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# NEW EIGENVALUE PINCHING RESULTS FOR EUCLIDEAN DOMAINS

JULIEN ROTH AND ABHITOSH UPADHYAY

ABSTRACT. We prove stability results associated with sharp eigenvalue upper bounds for several operators on embedded hypersurfaces and boundary problems on smooth domains of the Euclidean space. These upper bounds involve isoperimetric ratio and mean curvature terms. The stability results derive from a general pinching result for the moment of inertia.

## 1. INTRODUCTION

There is a wide literature concerning estimates of the eigenvalues of the Laplacian (and more generally divergence-type second order operators) on hypersurfaces of space forms. The first upper bound for the first eigenvalue of the Laplacian on compact hypersurfaces of  $\mathbb{R}^{n+1}$  was obtained by Bleecker and Weiner [4] who showed

$$(1) \quad \lambda_1 \leq \frac{1}{\text{Vol}(M)} \int_M \|B\|^2 dv_g,$$

where  $B$  is the (real-valued) second fundamental form of the hypersurface  $M$  and  $\text{Vol}(M)$  its volume. After that, Reilly [15] improved this upper bound by getting the norm of the mean curvature instead of the second fundamental form. Precisely, he proved the following estimate:

$$(2) \quad \lambda_1 \leq \frac{n}{\text{Vol}(M)} \int_M H^2 dv_g,$$

where  $H = \frac{1}{n} \text{tr}(B)$ . For these two inequalities, the limiting case is attained if and only if the hypersurface is a geodesic hypersphere. In fact, in [15] Reilly proved a sequence of upper bounds involving higher order mean curvatures, for which (2) is just the particular case  $r = 0$ :

$$(3) \quad \lambda_1 \left( \int_M H_r dv_g \right)^2 \leq n \text{Vol}(M) \int_M H_{r+1}^2 dv_g.$$

We recall here that the higher order mean curvatures  $H_k$ ,  $k \in \{1, \dots, n\}$  are defined by  $H_k = \frac{1}{\binom{n}{k}} \sigma_k(B)$ , where  $\sigma_k(B)$  is the  $k$ -th elementary symmetric polynomial computed for the principal curvatures  $\kappa_1, \dots, \kappa_n$ . As the previous ones, these inequalities are sharp and the limiting cases are also characterized by geodesic hyperspheres. Note that the inequalities in (3) are not in general better than (2);

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they are not comparable to the other.

The Laplacian is not the only fundamental operator for which extrinsic estimates have been proved. In particular, comparable upper bounds have been established for some divergence-type second order elliptic operators as the operators  $L_r$  associated with the higher order mean curvatures  $H_r$  and even for more general elliptic divergence-free operators, cf. [1, 2, 11].

Assume that  $M$  is endowed with two symmetric, positive definite and divergence-free  $(1, 1)$ -tensors  $S$  and  $T$ . In [16], the first author proved the following upper bound for the operator  $L_T = -\operatorname{div}(T\nabla(\cdot))$

$$(4) \quad \lambda_1(L_T) \left( \int_M \operatorname{tr}(S) dv_g \right)^2 \leq \left( \int_M \operatorname{tr}(T) dv_g \right) \left( \int_M H_S^2 dv_g \right),$$

where  $H_S$  is defined by  $\operatorname{tr}(A \circ S)$ , where  $A$  is the shape operator, that is, the symmetric endomorphism of  $TM$  associated with the second fundamental form  $B$ . The equality case is also characterized. Namely, if  $H_s$  does not vanish identically, then equality occurs in (4) if and only if  $M$  is a geodesic hypersphere and  $\operatorname{tr}(S)$  is constant. Note that if  $T$  and  $S$  are chosen as the operators  $T_r$  and  $T_s$  naturally associated with  $H_r$  and  $H_s$  respectively, then (4) is nothing else but the inequality of Alias-Malacarné [2]:

$$(5) \quad \lambda_1(L_r) \left( \int_M H_s dv_g \right)^2 \leq c(r) \left( \int_M H_r dv_g \right) \left( \int_M H_{s+1}^2 dv_g \right),$$

with  $L_r = -\operatorname{div}(T_r \nabla(\cdot))$ . This operator play a crucial role in the study of stable hypersurfaces with constant  $r$ -th mean curvature (see [1] for details).

Note that for all the previous inequalities, the hypersurface  $M$  is not supposed to be embedded but only immersed.

On the other hand, in [15] Reilly proved an upper bound for  $\lambda_1(\Delta)$  on embedded hypersurfaces of the Euclidean space in terms of some isoperimetric ratio. Namely, for  $M$  a smooth embedded hypersurface bounding a domain  $\Omega$  of  $\mathbb{R}^{n+1}$ , the first eigenvalue  $\lambda_1(\Delta)$  satisfies

$$\lambda_1(\Delta) \leq \frac{n}{(n+1)^2} \cdot \frac{\operatorname{Vol}(M)^2}{\operatorname{Vol}(\Omega)^2},$$

with equality if and only if  $\Omega$  is a ball. Other estimates of this type were obtain for the eigenvalues of second order of divergence type as well as for different Steklov-type problems by the first author in [17]. For instance, for  $\lambda_1(L_T)$ , we have

$$\lambda_1(L_T) \leq \frac{\operatorname{Vol}(M)}{(n+1)^2 \operatorname{Vol}(\Omega)^2} \int_M \operatorname{tr}(T) dv_g.$$

The common point between all these upper bounds, with mean curvature terms or isoperimetric ratio, is that they are obtained by using the coordinates functions  $X^i$  as test functions.

We recall that the moment of inertia of  $M$  with respect to the point  $x_0 \in \mathbb{R}^{n+1}$  is defined by

$$I_{x_0}(M) = \|X - x_0\|_2 = \left( \frac{1}{\operatorname{Vol}(M)} \int_M |X - x_0|^2 dv_g \right)^{\frac{1}{2}},$$

where  $X$  is the position of  $M$ .

The first aim of the article is to prove new eigenvalue upper bounds which are a mix

between mean curvature and isoperimetric terms. They will be derived in Section 3 from the following lower bound of the moment of inertia.

**Proposition 1.1.** *Let  $M$  be a closed hypersurface of  $\mathbb{R}^{n+1}$  bounding a domain  $\Omega$  and  $x_0 \in \mathbb{R}^{n+1}$  a fixed point. Assume that  $S$  is a symmetric, positive definite and divergence-free  $(1, 1)$ -tensor over  $M$  such that  $H_S$  does not vanish identically. Then, for any  $\alpha \in [0, 1]$ , the moment of inertia of  $M$  with respect to  $x_0$  satisfies*

$$I_{x_0}(M) \geq \frac{(n+1)^\alpha \text{Vol}(\Omega)^\alpha \left| \int_M \text{tr}(S) dv_g \right|^{1-\alpha}}{\text{Vol}(M) \|H_S\|_2^{1-\alpha}},$$

with equality if and only if  $\Omega = B(x_0, I_{x_0}(M))$  and  $\text{tr}(S)$  is constant.

Note that we assume that  $S$  is positive definite in order to avoid the possibility that

$$\int_M \text{tr}(S) dv_g = 0.$$

Moreover, for  $\alpha = 0$  and  $\alpha = 1$  respectively, we get the following estimates:

$$I_{x_0}(M) \geq \frac{\left| \int_M \text{tr}(S) dv_g \right|}{\text{Vol}(M) \|H_S\|_2} \quad \text{and} \quad I_{x_0}(M) \geq \frac{(n+1) \text{Vol}(\Omega)}{\text{Vol}(M)}.$$

Note that the first one has already been proven in [18] and there is no embeddedness assumption to get it. However, for  $\alpha \in (0, 1]$ , embeddedness is required.

In the sequel, for a lack of simplicity and without loss of generality, we will consider again, up to a possible translation, that  $\int_M X dv_g = 0$  and so we will deal with the moment of inertia with respect to 0, that is  $I_0(M) = \|X\|_2 = \left( \frac{1}{\text{Vol}(M)} \int_M |X|^2 dv_g \right)^{\frac{1}{2}}$ .

For  $\varepsilon > 0$ ,  $s \in (1, +\infty)$  and  $\alpha \in [0, 1]$ , we introduce the following pinching condition on the moment of inertia

$$(\mathcal{I}_{\varepsilon, s, \alpha}) \quad I_0(M)^2 \leq (1 + \varepsilon) \frac{(n+1)^{2\alpha} \text{Vol}(\Omega)^{2\alpha} \left| \int_M \text{tr}(S) dv_g \right|^{2(1-\alpha)}}{\text{Vol}(M)^2 \|H_S\|_{2s}^{2(1-\alpha)}}.$$

Note that we write the pinching condition with the square of the moment of inertia, which will be more convenient for getting applications to eigenvalue pinching in Section 3. Note also that we consider the pinching condition associated with the  $L^{2s}$ -norm of  $H_S$  instead of its  $L^2$ -norm. This is due to a technical reason that will appear in the proof of the main result.

**Theorem 1.2.** *Let  $n \geq 2$  and  $(M^n, g)$  be a connected, oriented and closed Riemannian manifold isometrically embedded into  $\mathbb{R}^{n+1}$  by  $X$ . We denote by  $\Omega$  the compact domain bounded by  $X(M)$ . Assume that  $S$  is a symmetric, positive definite and divergence-free  $(1, 1)$ -tensor over  $M$  such that  $H_S$  does not vanish identically. Let  $q > n$ ,  $s > 1$  and  $\alpha \in [0, 1]$  three real numbers. Then there exists three positive constants,  $\varepsilon_0$  and  $C$ , depending  $n$ ,  $s$ ,  $q$ ,  $\alpha$  and  $\text{Vol}(M) \|B\|_q^n$ , and  $\beta$  depending only on  $n$  and  $q$  such that if  $(\mathcal{I}_{\varepsilon, s, \alpha})$  holds with  $\varepsilon < \varepsilon_0$ , then  $M$  is diffeomorphic and  $C\varepsilon^\beta$ -almost-isometric to the sphere  $S(0, r)$  of radius*

$r = \frac{(n+1)^\alpha \text{Vol}(\Omega)^\alpha \left| \int_M \text{tr}(S) dv_g \right|^{1-\alpha}}{\text{Vol}(M) \|H_S\|_{2s}^{1-\alpha}}$ . That is, there exists a natural diffeomorphism

$$F: (M, d_1) \rightarrow (S(0, r), d_2)$$

we have

$$|d_2(F(x_1), F(x_2)) - d_1(x_1, x_2)| \leq Cr\varepsilon^\beta \quad \forall x_1, x_2 \in M.$$

After proving this theorem in Section 2, we obtain, in Section 3 several new optimal eigenvalue estimates and we derive from Theorem 1.2 several stability results associated with these estimates.

## 2. PROOF OF THE MAIN RESULT

**2.1. Proof of Proposition 1.1.** For a sake a simplicity, but without loss of generality, we can assume that  $x_0 = 0$ .

First, we have  $\text{div}_\Omega(X) = n+1$ , which give after integrating over  $\Omega$

$$(n+1)\text{Vol}(\Omega) = \int_\Omega \text{div}_\Omega(X) dv_{\bar{g}}$$

By the divergence theorem, we get

$$(6) \quad (n+1)\text{Vol}(\Omega) = \int_M \langle X, \nu \rangle dv_g$$

Second, since  $S$  is divergence free, we have the following generalized Hsiung-Minkowski formula (see [11, 16])

$$(7) \quad \int_M (\text{tr}(S) + H_S \langle X, \nu \rangle) dv_g = 0.$$

From (6) and (7), we have

$$\begin{aligned} (n+1)^\alpha \text{Vol}(\Omega)^\alpha \left| \int_M \text{tr}(S) dv_g \right|^{1-\alpha} &= \left( \int_M \langle X, \nu \rangle dv_g \right)^\alpha \left| \int_M H_S \langle X, \nu \rangle dv_g \right|^{1-\alpha} \\ &\leq \left( \int_M |X| dv_g \right)^\alpha \left( \int_M H_S^2 dv_g \right)^{\frac{1-\alpha}{2}} \left( \int_M \langle X, \nu \rangle^2 dv_g \right)^{\frac{1-\alpha}{2}} \\ &\leq \text{Vol}(M)^{\frac{\alpha}{2}} \left( \int_M |X|^2 dv_g \right)^{\frac{\alpha}{2}} \left( \int_M H_S^2 dv_g \right)^{\frac{1-\alpha}{2}} \left( \int_M |X|^2 dv_g \right)^{\frac{1-\alpha}{2}} \\ &= \text{Vol}(M) \|H_S\|_2^{1-\alpha} I_0(M), \end{aligned}$$

which gives

$$I_0(M) \geq \frac{(n+1)^\alpha \text{Vol}(\Omega)^\alpha \left| \int_M \text{tr}(S) dv_g \right|^{1-\alpha}}{\text{Vol}(M) \|H_S\|_2^{1-\alpha}}.$$

Note that we have used  $\langle X, \nu \rangle \leq |X|$  and Cauchy-Schwarz inequality. The inequality is proved. Moreover equality occurs if and only if equality occurs in all these inequalities, in particular, we have  $\langle X, \nu \rangle \leq |X|$  everywhere which implies that  $|X|$  is constant and  $M$  is contained in the sphere  $S(0, |X|)$  and  $I_0(M) = |X|$ . In addition, since  $M$  has no boundary,  $M = S(0, I_0(M))$  and so  $\Omega = B(0, I_0(M))$ .

Moreover, from equality in Cauchy-Schwarz inequality and since  $\langle X, \nu \rangle$  is constant, then we get that  $H_S$  is constant. But,  $M$  is a sphere, so  $M$  is totally umbilical. Hence, from the definition of  $H_S = \text{tr}(A \circ S)$ , this implies in fact that  $\text{tr}(S)$  is constant.

Conversely, if  $\Omega$  is a ball and  $\text{tr}(S)$  is constant, then first,  $M$  is a ball of radius  $r = I_0(M)$  (we still assume that  $x_0 = 0$ ). Hence, the shape operator of  $M$  is  $A = \frac{1}{r}\text{Id}$ ,  $\text{Vol}(\Omega) = \frac{r^{n+1}\omega_n}{n+1}$  and  $\text{Vol}(M) = r^n\omega_n$ , where  $\omega_n$  is the volume of the  $n$ -dimensional unit sphere. Moreover, we have  $H_S = \text{tr}(A \circ S) = \text{tr}\left(\frac{1}{r}S\right) = \frac{1}{r}\text{tr}(S)$ . So, we have

$$\begin{aligned} \frac{(n+1)^\alpha \text{Vol}(\Omega)^\alpha \left| \int_M \text{tr}(S) dv_g \right|^{1-\alpha}}{\text{Vol}(M) \|H_S\|_2^{1-\alpha}} &= \frac{(n+1)^\alpha \text{Vol}(\Omega)^\alpha \text{tr}(S)^{1-\alpha} \text{Vol}(M)^{1-\alpha}}{\text{Vol}(M) \frac{1}{r^{1-\alpha}} \text{tr}(S)^{1-\alpha}} \\ &= \frac{(n+1)^\alpha \frac{r^{\alpha(n+1)} \omega_n^\alpha}{(n+1)^\alpha} r^{n(1-\alpha)} \omega_n^{1-\alpha}}{r^n \omega_n \frac{1}{r^{1-\alpha}}} \\ &= r \\ &= I_0(M). \end{aligned}$$

Hence, equality occurs.  $\square$

**2.2. Proof of Theorem 1.2.** The proof is divided into three steps. First, we show that the pinching condition  $(\mathcal{I}_{\varepsilon, s, \alpha})$  implies an integral proximity between  $M$  and the sphere  $S(0, r)$ .

Second, we show that this integral proximity induces a pointwise proximity. This step does not require the use of the pinching condition and is based on an iteration process.

Finally, using the second step, we construct the appropriate diffeomorphism and conclude.

**Step 1 : Integral proximity.** We begin with the following two lemmas which give us an integral proximity of  $M$  to a sphere. The first lemma gives an estimates of the smallness of  $\|X\|_2 - \|X\|_1$  under the pinching condition  $(\mathcal{I}_{\varepsilon, s, \alpha})$ . This will be useful in particular in Step 2 to show that  $M$  is contained between two spheres of close radii. Namely, we have the following

**Lemma 2.1.** *There exists a positive constant  $K = K(\alpha, s)$  so that if  $(\mathcal{I}_{\varepsilon, s, \alpha})$  holds for  $\varepsilon < 1$ , then*

$$1 - \frac{\|X\|_1}{\|X\|_2} \leq K\varepsilon.$$

*Proof:* From the pinching condition  $(\mathcal{I}_{\varepsilon,s,\alpha})$  first and the divergence theorem after, we have

$$\begin{aligned} \|X\|_2^2 &\leq (1+\varepsilon) \frac{(n+1)^{2\alpha} \text{Vol}(\Omega)^{2\alpha} \left| \int_M \text{tr}(S) dv_g \right|^{1-\alpha}}{\text{Vol}(M) \|H_S\|_{2s}^{2(1-\alpha)}} \\ &\leq (1+\varepsilon) \left( \frac{(n+1) \text{Vol}(\Omega)}{\text{Vol}(M)} \right)^{2\alpha} \left( \frac{\left| \int_M \text{tr}(S) dv_g \right|}{\text{Vol}(M) \|H_S\|_{2s}} \right)^{2(1-\alpha)} \\ &\leq (1+\varepsilon) \left( \frac{\int_M \langle X, \nu \rangle dv_g}{\text{Vol}(M)} \right)^{2\alpha} \left( \frac{\left| \int_M H_S \langle X, \nu \rangle dv_g \right|}{\text{Vol}(M) \|H_S\|_{2s}} \right)^{2(1-\alpha)} \end{aligned}$$

where we have use  $\text{div}_\Omega(X) = n+1$  with the divergence theorem as well as the Hsiung-Minkowski formula (7). Now, using that  $\langle X, \nu \rangle \leq |X|$  and Hölder inequality, we have

$$\begin{aligned} \|X\|_2^2 &\leq (1+\varepsilon) \|X\|_1^{2\alpha} \left( \frac{\|H_S\|_{2s} \|X\|_{\frac{2s}{2s-1}} \text{Vol}(M)}{\text{Vol}(M) \|H_S\|_{2s}} \right)^{2(1-\alpha)} \\ &\leq (1+\varepsilon) \|X\|_1^{2\alpha} \|X\|_{\frac{2s}{2s-1}}^{2(1-\alpha)}. \end{aligned}$$

Moreover, since  $s > 1$ , we have  $\frac{2s}{2s-1} \in (1, 2)$  and by the log-convexity of  $\|\cdot\|_t$  with respect to  $t$ , we have:

$$\|X\|_{\frac{2s}{2s-1}} \leq \|X\|_1^{1-\frac{1}{s}} \|X\|_2^{\frac{1}{s}},$$

which reported in the last inequality gives

$$\|X\|_2^{\frac{2(s-1+\alpha)}{s}} \leq (1+\varepsilon) \|X\|_1^{\frac{2(s-1+\alpha)}{s}}$$

and so

$$1 - \frac{\|X\|_1}{\|X\|_2} \leq 1 - \left( \frac{1}{1+\varepsilon} \right)^{\frac{s}{2(s-1+\alpha)}}.$$

Moreover, there exists a positive constant  $K$  depending only on  $s$  and  $\alpha$  so that for  $\varepsilon < 1$ , we have  $\left( \frac{1}{1+\varepsilon} \right)^{\frac{s}{2(s-1+\alpha)}} \geq 1 - K\varepsilon$  which gives

$$1 - \frac{\|X\|_1}{\|X\|_2} \leq K\varepsilon.$$

□

Now, we have this second lemma that will allow us in particular to show that the radial projection of  $M$  onto  $S(0, r)$  is the desired diffeomorphism. Precisely, we have

**Lemma 2.2.** *If  $(\mathcal{I}_{\varepsilon,s,\alpha})$  holds, then*

$$\|X^\top\|_2^2 \leq \varepsilon \|X\|_2^2.$$

*Proof:* We have

$$\begin{aligned}
\|X^\top\|_2^2 \text{Vol}(M) &= \int_M \|X\|^2 dv_g - \int_M \langle X, \nu \rangle^2 dv_g \\
&\leq \|X\|_2^2 \text{Vol}(M) - \left( \int_M \langle X, \nu \rangle^2 dv_g \right)^\alpha \left( \int_M \langle X, \nu \rangle^2 dv_g \right)^{1-\alpha} \\
&\leq \|X\|_2^2 \text{Vol}(M) - \frac{\left( \int_M \langle X, \nu \rangle dv_g \right)^{2\alpha}}{\text{Vol}(M)^\alpha} \left( \int_M \langle X, \nu \rangle^2 dv_g \right)^{1-\alpha} \\
&\leq \|X\|_2^2 \text{Vol}(M) - \frac{(n+1)^{2\alpha} \text{Vol}(\Omega)^{2\alpha}}{\text{Vol}(M)^\alpha} \left( \int_M \langle X, \nu \rangle^2 dv_g \right)^{1-\alpha}
\end{aligned}$$

by Cauchy-Schwarz inequality and divergence theorem. Moreover, we have

$$\begin{aligned}
\int_M \langle X, \nu \rangle^2 dv_g &\geq \frac{\left( \int_M \langle X, \nu \rangle dv_g \right)^{1/2}}{\text{Vol}(M)^{1/2}} \\
&\geq \frac{\left( \int_M \langle X, \nu \rangle dv_g \right)^{1/2} \left( \int_M H^2 dv_g \right)^{1/2}}{\|H\|_{2s} \text{Vol}(M)} \\
&\geq \frac{\int_M H_S \langle X, \nu \rangle dv_g}{\|H\|_{2s} \text{Vol}(M)} \\
&\geq \frac{\left| \int_M \text{tr}(S) dv_g \right|}{\|H\|_{2s} \text{Vol}(M)}
\end{aligned}$$

where we have used the Cauchy-Schwarz inequality twice and finally the Hsiung-Minkowski formula (7). Thus, we get

$$\|X^\top\|_2^2 \text{Vol}(M) \leq \|X\|_2^2 \text{Vol}(M) - \frac{(n+1)^{2\alpha} \text{Vol}(\Omega)^{2\alpha}}{\text{Vol}(M)^\alpha} \left( \frac{\int_M H_S \langle X, \nu \rangle dv_g}{\|H\|_{2s} \text{Vol}(M)} \right)^{1-\alpha}$$

Now, using the pinching condition  $(\mathcal{I}_{\varepsilon, s, \alpha})$ , we obtain

$$\begin{aligned}
&\leq \|X\|_2^2 \text{Vol}(M) \left( 1 - \frac{1}{1 + \varepsilon} \right) \\
&\leq \|X\|_2^2 \text{Vol}(M) \varepsilon.
\end{aligned}$$

Hence, we get  $\|X^\top\|_2^2 \leq \varepsilon \|X\|_2^2$ .  $\square$

**Step 2: From integral to pointwise proximity.** Now, we recall without proof the two following lemmas which are independent of any pinching conditions and which are based on classical iteration processes.



**Lemma 2.3.** [12] *Let  $q > n$  be a real number. There exists a constant  $\Gamma(n, q) > 0$  so that for any isometrically immersed, compact hypersurface  $M^n$  of  $\mathbb{R}^{n+1}$  we have*

$$\||X| - \|X\|_2\|_\infty \leq \Gamma (\text{Vol}(M)\|H\|_q^n)^{\frac{\gamma}{n}} \|X\|_2 \left(1 - \frac{\|X\|_1}{\|X\|_2}\right)^{\frac{1}{2(\gamma+1)}},$$

where  $\gamma = \frac{nq}{2(q-n)}$ .

Note that this lemma implies in particular that

$$(8) \quad \|X\|_\infty \leq \Gamma (\text{Vol}(M)\|H\|_q^n)^{\frac{\gamma}{2}} + 1 \|X\|_2$$

and so

$$(9) \quad \|X\|_\infty \leq \Gamma (\text{Vol}(M)\|B\|_q^n)^{\frac{\gamma}{2}} + 1 \|X\|_2$$

**Lemma 2.4.** [19] *Let  $q > n$  be a real number. There exists a constant  $\Gamma' = \Gamma'(n, q)$ , so that for any isometrically immersed, compact hypersurface  $M^n$  of  $\mathbb{R}^{n+1}$  we have*

$$\|X^T\|_\infty \leq \Gamma' (\text{Vol}(M)\|B\|_q^n \|X\|_\infty)^{\frac{\gamma}{\gamma+1}} \|X^T\|_2^{\frac{1}{\gamma+1}}.$$

Note that  $\gamma$  is the same that in Lemma 2.3, that is,  $\gamma = \frac{nq}{2(q-n)}$ .

Now, using these lemmas, we can finish in a now classical way the proof of Theorem 1.2. First, combining Lemmas 2.1 and 2.3, we get

$$(10) \quad \||X| - \|X\|_2\|_\infty \leq \Gamma (\text{Vol}(M)\|H\|_q^n)^{\frac{\gamma}{2}} \|X\|_2 K^{\frac{1}{2(\gamma+1)}} \varepsilon^{\frac{1}{2(\gamma+1)}}$$

Now, assuming that  $\varepsilon < 1$ , we get from  $(\mathcal{I}_{\varepsilon, s, \alpha})$  that

$$\|X\|_2 \leq 2 \frac{(n+1)^\alpha \text{Vol}(\Omega)^\alpha \left| \int_M \text{tr}(S) dv_g \right|^{1-\alpha}}{\text{Vol}(M) \|H_S\|_{2s}^{1-\alpha}}.$$

For more compactness in the formulas, we will denote

$$r = \frac{(n+1)^\alpha \text{Vol}(\Omega)^\alpha \left| \int_M \text{tr}(S) dv_g \right|^{1-\alpha}}{\text{Vol}(M) \|H_S\|_{2s}^{1-\alpha}}$$

and so we have

$$(11) \quad \|X\|_2 \leq 2r.$$

Thus, we have

$$\begin{aligned} \||X| - r\|_\infty &\leq \||X| - \|X\|_2\|_\infty + \||X\|_2 - r| \\ &\leq 2r\Gamma (\text{Vol}(M)\|H\|_q^n)^{\frac{\gamma}{2}} K^{\frac{1}{2(\gamma+1)}} \varepsilon^{\frac{1}{2(\gamma+1)}} + r\varepsilon \\ &\leq r(2\Gamma (\text{Vol}(M)\|H\|_q^n)^{\frac{\gamma}{2}} K^{\frac{1}{2(\gamma+1)}} \varepsilon^{\frac{1}{2(\gamma+1)}} + \varepsilon) \\ (12) \quad &\leq C_1 r \varepsilon^{\frac{1}{2(\gamma+1)}}, \end{aligned}$$

where  $C_1 = 2\Gamma (\text{Vol}(M)\|H\|_q^n)^{\frac{\gamma}{2}} K^{\frac{1}{2(\gamma+1)}} + 1$ . Note that the last line comes from the fact that  $\frac{1}{2(\gamma+1)} < 1$  and we have assumed  $\varepsilon < 1$  so that  $\varepsilon \leq \varepsilon^{\frac{1}{2(\gamma+1)}}$ . We want to point out that the constant  $C_1$  depends on  $n, q, s$  and  $\text{Vol}(M)\|H\|_q^n$ . We deduce immediately from this that

$$(13) \quad \|X\|_\infty \leq r \left(1 + C_1 \varepsilon^{\frac{1}{2(\gamma+1)}}\right).$$

□

On the other hand, from Lemma 2.4, we have

$$\|X^\top\|_\infty \leq \Gamma'(\text{Vol}(M)\|B\|_q^n\|X\|_\infty)^{\frac{\gamma}{\gamma+1}} \|X^T\|_2^{\frac{1}{\gamma+1}}$$

which gives by Lemma 2.2

$$\begin{aligned} \|X^\top\|_\infty &\leq \Gamma'(\text{Vol}(M)\|B\|_q^n\|X\|_\infty)^{\frac{\gamma}{\gamma+1}} \|X\|_2^{\frac{1}{\gamma+1}} \varepsilon^{\frac{1}{2(\gamma+1)}} \\ &\leq \Gamma'(\text{Vol}(M)\|B\|_q^n)^{\frac{\gamma}{\gamma+1}} \|X\|_\infty^{\frac{\gamma}{\gamma+1}} \|X\|_2^{\frac{1}{\gamma+1}} \varepsilon^{\frac{1}{2(\gamma+1)}} \end{aligned}$$

Moreover, by (13), we get

$$\|X^\top\|_\infty \leq \Gamma'(\text{Vol}(M)\|B\|_q^n)^{\frac{\gamma}{\gamma+1}} r^{\frac{\gamma}{\gamma+1}} \left(1 + C_1 \varepsilon^{\frac{1}{2(\gamma+1)}}\right)^{\frac{\gamma}{\gamma+1}} \|X\|_2^{\frac{1}{\gamma+1}} \varepsilon^{\frac{1}{2(\gamma+1)}}$$

Using (11) and the fact that  $\varepsilon < 1$ , we get

$$\begin{aligned} \|X^\top\|_\infty &\leq \Gamma'(\text{Vol}(M)\|B\|_q^n)^{\frac{\gamma}{\gamma+1}} (1 + C_1)^{\frac{\gamma}{\gamma+1}} r^{\frac{\gamma}{\gamma+1}} (2r)^{\frac{1}{\gamma+1}} \varepsilon^{\frac{1}{2(\gamma+1)}} \\ &\leq 2r \Gamma'(\text{Vol}(M)\|B\|_q^n)^{\frac{\gamma}{\gamma+1}} (1 + C_1)^{\frac{\gamma}{\gamma+1}} \varepsilon^{\frac{1}{2(\gamma+1)}} \\ (14) \quad &\leq C_2 r \varepsilon^{\frac{1}{2(\gamma+1)}}, \end{aligned}$$

where  $C_2 = 2\Gamma'(\text{Vol}(M)\|B\|_q^n)^{\frac{\gamma}{\gamma+1}} (1 + C_1)^{\frac{\gamma}{\gamma+1}}$  is a constant depending only on  $n, p, q$  and  $\text{Vol}(M)\|B\|_q^n$ .

**Step 3 : Conclusion** Now, we set  $\varepsilon_1 = \inf \left\{ \frac{\sqrt{2}-1}{\sqrt{2}C_1}, \frac{1}{2C_2} \right\}^{2(\gamma+1)}$ . Note that  $\varepsilon_1$  depends only on  $n, q$  and  $\text{Vol}(M)\|B\|_q^n$  and if  $\varepsilon \leq \varepsilon_1$ , we have

$$C_1 \varepsilon^{\frac{1}{2(\gamma+1)}} < 1, \quad C_2 \varepsilon^{\frac{1}{2(\gamma+1)}} < 1 \quad \text{and} \quad \left(1 - C_1 \varepsilon^{\frac{1}{2(\gamma+1)}}\right)^2 - \left(C_2 \varepsilon^{\frac{1}{2(\gamma+1)}}\right)^2 \geq \frac{1}{4}.$$

From this last inequality, with (13) and (14), we deduce that

$$\begin{aligned} \langle X, \nu \rangle^2 &= |X|^2 - |X^\top|^2 \\ &\geq (r - \| |X| - r \|_\infty)^2 - \|X^T\|_\infty^2 \\ &\geq r^2 \left[ \left(1 - C_1 \varepsilon^{\frac{1}{2(\gamma+1)}}\right)^2 - \left(C_2 \varepsilon^{\frac{1}{2(\gamma+1)}}\right)^2 \right] \\ (15) \quad &\geq \frac{r^2}{4}. \end{aligned}$$

Now, we consider the map

$$\begin{aligned} F : M &\longrightarrow S(0, r) \\ x &\longmapsto r \frac{X(x)}{|X(x)|}. \end{aligned}$$

First of all, from (12) and the fact that  $C_1 \varepsilon^{\frac{1}{2(\gamma+1)}} < 1$ , we get  $\| |X| - r \|_\infty < r$ . Therefore  $|X|$  never vanishes and so  $F$  is well defined. Now, we will compute the differential of  $F$ . Let  $u \in T_x M$  be a unit vector. We get immediately that

$$dF_x(u) = \frac{r}{|X|} \left( u - \frac{\langle u, X \rangle}{|X|^2} X \right),$$

and so

$$\begin{aligned} \left| |dF_x(u)|^2 - 1 \right| &\leq \left| \frac{r^2}{\|X\|^2} - 1 \right| + \frac{r^2}{\|X\|^4} \langle u, X \rangle^2 \\ &\leq \frac{1}{|X|^2} \left| |X| - r \right| \cdot \left| |X| + r \right| + \frac{r^2 |X^\top|^2}{|X|^4} \end{aligned}$$

Now, from (12), (13), (14) and the fact that  $C_1 \varepsilon^{\frac{1}{2(\gamma+1)}} < 1$ , we get

$$\left| |dF_x(u)|^2 - 1 \right| \leq \frac{2r^2 C_1 \varepsilon^{\frac{1}{2(\gamma+1)}}}{|X|^2} + \frac{r^3 C_2 \varepsilon^{\frac{1}{2(\gamma+1)}}}{|X|^3}.$$

Finally, from (15), we deduce that  $\frac{1}{|X|} \leq \frac{2}{r}$  and so

$$(16) \quad \left| |dF_x(u)|^2 - 1 \right| \leq \frac{2C_1 + C_2}{4} \varepsilon^{\frac{1}{2(\gamma+1)}} = C_3 \varepsilon^{\frac{1}{2(\gamma+1)}}$$

We put  $\varepsilon_0 = \inf \left\{ \varepsilon_1, \left( \frac{1}{2C_3} \right)^{2(\gamma+1)} \right\}$ . We get from this that  $|dF_x(u)|^2 \geq \frac{1}{2}$  which implies that  $F$  is a local diffeomorphism. Since  $M$  and  $S(0, r)$  are connected,  $M$  is closed and  $S(0, r)$  is simply connected, then  $F$  is a global diffeomorphism.

Moreover, from (16), we get that for any  $x, y \in M$

$$|d_2(f(x), f(y)) - d(x, y)| \leq m d_1(x, y),$$

where  $d_1$  and  $d_2$  are the Riemannian distance on  $M$  and  $S(0, r)$  respectively and  $m = \sup \{ \left| |dF_x(u)|^2 - 1 \right|, x \in M, u \in U_x M \}$ . Hence, we deduce that

$$|d_2(f(x), f(y)) - d(x, y)| \leq 2\pi r C_3 \varepsilon^\beta,$$

where  $\beta = \frac{1}{2(\gamma+1)}$  and so depends only on  $n$  and  $q$ .

### 3. APPLICATIONS

In this section, we will prove as corollaries several applications of Proposition 1.1 and Theorem 1.2. Precisely, we will state new optimal estimates with ball as limiting domains and the associated stability results. Before stating the results, we will introduce in the following subsections the different operators and boundary value problems for which will obtain new eigenvalue estimates. The estimates as well as the associated stability results will be stated in Subsection 3.7.

**3.1. Extrinsic radius.** The extrinsic radius  $R(M)$  of a closed hypersurface of  $\mathbb{R}^{n+1}$  is the smallest radius of a ball containing  $M$ , that is

$$R(M) = \inf \{ r > 0 \mid \exists x \in \mathbb{R}^{n+1}, M \subset B(x, r) \}.$$

By compactness, there exists a point  $x_0$  so that  $M \subset B(x_0, R(M))$ . Thus, obviously, we have

$$R(M) \geq I_{x_0}(M).$$

where  $I_{x_0}(M)$  is the moment of inertia of  $M$  with respect to  $x_0$ .

**3.2.  $L_T$  operators.** For a positive definite symmetric and divergence-free  $(1, 1)$ -tensor  $T$  on  $M$ , we consider the operator  $L_T$  by

$$L_T u = -\operatorname{div}(T\nabla u).$$

This operator is positive, elliptic, self-adjoint and so, has a discrete spectrum of nonnegative eigenvalues

$$0 = \lambda_{0,T} < \lambda_{1,T} \leq \lambda_{2,T} \cdots \leq \lambda_{k,T} \leq \cdots \longrightarrow +\infty.$$

The eigenvalue 0 is simple and the corresponding eigenfunctions are the constants so that  $\lambda_{1,T}$  is the first positive eigenvalue.

**3.3. Generalized Steklov problems.** In this section, we will consider some generalized Steklov problems for a smooth domain  $\Omega$  of  $\mathbb{R}^{n+1}$  with boundary  $\partial\Omega = M$ . We consider  $\mathcal{T}$  a positive symmetric and divergence-free  $(1, 1)$ -tensor defined on  $\Omega$  and  $\mathcal{L}_{\mathcal{T}}$  defined for any  $\mathcal{C}^2$  function  $u$  on  $\Omega$  by  $\mathcal{L}_{\mathcal{T}}u = -\operatorname{div}(T\nabla u)$ , where  $\nabla$  is the Euclidean connection on  $\Omega$ . Then, we can consider the following generalized weighted Steklov problem

$$(S_T) \quad \begin{cases} \mathcal{L}_{\mathcal{T}}u = 0 & \text{on } \Omega, \\ \frac{\partial u}{\partial \nu_{\mathcal{T}}} = \sigma u & \text{on } M, \end{cases}$$

where  $\frac{\partial u}{\partial \nu_{\mathcal{T}}} = \langle \mathcal{T}(\nabla u), \nu \rangle$ . For  $\mathcal{T} = \operatorname{Id}$ , this problem reduces to the usual Steklov problem.

From [3], we know that this problem also have a discrete nonnegative spectrum

$$0 = \sigma_{0,\mathcal{T}} < \sigma_{1,\mathcal{T}} \leq \sigma_{2,\mathcal{T}} \cdots \leq \sigma_{k,\mathcal{T}} \leq \cdots \longrightarrow +\infty.$$

Here also, 0 is a simple eigenvalue with constant functions as corresponding eigenfunctions. Hence,  $\sigma_{1,\mathcal{T}}$  is the first positive eigenvalue.

**3.4. Steklov-Wentzell problem.** Let  $b$  be a nonnegative constant. We consider the following Steklov-type problem for the Laplacian  $\bar{\Delta}$  with the so-called Wentzell boundary condition. Namely, we consider

$$(SW) \quad \begin{cases} \bar{\Delta}u = 0 & \text{on } \Omega, \\ -b\Delta u - \frac{\partial u}{\partial \nu} = \alpha u & \text{on } M = \partial\Omega. \end{cases}$$

Here,  $\bar{\Delta}u$  and  $\Delta$  are the respective Laplacians on  $\Omega$  and  $M$ .

Obviously, if  $b = 0$ , then, we recover the classical Steklov problem. The spectrum of this problem is an increasing sequence of nonnegative eigenvalues denoted  $\{\alpha_j\}_{j \in \mathbb{N}}$  (see [8]). The first eigenvalue  $\alpha_0 = 0$  is simple and the corresponding eigenfunctions are the constant ones. We will consider  $\alpha_1$  its first positive eigenvalue.

**3.5. Biharmonic Steklov problem.** Let  $\tau$  be a positive constant. We consider the following biharmonic Steklov problem

$$(BS) \quad \begin{cases} \overline{\Delta}^2 u - \tau \overline{\Delta} u = 0 & \text{on } \Omega, \\ \frac{\partial^2 u}{\partial \nu^2} = 0 & \text{on } M = \partial\Omega, \\ \tau \frac{\partial u}{\partial \nu} - \operatorname{div}_M(P_M((\nabla^2 u)\nu)) - \frac{\partial \Delta u}{\partial \nu} = \beta u & \text{on } M = \partial\Omega. \end{cases}$$

where  $P_M$  is the projection over the tangent space of  $M$ . From the physical point of view, this problem raised when studying linear elastic structures (see [6] for instance). Here also, this problem has a discrete spectrum consisting in an increasing sequence of nonnegative numbers  $\{\beta_j\}_{j \in \mathbb{N}}$  (see [7]) with  $\beta_0 = 0$  is the (simple) first eigenvalue, with constant functions as corresponding eigenfunctions. Hence,  $\beta_1$  is the first positive eigenvalue.

**3.6.  $p$ -Laplacian and  $p$ -Steklov problem.** For  $p \in (1, +\infty)$ , we consider the so-called  $p$ -Laplacian defined by

$$\Delta_p u = -\operatorname{div}(\|\nabla u\|^{p-2} \nabla u)$$

for any  $\mathcal{C}^2$  function. For  $p = 2$ ,  $\Delta_2$  is nothing else than the Laplace-Beltrami operator of  $(M^n, g)$ .

Over the past years, this operator  $\Delta_p$ , and especially its spectrum, has been intensively studied, mainly for Euclidean domains with Dirichlet or Neumann boundary conditions (see for instance [13] and references therein) but also on Riemannian manifolds [9, 14, 20].

We will also consider the Steklov problem associated with the  $p$ -Laplacian on submanifolds with boundary of the Euclidean space. Namely, we consider the  $p$ -Steklov problem which is the following boundary value problem

$$(S_p) \quad \begin{cases} \overline{\Delta}_p u = 0 & \text{in } \Omega, \\ \|\nabla u\|^{p-2} \frac{\partial u}{\partial \nu} = \lambda |u|^{p-2} u & \text{on } M, \end{cases}$$

where  $\overline{\Delta}_p$  is the  $p$ -Laplacian on  $\mathbb{R}^{n+1}$  and  $\frac{\partial u}{\partial \nu}$  is the derivative of the function  $u$  with respect to the outward unit normal  $\nu$  to the boundary  $\partial M$ . Note that for  $p = 2$ ,  $(S_p)$  is the usual Steklov problem (the reader can for instance refer to [10] for an overview of results about the spectral geometry of the Steklov problem). Little is known about the spectra of the  $p$ -Laplacian on  $M$  or this  $p$ -Steklov problem. For both we know that there exists a sequence of positive eigenvalues

$$\lambda_{0,p} = 0 < \lambda_{1,p} \leq \lambda_{2,p} \leq \dots \leq \lambda_{k,p} \leq \dots$$

and

$$\sigma_{0,p} = 0 < \sigma_{1,p} \leq \sigma_{2,p} \leq \dots \leq \sigma_{k,p} \leq \dots$$

consisting in the variational spectrum and obtained by the Ljusternik-Schnirelmann theory (see [13, 21] for instance). One can refer to [5] for details about the Ljusternik-Schnirelmann principle. Note that, as mentioned in [14, Remark 1.1], the arguments used in [13] can be extended to domains on Riemannian manifolds

and we have that there exists a non-decreasing sequence of variational eigenvalues obtained by the Ljusternik–Schnirelman principle. Moreover, for both cases the eigenvalue 0 is simple with constant eigenfunctions and is isolated, that is there is no eigenvalue between 0 and  $\lambda_{1,p}$  or between 0 and  $\sigma_{1,p}$ . Note that all the other eigenvalues of each of both sequences also have a variational characterization but we do not know if all the spectrum is contained in this sequence.

**3.7. New eigenvalues estimates and associated pinching results.** Now, we are in position to state the following new upper bounds for all the first eigenvalues of the above operators and problems. These bounds hold for embedded hypersurfaces and are mix between isoperimetric bounds and extrinsic bounds in term of mean curvatures. Namely, we have

**Proposition 3.1.** *Let  $n \geq 2$  and  $(M^n, g)$  be a connected, oriented and closed Riemannian manifold isometrically embedded into  $\mathbb{R}^{n+1}$  by  $X$ . We denote by  $\Omega$  the compact domain bounded by  $X(M)$ .*

*Let  $b \geq 0$ ,  $\tau > 0$ ,  $p \geq 2$  and assume that  $T$  and  $\mathcal{T}$  are symmetric, positive definite and divergence-free  $(1, 1)$ -tensors on  $M$  and  $\Omega$  respectively.*

*Assume that  $S$  is a symmetric, positive definite and divergence-free  $(1, 1)$ -tensor over  $M$  so that  $H_S$  does not vanish identically.*

*Then, for any  $\alpha \in [0, 1]$ , the following inequalities hold:*

$$(17) \quad R(M) \geq \frac{(n+1)^\alpha \text{Vol}(\Omega)^\alpha \left| \int_M \text{tr}(S) dv_g \right|^{1-\alpha}}{\text{Vol}(M) \|H_S\|_2^{1-\alpha}},$$

$$(18) \quad \lambda_{1,T} \leq \left( \int_M \text{tr}(T) dv_g \right) \frac{\text{Vol}(M) \|H_S\|_2^{2(1-\alpha)}}{(n+1)^{2\alpha} \text{Vol}(\Omega)^{2\alpha} \left| \int_M \text{tr}(S) dv_g \right|^{2(1-\alpha)}}.$$

$$(19) \quad \sigma_{1,\mathcal{T}} \leq \left( \int_\Omega \text{tr}(\mathcal{T}) dv_{\bar{g}} \right) \frac{\text{Vol}(M) \|H_S\|_2^{2(1-\alpha)}}{(n+1)^{2\alpha} \text{Vol}(\Omega)^{2\alpha} \left| \int_M \text{tr}(S) dv_g \right|^{2(1-\alpha)}}.$$

$$(20) \quad \alpha_1 \leq \left[ (n+1) \text{Vol}(\Omega) + bn \text{Vol}(M) \right] \frac{\text{Vol}(M) \|H_S\|_2^{2(1-\alpha)}}{(n+1)^{2\alpha} \text{Vol}(\Omega)^{2\alpha} \left| \int_M \text{tr}(S) dv_g \right|^{2(1-\alpha)}}.$$

$$(21) \quad \beta_1 \leq \frac{\text{Vol}(M) \|H_S\|_2^{2(1-\alpha)} (n+1)^{1-2\alpha} \text{Vol}(\Omega)^{1-2\alpha}}{\left| \int_M \text{tr}(S) dv_g \right|^{2(1-\alpha)}}.$$

$$(22) \quad \lambda_{1,p} \leq \frac{n^{p/2} (n+1)^{\frac{p-2}{2} - \alpha p} \text{Vol}(M)^p \|H_S\|_2^{p(1-\alpha)}}{\text{Vol}(\Omega)^{\alpha p} \left| \int_M \text{tr}(S) dv_g \right|^{p(1-\alpha)}}.$$

$$(23) \quad \sigma_{1,p} \leq \frac{(n+1)^{(1-\alpha)p-1} \text{Vol}(M)^{p-1} \|H_S\|_2^{p(1-\alpha)}}{\text{Vol}(\Omega)^{\alpha p-1} \left| \int_M \text{tr}(S) dv_g \right|^{p(1-\alpha)}}.$$

Moreover, if equality occurs in one of these inequalities, then  $\Omega$  is a ball or radius

$$r = \frac{(n+1)^\alpha \text{Vol}(\Omega)^\alpha \left| \int_M \text{tr}(S) dv_g \right|^{1-\alpha}}{\text{Vol}(M) \|H_S\|_{2s}^{1-\alpha}}$$

and  $\text{tr}(S)$  is constant.

*Proof:* First, (17) is a direct consequence of the fact that  $R(M) \geq I_{x_0}(M)$ , where  $x_0$  the the center of the ball of radius  $R(M)$  which contains  $X(M)$  together with the lower bound of  $I_0(M)$  given in Proposition 1.1.

The other bounds are also a consequence of Proposition 1.1 together with the following inequalities obtained by using the coordinates  $X_i$ ,  $1 \leq i \leq n+1$  into the respective variational characterizations of the corresponding eigenvalues (up to a possible translation to ensure that these coordinates are eligible to be test functions:

$$\begin{aligned} \lambda_{1,T} \int_M |X|^2 dv_g &\leq \int_M \text{tr}(T) dv_g, \\ \sigma_{1,\tau} \int_M |X|^2 dv_g &\leq \int_\Omega \text{tr}(\mathcal{T}) dv_{\bar{g}}, \\ \alpha_1 \int_M |X|^2 dv_g &\leq (n+1) \text{Vol}(\Omega) + bn \text{Vol}(M), \\ \beta_1 \int_M |X|^2 dv_g &\leq n\tau \text{Vol}(\Omega), \\ \lambda_{1,p} \left( \frac{1}{\text{Vol}(M)} \int_M |X|^2 dv_g \right)^{p/2} &\leq n^{\frac{p}{2}} (n+1)^{\frac{p-2}{2}}, \\ \sigma_{1,p} \left( \frac{1}{\text{Vol}(M)} \int_M |X|^2 dv_g \right)^{p/2} &\leq (n+1)^{p-1} \frac{\text{Vol}(\Omega)}{\text{Vol}(M)}. \end{aligned}$$

We can refer to [9, 17, 18] for a proof of these different inequalities. Once we have them, the proof of the different inequalities is immediate with the lower bound of  $I_0(M)$  given by Proposition 1.1. Therefore, the equality case is also immediate from Proposition 1.1.  $\square$

Now, we introduce the following seven pinching conditions in order to study the stability of the limiting case for the above optimal inequalities. For  $\varepsilon > 0$ ,  $s > 1$ ,  $\alpha \in [0, 1]$  and  $p \geq 2$ , we set

$$\begin{aligned} (\mathcal{R}_{\varepsilon,s,\alpha}) \quad R(M)^2 &\leq (1+\varepsilon) \frac{(n+1)^\alpha \text{Vol}(\Omega)^\alpha \left| \int_M \text{tr}(S) dv_g \right|^{1-\alpha}}{\text{Vol}(M) \|H_S\|_{2s}^{1-\alpha}} \\ (\mathcal{T}_{\varepsilon,s,\alpha}) \quad \lambda_{1,T} &\geq (1-\varepsilon) \left( \int_M \text{tr}(T) dv_g \right) \frac{\text{Vol}(M) \|H_S\|_{2s}^{2(1-\alpha)}}{(n+1)^{2\alpha} \text{Vol}(\Omega)^{2\alpha} \left| \int_M \text{tr}(S) dv_g \right|^{2(1-\alpha)}}. \end{aligned}$$

$$\begin{aligned}
(\mathcal{S}_{\varepsilon,s,\alpha}) \quad \sigma_{1,T} &\geq (1-\varepsilon) \left( \int_{\Omega} \operatorname{tr}(T) dv_{\bar{g}} \right) \frac{\operatorname{Vol}(M) \|H_S\|_{2s}^{2(1-\alpha)}}{(n+1)^{2\alpha} \operatorname{Vol}(\Omega)^{2\alpha} \left| \int_M \operatorname{tr}(S) dv_g \right|^{2(1-\alpha)}}. \\
(\mathcal{W}_{\varepsilon,s,\alpha}) \quad \alpha_1 &\geq (1-\varepsilon) \left[ (n+1) \operatorname{Vol}(\Omega) + bn \operatorname{Vol}(M) \right] \frac{\operatorname{Vol}(M) \|H_S\|_{2s}^{2(1-\alpha)}}{(n+1)^{2\alpha} \operatorname{Vol}(\Omega)^{2\alpha} \left| \int_M \operatorname{tr}(S) dv_g \right|^{2(1-\alpha)}}. \\
(\mathcal{B}_{\varepsilon,s,\alpha}) \quad \beta_1 &\geq (1-\varepsilon) \frac{\operatorname{Vol}(M) \|H_S\|_{2s}^{2(1-\alpha)} (n+1)^{1-2\alpha} \operatorname{Vol}(\Omega)^{1-2\alpha}}{\left| \int_M \operatorname{tr}(S) dv_g \right|^{2(1-\alpha)}}. \\
(\mathcal{L}_{p,\varepsilon,s,\alpha}) \quad \lambda_{1,p} &\geq (1-\varepsilon) \frac{n^{p/2} (n+1)^{\frac{p-2}{2}-\alpha p} \operatorname{Vol}(M)^p \|H_S\|_{2s}^{p(1-\alpha)}}{\operatorname{Vol}(\Omega)^{\alpha p} \left| \int_M \operatorname{tr}(S) dv_g \right|^{p(1-\alpha)}}. \\
(\Sigma_{p,\varepsilon,s,\alpha}) \quad \sigma_{1,p} &\geq (1-\varepsilon) \frac{(n+1)^{(1-\alpha)p-1} \operatorname{Vol}(M)^{p-1} \|H_S\|_2^{p(1-\alpha)}}{\operatorname{Vol}(\Omega)^{\alpha p-1} \left| \int_M \operatorname{tr}(S) dv_g \right|^{p(1-\alpha)}}.
\end{aligned}$$

Note that like for  $(\mathcal{I}_{\varepsilon,s,\alpha})$ , we consider  $L^{2s}$ -norm for  $H_S$  instead of its  $L^2$ -norm, for  $s > 1$ .

We have the following proposition which gives a link between the last seven pinching condition and the pinching of  $I_0(M)$ .

**Proposition 3.2.** *• If  $(\mathcal{R}_{\varepsilon,s,\alpha})$  holds then also holds  $(\mathcal{I}_{\varepsilon,s,\alpha})$   
• If one of the pinching conditions  $(\mathcal{T}_{\varepsilon,s,\alpha})$ ,  $(\mathcal{S}_{\varepsilon,s,\alpha})$ ,  $(\mathcal{W}_{\varepsilon,s,\alpha})$ ,  $(\mathcal{B}_{\varepsilon,s,\alpha})$ ,  $(\mathcal{L}_{p,\varepsilon,s,\alpha})$  or  $(\Sigma_{p,\varepsilon,s,\alpha})$  holds for  $\varepsilon \in (0, \frac{1}{2})$ , then also holds  $(\mathcal{I}_{2\varepsilon,s,\alpha})$ .*

*Proof:* First, obviously, for the extrinsic radius, we have  $R(M) \geq I_0(M)$ , where, up to a possible translation, we have 0 is the center of the ball of radius  $R(M)$  which contains  $M$ . Hence, we have  $(\mathcal{R}_{\varepsilon,s,\alpha})$  implies  $(\mathcal{I}_{\varepsilon,s,\alpha})$ .

Now, we consider the other cases without giving the details for all cases since they are similar. For instance, for  $\lambda_{1,T}$ , we have, from the variational characterization used with coordinates as test function, as seen in the proof of the last proposition, that

$$\lambda_{1,T} \int_M |X|^2 dv_g \leq \int_M \operatorname{tr}(T) dv_g,$$

or in other words

$$(24) \quad \lambda_{1,T} I_0(M)^2 V(M) \leq \int_M \operatorname{tr}(T) dv_g.$$

Moreover, from the assumption that  $(\mathcal{T}_{\varepsilon,s,\alpha})$  holds, we have

$$\lambda_{1,T} \geq (1-\varepsilon) \left( \int_M \operatorname{tr}(T) dv_g \right) \frac{\operatorname{Vol}(M) \|H_S\|_{2s}^{2(1-\alpha)}}{(n+1)^{2\alpha} \operatorname{Vol}(\Omega)^{2\alpha} \left| \int_M \operatorname{tr}(S) dv_g \right|^{2(1-\alpha)}},$$



which gives together with (24)

$$I_0(M) \leq \frac{1}{1-\varepsilon} \frac{(n+1)^{2\alpha} \text{Vol}(\Omega)^{2\alpha} \left| \int_M \text{tr}(S) dv_g \right|^{2(1-\alpha)}}{\text{Vol}(M)^2 \|H_S\|_{2s}^{2(1-\alpha)}}.$$

If we take  $\varepsilon < \frac{1}{2}$ , then we get

$$I_0(M) \leq (1+2\varepsilon) \frac{(n+1)^{2\alpha} \text{Vol}(\Omega)^{2\alpha} \left| \int_M \text{tr}(S) dv_g \right|^{2(1-\alpha)}}{\text{Vol}(M)^2 \|H_S\|_{2s}^{2(1-\alpha)}}.$$

which means that is  $\varepsilon < \frac{1}{2}$ , then the condition  $(\mathcal{T}_{\varepsilon,s,\alpha})$  implies  $(\mathcal{I}_{2\varepsilon,s,\alpha})$ . The proof is exactly the same for the other conditions  $(\mathcal{S}_{\varepsilon,s,\alpha})$ ,  $(\mathcal{W}_{\varepsilon,s,\alpha})$ ,  $(\mathcal{B}_{\varepsilon,s,\alpha})$ ,  $(\mathcal{L}_{p,\varepsilon,s,\alpha})$  or  $(\Sigma_{p,\varepsilon,s,\alpha})$ .  $\square$

We finish by the following corollary which answers to question of the stability of the limiting case of Proposition 3.1. It is an immediate consequence of the above proposition together with Theorem 1.2.

**Corollary 3.3.** *Let  $n \geq 2$  and  $(M^n, g)$  be a connected, oriented and closed Riemannian manifold isometrically embedded into  $\mathbb{R}^{n+1}$  by  $X$ . We denote by  $\Omega$  the compact domain bounded by  $X(M)$ .*

*Let  $q > n$ ,  $b \geq 0$ ,  $\tau > 0$ ,  $p \geq 2$ ,  $T$  and  $\mathcal{T}$  two symmetric, positive definite and divergence-free  $(1,1)$ -tensors on  $M$  and  $\Omega$  respectively.*

*Assume that  $S$  is a symmetric, positive definite and divergence-free  $(1,1)$ -tensor over  $M$  so that  $H_S$  does not vanish identically.*

*Then there exists three positive constants,  $\varepsilon_1$  and  $C'$ , depending  $n$ ,  $q$ , and  $\text{Vol}(M)\|B\|_q^n$ , and  $\alpha$  depending only on  $n$  and  $q$  such that if one of  $(\mathcal{R}_{\varepsilon,s,\alpha})$ ,  $(\mathcal{T}_{\varepsilon,s,\alpha})$ ,  $(\mathcal{S}_{\varepsilon,s,\alpha})$ ,  $(\mathcal{W}_{\varepsilon,s,\alpha})$ ,  $(\mathcal{B}_{\varepsilon,s,\alpha})$ ,  $(\mathcal{L}_{p,\varepsilon,s,\alpha})$  or  $(\Sigma_{p,\varepsilon,s,\alpha})$  holds with  $\varepsilon < \varepsilon_1$ , then  $M$  is diffeomorphic and  $C'\varepsilon^\alpha$ -almost-isometric to the sphere  $S(0,r)$  of radius*

$$r = \frac{(n+1)^\alpha \text{Vol}(\Omega)^\alpha \left| \int_M \text{tr}(S) dv_g \right|^{1-\alpha}}{\text{Vol}(M) \|H_S\|_{2s}^{1-\alpha}}.$$

*Proof:* We just apply Proposition 3.2 and then Theorem 1.2 with  $2\varepsilon$ . Hence  $\varepsilon$  should be smaller than  $\varepsilon_1 = \frac{\varepsilon_0}{2}$  and  $M$  is diffeomorphic and  $C(2\varepsilon)^\alpha = C'\varepsilon^\alpha$ , with  $C' = 2^\alpha C$ , where  $\varepsilon_0$ ,  $C$  and  $\alpha$  are given in Theorem 1.2.  $\square$

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