the standard Hermitian form of signature $(n,1)$ used to define $\mathbb{H}^n_{\mathbb{C}}$; moreover Φ induces an isomorphism $\Theta : \mathrm{U}(H_C) \to \mathrm{U}(n,1)$ defined by $\Theta(g) := \Phi g \Phi^{-1}$, and allows to define an arithmetic lattice $\Gamma_l < \mathrm{PU}(n,1)$ by $\Gamma_l := \pi(\Theta(U(H_C, \mathcal{O}_l)))$; and
- A subgroup Γ of $\Gamma_0 \cap \Gamma_l$ which is of finite index both in Γ_0 and Γ_l.

If we replace Γ by a finite index subgroup, we can also suppose that all its parabolic subgroups are unipotent. We define:

$$\mathbb{H}^n_l := \{[\Phi(x)] \mid x \in l^{n+1} \text{ and } H(x,x) < 0\},$$
$$\partial\mathbb{H}^n_l := \{[\Phi(x)] \mid x \in l^{n+1} \text{ and } H(x,x) = 0\}.$$

Let $\xi := [\Phi(\widetilde{\xi})] \in \partial\mathbb{H}^n_l$ and $o := [\Phi(\widetilde{o})] \in \mathbb{H}^n_l$, with $(\widetilde{\xi}, \widetilde{o}) \in l^{n+1} \times l^{n+1}$. Let C be the intersection of the stabilisers of $\widetilde{\xi}$ and $\widetilde{o}$ in $U(H_{\mathbb{C}})$, and C_l the subgroup of C consisting of the matrices with coefficients in l. By construction, $K(\xi, o) = \pi(\Theta(C))$.

Two commensurable lattices have the same commensurator and the same parabolic points. Therefore, the following three statements lead to the proposition with $S := \mathbb{H}^n_l$:

Assertion 1: let ξ be a parabolic point of Γ. Then $\xi \in \partial\mathbb{H}^n_l$.

Assertion 2: let $\xi \in \partial\mathbb{H}^n_l$ and $o \in \mathbb{H}^n_l$. Then $\pi(\Theta(C_l)) \subset \mathrm{Comm}(\Gamma_l) \cap K(\xi, o)$, with C_l defined as above.

Assertion 3: let $\xi \in \partial\mathbb{H}^n_l$ and $o \in \mathbb{H}^n_l$. Then C_l is dense in C, so $\pi(\Theta(C_l))$ is dense in $K(\xi, o)$.

The proofs of these three statements are independent of each other. Assertion 2 is classical, and we will omit its proof.

Proof of Assertion 1:

Let γ be a non-trivial parabolic element of Γ which fixes $\xi = [x]$. We must show that $\mathbb{C}x$ intersects non trivially l^{n+1}. Since $\Gamma \subset \Gamma_l$, we can write $\gamma = \pi(\Theta(M))$ with $M \in \mathrm{U}(H_{\mathbb{C}}, \mathcal{O}_l)$, and $\Theta(M)x = \alpha x$ for some $\alpha \in S^1$. Since γ is unipotent, it can also be represented by an upper triangular matrix T with diagonal entries 1, thus $\Theta(M)$ is a multiple $\alpha' T$ of T. Evaluating at

x gives $\alpha = \alpha'$, and in particular $\det(M) = \det(\Theta(M)) = \alpha^{n+1}$. We deduce that $\alpha^{n+1} \in \mathcal{O}_l$. The only integers of modulus 1 of purely imaginary quadratic extensions are roots of units, so α is a root of unit. Up to replacing γ by an appropriate power γ^k, we can thus suppose that $\gamma = [\Theta(M)]$ with $\Theta(M)x = x$.

The map M can also be seen as a linear map $M_l : l^{n+1} \to l^{n+1}$. The kernel E_l^1 of $M_l - \mathrm{Id}$ and the kernel E^1 of $M - \mathrm{Id}$ are related by $E^1 = E_l^1 \otimes \mathbb{C}$. Let $e := (e_1, \ldots, e_k)$ be an orthogonal basis of E_l^1. It is not possible to have $H(e_i, e_i) > 0$ for all i because otherwise, $H_{\mathbb{C}}$ would be positive definite on the $\mathbb{C}$-span of e, which is E^1, and this space contains the isotropic vector x. Thus $H(e_i, e_i) \leq 0$ for some i. Now $\xi = [x]$ and $[\Phi(e_i)]$ are two fixed points in $\overline{\mathbb{H}^n_{\mathbb{C}}}$ of the parabolic element $\gamma = \pi(\Theta(M))$, hence $\xi = [\Phi(e_i)] \in \partial\mathbb{H}^n_l$.

Proof of Assertion 3:

Let $\widetilde{\xi} \in l^{n+1}$ and $\widetilde{o} \in l^{n+1}$ be respectively isotropic and negative vectors of H, P the l-span of these two vectors and $P^\perp$ its orthogonal with respect to H. Then, since H has signature $(n, 1)$, $H(\widetilde{\xi}, \widetilde{o}) \neq 0$ and this easily implies that $P \cap P^\perp = \{0\}$. Hence $l^{n+1} = P \oplus P^\perp$, and H restricted to $P^\perp$ is a positive definite Hermitian form. In particular, there is a basis $(u_1, \ldots, u_{n-1})$ of $P^\perp$ orthogonal for H, and the matrix $B := (H(u_i, u_j))_{i,j}$ is diagonal, with positive rational diagonal entries. Now define:

$$\mathrm{U} := \{M \in \mathrm{Mat}(n-1, \mathbb{C}) \mid {}^tMB\overline{M} = B\}.$$

Then C is isomorphic to U by a Lie group isomorphism sending C_l to:

$$\mathrm{U}_l := \{M \in \mathrm{Mat}(n-1, l) \mid {}^tMB\overline{M} = B\}.$$

We will show that U_l is dense in U, using a classical argument involving the Cayley transform [Wey46, section II.10]. Here are the notations that we are going to use:

$$\begin{array}{cc} \mathrm{Mat}^* := \{M \in \mathrm{Mat}(n-1, \mathbb{C}) \mid \det(M + \mathrm{Id}) \neq 0\} & \mathrm{Mat}_l^* := \mathrm{Mat}^* \cap \mathrm{Mat}(n-1, l) \\ \mathrm{U}^* := \mathrm{U} \cap \mathrm{Mat}^* & U_l^* := \mathrm{U}^* \cap \mathrm{Mat}(n-1, l) \\ \mathrm{A}^* := \{M \in \mathrm{Mat}(n-1, \mathbb{C}) \mid {}^tMB = -B\overline{M}\} \cap \mathrm{Mat}^* & \mathrm{A}_l^* := \mathrm{A}^* \cap \mathrm{Mat}(n-1, l) \end{array}$$

The Cayley transform is the involution:

$$\begin{array}{rccc} S: & \mathrm{Mat}^* & \longrightarrow & \mathrm{Mat}^* \\ & N & \longmapsto & 2(I+N)^{-1} - I. \end{array}$$

We will use the two last facts about S of the following series, whose verification can be done using the first ones:

$$\begin{array}{l} S(N) = (I-N)(I+N)^{-1} = (I+N)^{-1}(I-N), \\ S(\overline{N}) = \overline{S(N)}, \\ S(N^{-1}) = -S(N) \text{ if } N \text{ is invertible}, \\ N \in \mathrm{Mat}_l^* \iff S(N) \in \mathrm{Mat}_l^*, \\ N \in \mathrm{U}^* \iff S(N) \in \mathrm{A}^*. \end{array}$$

Since U^* is dense in U and $S : \mathrm{U}^* \to \mathrm{A}^*$ is a homeomorphism sending U_l^* to A_l^*, it is enough to show that A_l^* is dense in A^*.

Let $S := X + iY$ be an element of A^* with X and Y real. Write $X := (x_{ij})$ and $Y := (y_{ij})$. Taking real and imaginary parts, the equation ${}^tSB = -B\overline{S}$ is equivalent to:

$$\left\{ \begin{array}{l} {}^tXB + BX = 0 \\ {}^tYB - BY = 0 \end{array} \right. \iff \left\{ \begin{array}{l} \forall j \geq i, x_{ji} = \frac{-b_{ii}}{b_{jj}} x_{ij} \\ \forall j \geq i, y_{ji} = \frac{b_{ii}}{b_{jj}} y_{ij}. \end{array} \right.$$

Since all b_{ii} are rational, we can approximate X and Y by matrices X_k, Y_k with rational coefficients and satisfying the above system. Then the sequence $S_k := X_k + iY_k$ is in A_l^* for k great enough, and converges to S. This concludes the proof. □

6 A strictly plurisubharmonic exhaustion function on the universal cover of X_Γ

In this section, we fix a non-uniform torsion-free lattice Γ of $\mathrm{PU}(n,1)$ verifying the following properties: the space $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ admits a toroidal compactification X_Γ with π_1-injected boundary tori, the Albanese map $A_\Gamma : X_\Gamma \to \mathrm{Alb}(X_\Gamma)$ is an immersion on the open set $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ and for any boundary torus T of X_Γ, $A_{\Gamma\,|T}$ is either an immersion or a constant. We will show that for such a lattice Γ, the universal cover $\widetilde{X_\Gamma}$ of X_Γ is Stein. This statement implies Theorem 5: indeed, in Hummel and Schroeder's article, it is noticed that boundary tori are totally geodesic in X_Γ for the metric of non-positive Riemannian curvature constructed there, and they are thus π_1-injected. In combination with Theorem 4, we deduce that for any torsion-free arithmetic lattice Γ_0 of $\mathrm{PU}(n,1)$, there exists a finite index subgroup $\Gamma' < \Gamma_0$ all of whose finite index subgroups satisfy these properties. We will also use that toroidal compactifications are projective manifolds.

Here is the global strategy of our proof. Let us denote by $\pi : \widetilde{X_\Gamma} \to X_\Gamma$ the universal cover of X_Γ. We can construct holomorphic functions on $\widetilde{X_\Gamma}$ of the form $f := gh$ where h is a meromorphic function obtained by the theorem of holomorphic convexity with respect to a line bundle [Nap90, Corollary 4.3] and g is a holomorphic function which kills the poles of h, obtained by post-composing the lift of the Albanese map of X_Γ by a holomorphic function from $\mathbb{C}^{b_1(X_\Gamma)/2}$ to $\mathbb{C}$. This strategy gives enough holomorphic functions to construct a strictly plurisubharmonic exhaustion ψ on $\widetilde{X_\Gamma} \setminus \pi^{-1}(U)$ where U is an open neighborhood of the set Z consisting of all points $x \in X_\Gamma$ whose fiber $A_\Gamma^{-1}(A_\Gamma(x))$ is not finite (Proposition 13-1). We also construct by hand a strictly plurisubharmonic exhaustion function ϕ on the lift of an open neighborhood of Z (Lemmas 11,12) and finally we glue ϕ and ψ (Lemma 14) to obtain a strictly plurisubharmonic exhaustion function on $\widetilde{X_\Gamma}$ (Proposition 13-2). We use the fact that A_Γ restricted to any boundary torus T is either an immersion or a constant in order to get a nice description of Z (Lemma 12 and its proof).

We now state our intermediary results. Lemmas 11 and 12 are about toroidal compactifications, and Lemma 12 says that the Albanese map of X_Γ satisfies all the hypotheses of Proposition 13. This proposition is stated for projective manifolds which are the source of a holomorphic map with values in a compact manifold with Stein universal cover, and we use it with the Albanese map of X_Γ to deduce that $\widetilde{X_\Gamma}$ is Stein. Lemma 14 is used to prove the second point of Proposition 13.

Lemma 11. *Let T be a boundary torus of X_Γ. If V is a sufficiently small open neighbourhood of T, its preimage $\widetilde{V} := \pi^{-1}(V)$ is a disjoint union of neighbourhoods of (copies of) $\mathbb{C}^{n-1}$. We can choose V so that each of these open neighbourhood admits a strictly plurisubharmonic function which is of exhaustion relatively to $\widetilde{X_\Gamma}$.*

This lemma does not use the assumptions on the Albanese map. When we say that a function is an exhaustion relatively to $\widetilde{X_\Gamma}$, it means that its sublevels are relatively compact in $\widetilde{X_\Gamma}$. Furthermore, recall that Z is the set of points $x \in X_\Gamma$ whose fiber $A_\Gamma^{-1}(A_\Gamma(x))$ is not finite.

Lemma 12. *The set Z is a finite number of fibers of the map A_Γ, and it consists of some boundary tori together with a finite number of points. In particular, there exists a neighbourhood*

V of Z as well as a smooth strictly plurisubharmonic function ϕ on $\widetilde{V} := \pi^{-1}(V)$ which is an exhaustion relatively to $\widetilde{X_\Gamma}$.

Proposition 13. *Let X be a projective manifold, $\pi : \widetilde{X} \to X$ its universal cover and $A : X \to Y$ a holomorphic map from X to a compact manifold Y whose universal cover $\widetilde{Y}$ is Stein.*

1. *Define:*
$$Z := \{x \in X \mid A^{-1}(A(x)) \text{ is not finite}\},$$
let U be an open neighborhood of Z and set $\widetilde{U} := \pi^{-1}(U)$. Then Z is an analytic subvariety of X and there is a smooth plurisubharmonic function $\psi : \widetilde{X} \to \mathbb{R}_+$, whose vanishing locus is exactly $\widetilde{Z} := \pi^{-1}(Z)$, that is strictly plurisubharmonic on $\widetilde{X} \setminus \widetilde{Z}$, and whose restriction to $\widetilde{X} \setminus \widetilde{U}$ is an exhaustion.

2. *In addition, if there is an open neighborhood V of Z and a smooth strictly plurisubharmonic function ϕ on $\widetilde{V} := \pi^{-1}(V)$ which is an exhaustion relatively to $\widetilde{X}$, then $\widetilde{X}$ is Stein.*

Lemma 14. *Let X be a metric space, Z a closed subspace of X and U, V two open sets of X containing Z with $\overline{U} \subset V$. Suppose that there exists:*

- *A continuous function $\phi : V \to (0, +\infty)$ which is an exhaustion relatively to X, and*
- *A continuous function $\psi : X \to \mathbb{R}_+$ such that $\psi^{-1}(\{0\}) \subset Z$ and $\psi_{|X\setminus U}$ is an exhaustion.*

Then for any open set V' containing U whose closure is included in V, there exists an increasing smooth convex function $\chi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\chi(0) = 0$ and $\chi \circ \psi > \phi$ on $V' \setminus U$.

Proof of Lemma 11. By hypothesis, for any boundary torus T of X_Γ, the map $\pi_1(T) \to \pi_1(X_\Gamma)$ induced by the inclusion $T \hookrightarrow X_\Gamma$ is injective, therefore the preimage of T in the universal cover $\widetilde{X_\Gamma}$ of X_Γ is a disjoint union of copies of $\mathbb{C}^{n-1}$.

Let $p : \mathbb{C}^{n-1} \to T$ be the universal cover of a boundary torus T of X_Γ, V a neighbourhood of T identified with the unit disk bundle of a holomorphic line bundle on T endowed with a Hermitian metric h as in Proposition 6, and $F : V \to T$ the application defining this bundle. Then we have the following commutative diagram:

$$\begin{array}{ccc} \widetilde{V} & \xrightarrow{\pi} & V \\ \downarrow{\scriptstyle \widetilde{F}} & & \downarrow{\scriptstyle F} \\ \mathbb{C}^{n-1} & \xrightarrow{p} & T, \end{array}$$

where $\pi : \widetilde{V} \to V$ is the universal cover of V and $\widetilde{F}$ the lift of F. We identify $\widetilde{V}$ with a connected component of the preimage of V in $\widetilde{X_\Gamma}$ and $\mathbb{C}^{n-1}$ with the connected component of the preimage of T in $\widetilde{X_\Gamma}$ contained in $\widetilde{V}$. We will show that some neighborhood of the zero section in $\widetilde{V}$ has a strictly plurisubharmonic function ϕ which is an exhaution relatively to $\widetilde{X_\Gamma}$. This function ϕ will be the sum of the pullback by $\widetilde{F}$ of a strictly plurisubharmonic exhaustion function on $\mathbb{C}^{n-1}$, for instance $z \mapsto \|z\|^2$, and a bounded function on the fibers, defined to be the pullback by π of the squared hermitian norm on V, which will be strictly plurisubharmonic in a neighborhood of the zero section. Explicitely, for $\widetilde{v} \in \widetilde{V}$, let us put $\phi(\widetilde{v}) := \|\widetilde{F}(\widetilde{v})\|^2 + N(\pi(\widetilde{v}))$, where $N : V \to \mathbb{R}_+$ is the function $v \mapsto h(v, v)$. Then the function ϕ is an exhaustion relatively to $\widetilde{X_\Gamma}$.

We now assert that ϕ is strictly plurisubharmonic on $\mathbb{C}^{n-1}$. Indeed, working in coordinates, this amount to saying that the map:

$$(a, v) \in \mathbb{C} \times \mathbb{C}^{n-1} \longmapsto \|v\|^2 + e^{h(v)}|a|^2$$

is strictly plurisubharmonic at all points of the form $(0, v)$, which is easily checked. Since $i\partial\bar{\partial}\phi$ is invariant under the action of $\pi_1(T)$ on $\widetilde{V}$, which is co-compact on some closed neighbourhood of $\mathbb{C}^{n-1}$, we deduce that ϕ is strictly plurisubharmonic on a "uniform" neighbourhood of $\mathbb{C}^{n-1}$, i.e. a set of the form $\pi^{-1}(V')$ for some open set $V' \subset V$. □

Proof of Lemma 12. Let $T_1, \dots, T_k$ be the boundary tori of X_Γ on which the Albanese map A_Γ is constant, of values $a_1, \dots, a_k$. We claim that $Z = A_\Gamma^{-1}(\{a_1, \dots, a_k\})$, which implies that it is an analytic set. Indeed, if $x \in Z$, there exists a smooth immersed curve $z \mapsto \gamma(z)$ in $A_\Gamma^{-1}(A_\Gamma(x))$. For all z, $dA_\Gamma \cdot \gamma'(z) = 0$ so γ is included in a torus. The Albanese map is an immersion in restriction to all tori except $T_1, \dots, T_k$ so γ must be included in one of the T_i, and $A_\Gamma(x) = a_i$. This proves one inclusion, the other being immediate. In particular, any smooth curve included in Z is either constant or included in one of the tori $T_1, \dots, T_k$ on which the Albanese map A_Γ is constant. We deduce that Z is the union of the tori $T_1, \dots, T_k$ with a discrete subset of $X_\Gamma \setminus \bigcup_{i=1}^k T_i$.

We claim that this discrete subset is finite, in other words that it does not accumulate on one of the tori $T_1, \dots, T_k$. This follows from the fact that Z is an analytic set: in the neighbourhood of any point of T_i, the set Z has a finite number of irreducible components. If it had a component distinct from T_i, Z would contain non-isolated points in the interior of X_Γ, which is not possible. Hence Z coincides locally with T_i.

The last part of the lemma then follows from Lemma 11 and the fact that each point has a Stein neighborhood. □

Proof of Proposition 13-1. This proof is mainly taken from [Nap90], with some modifications to take into account that we work in arbitrary dimensions.

Step 1. The set Z is an analytic subvariety of X.

Realise X as a submanifold of some projective space $\mathbb{CP}^d$. For all $x \in X \setminus Z$, the fiber $A^{-1}(A(x))$ is finite so there exists a hyperplane H_x of $\mathbb{CP}^d$ which does not intersect this set. We claim that:

$$Z = \bigcap_{x \in X \setminus Z} A^{-1}(A(H_x))$$

Indeed, for all $x \in X \setminus Z$, we have by construction $x \notin A^{-1}(A(H_x))$ thus $\bigcap_{x \in X \setminus Z} A^{-1}(A(H_x)) \subset Z$. On the other hand, if $x_0 \in Z$ and $x \in X \setminus Z$, then the compact analytic set of positive dimension $A^{-1}(A(x_0))$ cannot be included in $\mathbb{CP}^d \setminus H_x \simeq \mathbb{C}^d$. Hence $x_0 \in A^{-1}(A(H_x))$ and the above equality of sets holds. In particular, Z is an analytic subvariety of X.

Step 2. For any sequence $(\widetilde{x_k})_k$ of $\widetilde{X} \setminus \widetilde{U}$ with no accumulation point, there is a holomorphic function $f : \widetilde{X} \to \mathbb{C}$ which is not bounded on $(\widetilde{x_k})_k$ and vanishes on $\widetilde{Z} := \pi^{-1}(Z)$.

Let $\rho : \widetilde{Y} \to Y$ be the universal cover of Y and $\widetilde{A} : \widetilde{X} \to \widetilde{Y}$ the lift of A, so that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{A}} & \widetilde{Y} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \rho} \\ X & \xrightarrow{A} & Y. \end{array}$$

If $(\widetilde{A}(\widetilde{x_k}))_k$ has no accumulation point, then as $\widetilde{Y}$ is a Stein manifold, there is a holomorphic function $u : \widetilde{Y} \to \mathbb{C}$ which is unbounded on $(\widetilde{A}(\widetilde{x_k}))_k$ and vanishes on the variety $\rho^{-1}(A(Z))$. The function $u \circ \widetilde{A}$ has the required properties.

Otherwise, the sequence $\widetilde{y_k} := \widetilde{A}(\widetilde{x_k})$ has an accumulation point, and up to extracting a subsequence, we can assume, on the one hand, that it converges to an element $\widetilde{y_\infty} \in \widetilde{Y}$, and on

the other hand that the sequence $x_k := \pi(\widetilde{x_k})$ converges to a limit $x_\infty \in X \setminus Z$. We claim that there exists an ample line bundle $L \to X$ such that:

- There exists a holomorphic section $t \in H^0(\widetilde{X}, \pi^* L)$ which is unbounded on $(\widetilde{x_k})_k$.
- There exists a holomorphic section $s \in H^0(X, L)$ whose vanishing locus $S \subset X$ satisfies:
  - $S \cap A^{-1}(A(x_\infty)) = \emptyset$, in other words $A(x_\infty) \notin A(S)$. In particular, $\pi^* s(\widetilde{x_k}) \longrightarrow s(x_\infty) \neq 0$.
  - The complement of S does not contain a compact variety of positive dimension of X.

Indeed, since X is a projective manifold, it can be realized as a submanifold of some projective space $\mathbb{CP}^d$. Let $H \subset \mathbb{CP}^d$ be a projective hyperplane disjoint from the finite set $A^{-1}(A(x_\infty))$. Then $\mathcal{O}(1)_{|X}$ is positive so by [Nap90, Corollary 4.3], for a large enough integer p, the bundle $L := \mathcal{O}(p)_{|X}$ is sufficiently positive so that there exists a holomorphic section $t \in H^0(\widetilde{X}, \pi^* L)$ unbounded on $(\widetilde{x_k})_k$. For the second point, let $\widetilde{s_0}$ be a global holomorphic section of $\mathcal{O}(1)$ whose vanishing locus is H. Then the vanishing locus of the section $\widetilde{s} := \widetilde{s_0}^{\otimes p}$ of $\mathcal{O}(p)$ is H and its complement, biholomorphic to $\mathbb{C}^d$, is a Stein manifold and therefore contains no compact variety of positive dimension. The restriction $s \in H^0(X, L)$ of $\widetilde{s}$ to X, whose vanishing locus is $S := X \cap H$ has the required properties. Set also $\widetilde{S} := \pi^{-1}(S)$.

The set $E := \rho^{-1}(A(S))$ is an analytic subset of $\widetilde{Y}$, which is equal to $\rho^{-1}(\rho \circ \widetilde{A}(\widetilde{S}))$. Since $A(x_\infty) \notin A(S)$, the point $\widetilde{y_\infty}$, whose image by ρ is $A(x_\infty)$, does not belong to E. We deduce that there exists a holomorphic function $u : \widetilde{Y} \to \mathbb{C}$ which vanishes on E but does not vanish on $\widetilde{y_\infty}$. Indeed the sheaf $\mathcal{F}$ of holomorphic functions vanishing on E is coherent, so by Cartan's Theorem B, there exist holomorphic functions vanishing on E which generate the germ space $\mathcal{F}_{\widetilde{y_\infty}} = \mathcal{O}_{\widetilde{y_\infty}}$. Then, E contains $\widetilde{A}(\widetilde{S})$, and, we affirm that it also contains $\widetilde{A}(\widetilde{Z})$. Indeed, let $\widetilde{x} \in \widetilde{Z}$ and $x := \pi(\widetilde{x})$. By definition of Z_0, the analytic subset $A^{-1}(A(x))$ is not finite, so has a non-empty intersection with S. In other words, $\rho \circ \widetilde{A}(\widetilde{x}) = A(x) \in A(S)$, so $\widetilde{A}(\widetilde{x}) \in E$. Thus, $u \circ \widetilde{A}$ vanishes on $\widetilde{S} \cup \widetilde{Z}$. As the divisor of $\pi^* s$ is $p\widetilde{S}$, we deduce that the meromorphic function:

$$f := (u \circ \widetilde{A})^{p+1} \times (t / \pi^* s)$$

is in fact holomorphic. Since t is not bounded on $(\widetilde{x_k})_k$, and the sequences $u(\widetilde{y_k})$ and $\pi^* s(\widetilde{x_k})$ converge respectively to non-zero limits $u(\widetilde{y_\infty})$ and $s(x_\infty)$, the function f is not bounded on $(\widetilde{x_k})_k$. Finally it is divisible as a holomorphic function by $u \circ \widetilde{A}$, thus it vanishes on $\widetilde{Z}$.

Step 3. For any point $\widetilde{x_0} \in \widetilde{X} \setminus \widetilde{Z}$, there exists a holomorphic function $F : \widetilde{X} \to \mathbb{C}^n$ which vanishes on $\widetilde{Z}$, does not vanish at $\widetilde{x_0}$ and whose differential at $\widetilde{x_0}$ is invertible ($n = \dim X$).

Let us write $x_0 := \pi(\widetilde{x_0})$. We assert that there exists an ample line bundle $L \to X$ and holomorphic sections $s, t_1, \ldots, t_n$ of L such that:

- The section s does not vanish on $A^{-1}(A(x_0))$, and in particular does not vanish at x_0 so that the meromorphic functions t_i/s are holomorphic on a neighbourhood of x_0.
- In a neighbourhood of x_0, the holomorphic functions $(\frac{t_1}{s}, \ldots, \frac{t_n}{s})$ do not vanish and define coordinates of X.
- The complement of the vanishing locus $S \subset X$ of s does not contain a compact variety of positive dimension of X.

Indeed, embedding X in $\mathbb{CP}^d$ we choose $L := \mathcal{O}(1)_{|X}$, and $s := \widetilde{s}_{|X}$ where $\widetilde{s}$ is a section of $\mathcal{O}(1)$ whose vanishing hyperplane H is disjoint from $A^{-1}(A(x_0))$. On the complement of H, the

section $\widetilde{s}$ trivializes $\mathcal{O}(1)$, therefore it makes sense to speak of the differential of a section $\widetilde{t}$ of $\mathcal{O}(1)$ at a point of the complement of H: it is the differential of the function $\widetilde{t}/\widetilde{s}$. There are homogeneous coordinates $[z_1 : \cdots : z_{d+1}]$ for which $\widetilde{s}$ identifies with the linear form z_{d+1}, the point x_0 has coordinates $[1 : 1 : \cdots : 1]$ and, in affine coordinates $(z_1, \ldots, z_d) \mapsto [z_1 : \cdots : z_d : 1]$, the tangent space of X at x_0 is spanned by $(\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n})$. The linear forms $z_1, \ldots, z_n$ identify with global sections $\widetilde{t_1}, \ldots, \widetilde{t_n}$ of $\mathcal{O}(1)$. Let $t_1, \ldots, t_n$ denote the restriction of these sections to X. By construction, the holomorphic functions $(\frac{t_1}{s}, \ldots, \frac{t_n}{s})$ do not vanish and define coordinates of X.

Let $S := X \cap H$ be the vanishing locus of s, and $\widetilde{S} := \pi^{-1}(S)$. By construction, the functions $\pi^*(t_i/s)$ are meromorphic on $\widetilde{X}$, and their divisors satisfy $\mathrm{div}(\pi^*(t_i/s)) \geq -\widetilde{S}$. As above, $E := \rho^{-1}(\rho \circ \widetilde{A}(\widetilde{S}))$ is an analytic subset of $\widetilde{Y}$ containing $\widetilde{A}(\widetilde{S}) \cup \widetilde{A}(\widetilde{Z})$ and not containing $\widetilde{A}(\widetilde{x_0})$. By Cartan's Theorem B, there exists a holomorphic function $u : \widetilde{Y} \to \mathbb{C}$ which vanishes on $E := \rho^{-1}(\rho \circ \widetilde{A}(\widetilde{S}))$, does not vanish at $\widetilde{A}(\widetilde{x_0})$ and whose differential at $\widetilde{A}(\widetilde{x_0})$ is zero. In particular, $u \circ \widetilde{A}$ vanishes on $\widetilde{S} \cup \widetilde{Z}$ but not at $\widetilde{x_0}$. Let us then set:

$$f_i := (u \circ \widetilde{A})^2 \times \pi^* \frac{t_i}{s}.$$

Then $F := (f_1, \ldots, f_n)$ is holomorphic. We assert that F vanishes on $\widetilde{Z}$, does not vanish at $\widetilde{x_0}$ and that its differential is invertible at $\widetilde{x_0}$. Indeed, for all i, f_i is divisible by $u \circ \widetilde{A}$, thus it vanishes on $\widetilde{Z_0}$. However f_i does not vanish at $\widetilde{x_0}$ by construction. Finally, the differential of $u \circ \widetilde{A}$ is zero at $\widetilde{x_0}$ so:

$$(df_i)_{\widetilde{x_0}} = u(\widetilde{A}(\widetilde{x_0}))^2 \pi^* d \left(\frac{t_i}{s} \right)_{x_0}.$$

Therefore, the differential of F is invertible at $\widetilde{x_0}$.

Step 3. Construction of ψ from these two statements: we refer to [Nap90, pp. 470-472]. □

To prove the second point of Proposition 13, we need Lemma 14, that we prove here.

Proof of Lemma 14. First, we claim that $m := \inf_{V' \setminus U}(\psi)$ is positive. Indeed, seeking a contradiction, suppose that $m = 0$. Then there is a sequence (v_n) of elements of $V' \setminus U$ such that $\psi(v_n) \to 0$. By assumption, $\psi_{|X \setminus U}$ is an exhaustion, thus up to taking some subsequence, we can assume that (v_n) converges toward a limit $v_\infty \in X \setminus U$ such that $\psi(v_\infty) = 0$. This contradicts the hypothesis $\psi^{-1}(\{0\}) \subset Z$.

If ϕ is bounded on $V' \setminus U$ by some constant C, then $\chi(x) := \frac{C+1}{m} x$ is suitable. Assume now that ϕ is unbounded on $V' \setminus U$, and for all $j \in \mathbb{N}^*$, define the set $A_p := \{x \in V' \setminus U \mid p-1 \leq \phi(x) < p\}$. There is an increasing sequence of integers p_k such that $A_{p_k} \neq \emptyset$ for all k. For all k, the set A_{p_k} is relatively compact in X, so $\psi_{|\overline{A_{p_k}}}$ attains its minimum a_k at a point $x_k \in \overline{V'} \setminus U$. We claim that $a_k \underset{k \to +\infty}{\longrightarrow} +\infty$. Indeed, seeking a contradiction, suppose that there is a subsequence $(a_{k_j})_{j \in \mathbb{N}^*}$ of (a_k), bounded by some constant M. Then the sublevel $\{\psi_{|X \setminus U} \leq M\}$ is compact and contains x_{k_j} for all j so up to a subsequence, we can assume that x_{k_j} converges to a limit $\ell \in \overline{V'}$. We obtain a contradiction by noticing that $\phi(x_{k_j}) \to \phi(\ell)$ and $\phi(x_{k_j}) \geq p_{k_j} - 1 \to +\infty$.

Thus for all integer $n \in \mathbb{N}^*$ there exists $k_n \in \mathbb{N}^*$ such that $\forall k \geq k_n, \frac{a_k}{m} \geq n$. We can choose $k_1 = 1$ and arrange so that the sequence $(k_n)_{n \in \mathbb{N}^*}$ is increasing. Let χ be a smooth increasing convex function such that $\chi(0) = 0$ and $\chi(nm) \geq p_{k_{n+1}}$ $\forall n \in \mathbb{N}^*$. We claim that $\chi \circ \psi > \phi$ on $V' \setminus U$. Indeed, for all $k \in \mathbb{N}^*$, we can find some $n \in \mathbb{N}^*$ such that $k_n \leq k < k_{n+1}$. Then because $k \geq k_n$, we find $a_k \geq nm$ so $\forall x \in A_{p_k}, \psi(x) \geq a_k \geq nm$. We deduce that:

$$\forall x \in A_{p_k}, \chi \circ \psi(x) \geq \chi(nm) \geq p_{k_{n+1}} > \phi(x). \qquad \square$$

Proof of Proposition 13-2. By assumption, there is an open neighborhood V of Z and a smooth strictly plurisubharmonic function ϕ on $\widetilde{V} := \pi^{-1}(V)$ which is an exhaustion relatively to $\widetilde{X}$. Let U be an open neighbourhood of Z whose closure is included in V, and $\psi : \widetilde{X} \to \mathbb{R}$ the function given by the first point of the Proposition. Write $\widetilde{Z} := \pi^{-1}(Z)$.

After replacing $\widetilde{V}$ by a smaller open set $\widetilde{V'}$, let χ be an increasing smooth convex function, given by Lemma 14, such that $\chi(0) = 0$ and $\chi \circ \psi > \phi + 2$ on $\widetilde{V'} \setminus \widetilde{U}$. Set $\phi_2 := \phi + 1$ and $\psi_2 := \chi \circ \psi$. Setting $\eta = \frac{1}{2}$, let $M_{(\eta,\eta)}$ be the regularized maximum function defined in [Dem, Lemma I.5.18]. Recall that this is a smooth symmetric function on $\mathbb{R}^2$ with the following properties: $\max(x, y) \leq M_{(\eta,\eta)}(x, y)$, and $M_{(\eta,\eta)}(x, y) = y$ whenever $y \geq x + 2\eta$. Also, $M_{(\eta,\eta)}(u_1, u_2)$ is (strictly) plurisubharmonic whenever u_1, u_2 are (strictly) plurisubharmonic functions. Now let $\lambda : \widetilde{X} \to \mathbb{R}$ be the function defined by:

$$\lambda(x) := \begin{cases} M_{(\eta,\eta)}(\phi_2(x), \psi_2(x)) & \text{if } x \in \widetilde{V'}, \\ \psi_2(x) & \text{if } x \in \widetilde{X} \setminus \widetilde{V'}. \end{cases}$$

Notice that $\psi_2 \geq \phi_2 + 2\eta$ on $\widetilde{V'} \setminus \widetilde{U}$ and $\phi_2 \geq \psi_2 + 2\eta$ on some neighbourhood of $\widetilde{Z}$. Hence λ is a smooth strictly plurisubharmonic function. For any real number C, we have:

$$\{x \in \widetilde{X} \mid \lambda(x) \leq C\} \subset \{x \in \widetilde{X} \setminus \widetilde{U} \mid \psi_2(x) \leq C\} \cup \{x \in \widetilde{V} \mid \phi_2(x) \leq C\}.$$

The right-hand side being compact, we infer that λ is an exhaustion, which implies that $\widetilde{X}$ is a Stein manifold. □

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