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# Reconstructing discrete measures from projections. Consequences on the empirical Sliced Wasserstein Distance.

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## Abstract

This paper deals with the reconstruction of a discrete measure  $\gamma_Z$  on  $\mathbb{R}^d$  from the knowledge of its pushforward measures  $P_i\#\gamma_Z$  by linear applications  $P_i : \mathbb{R}^d \rightarrow \mathbb{R}^{d_i}$  (for instance projections onto subspaces). The measure  $\gamma_Z$  being fixed, assuming that the rows of the matrices  $P_i$  are independent realizations of laws which do not give mass to hyperplanes, we show that if  $\sum_i d_i > d$ , this reconstruction problem has almost certainly a unique solution. This holds for any number of points in  $\gamma_Z$ . A direct consequence of this result is an almost-sure separability property on the empirical Sliced Wasserstein distance.

## 1 Introduction

In this note, we are interested in the following question: for a given discrete probability measure  $\gamma_Z$  on  $\mathbb{R}^d$ , and  $r$  linear transformations  $P_i : \mathbb{R}^d \rightarrow \mathbb{R}^{d_i}$ , can we characterize the set of probability measures on  $\mathbb{R}^d$  with exactly the same images as  $\gamma_Z$  through all of the maps  $P_i$ ? Formally, this set writes

$$\mathcal{S} = \left\{ \gamma \in \mathcal{P}(\mathbb{R}^d) \mid \forall i \in \llbracket 1, r \rrbracket, P_i\#\gamma = P_i\#\gamma_Z \right\}, \quad (\text{RP})$$

where  $P_i\#\gamma$  denotes the push-forward of  $\gamma$  by  $P_i$ , *i.e.* the measure on  $\mathbb{R}^{d_i}$  such that for any Borelian  $A \subset \mathbb{R}^{d_i}$ ,  $(P_i\#\gamma)(A) = \gamma(P_i^{-1}(A))$ , and  $\mathcal{P}(\mathbb{R}^d)$  denotes the space of probability measures on  $\mathbb{R}^d$ . The set  $\mathcal{S}$  is nonempty since it contains at least  $\gamma_Z$ . A natural underlying question is to know when we get uniqueness, *i.e.* when  $\mathcal{S} = \{\gamma_Z\}$ . Indeed, in this case  $\gamma_Z$  can be exactly reconstructed from the knowledge of all the  $P_i\#\gamma_Z$ , which is why we refer to this problem as a reconstruction problem.

This reconstruction problem appears in many applied fields where a multidimensional measure is known only through a finite set of images or projections. This is the case for instance in medical or geophysical imaging problems such as tomography [8]. It is also strongly related to the separability properties of the empirical version of the Sliced Wasserstein distance [17, 1], which is frequently used in machine learning applications [12, 6, 22].

In our reconstruction problem, it is clear that if one of the  $P_i$  is injective (which implies  $d \leq d_i$ ), then  $\mathcal{S} = \{\gamma_Z\}$ , which is why we focus here on the cases where none of the  $P_i$  is injective. We will also assume in this note that the  $P_i$  are surjective and that all the  $d_i$  are strictly smaller than  $d$ , since we can always replace  $\mathbb{R}^{d_i}$  by the smaller subspace  $\text{Im}(P_i)$ . To the best of our knowledge, this problem has not been widely discussed in the literature, perhaps because of its apparent simplicity. A close and more discussed question is the one of the existence of probability measures  $\gamma$  with marginal constraints [13, 5, 14]. Existence results for such couplings are known for some families of measures [11], or measures exhibiting some specific correlation structures [4]. However, in the general case, even if marginal constraints are compatible with each other, the existence of solutions is not always ensured [9].

Our study case is different, since the constraints are all obtained as push-forwards of an unknown  $\gamma_Z$ , and the central question is not existence but uniqueness of solutions. It is well known that a

measure is uniquely determined by its projections on *all* lines of  $\mathbb{R}^d$  (Cramer-Wold theorem [3]), and more generally by its projections on a set of subspaces as soon as they cover the whole space together [18]. The problem for a non-discrete target measure has been studied by Chafai [2] which considers a similar reconstruction problem from the viewpoint of characteristic functions, using the equivalence  $\forall i \in [1, r], P_i \# \mu = P_i \# \nu = 0 \iff \forall i \in [1, r], \forall x \in \mathbb{R}^d, \phi_\mu(P_i x) = \phi_\nu(P_i x)$ , where  $\phi_\mu$  and  $\phi_\nu$  denote the respective characteristic functions of  $\mu$  and  $\nu$ . In general, the knowledge of the characteristic function on a finite union of strict subspaces is insufficient to determine a measure [2]. For a finite number of directions and in the case of a discrete measure  $\gamma_Z$ , simple linear algebra shows that if the number  $r$  of projections is large enough, we get  $\mathcal{S} = \{\gamma_Z\}$ . When the  $P_i$  are projections on different hyperplanes for instance, Heppes showed in 1956 [10] that a discrete distribution of at most  $n$  points  $\gamma_Z = \frac{1}{n} \sum_{\ell=1}^n \delta_{z_\ell}$  is uniquely characterized by its projections  $P_i \# \gamma_Z$  if the number  $r$  of these projections is larger than  $n + 1$ , and that simple counter-examples could be exhibited with only  $r = n$  hyperplanes. More recent works [7] show that uniqueness can be ensured with less projections as soon as the set of points is known to belong to a specific quadratic manifold. These results are deterministic, they hold for every set of points and hyperplanes with the appropriate cardinality. In this paper, we add some stochasticity to the problem, and assume that the lines of the matrices  $P_i$ <sup>1</sup> are i.i.d. following a law  $\mathbb{P}$  which does not give mass to hyperplanes. Under this assumption, we show that if  $\sum_{i=1}^r d_i > d$ , then  $\mathbb{P}$ -almost surely  $\mathcal{S} = \{\gamma_Z\}$ . This result is very different from the ones already present in the literature: it holds only a.s., but this permits a considerably weaker condition on the  $P_i$ , and the condition for the reconstruction surprisingly does not depend on the number of points.

## 2 Solutions of the Reconstruction Problem

In this section, we characterize the set  $\mathcal{S}$  of solutions defined in (RP) depending on the set of linear maps  $P_i$ . We write  $\gamma_Z = \sum_{\ell=1}^n b_\ell \delta_{z_\ell}$  with  $Z = (z_1, \dots, z_n) \in (\mathbb{R}^d)^n$ , and assume that all points are distinct ( $k \neq j \implies z_k \neq z_j$ ). We also always assume that  $n > 1$ . The weights  $(b_\ell) \in (\mathbb{R}_+^*)^n$  sum to one and are each nonzero.

As we shall see, given a discrete measure  $\gamma_Z$  with  $n$  points in dimension  $d$ , the Reconstruction Problem (RP) has a unique solution  $\mathcal{S} = \{\gamma_Z\}$  almost-surely when drawing the linear maps  $P_i$  randomly, and when the dimensions strictly exceed  $d$ , *i.e.* when  $D := d_1 + \dots + d_r > d$ .

### 2.1 Computing Linear Push-Forwards of Discrete Measures

Characterizing  $\mathcal{S}$  requires the following technical Lemma, which provides a geometrical viewpoint of the push-forward operation.

**Lemma 1** (Linear push-forward formula). *Let  $P \in \mathcal{M}_{h,d}(\mathbb{R})$  of rank  $h \leq d$  and  $B \subset \mathbb{R}^h$ . Then  $P^{-1}(B) = P^T (PP^T)^{-1} B + \text{Ker} P$ .*

Figure 1 shows a visualization of the set  $P^T (PP^T)^{-1} B + \text{Ker} P$ , first where  $B$  is comprised of two points of  $\mathbb{R}^2$  and  $\text{Ker} P$  is a horizontal plane in 3D, and second with  $B$  a measurable set of  $\mathbb{R}^2$ . This illustrates the ill-posedness of the problem when the dimension of the projections and number of projections is too small. In this case with  $r = 1$ ,  $d = 3$  and  $d_1 = 2$ , the condition  $P^{-1}(A) = P^{-1}(B)$  leaves a degree of freedom, which we can visualize as the vertical axis here.

<sup>1</sup>With a slight abuse of notation, we use the same notation here for the linear maps and their associated matrices.

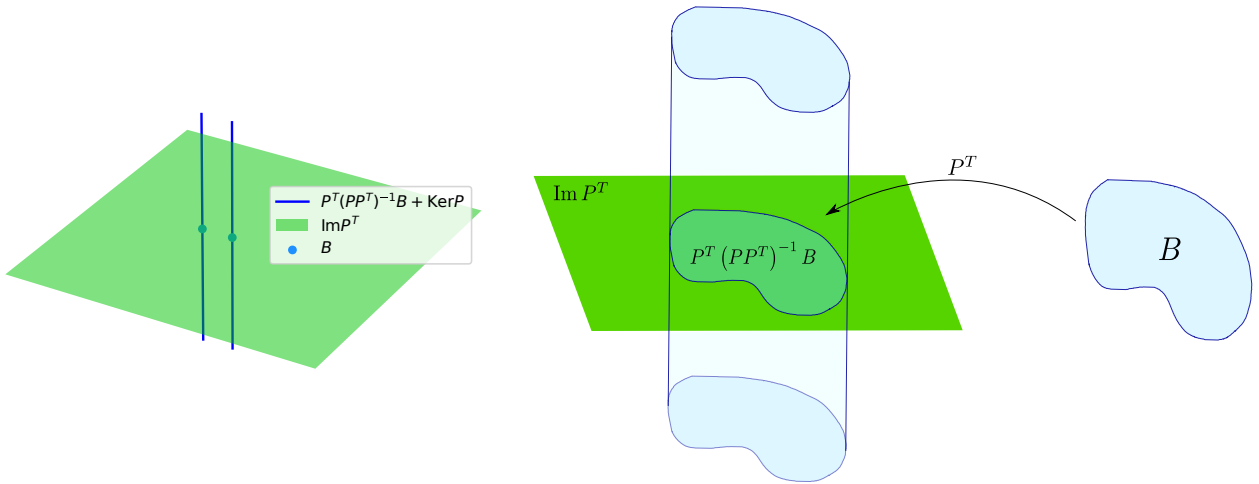


Figure 1: Illustrations of the linear push-forward formula  $P^{-1}(B)$  for a 3D to 2D projection  $P$ , (left) when  $B$  is a set of two points and (right) for a more general set  $B$ .

*Proof.* If  $a \in P^T(PP^T)^{-1}B + \text{Ker}P$ , then by writing  $a = P^T(PP^T)^{-1}b + x$  with  $b \in B$  and  $x \in \text{Ker}P$ , we have  $Pa = b \in B$ , thus  $a \in P^{-1}(B)$ .

For the opposite inclusion, consider  $a \in P^{-1}(B)$ . Since  $P$  is of full rank  $h$ , we have the decomposition  $\mathbb{R}^d = \text{Im}P^T \oplus \text{Ker}P$ , with  $Q := P^T(PP^T)^{-1}P$  the orthogonal projection on  $\text{Im}P^T$ .

Thus we can write  $a = Qa + (I - Q)a = P^T(PP^T)^{-1}Pa + (I - Q)a$ . Since  $Pa \in B$ , we conclude that  $a \in P^T(PP^T)^{-1}B + \text{Ker}P$ .  $\square$

## 2.2 Restraining the support of solutions of RP

The following theorem states that the support of any solution of (RP) is constrained to a set  $S$  obtained as the intersection of all sets  $Z + \text{Ker}P_i$ . Without loss of generality, we will assume that each  $P_i$  is of full rank  $d_i$ .

**Theorem 1** (Support of solutions of (RP)).

If  $\gamma$  is a solution of (RP), then  $\gamma(S) = 1$  with

$$S := \bigcap_{i=1}^r (Z + \text{Ker}P_i) = \bigcup_{(\ell_1, \dots, \ell_r) \in \llbracket 1, n \rrbracket^r} \bigcap_{i=1}^r (z_{\ell_i} + \text{Ker}P_i). \quad (1)$$

*Proof.* Using the same notations as in the proof of Lemma 1, we write  $Q_i := P_i^T(P_i P_i^T)^{-1}P_i$  the orthogonal projection on  $\text{Im}P_i^T$  and we recall the decomposition  $\mathbb{R}^d = \text{Im}P_i^T \oplus \text{Ker}P_i$ . Thus, for any borelian  $A$  of  $\mathbb{R}^d$ ,  $A \subset Q_i A + \text{Ker}P_i$ . Then  $\gamma(A) \leq \gamma(Q_i A + \text{Ker}P_i) = P_i \# \gamma(P_i A)$ , where the last equality is a direct consequence of Lemma 1.

Now, assume that  $\gamma \in \mathcal{S}$  and define  $S := \bigcap_{i=1}^r K_i$  with  $K_i = Z + \text{Ker}P_i$ . For each  $i$ , applying the previous inequality to  $K_i^c$  yields  $\gamma(K_i^c) \leq P_i \# \gamma(P_i K_i^c)$ . Since  $\gamma$  is a solution, we have  $P_i \# \gamma = P_i \# \gamma_Z$ . By construction,  $K_i = \{x, P_i x \in P_i Z\}$  thus  $K_i^c = \{x, P_i x \notin P_i Z\}$ . Since  $P_i \# \gamma$  is supported by  $P_i Z$ , it follows that  $P_i \# \gamma(P_i K_i^c) = 0$ . Finally,  $\gamma(S^c) = \gamma(\bigcup_{i=1}^r K_i^c) \leq \sum_{i=1}^r \gamma(K_i^c) = 0$  and thus  $\gamma(S) = 1$ .  $\square$

Figure 2 illustrates the previous result, with  $r = 2$  projections onto lines in  $\mathbb{R}^2$ , with  $n = 3$  points  $Z = (z_1, z_2, z_3)$ . The support of any solution is confined to the intersections between any two lines of the form  $z_\ell + \text{Ker}P_i$ . Here this corresponds to the intersecting points between an orange and a red line, allowing for 9 possible points, including the original 3. In this case any weighting of the 9 Dirac

masses that respect the marginal constraints will give a solution: there exists an infinity of possible solutions.

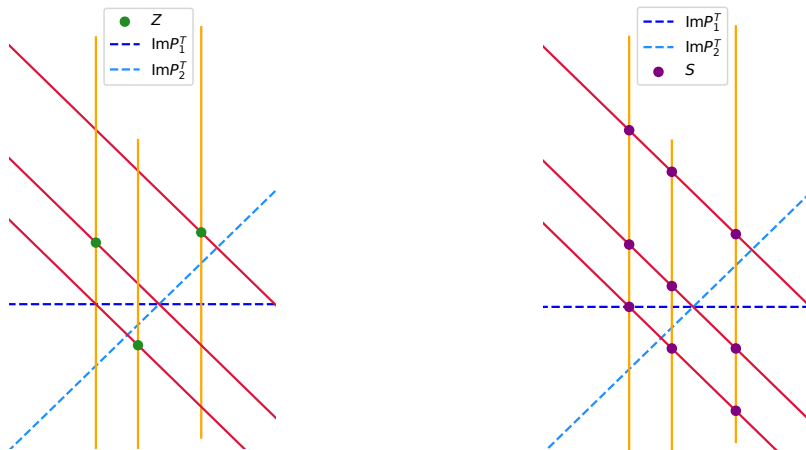


Figure 2: Illustration of the possible points for the support of a solution. On the left,  $Z$  is the original measure points, and on the right,  $S$  is the set of possible points for the support of a solution.

### 2.3 Conditions for unicity of solutions of RP

Leveraging the previous support restriction and elementary random affine geometry, we can further restrict the condition on the set of solutions  $\mathcal{S}$ . Theorem 2 below shows that if the random linear maps  $P_i$  cover the original space  $\mathbb{R}^d$  with redundancy (i.e. the sum of their target space dimensions strictly exceeds  $d$ ), then almost surely, the reconstruction problem has a unique solution  $\gamma_Z$ . We formalize this random setting by the following assumption.

**Assumption** ( $\mathcal{A}_{\mathbb{P}}$ ).

$$\forall i \in \llbracket 1, r \rrbracket, P_i = \begin{pmatrix} - & (u_i^{(1)})^T & - \\ - & \vdots & - \\ - & (u_i^{(d_i)})^T & - \end{pmatrix} \quad \text{where } u_i^{(j)} \sim \mathbb{P} \text{ i.i.d.},$$

where  $\mathbb{P}$  is a probability distribution over  $\mathbb{R}^d$  s.t. for any hyperplane  $H \subset \mathbb{R}^d$ ,  $\mathbb{P}(H) = 0$ .

The condition on the probabilities is verified in particular if  $\mathbb{P}$  is absolutely continuous w.r.t. the Lebesgue measure of  $\mathbb{R}^d$ , or w.r.t.  $\sigma$ , the uniform measure over  $S^d$  (the unit sphere of  $\mathbb{R}^d$ ). These two examples are the most common for practical reconstruction problems, which is why we formulate  $\mathcal{A}_{\mathbb{P}}$  in this manner.

The next theorems use assumption  $\mathcal{A}_{\mathbb{P}}$  but still hold true under milder hypotheses, where the lines  $(u_i^{(j)})^T$  of the matrices  $P_i$  are assumed independent with (possibly different) probability laws giving no mass to hyperplanes.

**Theorem 2** (Almost-sure unicity in (RP)). *Let  $\gamma_Z$  be a fixed discrete probability measure. Assume that the matrices  $P_i$  follow assumption  $\mathcal{A}_{\mathbb{P}}$ , and that  $D := \sum_{i=1}^r d_i > d$ . Then  $\mathbb{P}$ -almost surely,  $S = Z$  and  $\mathcal{S} = \{\gamma_Z\}$ .*

The idea behind the proof of Theorem 2 is that  $S$  is the union of sets of the form  $\bigcap_{i=1}^r (z_{\ell_i} + \text{Ker } P_i)$ , which can be rewritten as intersections of more than  $d$  affine subspaces in dimension  $d$ , thus are  $\mathbb{P}$ -almost surely either singletons or empty.

*Proof.* — *Step 1:*  $S = Z$

Let  $\ell := (\ell_1, \dots, \ell_r) \in \llbracket 1, n \rrbracket^r$  and  $S_\ell := \bigcap_{i=1}^r (z_{\ell_i} + \text{Ker}P_i)$ . We want to show  $S_\ell \subset Z$ .

First, observe that  $x \in S_\ell \iff \forall i \in \llbracket 1, r \rrbracket, \forall j \in \llbracket 1, d_i \rrbracket, (u_i^{(j)})^T x = (u_i^{(j)})^T z_{\ell_i}$ .

We write  $D = \sum_{i=1}^r d_i$ . For the sake of simplicity, we rewrite the  $k^{\text{th}}$  vector  $u_i^{(j)}$  as  $v_k$ , where  $k \in \llbracket 1, D \rrbracket$ , and in the same way we write  $(w_k)_{k=1 \dots D}$  the vectors  $(z_{\ell_1}, \dots, z_{\ell_1}, z_{\ell_2}, \dots, z_{\ell_2}, \dots, z_{\ell_r}, \dots, z_{\ell_r})$  with each  $z_{\ell_i}$  repeated  $d_i$  times. With these notations, we get

$$x \in S_\ell \iff v_k^T x = v_k^T w_k, \quad \forall k \in \llbracket 1, D \rrbracket. \quad (2)$$

Let us call (LS) the linear system on the right of (2). (LS) has  $D$  equations and  $d$  unknowns, with  $D > d$ , it is therefore overdetermined. When all  $w_k$  are equal, *i.e.* when  $\ell := (\ell, \dots, \ell)$ , clearly  $x = z_\ell$  is a solution, which shows that  $z_\ell \in S$  and thus  $Z \subset S$ .

If  $\mathcal{A}_\mathbb{P}$  is satisfied, the matrix  $U^{(d)} = (v_1, \dots, v_d)^T$  is almost surely of full rank and the linear system  $v_k^T x = v_k^T w_k$  for  $k \in \llbracket 1, d \rrbracket$  almost surely has a unique solution  $x^*$ . The  $(d+1)^{\text{th}}$  equality of (LS) is  $v_{d+1}^T (x^* - w_{d+1}) = 0$ , which happens *iff*  $x^* = w_{d+1}$ , or  $x^* \neq w_{d+1}$  and  $v_{d+1} \in (x^* - w_{d+1})^\perp$ . In the first case, the solution  $x^*$  belongs to  $Z$  since  $w_{d+1}$  is one of the  $z_{\ell_i}$ . If  $x^* \neq w_{d+1}$ , since all the  $\{v_k\}$  are i.i.d. of law  $\mathbb{P}$ , conditionally to  $U^{(d)}$  the probability that  $v_{d+1}$  is orthogonal to  $(x^* - w_{d+1})$  is null and (LS) has almost surely no solution. We conclude that  $S = Z$  almost surely.

— *Step 2:* The set of solutions of (RP) is  $\{\gamma_Z\}$  a.s.

We have proven that  $S = Z$  a.s., and thus that any solution  $\gamma \in \mathcal{S}$  is supported by  $Z$  a.s.. Let us write  $\gamma = \sum_{\ell=1}^n a_\ell \delta_{z_\ell}$  and  $\gamma_Z = \sum_{\ell=1}^n b_\ell \delta_{z_\ell}$ . It follows in particular that  $\sum_{\ell=1}^n (a_\ell - b_\ell) \delta_{P_1 z_\ell} = 0$ , and since for  $k \neq \ell$ ,  $z_k \neq z_\ell$ , hence  $\mathbb{P}(P_1 z_\ell = P_1 z_k) \leq \mathbb{P}(u_1^{(1)} \in (z_\ell - z_k)^\perp) = 0$ , thus  $\forall \ell, a_\ell = b_\ell$  a.s.  $\square$

The previous Theorem 2 only holds almost-surely, however "improbable" counter-examples do exist with excessive symmetry. Below we present a counter-example adapted from [7]. Let  $d := 2$ ,  $r := n > d$  and  $\forall i \in \llbracket 1, r \rrbracket, d_i := 1$ .

Consider  $z_\ell := \left( \cos\left(\frac{(2\ell+1)\pi}{n}\right), \sin\left(\frac{(2\ell+1)\pi}{n}\right) \right)^T$ ,  $P_\ell := \left( \cos\left(\frac{(2\ell+1)\pi}{2n}\right), \sin\left(\frac{(2\ell+1)\pi}{2n}\right) \right)$ .

As can be seen below (Figure 3), for  $n = 3$ , this corresponds to placing the  $(z_\ell)$  on every other vertex of a regular  $2n$ -gon, and defining the  $P_\ell$  such that  $\text{Im}P_\ell^T$  is the  $\ell$ -th bisector of the  $2n$ -gon.

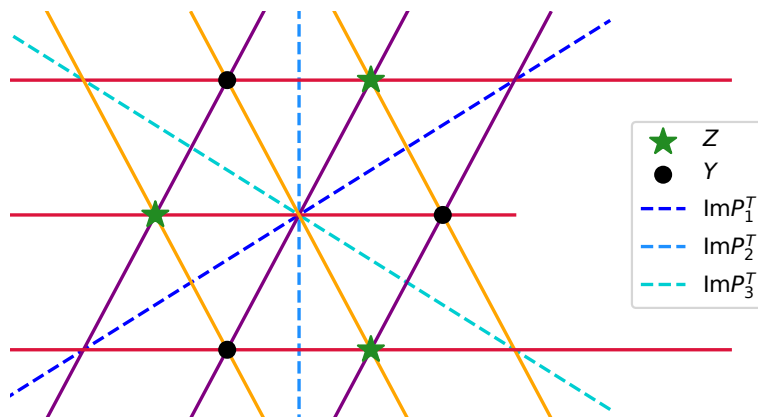


Figure 3: Illustration of a pathological super-critical case without unicity for specific projections  $P_i$ . In this case,  $Y$  and  $Z$  are distinct solutions with the same projections.

The points of  $S$  are the points of the form  $\bigcap_{i=1}^3 (z_{\ell_i} + \text{Ker}P_i)$ , or visually the intersection points of a yellow line, a red line and a purple line. We can see that the remaining vertices of the polygon constitute another valid measure  $\gamma_Y$  whose push-forwards  $P_i \# \gamma_Y$  are all the same as those of the original measure.

As mentioned in [7], for a fixed list of hyperplanes, there always exists two sets of points with the same projections on all of these hyperplanes. Theorem I.2 from [7] indicates that a necessary condition to ensure uniqueness in this case is  $r > n$ . In our Theorem 2, the points are fixed and uniqueness of the reconstruction holds almost surely when the  $P_i$  follow assumption  $\mathcal{A}_{\mathbb{P}}$  and as soon as  $D > d$ , whatever the number  $n$  of points in the discrete measure.

## 2.4 Details on the critical case $\sum_i d_i = d$

In the theorem below, we show that the example in Figure 2 is representative of the critical case.

**Theorem 3** (Number of admissible points in the critical case).

Let  $\gamma_Z$  be a fixed discrete probability measure. Assume that the matrices  $P_i$  follow assumption  $\mathcal{A}_{\mathbb{P}}$ , and that  $D := \sum_{i=1}^r d_i = d$ . Then the cardinality of  $S$  is exactly  $n^r$ ,  $\mathbb{P}$ -a.s..

*Proof.* Using the notation  $\ell := (\ell_1, \dots, \ell_r) \in \llbracket 1, n \rrbracket^r$ , we know that  $S = \bigcup_{\ell \in \llbracket 1, n \rrbracket^r} S_{\ell}$  where  $S_{\ell} = \bigcap_{i=1}^r (z_{\ell_i} + \text{Ker} P_i)$ . Following the proof of Theorem 2, in the case  $D = d$ , we see that assumption  $\mathcal{A}_{\mathbb{P}}$  implies that  $S_{\ell}$  is almost surely a singleton  $\{x_{\ell}\}$ . It follows that  $S$  is almost surely the union of at most  $n^r$  singletons. Let us show that if  $\ell \neq \ell'$  then  $S_{\ell} \cap S_{\ell'} = \emptyset$  a.s.. Indeed, if  $x$  belongs to  $S_{\ell} \cap S_{\ell'}$  then  $x$  is solution of a linear system of  $2d$  equations:

$$\forall i \in \llbracket 1, r \rrbracket, \forall j \in \llbracket 1, d_i \rrbracket, \quad \begin{cases} (u_i^{(j)})^T x = (u_i^{(j)})^T z_{\ell_i} \\ (u_i^{(j)})^T x = (u_i^{(j)})^T z_{\ell'_i} \end{cases},$$

which implies  $\forall i \in \llbracket 1, r \rrbracket, \forall j \in \llbracket 1, d_i \rrbracket, \ell_i = \ell'_i$ , or  $\ell_i \neq \ell'_i$  and  $u_{\ell_i}^{(j)} \in (z_{\ell_i} - z_{\ell'_i})^{\perp}$ .

Now, under  $\mathcal{A}_{\mathbb{P}}$ , if  $\ell_i \neq \ell'_i$ , then  $\mathbb{P}(u_{\ell_i}^{(j)} \in (z_{\ell_i} - z_{\ell'_i})^{\perp}) = 0$ , and thus  $\ell = \ell'$  a.s.. □

Let us clarify what the set of solutions  $\mathcal{S}$  looks like in this critical case  $D = d$ . Let  $\gamma$  be a solution of (RP) and denote  $S = (x_{\ell})_{\ell \in \llbracket 1, n \rrbracket^r}$ . By Theorem 3,  $\gamma$  is of the form  $\gamma = \sum_{\ell \in \llbracket 1, n \rrbracket^r} a_{\ell} \delta_{x_{\ell}}$ . Now, since  $\gamma$  is

a solution, we have for  $i \in \llbracket 1, r \rrbracket$ ,  $P_i \# \gamma = P_i \# \gamma_Z$ , thus  $\sum_{\ell \in \llbracket 1, n \rrbracket^r} a_{\ell} \delta_{P_i z_{\ell_i}} = \sum_{k=1}^n b_k \delta_{P_i z_k}$ . Since the  $(P_i z_{\ell})_{\ell}$

are all distinct a.s., this entails for all  $k \in \llbracket 1, n \rrbracket$ :  $\sum_{\ell_{-i} \in \llbracket 1, n \rrbracket^{r-1}} a_{\ell_1, \dots, \ell_{i-1}, k, \ell_{i+1}, \dots, \ell_r} = b_k$ , where  $\ell_{-i}$

indicates that we index this  $(r-1)$ -tuple on  $\llbracket 1, n \rrbracket \setminus \{i\}$ . We can re-write this condition as  $a \in \Pi_n^r(b)$ , the set of  $n$ -dimensional  $r$ -tensors on  $\mathbb{R}_+$  ( $\mathbb{R}_+^{n^r}$ ) with all  $r$  marginals equal to  $b$ . Conversely, if  $\gamma$  is of

the form  $\gamma = \sum_{\ell \in \llbracket 1, n \rrbracket^r} a_{\ell} \delta_{x_{\ell}}$  with  $a \in \Pi_n^r(b)$ , then we have by construction  $\forall i \in \llbracket 1, r \rrbracket$ ,  $P_i \# \gamma = P_i \# \gamma_Z$

and thus  $\gamma$  is a solution.

In the particular case where  $\gamma_Z$  is uniform, if we restrain  $\gamma$  to be also a uniform measure, the problem in this critical case has a combinatorial amount of solutions. Without this restriction, the problem has an infinite amount of solutions, as is discussed in the particular case of Figure 2.

## 3 Consequence for the empirical Sliced Wasserstein Distance

The Sliced Wasserstein (SW) distance between probability measures is frequently used in applied fields such as image processing or machine learning, as an efficient alternative to the Wasserstein distance. It was introduced in [17] to generate barycenters between images of textures, and it is

commonly used nowadays as a loss [12, 6, 22] to train generative networks. This distance writes:

$$\forall \alpha, \beta \in \mathcal{P}_2(\mathbb{R}^d), \quad \text{SW}^2(\alpha, \beta) = \int_{\theta \in \mathbb{S}^d} \text{W}_2^2(P_\theta \# \alpha, P_\theta \# \beta) d\sigma(\theta),$$

where  $\sigma$  is the uniform distribution over the unit sphere  $\mathbb{S}^d$  of  $\mathbb{R}^d$ , and  $P_\theta$  denotes the linear projection on the line of direction  $\theta$ . In its empirical (Monte-Carlo) approximation, used for numerical applications, it becomes:

$$\forall \alpha, \beta \in \widehat{\mathcal{P}}_2(\mathbb{R}^d), \quad \widehat{\text{SW}}_r^2(\alpha, \beta) := \frac{1}{r} \sum_{i=1}^r \text{W}_2^2(P_{\theta_i} \# \alpha, P_{\theta_i} \# \beta). \quad (3)$$

The main advantage of SW over the usual Wasserstein distance is computational: for two  $d$ -dimensional uniform discrete measures with  $n$  samples, the empirical estimation with  $r$  projections (3) can be computed in  $\mathcal{O}(rdn + rn \log(n))$  ([15], §2.6), leveraging the fact that the 1D projected Wasserstein distances can be computed by a sorting algorithm. In the same setting, the Wasserstein distance can only be computed in super-quadratic complexity with respect to  $n$  [16].

Since  $\text{W}_2$  is a distance on  $\mathcal{P}_2(\mathbb{R}^d)$  (the space of probability measures over  $\mathbb{R}^d$  admitting a finite second-order moment),  $\widehat{\text{SW}}_r$  is non-negative, homogeneous and satisfies the triangle inequality. However,  $\widehat{\text{SW}}_r$  is only a pseudo-distance since it does not satisfy the separation property: whatever the  $r$  directions chosen, it is always possible to find two different distributions  $\alpha$  and  $\beta$  such that  $\widehat{\text{SW}}_r(\alpha, \beta) = 0$ . Now, our previous reconstruction results show that when the  $r$  directions are drawn from  $\sigma$  and  $\beta$  is a fixed discrete measure, then  $\forall \alpha \in \mathcal{P}_2(\mathbb{R}^d)$ ,  $\widehat{\text{SW}}_r^2(\alpha, \beta) = 0 \implies \alpha = \beta$  almost surely provided that  $r > d$ . Indeed,  $\widehat{\text{SW}}_r(\alpha, \beta) = 0$  if and only if  $\alpha$  belongs to the set  $\mathcal{S}$  (for  $\gamma_Z = \beta$ ). On the contrary, when the number of projections is too small, the set of discrete measures at distance 0 from a given one is infinite, as stated in the theorem below.

**Theorem 4.** Let  $\gamma_Z := \sum_{\ell=1}^n b_\ell \delta_{z_\ell}$ , where the  $(z_\ell)$  are fixed and distinct. Assume  $\theta_1, \dots, \theta_r \sim \sigma^{\otimes r}$ .

- if  $r \leq d$ , there exists  $\sigma$ -a.s. an infinity of measures  $\gamma \neq \gamma_Z \in \mathcal{P}_2(\mathbb{R}^d)$  s.t.  $\widehat{\text{SW}}_r(\gamma, \gamma_Z) = 0$ .
- if  $r > d$ , we have  $\sigma$ -almost surely  $\{\gamma_Z\} = \underset{\gamma \in \mathcal{P}_2(\mathbb{R}^d)}{\text{argmin}} \widehat{\text{SW}}_r(\gamma, \gamma_Z)$ .

In the limit case  $r = d$ , the distance can be grown by scaling the points of  $\gamma_Z$  further away from the origin. In the case  $r < d$ , the supports of solution measures can be infinitely far from the support of  $\gamma_Z$ , as illustrated in Figure 1. To conclude, if  $r \leq d$ , the information  $\widehat{\text{SW}}_r(\gamma, \gamma_Z) \approx 0$  yields no valuable information regarding the closeness of  $\gamma$  and  $\gamma_Z$ . If  $r > d$ , the information  $\widehat{\text{SW}}_r(\gamma, \gamma_Z) = 0$  yields  $\gamma = \gamma_Z$  almost-surely with random projections. Due to possible local optima of  $Y \mapsto \widehat{\text{SW}}_r(\gamma_Y, \gamma_Z)$ , knowing only  $\widehat{\text{SW}}_r(\gamma, \gamma_Z) \approx 0$  may be too weak to conclude that the measures are close even in the favorable case  $r > d$ , although these considerations are beyond the scope of this work.

## 4 Conclusion: Discussion on SW as a Loss in Machine Learning

In Sliced-Wasserstein-based Machine Learning, the question of global optima is paramount since in practice, one must default to a surrogate of SW: be it through stochastic gradient descent (drawing a batch of  $\theta_i$  at each iteration), or directly through the estimation  $\widehat{\text{SW}}_r$ . To be precise, generative models such as [6] minimize  $\theta \mapsto \text{SW}(T_\theta \# \mu_0, \mu)$  - or a surrogate thereof - where  $\mu_0$  is a low-dimensional input distribution (often chosen as realizations of Gaussian noise), where  $\mu$  is the target distribution (the discrete dataset), and where  $T_\theta$  is the model of parameters  $\theta$ . In this case, the dimension  $d$  of the data, which for images can easily exceed one million, can be very large. Our results suggest that performing optimisation with less than  $d$  projections is unsound in this case, since it leads to solutions that can be arbitrarily far away from the true data distribution. However, these results do not take into account the intrinsic dimension of the target measure  $\gamma_Z$ , which in



the case of images is probably supported on a  $d'$ -dimensional manifold with  $d' \ll d$ . This question has been addressed very recently in [19], which was unknown to us at the time of working on this paper, and provides insightful reconstruction results when  $\gamma_Z$  is known to belong to a  $d'$ -dimensional manifold.

Furthermore, it is important to underline that having the guarantee that the global optima are the desired original measure is insufficient in practice. Indeed, the landscape  $Y \mapsto \widehat{\text{SW}}_r(\mu_Y, \mu_Z)$  can present numerous local optima, which can limit the usefulness of SW as a loss function. For practical considerations, this study on global optima could be complemented by an analysis of the aforementioned landscape, which we leave for future work. Namely, one may find a study of the optimization properties of  $Y \mapsto \widehat{\text{SW}}_r(\mu_Y, \mu_Z)$  and  $Y \mapsto \text{SW}(\mu_Y, \mu_Z)$  in [21], and a related extension to the study of Stochastic Gradient Descent for SW generative networks in [20].

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