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Hyperbolic Radial Spanning Tree*

David Coupier [†], Lucas Flammant [‡] and Viet Chi Tran [§]

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We define and analyze an extension to the d -dimensional hyperbolic space of the Radial Spanning Tree (RST) introduced by Baccelli and Bordenave in the two-dimensional Euclidean space (2007). In particular, we will focus on the description of the infinite branches of the tree. The properties of the two-dimensional Euclidean RST are extended to the hyperbolic case in every dimension: almost surely, every infinite branch admits an asymptotic direction and each asymptotic direction is reached by at least one infinite branch. Moreover, the branch converging to any deterministic asymptotic direction is unique almost surely. To obtain results for any dimension, a completely new approach is considered here. Our strategy mainly involves the two following ingredients, that rely on the hyperbolic Directed Spanning Forest (DSF) introduced and studied in Flammant (2019). First, the hyperbolic metric allows us to obtain fine control of the branches' fluctuations in the hyperbolic DSF without using planarity arguments. Then, we couple the hyperbolic RST with the hyperbolic DSF and conclude thanks to the precise estimates mentioned before.

Key words: continuum percolation, hyperbolic space, stochastic geometry, random geometric tree, Radial Spanning Tree, Directed Spanning Forest, Poisson point processes.

AMS 2010 Subject Classification: Primary 60D05, 60K35, 82B21.

1. Introduction

Geometric random trees are well studied in the literature since they interact with many other fields, such as communication networks, particles systems or population dynamics (e.g. [15]). Several works have established scaling limits for two-dimensional radial trees [10, 9] and translation invariant forests [11, 27, 16]. In addition, random spanning trees appear in the context of first passage percolation [17]. A complete introduction to geometric random graphs is given in [23].

Here we are interested in the Radial Spanning Tree (RST), introduced by Baccelli and Bordenave [1] in the Euclidean plane and with motivations from communication networks. The construction of this tree is the same on the plane \mathbb{R}^2 or on the hyperbolic space \mathbb{H}^{d+1} (presented below). The set of vertices is given by a homogeneous Poisson Point Process (PPP) \mathcal{N} of intensity λ . The RST rooted at the origin 0 is the graph obtained by connecting each point $z \in \mathcal{N}$ to its parent $A(z)$, defined as the closest point to z among all points $z' \in \mathcal{N} \cup \{0\}$ that are closer to the origin than z . This defines a random tree rooted at the origin with a radial structure. Given a path, we will say that the forward direction is towards 0 and the backward direction is towards infinity. An infinite

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backward path is defined as a sequence of Poisson points $(z_n)_{n \geq 0} \in (\mathcal{N} \cup \{0\})^{\mathbb{N}}$ with $z_0 = 0$ and $z_n = A(z_{n+1})$ for any $n \geq 0$. Because a vertex can be the ancestor of no other vertex, all backward paths are not infinite.

The topological properties of the bi-dimensional Euclidean RST are well-understood. Baccelli and Bordenave showed that almost surely (a.s.), any infinite backward path admits an asymptotic direction, i.e. that a.s. for any infinite path $(z_n)_{n \geq 0}$, the sequence $(z_n/|z_n|)_{n \geq 0}$ converges to a limit in the unit sphere \mathbb{S}^1 of \mathbb{R}^2 . The limit is called the asymptotic direction of the infinite path. Moreover, a.s., every asymptotic direction of \mathbb{S}^1 is reached by at least one infinite backward path and there exists a.s. a unique infinite path in any given deterministic asymptotic direction [1]. These results on the infinite paths are completed by Baccelli, Coupier & Tran [2].

For any integer $d \geq 1$, the hyperbolic space \mathbb{H}^{d+1} is a $(d+1)$ -dimensional Riemannian manifold with constant negative curvature, that can be chosen equal to -1 without loss of generality. It admits a set of ideal boundary points $\partial\mathbb{H}^{d+1}$, and $\overline{\mathbb{H}^{d+1}} := \mathbb{H}^{d+1} \cup \partial\mathbb{H}^{d+1}$ denotes the hyperbolic space endowed with its boundary. It is a non-amenable space, i.e. the measure of the boundary of a large subset is not negligible with respect to its volume. The hyperbolic space is defined in more details in [7, 8, 22, 26].

There is a growing interest for the study of random models in a hyperbolic setting. Benjamini and Schramm establish percolation results on regular tilings and Voronoï tessellations in the hyperbolic plane [3]. Mean characteristics of the Poisson-Voronoi tessellation have also been considered in a general Riemannian manifold by Calka et al. [6]. This interest is explained by at least two reasons. First, hyperbolic random graphs are well-fitted to model social networks [5]. Secondly, strong differences have been noticed for properties of random models depending on whether they are considered in an Euclidean or in a hyperbolic setting. For example, some hyperbolic random graphs admit a non-degenerate regime with infinitely many unbounded components in the hyperbolic space [28, 18], which is generally not the case in the Euclidean space. In addition, behaviors of non-amenable spaces are well studied in a discrete context [4, 20, 24].

Thus it is natural to consider and study the hyperbolic RST, which we define in the same way as the Euclidean RST. A simulation of the two-dimensional hyperbolic RST is given in Figure 1. In this paper, we extend the results of Baccelli and his coauthors to hyperbolic geometry **in every dimension**. Here is our main result:

Theorem 1.1. *For any dimension $d \geq 1$ and any intensity λ , the following happens:*

- (i) *almost surely, any infinite backward path $(z_n)_{n \in \mathbb{N}}$ admits an asymptotic direction, i.e. there exists $z_\infty \in \partial\mathbb{H}^{d+1}$ such that $\lim_{n \rightarrow \infty} z_n = z_\infty$ (in the sense of the topology of $\overline{\mathbb{H}^{d+1}}$);*
- (ii) *almost surely, for any $z_\infty \in \partial\mathbb{H}^{d+1}$, there exists an infinite backward path (z_n) with asymptotic direction z_∞ ;*
- (iii) *for any deterministic boundary point $z_\infty \in \partial\mathbb{H}^{d+1}$, the path with asymptotic direction z_∞ is almost surely unique;*
- (iv) *almost surely, the random set of boundary points with at least two infinite backward paths is dense in $\partial\mathbb{H}^{d+1}$;*
- (v) *almost surely, this random set is countable in the bi-dimensional case (i.e. $d = 1$).*

Theorem 1.1 describes the infinite branches of the hyperbolic RST: every infinite branch admits an asymptotic direction and Point (ii) and (iii) say that for any fixed and deterministic boundary point z_∞ , there exists a unique infinite path having z_∞ as asymptotic direction. But there is a random dense set of boundary points having more than one backwards paths, and in dimension 2 (for $d = 1$) this set is countable.

Establishing the results announced in Theorem 1.1 in every dimension constitutes the main originality of this paper. For the two reasons explained further, the proofs of Baccelli and Bordenave in the 2D-Euclidean setting [1] cannot be generalized to higher dimensions.

In both contexts \mathbb{R}^2 and \mathbb{H}^{d+1} , for any $d \geq 1$, the proofs of (i), (ii), (iv) and (v) of Theorem 1.1 follow the strategy of Howard and Newman [17], which is to show that the tree is straight, that is,

the descendant subtree of a vertex far from the origin is included in a thin cone. To prove that the 2D-Euclidean RST is straight, Baccelli and Bordenave used a translation invariant model derived from the RST: the Directed Spanning Forest (DSF), which constitutes a local approximation of the RST far from the origin [1]. They exploit the theory of Markov chains to bound from above fluctuations of trajectories in the DSF and then, they deduce the straightness of the RST via planarity. This strategy cannot be generalized to higher dimensions. However, in \mathbb{H}^{d+1} , we manage to control the angular deviations of branches in the RST without resorting to an auxiliary model that requires planarity as in the Euclidean setting. The hyperbolic metric guarantees that angular deviations decay exponentially fast with the distance to the origin, which is strong enough to show straightness.

In addition, in the Euclidean context, the uniqueness part (point (iii) in Theorem 1.1) is only proved in dimension 2 since it strongly uses planarity [17, 1], and the strategy of proof cannot be generalized to higher dimensions. To prove (iii) in \mathbb{H}^{d+1} , our strategy consists in exploiting the link existing between the hyperbolic RST and the hyperbolic DSF, defined and studied in Flammant [12], which is the hyperbolic counterpart of the Euclidean DSF used by Baccelli and Bordenave. Roughly speaking, the hyperbolic DSF can be defined as the limit of the hyperbolic RST when the origin point tends to an ideal boundary point. Similarly to the Euclidean setting, it constitutes a local approximation of the RST far from the origin. The proof of (iii) exploits the coalescence of the hyperbolic DSF (i.e. it is almost surely a tree) [12, Theorem 1.1], which is a non-trivial fact obtained by exploiting the mass-transport principle, and a local coupling between the two models.

After defining the hyperbolic RST and giving its basic properties, we define two quantities that encode angular fluctuations along trajectories, the Cumulative angular Forward Deviations (CFD) and the Maximal Backward Deviations (MBD). We then establish upper bounds of these quantities: first, we upper-bound the Maximal Backward Deviations in a thin annulus of width $\delta > 0$ (Proposition 2.5) and then we deduce a global control of MBD in the whole space (Proposition 2.6), that roughly says that angular deviations decay exponentially fast with the distance to the origin. From this upper-bound, we deduce that the RST is straight in the sense of Howard & Newman (Proposition 2.7). The points (i), (ii), (iv) and (v) in Theorem 1.1 can be deduced from straightness and the upper-bound of MBD given by Proposition 2.6. The point (iii) (the uniqueness part) is done by exploiting a local coupling existing between the RST and the DSF far from the origin.

The rest of paper is organized as follows. In Section 2, we set some reminders of hyperbolic geometry and we define the hyperbolic RST. Then, we give its basic properties and a road-map of the proofs. We also announce the upper bounds of angular deviations (Propositions 2.5 and 2.6) and the straightness property (Proposition 2.7). The proof of Theorem 1.1 is done in Section 3. Proposition 2.5 is proved in Section 4 and the proofs of Propositions 2.6 and 2.7 are done in Section 5.

2. Definitions, notations and basic properties

We denote by \mathbb{N} the set of non-negative integers and by \mathbb{N}^* the set of positive integers. In the rest of the paper, c (resp. C) will be some small (resp. large) constant whose value can change from a line to another.

2.1. The hyperbolic space

We refer to [26] for a complete introduction to hyperbolic geometry. For $d \in \mathbb{N}^*$, the $(d+1)$ -dimensional hyperbolic space, denoted by \mathbb{H}^{d+1} , is a $(d+1)$ -dimensional Riemannian manifold of constant negative curvature -1 that can be defined by several isometric models. One of them is the open-ball model consisting in the unit open ball

$$I = \{(x_1, \dots, x_{d+1}) \in \mathbb{R}^{d+1}, x_1^2 + \dots + x_{d+1}^2 < 1\} \quad (2.1)$$

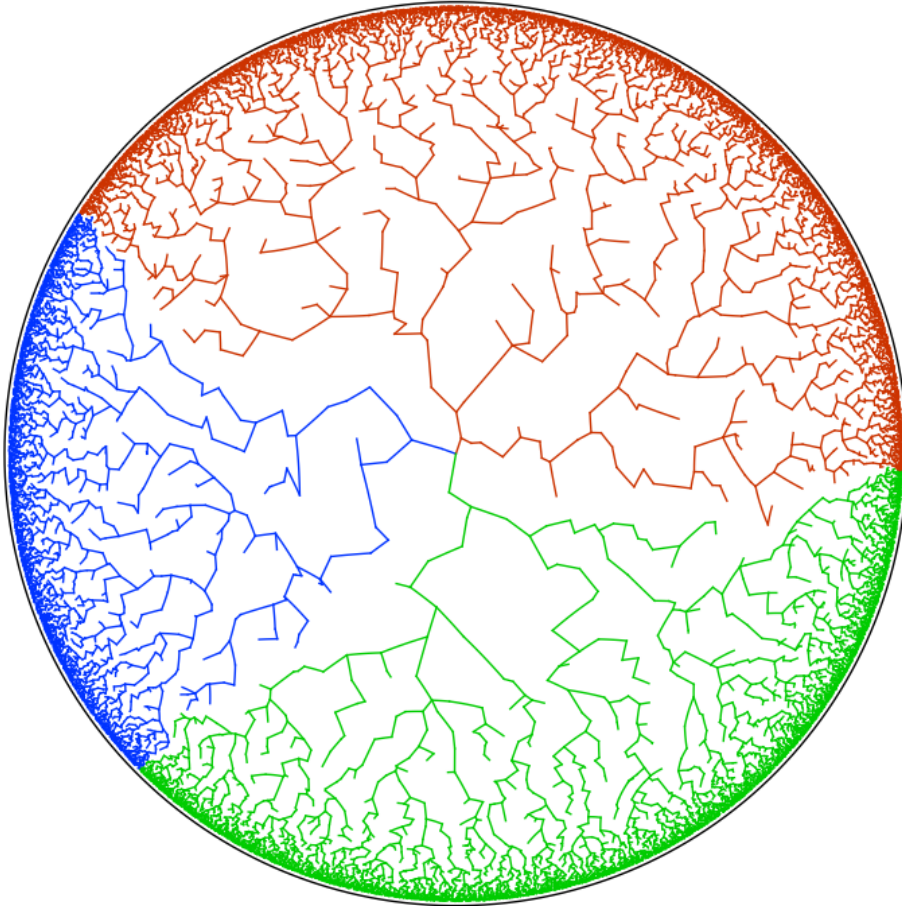


Figure 1: Simulation of the two-dimensional hyperbolic RST, with $\lambda = 30$, in the Poincaré disc model. The edges are represented by geodesics. The different connected components of the RST (apart from the root) are represented with different colors.

endowed with the following metric:

$$ds_I^2 := 4 \frac{dx_1^2 + \dots + dx_{d+1}^2}{(1 - x_1^2 - \dots - x_{d+1}^2)^2}. \quad (2.2)$$

We denote by $d(\cdot, \cdot)$ the hyperbolic distance in \mathbb{H}^{d+1} , and by $\|\cdot\|$ the Euclidean norm in \mathbb{R}^d , with the convention $\|\infty\| = \infty$.

The volume measure on (I, ds_I^2) , denoted by Vol_I , is given by

$$d\text{Vol}_I = 2^{d+1} \frac{dx_1 \dots dx_{d+1}}{(1 - x_1^2 - \dots - x_{d+1}^2)^{d+1}}. \quad (2.3)$$

An important fact about hyperbolic geometry is that \mathbb{H}^{d+1} is homogeneous, isotropic and rotation invariant. It means that the group of isometries of \mathbb{H}^{d+1} acts transitively on the unit tangent bundle of \mathbb{H}^{d+1} : given two points $x, y \in \mathbb{H}^{d+1}$ and two unit tangent vectors $u \in T_x \mathbb{H}^{d+1}$, $v \in T_y \mathbb{H}^{d+1}$, there exists an isometry g of \mathbb{H}^{d+1} such that $g(x) = y$ and that pushes forward u on v . The notations T_x, T_y and the vocabulary relating to Riemannian geometry are defined in [19]. We refer to [22, Proposition 1.2.1 p.5] for a proof.

Let $0 \in \mathbb{H}^{d+1}$ be some arbitrary origin point (it can be thought as the center of the ball in the open-ball representation), which will play the role of the root of the RST. The hyperbolic space \mathbb{H}^{d+1} is naturally equipped with a set of points at infinity, denoted by $\partial \mathbb{H}^{d+1}$.

In the open-ball model (I, ds_I^2) , the set of points at infinity is identified by the boundary unit sphere. Let us denote by \mathbb{S}^d the unit Euclidean sphere in \mathbb{R}^{d+1} and by ν its d -dimensional volume measure. We denote by $\overline{\mathbb{H}^{d+1}} := \mathbb{H}^{d+1} \cup \partial\mathbb{H}^{d+1}$ the hyperbolic space \mathbb{H}^{d+1} plus the set of points at infinity, with the topology given by the closed ball. A point $z_\infty \in \partial\mathbb{H}^{d+1}$ is called ideal point or point at infinity.

The metric becomes smaller as we get closer to the boundary unit sphere ∂I , and this boundary is at infinite distance from the center 0.

For any subset $E \subset \overline{\mathbb{H}^{d+1}}$, \overline{E} denotes the closure of E in $\overline{\mathbb{H}^{d+1}}$.

In the open ball model (I, ds_I^2) , the geodesics are of two types: the diameters of I and the arcs that are perpendicular to the boundary unit sphere \mathbb{S}^d , see Figure 2. We refer to discussion [7, p.80] for a proof. This model is conformal, which means that the hyperbolic angle between two geodesics corresponds to their Euclidean angle in the open-ball representation. For $z_1, z_2, z_3 \in \overline{\mathbb{H}^{d+1}}$, $\widehat{z_1 z_2 z_3}$ is the measure of the corresponding (non-oriented) angle.

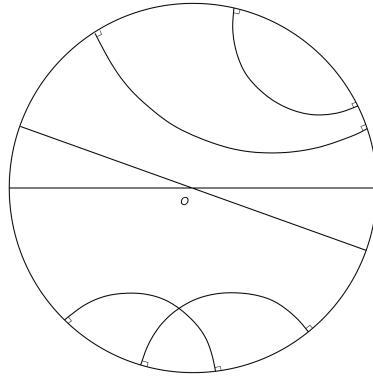


Figure 2: Geodesics in the open ball model

For $z_1, z_2 \in \overline{\mathbb{H}^{d+1}}$, let us denote by $[z_1, z_2]$ the geodesic between z_1 and z_2 . Moreover, we set the notations:

$$[z_1, z_2[:= [z_1, z_2] \setminus \{z_2\}, \quad]z_1, z_2] := [z_1, z_2] \setminus \{z_1\}, \quad]z_1, z_2[\setminus (\{z_2\} \cup \{z_1\}).$$

Let us denote by $]z_1, z_2[$ (resp. $(z_1, z_2]$) the semi-geodesic passing through z_2 (resp. z_1) and ending at z_1 (resp. z_2).

For $z \in \mathbb{H}^{d+1}$ and $r > 0$, we denote by $B(z, r) := \{z' \in H, d(z, z') < r\}$ (resp. $S(z, r) := \{z' \in H, d(z, z') = r\}$) the hyperbolic ball (resp. *sphere*) centered at z of radius r , and we set $B(r) := B(0, r)$ (resp. $S(r) := S(0, r)$). For $x \in \mathbb{R}^d$ and $r > 0$, let us also denote by $B_{\mathbb{R}^d}(x, r) := \{x' \in \mathbb{R}^d, \|x' - x\| < r\}$ the Euclidean ball centered at x of radius r .

For any point $z \in \overline{\mathbb{H}^{d+1}}$ and $\theta > 0$, $\text{Cone}(z, \theta) := \{z' \in \mathbb{H}^{d+1}, \widehat{z0z'} \leq \theta\}$ is defined as the cone of apex 0 and aperture θ (if $\theta \geq \pi$ then $\text{Cone}(z, \theta)$ is the whole space \mathbb{H}^{d+1}).

Since the RST is a graph rooted at 0, a convenient way to represent points in \mathbb{H}^{d+1} is to use polar coordinates. Recall that 0 is the origin point. For any point $z \in \mathbb{H}^{d+1}$, we denote by $z = (r; u)$ its polar coordinates w.r.t. 0: r is its distance to 0 and $u \in UT_0\mathbb{H}^{d+1} \simeq \mathbb{S}^d$ is its direction ($UT_0\mathbb{H}^{d+1}$ is the unitary tangent space of 0 in \mathbb{H}^{d+1} , consisting of tangent vectors of norm 1). In polar coordinates, the volume measure Vol is given by

$$d\text{Vol}(r; u) = \sinh(r)^d dr d\nu(u). \quad (2.4)$$

A direct consequence is that the volume of a ball of radius r is given by:

$$\text{Vol}(B(r)) = \int_0^r \sinh(t)^d dt \times \nu(\mathbb{S}^d) \asymp e^{dr} \text{ when } r \rightarrow \infty, \quad (2.5)$$

where \asymp means that $\text{Vol}(B(r))$ is asymptotically proportional to e^{dr} when $r \rightarrow +\infty$. We refer the reader to [8, Section III.3 and (III.4.1)].

The hyperbolic law of cosines [25, (6.3-5) p.197] is a well adapted tool to compute distances using polar coordinates. Given $z_1 = (r_1; u_1)$, $z_2 = (r_2, u_2) \in \mathbb{H}^{d+1}$, the hyperbolic law of cosines gives,

$$\cosh d(z_1, z_2) = \cosh(r_1) \cosh(r_2) - \langle u_1, u_2 \rangle \sinh(r_1) \sinh(r_2). \quad (2.6)$$

2.2. The hyperbolic RST

In the rest of the paper, the dimension $d \geq 1$ and the intensity $\lambda > 0$ are fixed. Let \mathcal{N} be a homogeneous PPP of intensity λ in \mathbb{H}^{d+1} . The definition of the hyperbolic RST is similar to the Euclidean case. The set of vertices is $\mathcal{N} \cup \{0\}$. Each vertex $z \in \mathcal{N}$ has a unique outgoing edge that connects z to the closest Poisson point among those that are closer to the origin:

Definition 2.1 (Radial Spanning Tree in \mathbb{H}^{d+1}). For any $z = (r; u) \in \mathcal{N}$, the parent (or ancestor) of z is defined as

$$A(z) := \underset{z' \in (\mathcal{N} \cup \{0\}) \cap B(r)}{\text{argmin}} \quad d(z', z).$$

We call Radial Spanning Tree (RST) in \mathbb{H}^{d+1} rooted at 0 the oriented graph (V, \vec{E}) where

$$V := \mathcal{N} \cup \{0\}, \quad \vec{E} := \{(z, A(z)), z \in \mathcal{N}\}.$$

It is possible to assume that $\mathcal{N} \cup \{0\}$ does not contain isosceles triangles, since this event has probability 1. Thus the ancestor $A(z)$ is well-defined, and clearly the RST is a tree rooted at 0.

For $z \in \mathcal{N} \cup \{0\}$ and $k \in \mathbb{N}$, let us define the k -th ancestor of z by $A^{(k)}(z) = A \circ \dots \circ A(z)$, where composition is taken k times, and their descents after k generations by $A^{(-k)}(z) = \{z' \in \mathcal{N}, A^{(k)}(z') = z\}$ (in particular $A^{(-1)}(z)$ is the set of daughters of z). For $z \in \mathcal{N}$ and $r \geq 0$, let us define

$$B^+(z, r) := B(z, r) \cap B(0, d(0, z)) \text{ and } B^+(z) := B^+(z, d(z, A(z))). \quad (2.7)$$

By definition of the parent, $B^+(z) \cap \mathcal{N} = \emptyset$ for all $z \in \mathcal{N}$. This fact is responsible for many difficulties when studying the RST. Indeed, when restarting from $A(z) = (r'; u')$ and constructing the path forward (towards 0), with probability one $B(r') \cap B^+(z, d(z, A(z))) \neq \emptyset$. This means that the geometric information used to determine $A(z)$ is still involved for later steps of the process, generating statistical and geometrical dependencies. **Properties such as random walks or martingales can not be used in our context, and we resort here to the control of maximal backward deviations permitted by our hyperbolic setting.**

To end this section, let us mention the following basic properties about RST proved in Appendix A.

Proposition 2.2. *The RST is a tree and it has finite degree a.s. Moreover, in the bi-dimensional case ($d = 1$), the representation of the RST obtained by connecting each vertex $z \in \mathcal{N}$ to its parent $A(z)$ by the geodesic $[z, A(z)]$ is planar, i.e. there is no two points $z_1, z_2 \in \mathcal{N}$ such that $[z_1, A(z_1)] \cap [z_2, A(z_2)] \neq \emptyset$.*

In the above proposition, the edges are geodesics. However, Definition 2.1 does not specify the shape of edges. Since the results announced in Theorem 1.1 only concern the graph structure of the hyperbolic RST, their veracity does not depend on the geometry of edges. Although it is more natural to represent edges with hyperbolic geodesics, we do another choice in the sequel which will appear more convenient for the proofs. Given $z_1 = (r_1; u_1), z_2 = (r_2; u_2) \in \mathbb{H}^{d+1}$ such that $0 \notin [z_1, z_2]$, we define a path $[z_1, z_2]^*$, in an isotropic way, verifying the two following conditions:

- i) the distance to the origin 0 is monotone along the path $[z_1, z_2]^*$,
- ii) the distance to z_1 is also monotone along this path.

It is a **simplification** for the proofs that the shape of edges satisfy conditions (i) and (ii): remark that the geodesic $[z_1, z_2]$ or the Euclidean segment between z_1 and z_2 do not satisfy condition (i) in general. Since $0 \notin [z_1, z_2]$, u_1 and u_2 are not antipodal. Thus one can consider the unique geodesic path $\gamma_{u_1, u_2} : [0, 1] \rightarrow UT_0\mathbb{H}^{d+1}$ on the sphere with constant speed connecting u_1 to u_2 , and define the path $[z_1, z_2]^*$ as

$$\begin{aligned} [0, 1] &\rightarrow \mathbb{H}^{d+1} \\ t &\mapsto ((1-t)r_1 + tr_2; \gamma_{u_1, u_2}(\widehat{\phi_{r_1, r_2, \widehat{u_1, u_2}}}(t))), \end{aligned} \quad (2.8)$$

where $\widehat{\phi_{r_1, r_2, \widehat{u_1, u_2}}} : [0, 1] \rightarrow [0, 1]$ is defined as:

$$\widehat{\phi_{r_1, r_2, \widehat{u_1, u_2}}}(t) := \frac{1}{\widehat{u_1, u_2}} \arccos \left(\frac{(1-t)\sinh(r_1) + t\cos(\widehat{u_1, u_2})\sinh(r_2)}{\sinh((1-t)r_1 + tr_2)} \right).$$

This function $\widehat{\phi_{r_1, r_2, \widehat{u_1, u_2}}}$ is built to ensure that the distance to the origin z_1 is monotone along the path $[z_1, z_2]^*$. Indeed, by the hyperbolic law of cosines (2.6),

$$\begin{aligned} &\cosh d(z_1, ((1-t)r_1 + tr_2; \gamma_{u_1, u_2}(\widehat{\phi_{r_1, r_2, \widehat{u_1, u_2}}}(t)))) \\ &= \cosh(r_1) \cosh((1-t)r_1 + tr_2) - \cos(\widehat{\phi_{r_1, r_2, \widehat{u_1, u_2}}}(t)(\widehat{u_1, u_2})) \sinh(r_1) \sinh((1-t)r_1 + tr_2) \\ &= t [\cosh(r_1) \cosh(r_2) - \cos(\widehat{u_1, u_2}) \sinh(r_1) \sinh(r_2)] \end{aligned}$$

is monotone in t .

We define $[z_1, z_2]^* := [z_1, z_2]^* \setminus \{z_2\}$ and $]z_1, z_2]^* := [z_1, z_2]^* \setminus \{z_1\}$. It is possible to assume that \mathcal{N} does not contain two points z_1, z_2 such that $0 \in [z_1, z_2]$ since this event has probability 1. Let us now define the random set RST by connecting each point $z \in \mathcal{N}$ to $A(z)$ by the path $[z, A(z)]^*$:

$$\text{RST} := \bigsqcup_{z \in \mathcal{N}} [z, A(z)]^*.$$

It may exist some points z belonging to several paths $[z_1, A(z_1)]^*, \dots, [z_k, A(z_k)]^*$; in that case, z is counted with multiplicity k in RST. Formally, we should define the RST as $\bigcup_{z \in \mathcal{N}} [z, A(z)]^* \times \{z\} \subset \mathbb{H}^{d+1} \times \mathbb{H}^{d+1}$, i.e. an element $(z, z') \in \text{RST}$ is a couple where $z \in \mathbb{H}^{d+1}$ is a point of the RST and z' is the root of an edge containing z . For $(z, z') \in \text{RST}$, we define

$$z_{\downarrow} = z', \quad z_{\uparrow} = A(z'). \quad (2.9)$$

With this formalism, edges of the RST do not cross anymore, apart at vertices. For the sake of simplification, we will commit an abuse of notations by considering that $\text{RST} \subset \mathbb{H}^{d+1}$ and identifying an element $(z, z') \in \text{RST}$ to the corresponding point $z \in \mathbb{H}^{d+1}$. Given $z \in \text{RST}$, let $n := \min\{k \geq 0, A^{(k)}(z_{\uparrow}) = 0\}$ be the number of steps required to reach the origin from z_{\uparrow} ; we define the trajectory from z as

$$\pi(z) := [z, z_{\uparrow}]^* \cup \bigcup_{k=0}^{n-1} [A^{(k)}(z_{\uparrow}), A^{(k+1)}(z_{\uparrow})]^*.$$

For $r > 0$, we define the level r as

$$\mathcal{L}_r := \text{RST} \cap S(r).$$

For $0 < r \leq r'$ and for $z' \in \mathcal{L}_{r'}$, the ancestor at level r of z' , denoted by $\mathcal{A}_r^{r'}(z')$ is the intersection point of $\pi(z')$ and $S(r)$. For $0 < r \leq r'$ and for $z \in \mathcal{L}_r$, the set of descendants at level r' is defined as

$$\mathcal{D}_r^{r'}(z) := \{z' \in \mathcal{L}_{r'}, z \in \pi(z')\}$$

(we extend the notation for $z \notin \mathcal{L}_r$ by setting $\mathcal{D}_r^{r'}(z) := \emptyset$). For $z = (r; u) \in \text{RST}$, the descendant subtree of z is defined as $\mathcal{D}(z) := \bigcup_{r' \geq r} \mathcal{D}_r^{r'}(z)$, see Figure 3. Recall that an infinite backward path is a sequence $(z_n)_{n \in \mathbb{N}} \in (\mathbb{H}^{d+1})^{\mathbb{N}}$ such that $z_0 = 0$ and $z_n = A(z_{n+1})$ for all $n \geq 0$.

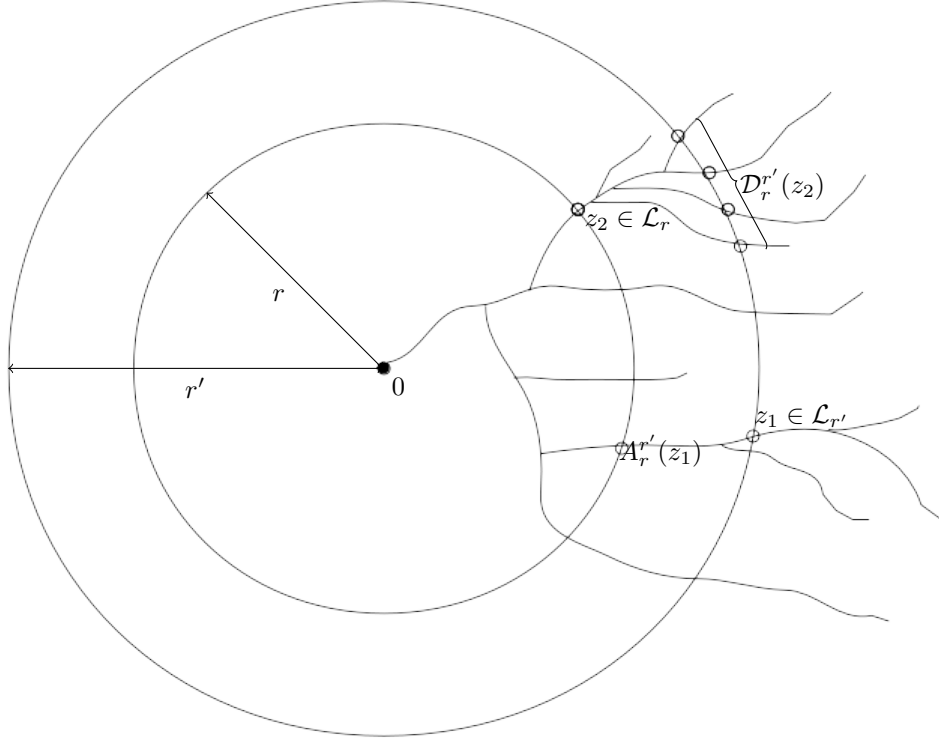


Figure 3: Representation of levels r and r' , the ancestor $A_r^{r'}(\cdot)$ and the set of descendants $\mathcal{D}_r^{r'}(\cdot)$. Note that ancestors are towards the root 0 of the RST, which is the “forward” direction. Descendants are seen when moving to higher radii, which is the “backward direction”.

2.3. Sketch of proofs

In order to prove our main result (Theorem 1.1), the key point is to upper-bound angular deviations of trajectories. We first introduce two quantities, the Cumulative Forward angular Deviations (CFD) and Maximal Backward Deviations (MBD) to quantify those fluctuations.

Definition 2.3 (Cumulative Forward angular Deviations). Let $0 < r \leq r'$ and $z' \in S(r')$. If $z' \notin \text{RST}$, we set $\text{CFD}_r^{r'}(z') = 0$ by convention and we now suppose that $z' \in \text{RST}$. Let $z := \mathcal{A}_r^{r'}(z')$. We define the Cumulative Forward angular Deviations of z' between levels r and r' as

$$\text{CFD}_r^{r'}(z') := \begin{cases} \widehat{z'0z} & \text{if } z_{\downarrow} = z'_{\downarrow}, \\ \widehat{z'0z'_{\uparrow}} + \sum_{k=0}^{n-1} A^{(k)}(z'_{\uparrow}) \widehat{0A^{(k+1)}(z'_{\uparrow})} + \widehat{z_{\downarrow}0z} & \text{otherwise,} \end{cases}$$

where n is the unique non negative integer such that $A^{(n)}(z'_{\uparrow}) = z_{\downarrow}$.

Definition 2.4 (Maximal Backward angular Deviations). Let $0 < r \leq r'$ and $z \in S(r)$. We define the Maximal Backward angular Deviations between levels r and r' as

$$\text{MBD}_r^{r'}(z) := \begin{cases} 0 & \text{if } z \notin \text{RST}, \\ \sup_{r'' \in [r, r']} \max_{z'' \in \mathcal{D}_r^{r''}(z)} \text{CFD}_r^{r''}(z'') & \text{if } z \in \text{RST}. \end{cases}$$

We extend the definition to $r' = \infty$ by setting:

$$\text{MBD}_r^{\infty}(z) := \lim_{r' \rightarrow \infty} \text{MBD}_r^{r'}(z),$$

the limit exists since $r' \mapsto \text{MBD}_r^{r'}(z)$ is non-decreasing.

All forward paths of the RST end at the root 0, but they can be finite in the backward direction if a vertex is the ancestor of no other vertex. The quantity $\text{MBD}_r^{r'}(z)$ takes into account backward paths from z that either end (in the backward direction) before level r' or reach level r' .

These quantities will be upper-bounded in two steps. First, a percolation argument is used to control angular deviations in any annulus of width $\delta > 0$ for some small $\delta > 0$ (Proposition 2.5) and then we deduce a global control of angular deviations (Proposition 2.6). These two propositions are proved in Sections 4 and 5.

Let us introduce some further notations. For $r > 0$, $z \in \mathbb{H}^{d+1}$ and $\theta > 0$, we define

$$B_{S(r)}(z, \theta) := \text{Cone}(z, \theta) \cap S(r). \quad (2.10)$$

We will control moments of the maximal backward angular deviations between close radii r and $r + \delta$ when z belongs to such arc $B_{S(r)}(z, \theta)$.

Proposition 2.5. *There exists $\delta > 0$ such that, for any $p \geq 1$, there exists $C = C(d, p) > 0$ such that for any $r > 0$, $\theta \geq 0$ and any direction $u \in \mathbb{S}^d$,*

$$\mathbb{E} \left[\sum_{z \in B_{S(r)}(u, \theta) \cap \text{RST}} \left(\text{MBD}_r^{r+\delta}(z) \right)^p \right] \leq C \theta^d e^{r(d-p)}, \quad (2.11)$$

Proposition 2.6. *For any $p \geq 3d/2$, there exists some constant $C_{\beta} > 0$ such that, for any $0 < r_0 < \infty$, $A > 0$ and any direction $u \in \mathbb{S}^d$,*

$$\mathbb{E} \left[\sum_{z \in B_{S(r_0)}(u, Ae^{-r_0}) \cap \text{RST}} \left(\text{MBD}_{r_0}^{\infty}(z) \right)^p \right] \leq C_{\beta} A^d e^{-r_0 p}.$$

These controls of angular deviations will be first used to show that the RST is straight (Proposition 2.7). The straightness property is the key to show (i), (ii) and (iv) in Theorem 1.1.

Proposition 2.7 (straightness property). *Almost surely, the following happens. For any $\varepsilon > 0$, there exists some $R_0 > 0$, such that, for any radius $r_0 \geq R_0$, for any $z \in \text{RST}$ with $d(0, z) \geq r_0$, the descendant subtree $\mathcal{D}(z)$ is contained in a cone of apex 0 and aperture $e^{-(1-\varepsilon)r_0}$, i.e. for any $z', z'' \in \mathcal{D}(z)$, $\widehat{z'0z''} \leq e^{-(1-\varepsilon)r_0}$.*

The proof of (iii) in Theorem 1.1 exploits the controls of angular deviations (Proposition 2.6) and the link existing between the RST and the hyperbolic Directed Spanning Forest introduced: the DSF approximates locally the RST far from the origin. The uniqueness of the infinite backward path with some given deterministic asymptotic direction has been shown for the DSF [14], and the local coupling existing between the two models, together with the controls of angular deviations, permits to show that this property remains true for the RST.

3. Proof of Theorem 1.1

Here we assume that Propositions 2.6 and 2.7 are proved and we show that it implies Theorem 1.1.

3.1. The existence part: proof of (i),(ii),(iv) and (v)

Proof of (i) and (ii) We first show that any infinite backward path admits an asymptotic direction and that any ideal boundary point is the asymptotic direction of an infinite backward path. The strategy consists in exploiting the straightness property (Proposition 2.7).

Let $(z_n)_{n \geq 0}$ be an infinite backward path, we prove that $(z_n)_{n \geq 0}$ admits an asymptotic direction. For $n \geq 0$, let us decompose z_n in polar coordinates: $z_n = (r_n; u_n)$. Proposition 2.7 immediately implies that the sequence $(u_n)_{n \geq 0}$ is a Cauchy sequence in $UT_0 \mathbb{H}^{d+1} \simeq \mathbb{S}^d$. To see this, let $\varepsilon > 0$. Since the path (z_n) is infinite, the sequence (r_n) converges to infinity. Let n_ε be such that $e^{-r_{n_\varepsilon}/2} \leq \varepsilon$. By the straightness property, there exists N_0 such that for $n_0 \geq N_0$, the path

$(z_n)_{n \geq n_0}$ remains inside $\text{Cone}(z_{n_0}, e^{-r_{n_0}/2}) \cap B(r_{n_0})^c$. Thus, for $n_0 \geq N_0 \vee n_\varepsilon$, the path $(z_n)_{n \geq n_0}$ remains in a cone of aperture ε , proving that $(u_n)_{n \geq 0}$ is a Cauchy sequence. Thus the sequence $(u_n)_{n \geq 0}$ converges, and so $(z_n)_{n \geq 0}$ converges to some boundary point $z_\infty \in \partial\mathbb{H}^{d+1}$, which proves (i).

Let $\Psi = \{\lim_{n \rightarrow \infty} z_n, (z_n) \text{ is an infinite backward path}\} \subset \partial\mathbb{H}^{d+1}$ be the set of asymptotic directions reached by at least one infinite backward path. In order to prove that $\Psi = \partial\mathbb{H}^{d+1}$, we proceed in two steps: we first show that Ψ is dense in $\partial\mathbb{H}^{d+1}$, then we show that Ψ is closed in $\partial\mathbb{H}^{d+1}$.

Since the RST is an infinite tree with finite degree a.s. (Proposition 2.2), there exists an infinite backward path from 0 and the corresponding infinite backward path converges to an ideal boundary point by the previous paragraph, thus $\Psi \neq \emptyset$ almost surely.

We denote by $\text{Stab}(0)$ the set of isometries that fix 0, in particular it contains rotations centred at 0. Let B be an open subset of $\partial\mathbb{H}^{d+1}$. Since $\partial\mathbb{H}^{d+1} \simeq \mathbb{S}^d$ is compact, there exists finitely many isometries $\gamma_1, \dots, \gamma_k \in \text{Stab}(0)$ such that $\bigcup_{i=1, \dots, k} \gamma_i B = \partial\mathbb{H}^{d+1}$. The random set RST is invariant in distribution by $\text{Stab}(0)$, so the events $\{\Psi \cap \gamma_i B \neq \emptyset\}$ all have the same probability. Since $\Psi \neq \emptyset$ almost surely, $\mathbb{P}\left[\bigcup_{i=1, \dots, k} \{\Psi \cap \gamma_i B \neq \emptyset\}\right] = 1$ therefore $\mathbb{P}(\Psi \cap B \neq \emptyset) > 0$. In addition, for any neighbourhood $\Phi \subset \overline{\mathbb{H}^{d+1}}$ of B , the event $\{\Psi \cap B \neq \emptyset\}$ is entirely determined by $\mathcal{N} \cap \Phi$, therefore it has probability 0 or 1, by Kolmogorov's 0-1 law. Thus $\Psi \cap B \neq \emptyset$ almost surely, since we already showed that $\mathbb{P}(\Psi \cap B \neq \emptyset) > 0$. Since the topology on $\partial\mathbb{H}^{d+1}$ admits a countable basis, Ψ is almost surely dense in $\partial\mathbb{H}^{d+1}$.

It remains to show that Ψ is a closed subset of $\partial\mathbb{H}^{d+1}$. Let $I \in \overline{\Psi}$ (recall that $\overline{\Psi}$ is the closure of Ψ in $\overline{\mathbb{H}^{d+1}}$). We construct by induction an infinite backward path $(z_n)_{n \geq 0} \in \mathcal{N}^{\mathbb{N}^*}$ such that, for any $i \in \mathbb{N}$, $I \in \overline{\mathcal{D}(z_i)}$. Suppose $0, \dots, z_{i-1}$ already defined such that $z_j = A(z_{j+1})$ for $0 \leq j \leq i-2$ and $I \in \overline{\mathcal{D}(z_{i-1})}$. Since the vertex z_{i-1} has finitely many daughters, there exists some $z \in A^{(-1)}(z_{i-1})$ such that $I \in \overline{\mathcal{D}(z)}$. We define z_i as such a z .

We now use straightness to show that the infinite backward path (z_n) constructed above converges to I (and thus $I \in \Psi$). This infinite backward path converges to some $I' \in \partial\mathbb{H}^{d+1}$ by (i). Let $\varepsilon > 0$, by Proposition 2.7 there exists some $i \geq 0$ such that $\mathcal{D}(z_i)$ (and thus $\overline{\mathcal{D}(z_i)}$) is contained in a cone of apex 0 and aperture at most ε . Since both I and I' belong to $\overline{\mathcal{D}(z_i)}$, $\widehat{II'} \leq \varepsilon$. Thus $I = I'$, which achieves the proof of (ii).

Proof of (iv) and (v) Let us denote by $\Psi' \subset \partial\mathbb{H}^{d+1}$ the set of asymptotic directions with two infinite backward paths. To show (iv), we first show that, a.s. $\Psi' \neq \emptyset$. For $z \in \text{RST}$, let us define $\Psi_z \subset \partial\mathbb{H}^{d+1}$ as the set of asymptotic directions of infinite backward paths from z (empty or not). By the same argument as above for the proof of (ii), Ψ_z is a closed subset of $\partial\mathbb{H}^{d+1}$. By (ii), a.s., there exists at least two infinite backward paths, so there exists a.s. some level $r_0 > 0$ with two points connected to infinity. Thus $\{\Psi_z, z \in \mathcal{L}(r_0)\}$ is a covering of $\partial\mathbb{H}^{d+1}$ by closed subsets, where at least two of them are nonempty. Since $\partial\mathbb{H}^{d+1}$ is connected, it implies that there exists $z_1, z_2 \in \mathcal{L}_{r_0}$ such that $\Psi_{z_1} \cap \Psi_{z_2} \neq \emptyset$. Thus $\Psi' \neq \emptyset$ a.s.

We use the same argument as in the proof of (ii) to deduce that Ψ' is dense. Let B be an open subset of $\partial\mathbb{H}^{d+1}$. Since $\partial\mathbb{H}^{d+1} \simeq \mathbb{S}^d$ is compact, there exist finitely many isometries $\gamma_1, \dots, \gamma_k \in \text{Stab}(0)$ such that $\bigcup_{i=1, \dots, k} \gamma_i B = \partial\mathbb{H}^{d+1}$. The random set RST is invariant in distribution by $\text{Stab}(0)$, so the events $\Gamma_i := \{\Psi' \cap \gamma_i B \neq \emptyset\}$ all have the same probability. Since $\Psi' \neq \emptyset$ almost surely, $\mathbb{P}\left[\bigcup_{i=1, \dots, k} \Gamma_i\right] = 1$ and therefore $\mathbb{P}(\Gamma_i) > 0$. In addition, for any neighbourhood $\Phi \subset \overline{\mathbb{H}^{d+1}}$ of B , the event Γ_i is entirely determined by $\mathcal{N} \cap \Phi$, which implies that it has probability 0 or 1. Thus $\Psi' \cap B \neq \emptyset$ almost surely. Since the topology on $\partial\mathbb{H}^{d+1}$ admits a countable basis, Ψ' is almost surely dense in $\partial\mathbb{H}^{d+1}$.

The proof of (v) is done by exploiting the planarity in the bi-dimensional case (Proposition 2.2). Let us associate to any $z_\infty \in \Psi'$ a couple of vertices $P(z_\infty) = (z_1, z_2) \in \mathcal{N}^2$ with $z_1 \neq z_2$ such that $z_\infty \in \Psi_{z_1} \cap \Psi_{z_2}$. By planarity, such an application P must be injective. Indeed, if $z_\infty \neq z'_\infty$ are such that $P(z_\infty) = P(z'_\infty) = (z_1, z_2)$, then there exist four distinct backward infinite paths joining z_∞ to z_1 , z_∞ to z_2 , z'_∞ to z_1 and z'_∞ to z_2 . This implies that two paths among them intersect each other, even if the representation of edges are replaced by geodesics, which contradicts planarity. Therefore Ψ' is a.s. countable in the case $d = 1$.

3.2. The uniqueness part: proof of (iii)

The strategy consists in exploiting a coupling between the hyperbolic RST and the hyperbolic Directed Spanning Forest (DSF), stated in Proposition 3.1 below, and the fact that there exists a.s. a unique infinite backward path in the DSF that converges to any given deterministic direction (see [14, Theorem 1.2]).

First, recall that the space \mathbb{H}^{d+1} can be described with several isometric models. We have already seen the open-ball model (I, ds_I^2) and we introduce here another one, called the upper half-space model, (H, ds^2) , where:

$$H = \{z = (x_1, \dots, x_d, y) \in \mathbb{R}^{d+1}, y > 0\}, \quad ds^2 = \frac{dx_1^2 + \dots + dx_d^2 + dy^2}{y^2}.$$

In the following, we will identify the point $z = (x_1, \dots, x_d, y) \in H$ with the couple $(x, y) \in \mathbb{R}^d \times \mathbb{R}_+$ with $x := (x_1, \dots, x_d)$ and $y := x_{d+1}$. The coordinate x is referred as the *abscissa* and y as the *ordinate*. The coordinate y plays a special role and will be the direction in which the DSF is built. Let us remind that, in the half-space representation, the boundary set $\partial\mathbb{H}^{d+1}$ is identified as the boundary hyper-plane $\mathbb{R}^d \times \{0\}$, plus an additional point at infinity denoted by ∞ , obtained by compactifying the closed half-space $\mathbb{R}^d \times \mathbb{R}_+$.

Let us now define the **hyperbolic DSF with direction ∞** in the half-space model (H, ds^2) . The hyperbolic DSF with direction ∞ is a graph with vertex set given by the points of the PPP \mathcal{N} and in which each vertex has outdegree one. Thus we connect each vertex $z \in \mathcal{N}$ to the closest Poisson point in the direction of ∞ : this point, denoted by $A_{\text{DSF}}(z)$, is the parent of $z = (x, y) \in \mathcal{N}$. It is defined as:

$$A_{\text{DSF}}(z) := \underset{z'=(x',y') \in \mathcal{N}, y' > y}{\operatorname{argmin}} d(z', z) \quad (3.1)$$

where $d(\cdot, \cdot)$ is the hyperbolic distance.

To compare the DSF (with direction ∞) and the RST, we also define the RST in the half-space model (H, ds^2) whose root $O(h)$ is devoted to tend to ∞ . Precisely, for any $h \geq 0$, let us set $\text{RST}(h)$ as the Radial Spanning Tree of \mathcal{N} with origin $O(h) := (0, e^h)$ similarly as in Definition 2.1. This is a tree rooted at $O(h)$ with vertex set \mathcal{N} and in which each vertex has outdegree one. In $\text{RST}(h)$, each vertex $z \in \mathcal{N}$ is connected to its parent

$$A_{\text{RST}(h)}(z) := \underset{z' \in (\mathcal{N} \cup \{O(h)\}) \cap B(O(h), r)}{\operatorname{argmin}} d(z', z) \quad (3.2)$$

with $r = d(O(h), z)$. To avoid confusion, the parent of z in the hyperbolic RST is now denoted by $A_{\text{RST}}(z)$ instead of $A(z)$ when necessary.

When $h \rightarrow \infty$, i.e. when $O(h)$ tends to the point ∞ , the graph $\text{RST}(h)$ mimics the DSF with direction ∞ in any compact set with probability tending to 1. Proposition 3.1 is proved at the end of the section.

Proposition 3.1 (Coupling between RST and DSF). *Let $K \subset \mathbb{H}^{d+1}$ be a compact set. Then*

$$\lim_{h \rightarrow \infty} \mathbb{P}[\forall z \in \mathcal{N} \cap K, A_{\text{RST}(h)}(z) = A_{\text{DSF}}(z)] = 1.$$

Let us now prove Item (iii) of Theorem 1.1. Since the law of $\text{RST}(h)$ is invariant w.r.t. horizontal translations, it is sufficient to prove the uniqueness result with the deterministic boundary point z_∞ , represented by $(0, 0)$ in H , and defined as $(0, \dots, -1) \in UT_{O(h)}\mathbb{H}^{d+1} \simeq \mathbb{S}^d$. To do so, let us introduce, for any $h > 0$, the event

$$U(h) := \{ \text{RST}(h) \text{ admits only one infinite path towards } z_\infty \}.$$

We proceed by contradiction with assuming that $\mathbb{P}[U(h)] < 1$. Since the law of $\text{RST}(h)$ is invariant w.r.t. dilatations $(x, y) \rightarrow (e^\alpha x, e^\alpha y)$, the probability $q := \mathbb{P}[U(h)^c]$ does not depend on h , and by our assumption,

$$q = \mathbb{P}[\text{RST}(h) \text{ admits two infinite paths towards } z_\infty] > 0.$$

The two infinite paths of $\text{RST}(h)$ mentioned above eventually bifurcate at some horizontal axis with (random) ordinate. Using one more time the invariance by dilatations of $\text{RST}(h)$, we get that the event

$$V(h) := \left\{ \begin{array}{l} \text{RST}(h) \text{ admits two infinite paths towards } z_\infty \\ \text{bifurcating before the horizontal axis } y = 1 \end{array} \right\}$$

has probability $q' > 0$ (uniform on h).

In the next step, we need to control fluctuations of RST paths (see Proposition 3.3 below). Hence, let $\text{Cone}_{z_0}(z, \theta)$ be the cone with apex z_0 , direction z and opening angle θ . For any $A, a, h \geq 0$, let us set:

- $\text{Vois}(A, h) := \text{Cone}_{O(h)}(z_\infty, Ae^{-h}) \setminus B(O(h), h)$;
- $\text{Vois}'(A, a, h) := (B(O(h), h+a) \cap \text{Cone}_{O(h)}(z_\infty, Ae^{-h})) \setminus B(O(h), h)$;
- $\text{Vois}''(A, a, h) := \text{Cone}_{O(h)}(z_\infty, Ae^{-h-a}) \setminus B(O(h), h+a)$.

When h is large, these three sets are included in some cylinders:

Lemma 3.2. *For any $A, a \geq 0$, the parameter h can be chosen large enough such that*

- $\text{Vois}(A, h) \subset \text{Cyl}(A) := B_{\mathbb{R}^d}(0, A) \times \left(0, \frac{3}{2}\right]$;
- $\text{Vois}'(A, a, h) \subset \text{Cyl}'(A, a) := B_{\mathbb{R}^d}(0, A) \times \left[\frac{1}{2}e^{-a}, \frac{3}{2}\right]$;
- $\text{Vois}''(A, a, h) \subset \text{Cyl}''(A, a) := B_{\mathbb{R}^d}(0, Ae^{-a}) \times \left(0, \frac{3}{2}e^{-a}\right]$.

The sets $\text{Vois}(A, h)$, $\text{Vois}'(A, a, h)$, $\text{Vois}''(A, a, h)$ and $\text{Cyl}(A)$, $\text{Cyl}'(A, a)$, $\text{Cyl}''(A, a)$ are represented in Figure 4.

For $A, a, h \geq 0$, let us introduce the event $E(A, h)$ saying that each infinite backward path in $\text{RST}(h)$ converging to z_∞ and restricted to $\mathbb{H}^{d+1} \setminus B(O(h), h)$ is contained in $\text{Vois}(A, h)$:

$$E(A, h) := \left\{ \forall z \in \text{RST}(h) \cap (\mathbb{H}^{d+1} \setminus B(O(h), h)), z_\infty \in \overline{\mathcal{D}_{\text{RST}(h)}(z)} \implies z \in \text{Vois}(A, h) \right\},$$

where $\mathcal{D}_{\text{RST}(h)}(z)$ denotes the descendant subtree of z in $\text{RST}(h)$. Let us also introduce a slight modification of $E(A, h)$. The event $E''(A, h, a)$ says that each infinite backward path in $\text{RST}(h)$ converging to z_∞ and restricted to $\mathbb{H}^{d+1} \setminus B(O(h), h+a)$ is contained in $\text{Vois}''(A, a, h)$:

$$E''(A, a, h) := \left\{ \forall z \in \text{RST}(h) \cap (\mathbb{H}^{d+1} \setminus B(O(h), h+a)), z_\infty \in \overline{\mathcal{D}_{\text{RST}(h)}(z)} \implies z \in \text{Vois}''(A, a, h) \right\}.$$

Here is the announced control of RST paths, i.e. the events have probability tending to 1 uniformly on h, a . Lemma 3.2 and Proposition 3.3 are proved at the end of the section.

Proposition 3.3. *The following limits hold uniformly on parameters $h, a > 0$:*

$$\lim_{A \rightarrow \infty} \mathbb{P}[E(A, h)] = \lim_{A \rightarrow \infty} \mathbb{P}[E''(A, a, h)] = 1.$$

Let $a > 0$. Recall that q' denotes the probability of $V(h)$, we first choose A large enough such that $\mathbb{P}[E(A, h)]$ and $\mathbb{P}[E''(A, a, h)]$ are both larger than $1 - q'/4$ (Proposition 3.3). Thus, applying Proposition 3.1 to the compact set $K := \text{Cyl}'(A, a)$, we choose $h = h(a, A)$ large enough such that the event

$$\text{CO}(A, a, h) := \left\{ \forall z \in \mathcal{N} \cap \text{Cyl}'(A, a), A_{\text{RST}(h)}(z) = A_{\text{DSF}}(z) \right\}$$

occurs with probability larger than $1 - q'/4$. As a consequence, the event $Z(a) = Z(a, h)$ defined by

$$Z(a) := V(h) \cap E(A, h) \cap E''(A, a, h) \cap \text{CO}(A, a, h)$$

satisfies $\mathbb{P}[Z(a)] \geq q'/4$. Let us add that, in the event $Z(a) = Z(a, h)$, h can be chosen large enough so that inclusions of Lemma 3.2 hold. Finally we set

$$Z := \bigcap_{a_0 > 0} \bigcup_{a \geq a_0} Z(a).$$

which has probability at least $q'/4$.

We need to add to the event Z an extra property concerning this time the DSF paths. Let $G(A, \alpha)$ be the event that each DSF path crossing the cylinder $C := B_{\mathbb{R}^d}(0, A) \times [e^{-\alpha}/2, 1]$ from the bottom face to the top one while staying inside, admits at least one vertex inside C . Because a large edge in the DSF implies a large region empty of Poisson points which is very unlikely, the probability of $G(A, \alpha)$ tends to 1 as $\alpha \rightarrow \infty$. Henceforth we choose α_0 large such that the event $Z \cap G_0$, with $G_0 := G(A, \alpha_0)$, has positive probability.

Now we are ready to get a contradiction on the event $Z \cap G_0$. We a.s. have an unbounded and increasing sequence $(a_n)_{n \geq 0}$ with $a_0 > \alpha_0$ such that the events $Z(a_n) \cap G_0$'s occur. We also denote by $h_n := h(A, a_n)$ the associated parameter to a_n in $Z(a_n)$. So $V(h_n)$ gives us the existence of two infinite paths in $RST(h_n)$, say $\pi_{n,1}$ and $\pi_{n,2}$, converging to z_∞ and bifurcating before the horizontal axis $y = 1$. Be careful that we work here with a sequence of different RST trees, $RST(h_n)$, whose roots $O(h_n)$ tend to infinity, but defined on the same set of vertices \mathcal{N} . By $E(A, h_n) \cap E''(A, a_n, h_n)$, these two paths are trapped inside $\text{Vois}(A, h_n)$ and $\text{Vois}''(A, a_n, h_n)$, then also inside $\text{Vois}'(A, a_n, h_n) \subset \text{Cyl}'(A, a_n)$ (by Lemma 3.2). Thus, $\text{CO}(A, a_n, h_n)$ indicates that $\pi_{n,1}$ and $\pi_{n,2}$ respectively coincide in $\text{Vois}'(A, a_n, h_n)$ with two DSF paths, say $\hat{\pi}_{n,1}$ and $\hat{\pi}_{n,2}$. The event G_0 allows us to assume w.l.o.g. that both backward (and disjoint) DSF paths $\hat{\pi}_{n,1}$ and $\hat{\pi}_{n,2}$ start from two different Poisson points in C , say respectively $z_{n,1}$ and $z_{n,2}$. Since the PPP is locally finite, we can assume w.l.o.g. that all the $\hat{\pi}_{n,1}$'s have the same starting Poisson point z_1 . Then, combining the fact that $\hat{\pi}_{n,1}$ reaches the axis $y = e^{-a_n}/2$ for any n , with the finite degree property of the DSF (Proposition 2.10 of [14]), we can build from the collection $\{\hat{\pi}_{n,1}\}$ an infinite backward path starting at z_1 . Let us denote by $\hat{\pi}_1$ this path. Theorem 1.2, Item (ii) of [14] asserts that $\hat{\pi}_1$ a.s. converges towards the past to an element of $\partial\mathbb{H}^{d+1}$ which necessarily is the deterministic point z_∞ thanks to the $E''(A, a_n, h_n)$'s. In the same way, from the collection $\{\hat{\pi}_{n,2}\}$, we can build an infinite backward path $\hat{\pi}_2$ starting at some $z_2 \neq z_1$. As previously, $\hat{\pi}_2$ a.s. converges towards the past to z_∞ . In conclusion, we just have exhibited two different infinite backward paths converging to the same deterministic point z_∞ : this is in contradiction with Theorem 1.2 Item (iv) of [14]. This achieves the proof.

Proof of Proposition 3.1. Given $z \in \mathbb{H}^{d+1}$, let us first assume that

$$\lim_{h \rightarrow \infty} \mathbb{P}[A_{RST(h)}(z) \neq A_{DSF}(z)] = 0. \quad (3.3)$$

The Campbell formula [13, Prop. 13.1.IV] and Lebesgue's dominated convergence theorem then allow to conclude. Let $K \subset \mathbb{H}^{d+1}$ be a compact set:

$$\begin{aligned} \mathbb{P}[\exists z \in \mathcal{N} \cap K, A_{RST(h)}(z) \neq A_{DSF}(z)] &\leq \mathbb{E}[\#\{z \in \mathcal{N} \cap K, A_{RST(h)}(z) \neq A_{DSF}(z)\}] \\ &= \lambda \int_K \mathbb{P}[A_{RST(h)}(z) \neq A_{DSF}(z)] d\text{Vol}(z) \end{aligned}$$

which tends to 0 as $h \rightarrow \infty$ (where Vol denotes the hyperbolic volume).

It then remains to prove (3.3). Let $z = (x, y)$ in \mathbb{H}^{d+1} and $\alpha > 0$. Let us consider the set

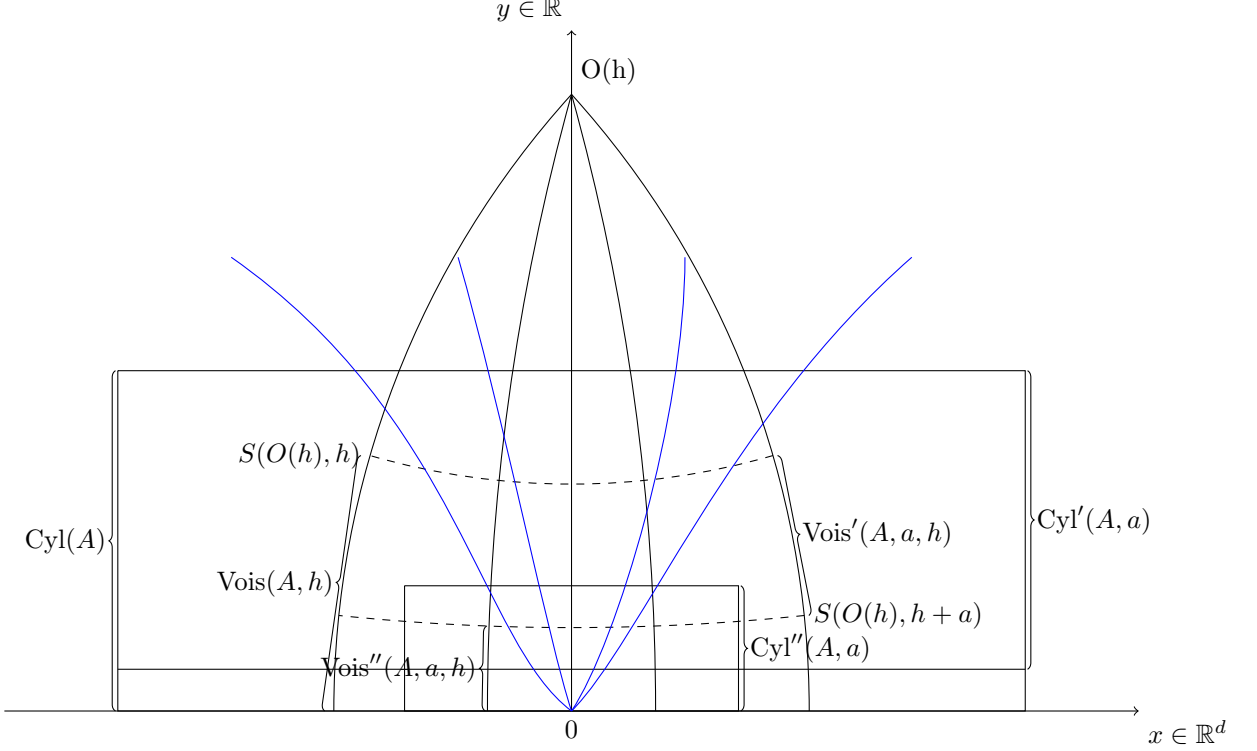
$$\Delta_h := B(z, h^\alpha) \cap (B(O(h), d(O(h), z)) \Delta (\mathbb{R}^d \times (y, \infty))),$$

where Δ denotes the symmetric difference, and we choose α small enough such that $\text{Vol}(\Delta_h) \rightarrow 0$ as $h \rightarrow \infty$. Now, the fact that $A_{RST(h)}(z) \neq A_{DSF}(z)$ combined with $\mathcal{N} \cap \Delta_h = \emptyset$ imply in particular that $A_{DSF}(z)$ is at distance at least h^α from z . So,

$$\mathbb{P}[A_{RST(h)}(z) \neq A_{DSF}(z)] \leq \mathbb{P}[\mathcal{N} \cap \Delta_h \neq \emptyset] + \mathbb{P}[d(z, A_{DSF}(z)) > h^\alpha]$$

where both terms of the upperbound tend to 0 as $h \rightarrow \infty$. □

Figure 4: Representation of the sets $\text{Vois}(A, h)$, $\text{Vois}'(A, a, h)$, $\text{Vois}''(A, a, h)$ and $\text{Cyl}(A)$, $\text{Cyl}'(A, a)$, $\text{Cyl}''(A, a)$. The backward paths of $\text{RST}(h)$ converging to 0 (in blue) are all contained in $\text{Vois}''(A, a, h)$ up to level $h + a$ and contained in $\text{Vois}(A, a, h)$ up to level h . In the dashed area ($\text{Cyl}'(A, a)$), the DSF and $\text{RST}(h)$ coincide.



Proof of Proposition 3.3. Let $n \in \mathbb{N}$ and $h \geq 0$. Recall that $B_{S(h)}(z_\infty, 2^{n+1}e^{-h})$, with $S(h) := S(O(h), h)$, denotes the sphere $S(O(h), h)$ intersected with $\text{Cone}_{O(h)}(z_\infty, 2^{n+1}e^{-h})$. Let us define the event

$$F_n(h) := \{ \exists z \in B_{S(h)}(z_\infty, 2^{n+1}e^{-h}), \text{MBD}_h^\infty(z) > 2^n e^{-h} \} .$$

Let us first show that $E(2^n, h)^\complement \subset \bigcup_{m \geq n} F_m(h)$, which will imply:

$$\mathbb{P}[E(2^n, h)] \geq 1 - \sum_{m \geq n} \mathbb{P}[F_m(h)] . \quad (3.4)$$

If $E(2^n, h)$ does not occur, then there exists some $z' \notin \text{Vois}(2^n, h)$ such that $z_\infty \in \overline{\mathcal{D}_{\text{RST}(h)}(z')}$. Let z be the ancestor of z' on $S(O(h), h)$. Either $z \widehat{O(h)} z_\infty < 2^{n+1}e^{-h}$ and we set $m = n$, or there exists $m > n$ such that $2^m e^{-h} \leq z \widehat{O(h)} z_\infty < 2^{m+1}e^{-h}$. Then $z \in S(O(h), h) \cap \text{Cone}_{O(h)}(z_\infty, 2^{m+1}e^{-h})$. If $m = n$, $\text{MBD}_h^\infty(z) \geq z \widehat{O(h)} z_\infty > 2^n e^{-h}$ and $F_n(h)$ occurs. If $m > n$, $\text{MBD}_h^\infty(z) \geq z \widehat{O(h)} z_\infty \geq 2^m e^{-h}$ and $F_m(h)$ occurs. This proves (3.4).

To conclude, we use Proposition 2.6 to upperbound $\mathbb{P}[F_m(h)]$. On $F_m(h)$, the following occurs:

$$\sum_{z \in B_{S(h)}(z_\infty, 2^{n+1}e^{-h}) \cap \text{RST}(h)} (\text{MBD}_h^\infty(z))^p > 2^{mp} e^{-ph} \quad (3.5)$$

which by Markov inequality leads to

$$\mathbb{P}[F_m(h)] \leq 2^{-mp} e^{ph} \mathbb{E} \left[\sum_{z \in B_{S(h)}(z_\infty, 2^{n+1}e^{-h}) \cap \text{RST}(h)} (\text{MBD}_h^\infty(z))^p \right] \leq C_{\text{fl}} 2^d 2^{m(d-p)}$$

where we used Proposition 2.6 with $r_0 = h$, $A = 2^{m+1}$. The constant $C_{\text{fl}} > 0$ appearing in Proposition 2.6 depends only on p . **At this step, notice that the dependence in h of the previous upperbound has disappeared, providing the announced uniformity on h .** Thus, for some $p > d$,

$$\mathbb{P}[E(2^n, h)] \geq 1 - 2^d C_{\text{fl}} \frac{2^{n(d-p)}}{1 - 2^{d-p}}.$$

The right hand side converges to 1 when $n \rightarrow \infty$ which proves the first part of Proposition 3.3. **The second part is treated exactly in the same way replacing the sphere $S(O(h), h)$, the opening angle $2^{n+1}e^{-h}$ and $\text{MBD}_h^\infty(\cdot)$ with resp. $S(O(h), h+a)$, the opening angle $2^{n+1}e^{-h-a}$ and $\text{MBD}_{h+a}^\infty(\cdot)$, leading to the searched limit uniformly in h and a .** \square

Proof of Lemma 3.2. Let $A, a \geq 0$ be fixed. For $h \geq 0$, let $z = (x, y) \in \text{Vois}(A, h)$. Considering the totally geodesic plane containing z_∞, z and $O(h)$ (represented by a half-plane in H), it is possible to suppose $d = 1$ without loss of generality. We apply the distance and angle formulas in (H, ds^2) (see e.g. [14, Proposition 2.1]).

Let $z = (x, y) \in \text{Vois}(A, h)$. We recall that the angle $z_\infty \widehat{O(h)} z$ can be computed as a function of x, y and h :

$$z_\infty \widehat{O(h)} z = \arctan \left| \frac{2xe^h}{e^{2h} - x^2 - y^2} \right|. \quad (3.6)$$

This formula can be obtained by noticing that the Poincaré open-ball model is conform and by using the fact that the application $\phi : H \rightarrow I$, sending the half-space model to the open-ball model, and defined as:

$$(x, y) \mapsto \frac{1}{x^2 + (y+1)^2} (x^2 + y^2 - 1, -2x)$$

is an isometry sending $(0, 1)$ on $(0, 0)$. Then, the angle $z \widehat{O(h)} z_\infty = \phi'(z) \widehat{0} \phi'(z_\infty)$, where the second angle is taken in the disc model. Since $\phi'(z_\infty) = (-1, 0)$, if $y < e^h$ then $z \widehat{O(h)} z_\infty < \frac{\pi}{2}$ and we can establish (3.6).

On the one hand, $z \widehat{O(h)} z_\infty \leq Ae^{-h}$, so, taking h large enough such that $Ae^{-h} < \pi/2$, we have

$$\arctan \left| \frac{2xe^h}{e^{2h} - x^2 - y^2} \right| \leq Ae^{-h}.$$

Thus, for h large enough,

$$|x|e^{-h} \leq \arctan |2xe^{-h}| \leq \arctan \left| \frac{2xe^h}{e^{2h} - x^2 - y^2} \right| \leq Ae^{-h},$$

so $|x| \leq A$.

On the other hand, $d(O(h), z) \geq h$. Recall that for $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2) \in H$,

$$d(z_1, z_2) = 2 \tanh^{-1} \left(\sqrt{\frac{\kappa^2 + (v-1)^2}{\kappa^2 + (v+1)^2}} \right) = 2 \tanh^{-1} \left(\sqrt{1 - \frac{4v}{\kappa^2 + (v+1)^2}} \right), \quad (3.7)$$

where $\kappa = \|x_1 - x_2\|/y_1$ and $v = y_2/y_1$. Applying this formula for $z = (x, y)$ and $O(h) = (0, e^h)$,

$$\begin{aligned} & 2 \tanh^{-1} \left(\sqrt{1 - \frac{4ye^h}{A^2 + (y+e^h)^2}} \right) \\ & \stackrel{|x| \leq A}{\geq} 2 \tanh^{-1} \left(\sqrt{1 - \frac{4ye^{-h}}{(xe^{-h})^2 + (ye^{-h} + 1)^2}} \right) = d(O(h), z) \geq h. \end{aligned}$$

This implies that

$$\sqrt{1 - \frac{4ye^{-h}}{(xe^{-h})^2 + (ye^{-h} + 1)^2}} \geq \tanh\left(\frac{h}{2}\right) = \frac{e^{h/2} - e^{-h/2}}{e^{h/2} + e^{-h/2}},$$

from which we deduce that:

$$e^h \leq \frac{1 + \sqrt{1 - \frac{4ye^h}{A^2 + (y+e^h)^2}}}{1 - \sqrt{1 - \frac{4ye^h}{A^2 + (y+e^h)^2}}} = \frac{e^h}{y} + o(e^h) \text{ when } h \rightarrow \infty,$$

for h large enough this implies $y \leq 3/2$. The two other inclusions are shown by similar computations. \square

4. Proof of Proposition 2.5

4.1. Step 0: Preliminaries

Let $r_0 > 0$, $\delta > 0$ (small) and $z \in S(r_0) \cap \text{RST}$. In order to control $\text{MBD}_{r_0}^{r_0+\delta}(z)$, we use a block control argument similar to [14, Section 4.3, Proof of Prop. 4.6]. For any $A > 0$ (large)– parameters A, δ will be properly chosen later –, let us set

$$\Psi_1(r_0, z) := \text{Cone}(z, 3Ae^{-r_0}) \cap (B(r_0 + \delta) \setminus B(r_0))$$

and

$$\Psi_2(r_0, z) := \text{Cone}(z, Ae^{-r_0}) \cap (B(r_0) \setminus B(r_0 - 1)).$$

An element $z \in S(r_0)$ is said to be *good* if the event

$$G(r_0, z) := \{\mathcal{N} \cap \Psi_1(r_0, z) = \emptyset \text{ and } \mathcal{N} \cap \Psi_2(r_0, z) \neq \emptyset\} \quad (4.1)$$

occurs. See Fig. 5. Thus let us introduce the set $\hat{\chi}(r_0) \subset S(r_0)$ of good points, i.e.

$$\hat{\chi}(r_0) := \{z \in S(r_0) : G(r_0, z) \text{ occurs}\}$$

and the subset of the annulus $B(r_0 + \delta) \setminus B(r_0)$ that $\hat{\chi}(r_0)$ generates:

$$\chi(r_0) := \{(s; u) : r_0 \leq s \leq r_0 + \delta \text{ and } (r_0; u) \in \hat{\chi}(r_0)\}.$$

The random set $\chi(r_0)$ is called the *controlled region* in which the cumulative forward deviations will be well controlled. This is the statement of Lemma 4.1, proved in Section 4.5.

Lemma 4.1. *For any $A > 2$, $\delta > 0$, $r_0 > 0$, the following holds:*

$$\sup_{s \in [r_0, r_0 + \delta]} \sup_{z \in \mathcal{L}_s \cap \chi(r_0)} \text{CFD}_{r_0}^s(z) \leq Ae^{-r_0}.$$

Thus we need to cover the sphere $S(r_0)$ with caps of angular radius e^{-r_0} such that the number of caps overlapping a given point never exceeds some constant K .

Lemma 4.2. *There exists an integer $K = K(d)$ such that, for any radius $r_0 > 0$, there exist an integer $N(r_0)$ and a family of points $z_1, \dots, z_{N(r_0)} \in S(r_0)$ such that:*

- (a) $\bigcup_{1 \leq i \leq N(r_0)} B_{S(r_0)}(z_i, e^{-r_0}) = S(r_0)$;
- (b) $\forall z \in S(r_0), \#\{1 \leq i \leq N(r_0) : z \in B_{S(r_0)}(z_i, e^{-r_0})\} \leq K$.

Moreover, there exists a constant $C_{\text{ball}} = C_{\text{ball}}(K, d) > 0$ such that for any $r_0 > 0$, $A \geq 1$,

$$\sup_{z \in S(r_0)} \#\left\{1 \leq i \leq N(r_0) : B_{S(r_0)}(z_i, e^{-r_0}) \cap B_{S(r_0)}(z, Ae^{-r_0}) \neq \emptyset\right\} \leq C_{\text{ball}} A^d. \quad (4.2)$$

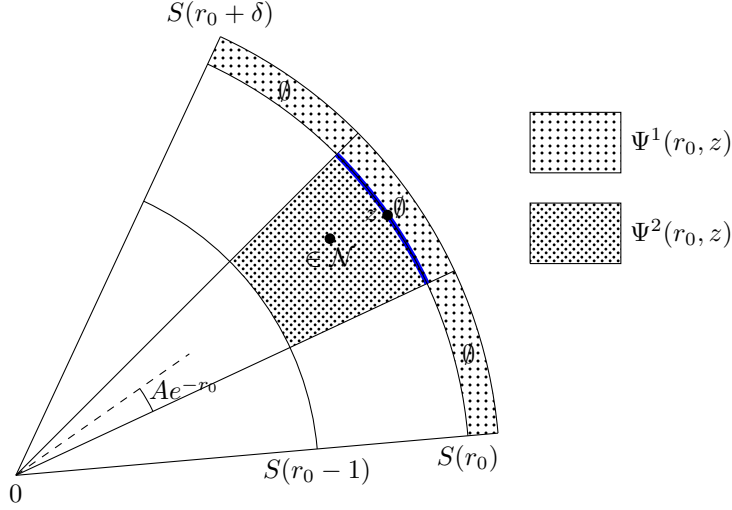


Figure 5: The element z is *good*; the fluctuations of trajectories crossing the cap $B_{S(r_0)}(z, Ae^{-r_0})$ (depicted in blue) are well controlled.

The $B_{S(r_0)}(z_i, e^{-r_0})$'s given by Lemma 4.2 will act as a discretization of the sphere $S(r_0)$. For $1 \leq i \leq N(r_0)$, the cap $B_{S(r_0)}(z_i, e^{-r_0})$ (and by extension its center z_i) is said *inhibited* if it overlaps $S(r_0) \setminus \chi(r_0)$. Let $\Psi(r_0) \subset S(r_0)$ be the union of all inhibited spherical caps:

$$\Psi(r_0) := \bigcup_{i: z_i \text{ inhibited}} B_{S(r_0)}(z_i, e^{-r_0}) .$$

The random set $\Psi(r_0)$ is called the *augmented uncontrolled region*. While Lemma 4.1 allows to bound the cumulative forward deviations starting at a point in the controlled region $\chi(r_0)$, we also need to control what happens in the augmented uncontrolled region $\Psi(r_0)$. Lemma 4.3 (below) is a first step in this direction.

Let us introduce some notations. Let z_i be an element given by Lemma 4.2 which is inhibited. We denote by $\hat{\text{Cl}}(z_i)$ its cluster in $\Psi(r_0)$ and we set

$$\text{Rad}(z_i) := \sup_{z' \in \hat{\text{Cl}}(z_i)} \widehat{z_i 0 z'} . \quad (4.3)$$

We extend this definition with setting $\text{Rad}(z_i) = 0$ when z_i is not inhibited. The next lemma, proved in Section 4.5, asserts that the radii of the clusters of the augmented uncontrolled region $\Psi(r_0)$ admit exponential tail decay:

Lemma 4.3. *There exist $A > 0$ large enough, $\delta > 0$ small enough and a constant $c_{\text{dec}} > 0$ such that, for any $B > 0$ large enough, for any $r_0 > 0$ and any z_i given by Lemma 4.2:*

$$\mathbb{P}[e^{r_0} \text{Rad}(z_i) > B] \leq e^{-c_{\text{dec}} B} .$$

Finally, a uniform bound for the moments of the r.v. $\#(\mathcal{L}_{r_0} \cap B_{S(r_0)}(z, e^{-r_0}))$ will be required:

Lemma 4.4. *For any $p \geq 1$, there exists a constant $C = C(d, p) > 0$ such that, for any $r_0 > 0$,*

$$\sup_{z \in S(r_0)} \mathbb{E} \left[\#(\mathcal{L}_{r_0} \cap B_{S(r_0)}(z, e^{-r_0}))^p \right] \leq C .$$

The proof of Proposition 2.5 is now divided into three steps. Let us choose A and δ as in Lemma 4.3. We first state in Step 1 an a.s. upper-bound for the maximal backward deviations between the radii r_0 and $r_0 + \delta$, using Lemmas 4.1, 4.2 and 4.3. This is the heart of the proof. The conclusion in Step 3 will follow from expectation considerations and Lemma 4.4.

4.2. Step 1: an almost-sure upper-bound for $\text{MBD}_{r_0}^{r_0+\delta}(\cdot)$

Let z_i be an element given by Lemma 4.2 and let $z \in B_{S(r_0)}(z_i, e^{-r_0}) \cap \text{RST}$. In this first step, we aim to get an a.s. upperbound for the maximal backward deviations $\text{MBD}_{r_0}^{r_0+\delta}(z)$ that takes into account all the paths starting at radius $r_0 + \delta$ (or before) and ending at z . To do it let us consider the set $\mathcal{D}[r_0, \delta, z]$ defined as the union of $\mathcal{D}_{r_0}^{r_0+\delta}(z)$ and Poisson points $z' = (r'; \cdot)$ in $\mathcal{N} \cap \mathcal{D}_{r_0}^{r_0+\delta}(z)$ with $r_0 < r' < r_0 + \delta$ and $A^{(-1)}(z') = \emptyset$. Hence, a.s.

$$\text{MBD}_{r_0}^{r_0+\delta}(z) = \max_{z'=(r'; \cdot) \in \mathcal{D}[r_0, \delta, z]} \text{CFD}_{r_0}^{r'}(z'). \quad (4.4)$$

Let $z' = (r'; \cdot) \in \mathcal{D}[r_0, \delta, z]$. Two cases must be distinguished: either z' belongs to the controlled region $\chi(r_0)$ or not. In the first case, it suffices to apply Lemma 4.1 to get that $\text{CFD}_{r_0}^{r'}(z')$ is smaller than Ae^{-r_0} . The second case requires more work. Let us define $z'' = (r''; u'')$ as the first element of the RST path from z' to z hitting $\chi(r_0)$. In other words, its radius satisfies $r'' := \sup\{s \in [r_0, r_0 + \delta] : \mathcal{A}_s^{r'}(z') \in \chi(r_0)\}$. In the case where the whole RST path from z' to z avoids $\chi(r_0)$, we set $z'' := z$. Using Lemma 4.1, we then can write:

$$\text{CFD}_{r_0}^{r'}(z') \leq \text{CFD}_{r_0}^{r''}(z'') + \text{CFD}_{r''}^{r'}(z') \leq Ae^{-r_0} + \text{CFD}_{r''}^{r'}(z').$$

By construction the projection $\bar{z}'' := (r_0; u'')$ of z'' onto $S(r_0)$ belongs to the augmented uncontrolled region $\Psi(r_0)$, and then to some $B_{S(r_0)}(z_j, e^{-r_0})$ with z_j one of the points introduced in Lemma 4.2. Its cluster in $\Psi(r_0)$, i.e. $\hat{\text{Cl}}(z_j)$ (see the definition before (4.3)), generates a subset of the annulus $B(r_0 + \delta) \setminus B(r_0)$, namely

$$\text{Cl}(z_j) := \{(s; v) : r_0 \leq s \leq r_0 + \delta \text{ and } (r_0; v) \in \hat{\text{Cl}}(z_j)\}.$$

So, the RST path from z' to z'' remains inside $\text{Cl}(z_j)$ and visits at most

$$M(z_j) := \#(\mathcal{N} \cap \text{Cl}(z_j))$$

Poisson points (if z_j is not inhibited, we set $M(z_j) := 0$). Moreover, between two consecutive such visited Poisson points, the angular deviation is at most $2\text{Rad}(z_j)$. It then follows that $\text{CFD}_{r''}^{r'}(z')$ is smaller than $2\text{Rad}(z_j)(M(z_j) + 1)$. We then have stated that

$$\text{CFD}_{r_0}^{r'}(z') \leq Ae^{-r_0} + 2\text{Rad}(z_j)(M(z_j) + 1). \quad (4.5)$$

However the above upperbound still depends on z' through the element z_j . To overcome this obstacle, let us consider the (deterministic) set

$$J_i := \{j \in \{1, \dots, N(r_0)\} : B_{S(r_0)}(z_j, e^{-r_0}) \cap B_{S(r_0)}(z_i, (A+1)e^{-r_0}) \neq \emptyset\}$$

and let us prove that the element $z_j = z_j(z')$ occurring in (4.5) has necessarily its index j in J_i . First,

$$\widehat{z_j 0 \bar{z}''} \leq e^{-r_0} \quad \text{and} \quad \widehat{z 0 \bar{z}_i} \leq e^{-r_0}$$

since $\bar{z}'' \in B_{S(r_0)}(z_j, e^{-r_0})$ and $z \in B_{S(r_0)}(z_i, e^{-r_0})$. Thus, by Lemma 4.1,

$$\widehat{\bar{z}'' 0 z} = \widehat{z'' 0 z} \leq \text{CFD}_{r_0}^{r''}(z'') \leq Ae^{-r_0}.$$

We can conclude that $j \in J_i$ since

$$\widehat{z_j 0 z_i} \leq \widehat{z_j 0 \bar{z}''} + \widehat{\bar{z}'' 0 z} + \widehat{z 0 z_i} \leq (A+2)e^{-r_0}.$$

Henceforth, we can write

$$\text{MBD}_{r_0}^{r_0+\delta}(z) = \max_{z'=(r'; \cdot) \in \mathcal{D}[r_0, \delta, z]} \text{CFD}_{r_0}^{r'}(z') \leq Ae^{-r_0} + 2 \max_{j \in J_i} \text{Rad}(z_j)(M(z_j) + 1),$$

and notice that the upperbound does not longer depend on z' , nor on z , but only on z_i . Also, for any $p \geq 1$, the previous inequality becomes

$$\left(\text{MBD}_{r_0}^{r_0+\delta}(z)\right)^p \leq 2^{p-1} \left(A^p e^{-pr_0} + 2^p \max_{j \in J_i} \text{Rad}(z_j)^p (M(z_j) + 1)^p\right). \quad (4.6)$$

4.3. Step 2: Moments of $M(z_j)$

For $j \leq N(r_0)$, recall that $M(z_j) = \#(\mathcal{N} \cap \text{Cl}(z_j)) \mathbf{1}_{z_j \text{ inhibited}}$. The goal of this second step is to show that for any $p \geq 1$ there exists $C_0 = C_0(d, p) > 0$ such that

$$\mathbb{E}[M(z_j)^{4p}] \leq C_0. \quad (4.7)$$

Let us set, for $R > 0$, $\text{Reg}(R) := \text{Cone}(z_j, Re^{-r_0}) \cap (B(r_0 + \delta) \setminus B(r_0))$. Either $\text{Rad}(z_j) > Re^{-r_0}/2$ or $\text{Cl}(z_j) \subset \text{Reg}(R)$. In any words, for any integer m ,

$$\mathbb{P}[M(z) > m] \leq \mathbb{P}[\text{Rad}(z_j) > Re^{-r_0}/2] + \mathbb{P}[\#(\mathcal{N} \cap \text{Reg}(R)) > m]. \quad (4.8)$$

The first term in the upperbound in (4.8) is smaller than $e^{-c_{\text{dec}}R/2}$ by Lemma 4.3. Let us focus now on the second term. The r.v. $\#(\mathcal{N} \cap \text{Reg}(R))$ is distributed according to the Poisson law with parameter

$$\lambda \text{Vol}(\text{Reg}(R)) = \lambda \nu(\{u' : \widehat{u\theta u'} < Re^{-r_0}\}) \text{Vol}(B(r_0 + \delta) \setminus B(r_0)) \leq CR^d,$$

for some constant $C = C(d, \lambda) > 0$. Hence, $\#(\mathcal{N} \cap \text{Reg}(R))$ is stochastically dominated by the Poisson distribution with parameter CR^d . Then the Chernoff bound for the Poisson distribution gives

$$\mathbb{P}[\#(\mathcal{N} \cap \text{Reg}(R)) \geq m] \leq \left(\frac{CeR^d}{m}\right)^m \quad (4.9)$$

for any $m \geq CR^d$.

Let us choose R such that $m = 2eCR^d$ (larger than CR^d). We then can use (4.9) which, combined to (4.8), leads to

$$\mathbb{P}[M(z_j) > m] \leq e^{-(c_{\text{dec}}/2)(m/2eC)^{1/d}} + 2^{-m} \leq e^{-cm^{1/d}}.$$

for a constant $c > 0$ small enough. At last, we obtain (4.7) as follows:

$$\mathbb{E}[M(z_j)^{4p}] = \int_0^\infty \mathbb{P}[M(z_j) > m^{1/(4p)}] dm \leq \int_0^\infty e^{-cm^{1/(4p)}} dm < \infty.$$

4.4. Step 3: conclusion

Our goal is to control the quantity

$$\mathbb{E}\left[\sum_{z \in B_{S(r_0)}(z_i, e^{-r_0}) \cap \text{RST}} (\text{MBD}_{r_0}^{r_0 + \delta}(z))^p\right] \quad (4.10)$$

for any $1 \leq i \leq N(r_0)$. Using the a.s. inequality (4.6), the expectation (4.10) is upperbounded by the sum of the following terms

$$\text{I} := 2^{p-1} A^p e^{-pr_0} \mathbb{E}[\#(B_{S(r_0)}(z_i, e^{-r_0}) \cap \text{RST})]$$

$$\text{II} := 2^{2p-1} \mathbb{E}\left[\sum_{z \in B_{S(r_0)}(z_i, e^{-r_0}) \cap \text{RST}} \max_{j \in J_i} \text{Rad}(z_j)^p (M(z_j) + 1)^p\right].$$

The first term is easily bounded using Lemma 4.4 applied with $p = 1$: $\text{I} \leq 2^{p-1} CA^p e^{-pr_0}$. For the second term, we first use the Cauchy-Schwarz inequality:

$$\begin{aligned} \text{II} &= 2^{2p-1} \mathbb{E}\left[\#(B_{S(r_0)}(z_i, e^{-r_0}) \cap \text{RST}) \max_{j \in J_i} \text{Rad}(z_j)^p (M(z_j) + 1)^p\right] \\ &\leq 2^{2p-1} \mathbb{E}\left[\#(B_{S(r_0)}(z_i, e^{-r_0}) \cap \text{RST})^2\right]^{\frac{1}{2}} \mathbb{E}\left[\max_{j \in J_i} \text{Rad}(z_j)^{2p} (M(z_j) + 1)^{2p}\right]^{\frac{1}{2}} \\ &\leq 2^{2p-1} C^{\frac{1}{2}} \left(\sum_{j \in J_i} \mathbb{E}[\text{Rad}(z_j)^{4p}]\right)^{\frac{1}{2}} \mathbb{E}[(M(z_j) + 1)^{4p}]^{\frac{1}{2}}. \end{aligned} \quad (4.11)$$

Using Step 2, we bound $\mathbb{E}[(M(z_j) + 1)^{4p}]$ by a constant C_1 . Moreover, using Lemma 4.3, we can write:

$$\begin{aligned}\mathbb{E}[\text{Rad}(z_j)^{4p}] &= e^{-4pr_0} \int_0^\infty \mathbb{P}[e^{r_0} \text{Rad}(z_i) > B^{\frac{1}{4p}}] dB \\ &\leq e^{-4pr_0} \int_0^\infty e^{-c_{\text{dec}} B^{\frac{1}{4p}}} dB \\ &= C_2 e^{-4pr_0}\end{aligned}$$

where C_2 is a positive real number. Let us plug the previous upperbounds in (4.11), we get:

$$\text{II} \leq 2^{2p-1} C^{\frac{1}{2}} (C_1 C_2)^{\frac{1}{4}} e^{-pr_0} (\#J_i)^{\frac{1}{2}} .$$

Lemma 4.2 asserts that the set J_i counts at most $C_{\text{ball}}(A+1)^d$ elements. We finally obtain that there exists a constant $C = C(p, d) > 0$ satisfying

$$\mathbb{E}\left[\sum_{z \in B_{S(r_0)}(z_i, e^{-r_0}) \cap \text{RST}} (\text{MBD}_{r_0}^{r_0+\delta}(z))^p\right] = \text{I} + \text{II} \leq C e^{-pr_0} . \quad (4.12)$$

The final step is to sum over all the indices i such that $B_{S(r_0)}(z_i, e^{-r_0})$ intersects $B_{S(r_0)}(u, \theta)$ for some given $u \in \mathbb{S}^d$ and $\theta > 0$. By Lemma 4.2, this sum concerns at most $C_{\text{ball}} e^{dr_0} \theta^d$ terms which are all uniformly bounded according to (4.12). This achieves the proof of Proposition 2.5:

$$\begin{aligned}\mathbb{E}\left[\sum_{z \in B_{S(r_0)}(u, \theta) \cap \text{RST}} (\text{MBD}_{r_0}^{r_0+\delta}(z))^p\right] &\leq \sum_{\substack{1 \leq i \leq N(r_0), \\ B_{S(r_0)}(z_i, e^{-r_0}) \\ \cap B_{S(r_0)}(u, \theta) \neq \emptyset}} \mathbb{E}\left[\sum_{z \in B_{S(r_0)}(z_i, e^{-r_0}) \cap \text{RST}} (\text{MBD}_{r_0}^{r_0+\delta}(z))^p\right] \\ &\leq C e^{(d-p)r_0} \theta^d .\end{aligned}$$

4.5. Proof of Lemmas 4.1, 4.2, 4.3, 4.4

Proof of Lemma 4.1. Let $z = (s; u) \in \chi(r_0) \cap \mathcal{L}(s)$ with $r_0 \leq s \leq r_0 + \delta$. Recall that the edge of RST containing z is $[z_\uparrow, z_\downarrow]^*$ with $z_\uparrow = A(z)$ the ancestor of z .

Let us first prove that z_\uparrow belongs to $B(r_0)$. To do so, let us consider the elements $z_1 := (r_0; u)$, $z_2 := (r_0 + \delta; u)$ and $R := \inf\{\rho > 0 : B(z_2, \rho) \supset \Psi_2(r_0, z_1)\}$. Whenever $A > 2$, the set $V := B(z_2, R) \cap (B(r_0 + \delta) \setminus B(r_0))$ is included in $\Psi_1(r_0, z_1)$. Since z belongs to the controlled region $\chi(r_0)$, the event $G(r_0, z_1)$ holds: $\Psi_1(r_0, z_1)_-$ and so V_- is empty of Poisson points while $\Psi_2(r_0, z_1)$ is not. Consequently, the ancestor z_\uparrow of z is necessarily in $B(z_2, R)$, but not in V . So, it is in $B(r_0)$.

Henceforth, the points $z_\uparrow, z_3 := \mathcal{A}_{r_0}^s(z)$, z and z_\downarrow are all on the edge $[z_\uparrow, z_\downarrow]^*$ of RST, and in this order when $[z_\uparrow, z_\downarrow]^*$ is traveled from z_\uparrow to z_\downarrow . Our goal consists now in proving that

$$\text{CFD}_{r_0}^s(z) = \widehat{z_0 z_3} \leq A e^{-r_0} . \quad (4.13)$$

Let us proceed by contradiction with assuming that (4.13) is false. On the one hand, we can write

$$\widehat{z_\downarrow 0 z_3} = \widehat{z_\downarrow 0 z} + \widehat{z_0 z_3} > \widehat{z_\downarrow 0 z} + A e^{-r_0}$$

and on the other hand, denoting by $z' = (r'; \cdot)$ a Poisson point in $\Psi_2(r_0, z_1)$ (whose existence is ensured by $G(r_0, z_1)$)

$$\widehat{z_\downarrow 0 z'} \leq \widehat{z_\downarrow 0 z} + \widehat{z_0 z'} \leq \widehat{z_\downarrow 0 z} + A e^{-r_0} .$$

We then get

$$\widehat{z_\downarrow 0 z_3} > \widehat{z_\downarrow 0 z'}$$

which combined with $r' \leq r_0$ implies

$$(1 - \cos(\widehat{z_\downarrow 0 z'})) \sinh(r_\downarrow) \sinh(r') < (1 - \cos(\widehat{z_\downarrow 0 z_3})) \sinh(r_\downarrow) \sinh(r_0) .$$

Using the hyperbolic law of cosines, we finally obtain $\cosh(d(z_\downarrow, z')) < \cosh(d(z_\downarrow, z_3))$, i.e. $d(z_\downarrow, z') < d(z_\downarrow, z_3)$. We can now conclude. Recall that the distance to z_\downarrow is increasing along the edge $[z_\downarrow, z_\uparrow]^*$ by the construction (2.8). Hence,

$$d(z_\downarrow, z') < d(z_\downarrow, z_3) \leq d(z_\downarrow, z_\uparrow) .$$

In other words, z_\downarrow is closer to the Poisson point z' than to its ancestor $z_\uparrow = A(z)$. This contradicts the construction of the RST. \square

Proof of Lemma 4.2. Proving Items (a) and (b) of Lemma 4.2 is equivalent to show that there exists an integer K such that, for any $\varepsilon > 0$, the Euclidean unit sphere \mathbb{S}^d can be covered by balls of radius ε such that the number of balls overlapping some given point is bounded by K . This is a standard fact.

We move on to show the the existence of $C_{\text{ball}} > 0$ such that, for any $r_0 > 0$, $z \in S(r_0)$ and $A \geq 1$, the number of caps intersecting $B_{S(r_0)}(z, Ae^{-r_0})$ is upper-bounded by $C_{\text{ball}}A^d$. Let $u_0 \in \mathbb{S}^d$ be the direction of z and $A \geq 1$. For $i \in \{1, \dots, N(r_0)\}$, the cap $B_{S(r_0)}(z_i, e^{-r_0})$ overlaps $B_{S(r_0)}(z, Ae^{-r_0})$ if and only if $\widehat{z_i 0 z} \leq (A+1)e^{-r_0}$. Hence the union of caps $B_{S(r_0)}(z_i, e^{-r_0})$, for $1 \leq i \leq N(r_0)$, overlapping $B_{S(r_0)}(z, Ae^{-r_0})$ is included in $B_{S(r_0)}(z, (A+2)e^{-r_0})$.

Recall that ν denotes the d -dimensional volume measure on \mathbb{S}^d . We have that

$$\nu\{u : \widehat{u 0 u_0} \leq (A+2)e^{-r_0}\} \leq ((A+2)e^{-r_0})^d$$

and

$$\nu\{u : \widehat{u 0 u_0} \leq e^{-r_0}\} \geq ce^{-r_0 d}$$

where the constant $c > 0$ is uniform on r_0 . Combining these previous facts, we can deduce that the number of caps overlapping $B_{S(r_0)}(z, Ae^{-r_0})$ is upper-bounded by:

$$K \frac{\nu\{u : \widehat{u 0 u_0} \leq (A+2)e^{-r_0}\}}{\nu\{u : \widehat{u 0 u_0} \leq e^{-r_0}\}} \leq \frac{K((A+2)e^{-r_0})^d}{ce^{-r_0 d}} \leq C_{\text{ball}}A^d$$

with $C_{\text{ball}} := (3^d K)/c$ and the conclusion follows. \square

Proof of Lemma 4.3. Let us start with estimating the probability that any z_i (given by Lemma 4.2) is inhibited. Recall that z_i is said inhibited if the spherical cap $B_{S(r_0)}(z_i, e^{-r_0})$ overlaps $S(r_0) \setminus \hat{\chi}(r_0)$. Let us introduce the following events:

$$E(i) := \{\mathcal{N} \cap \text{Cone}(z_i, (3A+1)e^{-r_0}) \cap (B(r_0 + \delta) \setminus B(r_0)) = \emptyset\} ,$$

$$F(i) := \{\mathcal{N} \cap \text{Cone}(z_i, (A-1)e^{-r_0}) \cap (B(r_0) \setminus B((r_0-1) \wedge 0)) \neq \emptyset\} .$$

For any $z \in B_{S(r_0)}(z_i, e^{-r_0})$, the inclusions

$$\Psi_1(r_0, z) \subset \text{Cone}(z_i, (3A+1)e^{-r_0}) \cap (B(r_0 + \delta) \setminus B(r_0))$$

and

$$\Psi_2(r_0, z) \supset \text{Cone}(z_i, (A-1)e^{-r_0}) \cap (B(r_0 + \delta) \setminus B(r_0))$$

hold by triangular inequality which means that, on the event $E(i) \cap F(i)$, the event $G(r_0, z)$ occurs. Consequently, the cap $B_{S(r_0)}(z_i, e^{-r_0})$ is included in the controlled region $\hat{\chi}(r_0)$, i.e. z_i is not inhibited. We then have proved that $\mathbb{P}[z_i \text{ inhibited}] \leq \mathbb{P}[E(i)^c] + \mathbb{P}[F(i)^c]$.

With $z_i = (r_0; u_i)$, let us write

$$\begin{aligned} \text{Vol}(\text{Cone}(z_i, (3A+1)e^{-r_0}) \cap (B(r_0 + \delta) \setminus B(r_0))) &= \int_{r_0}^{r_0 + \delta} \nu\{u : \widehat{u_i 0 u} \leq (3A+1)e^{-r_0}\} \sinh(r)^d dr \\ &\leq C\delta A^d , \end{aligned}$$

for some $C = C(d) > 0$ not depending on A, r_0, δ . So, $\mathbb{P}[E(i)^c] \leq 1 - e^{-\lambda C \delta A^d}$. A similar computation for the event $F(i)$ leads to $\mathbb{P}[F(i)^c] \leq e^{-\lambda c A^d}$ where $c = c(d) > 0$ does not depend

on A, r_0, δ . Thenceforward we choose A large enough, thus $\delta > 0$ small enough so that the upperbounds $e^{-\lambda c A^d}$ and $1 - e^{-\lambda C \delta A^d}$ are both smaller than $(3C_{\text{ball}}(6A + 4)^d)^{-1}$. It then follows:

$$\mathbb{P}[z_i \text{ inhibited}] \leq 1 - e^{-\lambda c \delta A^d} + e^{-\lambda c A^d} \leq \frac{2}{3} C_{\text{ball}}^{-1} (6A + 4)^{-d}. \quad (4.14)$$

Let $k > 0$ be an integer. Let us pick some z_i and assume that $\text{Rad}(z_i) \geq k(6A + 4)e^{-r_0}$. This traduces the fact that z_i is inhibited, i.e. the corresponding cap $B_{S(r_0)}(z_i, e^{-r_0})$ belongs to the augmented uncontrolled region $\Psi(r_0)$, and its cluster $\hat{\text{Cl}}(z_i)$ in $\Psi(r_0)$ is quite large. Precisely, there exists a sequence $z_{i_0} = z_i, z_{i_1}, \dots, z_{i_k}$ among the z_j 's such that:

1. For any $0 \leq j < k$, $B_{S(r_0)}(z_{i_{j+1}}, e^{-r_0})$ overlaps $B_{S(r_0)}(z_{i_j}, (6A + 4)e^{-r_0})$;
2. For any $0 \leq j, j' \leq k$, $B_{S(r_0)}(z_{i_j}, (3A + 1)e^{-r_0})$ and $B_{S(r_0)}(z_{i_{j'}}, (3A + 1)e^{-r_0})$ do not overlap;
3. z_{i_0}, \dots, z_{i_k} are inhibited.

If we denote by E_k the set of such sequences $(z_{i_0} = z_i, z_{i_1}, \dots, z_{i_k})$ satisfying the first two items, then

$$\mathbb{P}[\text{Rad}(z_i) \geq k(6A + 4)e^{-r_0}] \leq \sum_{(z_{i_0}, \dots, z_{i_k}) \in E_k} \mathbb{P}[z_{i_0}, \dots, z_{i_k} \text{ are inhibited}]. \quad (4.15)$$

Let $(z_{i_0}, \dots, z_{i_k}) \in E_k$. For $0 \leq j \leq k$, the event $\{z_{i_j} \text{ is inhibited}\}$ only depends on the PPP \mathcal{N} inside $\text{Cone}(z_{i_j}, (3A + 1)e^{-r_0})$. So by Item 2. the events $\{z_{i_j} \text{ is inhibited}\}$'s are mutually independent. Using (4.14), we obtain

$$\mathbb{P}[z_{i_0}, \dots, z_{i_k} \text{ are inhibited}] = \mathbb{P}[z_i \text{ is inhibited}]^{k+1} \leq \left(\frac{2}{3}\right)^{k+1} C_{\text{ball}}^{-(k+1)} (6A + 4)^{-d(k+1)}.$$

Furthermore, by Lemma 4.2, the number of $B_{S(r_0)}(z_j, e^{-r_0})$'s overlapping $B_{S(r_0)}(z, \theta)$ is upperbounded by $C_{\text{ball}} e^{d r_0 \theta^d}$. This implies that $\#E_k$ is smaller than $C_{\text{ball}}^{k+1} (6A + 4)^{d(k+1)}$.

Finally, with (4.15), we prove that

$$\mathbb{P}[\text{Rad}(z_i) \geq B e^{-r_0}] = \mathbb{P}[\text{Rad}(z_i) \geq k(6A + 4)e^{-r_0}] \leq \left(\frac{2}{3}\right)^{k+1} \leq e^{-c_{\text{dec}} B}$$

with $B = k(6A + 4)$ and $c_{\text{dec}} = -\ln(2/3)/(6A + 4)$. □

Proof of Lemma 4.4. Let $r_0, M > 0$ and $z \in S(r_0)$. Let $h \geq 0$ that will be fixed later. We divide the set $L = \{z' \in \mathcal{N} : [z', A(z')]^* \cap B_{S(r_0)}(z, e^{-r_0}) \neq \emptyset\}$ into two subsets $L_{\leq h}$ and $L_{> h}$ according to the length of $[z', A(z')]^*$:

$$L_{\leq h} := \{z' \in L : d(z', z) \leq h\} \quad \text{and} \quad L_{> h} := \{z' \in L : d(z', z) > h\}. \quad (4.16)$$

Thus $L = L_{\leq h} \cup L_{> h}$ and

$$\mathbb{P}[\#L \geq M] \leq \mathbb{P}[\#L_{\leq h} \geq M] + \mathbb{P}[L_{> h} \neq \emptyset]. \quad (4.17)$$

We first upperbound $\mathbb{P}[\#L_{\leq h} \geq M]$. Since $L_{\leq h} \subset B(z, h)$,

$$\mathbb{P}[\#L_{\leq h} \geq M] \leq \mathbb{P}[\#(\mathcal{N} \cap B(z, h)) \geq M].$$

By (2.5), $\text{Vol}(B(z, h)) \leq C e^{dh}$, for some $C > 0$ independent of r_0 . So the r.v. $\#(\mathcal{N} \cap B(z, h))$ is stochastically dominated by a Poisson law with parameter $C \lambda e^{dh}$ thus, by the Chernoff bound for the Poisson distribution [21],

$$\mathbb{P}[\#L_{\leq h} \geq M] \leq \frac{e^{-C \lambda e^{dh}} (C \lambda e^{dh+1})^M}{M^M}. \quad (4.18)$$

The second step is to upperbound $\mathbb{P}[L_{>h} \neq \emptyset]$. Recall that $\mathbb{P}_{z'}$ denotes the Palm distribution of \mathcal{N} conditionally on having a point at $z' \in \mathbb{H}^{d+1}$, and that any $z' \in L$ is necessarily outside $B(r_0)$. By Campbell formula [13]:

$$\mathbb{P}[L_{>h} \neq \emptyset] \leq \mathbb{E}[\#L_{>h}] = \mathbb{E}\left[\sum_{z' \in \mathcal{N}} \mathbf{1}_{z' \in L_{>h}}\right] \leq \lambda \int_{B(r_0)^c} \mathbb{P}_{z'}[z' \in L_{>h}] dz'. \quad (4.19)$$

Let n be an integer. We will control the integrand in (4.19) on each of the following annuli centred at z :

$$\mathcal{C}_n := (B(z, n+1) \setminus B(z, n)) \cap B(r_0)^c.$$

Consider $z' \in L_{>h} \cap \mathcal{C}_n$. Let us first show that h can be chosen large enough so that $d(z', A(z')) > n/2$. To do so, let us consider the intersection point between $[z', A(z')]^*$ and $S(r_0)$, say z^* . The hyperbolic law of cosines (2.6) says

$$\begin{aligned} d(z^*, z) &= \operatorname{arcosh}(\cosh(r_0)^2 - \cos(\widehat{z^*0z}) \sinh(r_0)^2) \\ &\leq \operatorname{arcosh}(\cosh(r_0)^2 - \cos(e^{-r_0}) \sinh(r_0)^2) \\ &= \operatorname{arcosh}(1 + (1 - \cos(e^{-r_0})) \sinh(r_0)^2) \end{aligned}$$

which is smaller than some constant $C_{\text{dis}} > 0$ independent of r_0 . Because distances grow along edges— see (2.8) –, we get

$$d(z', A(z')) \geq d(z', z^*) \geq d(z, z') - d(z, z^*) \geq n - C_{\text{dis}}.$$

Besides $z' \in L_{>h} \cap \mathcal{C}_n$ also means $h < d(z', z) < n+1$ and in particular $n \geq h$. So $h > 2C_{\text{dis}}$ implies that $d(z', A(z')) > n/2$. Notice that this later inequality forces z' to be outside $B(n/2)$.

These geometric considerations combined with (4.19) give

$$\begin{aligned} \mathbb{P}[L_{>h} \neq \emptyset] &\leq \lambda \sum_{n=0}^{+\infty} \int_{\mathcal{C}_n} \mathbb{P}_{z'}[z' \in L_{>h}] dz' \\ &\leq \lambda \sum_{n \geq h} \int_{\mathcal{C}_n \cap B(n/2)^c} \mathbb{P}_{z'}[B^+(z', n/2) \cap \mathcal{N} = \emptyset] dz' \\ &\leq \lambda \sum_{n \geq h} \int_{\mathcal{C}_n \cap B(n/2)^c} e^{-\lambda \operatorname{Vol}(B^+(z', n/2))} dz'. \end{aligned}$$

The estimate (A.1) for the volume of balls with r' (the radius of z') larger than $n/2$ allows us to write:

$$\mathbb{P}[L_{>h} \neq \emptyset] \leq \lambda \sum_{n=h}^{+\infty} e^{-\lambda c e^{dn/4}} \operatorname{Vol}(\mathcal{C}_n) \leq \lambda \sum_{n=h}^{+\infty} e^{-\lambda c e^{dn/4} + d(n+1)} \leq C e^{-ch} \quad (4.20)$$

for suitable positive constants c, C and h large enough.

Let us conclude with choosing $h = (K/c) \ln(M)$ where c is the constant appearing in (4.20) and K is any large integer. Hence the upperbound in (4.20) becomes M^{-K} while the one in (4.18) is negligible w.r.t. M^{-K} . So, by (4.17), $\mathbb{P}[\#L \geq M]$ is smaller than M^{-K} , for any K . This proves Lemma 4.4. \square

5. Proof of Propositions 2.6 and 2.7

We first prove Proposition 2.6.

Step 1: Let us fix $p > 3d/2$. For any $r_0 > 0$, $n \in \mathbb{N}$, let us define

$$S_n(A, r_0) := \sum_{z \in B_{S(r_0)}(u, Ae^{-r_0}) \cap \text{RST}} \left(\text{MBD}_{r_0}^{r_0+n\delta}(z) \right)^{2p}.$$

The strategy of the proof is to construct a family of non-negative random variables $(Y_n^M(A, r_0))_{r_0, A, M \geq 0, n \in \mathbb{N}}$ and $(Y^M(A, r_0))_{r_0, A, M \geq 0}$ such that

- (1) almost surely, $Y_n^M(A, r_0) \uparrow Y^M(A, r_0)$ when $n \rightarrow \infty$ for any $M, A, r_0 \geq 0$;
- (2) $\sup_{A, r_0} \mathbb{P}[Y^M(A, r_0) \geq M] = O(M^{-2/3})$ when $M \rightarrow \infty$;
- (3) the following implication holds almost surely:

$$S_n(A, r_0) \leq (M \wedge Y_n^M(A, r_0))A^d e^{-2r_0 p} \implies S_{n+1}(A, r_0) \leq Y_{n+1}^M(A, r_0)A^d e^{-2r_0 p}.$$

Let us suppose for the moment that such random variables $Y_n^M(A, r_0)$ and $Y^M(A, r_0)$ exist. Let $A, r_0 \geq 0$ and $M \geq 0$. On the event $\{Y^M(A, r_0) \leq M\}$, it can be shown by induction that $S_n(A, r_0) \leq MA^d e^{-2r_0 p}$ for any $n \geq 0$. Indeed, $S_0 = 0$, and if $S_n(A, r_0) \leq MA^d e^{-2r_0 p}$, then, by (3),

$$S_{n+1}(A, r_0) \leq Y_{n+1}^M(A, r_0)A^d e^{-2r_0 p} \leq Y^M(A, r_0)A^d e^{-2r_0 p} \leq MA^d e^{-2r_0 p},$$

since we are on the event $\{Y^M(A, r_0) \leq M\}$. This achieves the induction.

Thus, for any $A, r_0, M \geq 0$,

$$\mathbb{P}[S_n(A, r_0) \geq MA^d e^{-2r_0 p}] \leq \mathbb{P}[Y^M(A, r_0) \geq M] \leq CM^{-2/3} \text{ by item (2) above,} \quad (5.1)$$

for M large enough and some constant $C > 0$ independent of A, r_0, M . It follows that

$$\begin{aligned} C' &:= \sup_{A, r_0} \mathbb{E}[S_n(A, r_0)A^{-d}e^{2r_0 p}] = \sup_{A, r_0} \int_0^\infty \mathbb{P}[S_n(A, r_0) \geq MA^d e^{-2r_0 p}] dM \\ &\stackrel{(5.1)}{\leq} \int_0^\infty CM^{-2/3} dM < \infty. \end{aligned} \quad (5.2)$$

Recall the notation in (2.10). Let us apply Cauchy-Schwarz with the inner product defined by $\langle X, Y \rangle = \mathbb{E}[\sum_i X_i Y_i]$,

$$\begin{aligned} &\mathbb{E} \left[\sum_{z \in B_{S(r_0)}(u, Ae^{-r_0}) \cap \text{RST}} \left(\text{MBD}_{r_0}^{r_0+n\delta}(z) \right)^p \right] \\ &\leq \mathbb{E} \left[\sum_{z \in B_{S(r_0)}(u, Ae^{-r_0}) \cap \text{RST}} \left(\text{MBD}_{r_0}^{r_0+n\delta}(z) \right)^{2p} \right]^{1/2} \mathbb{E} [\#B_{S(r_0)}(u, Ae^{-r_0})]^{1/2} \\ &= \mathbb{E}[S_n(A, r_0)]^{1/2} \mathbb{E}[\#B_{S(r_0)}(u, Ae^{-r_0})]^{1/2} \\ &\stackrel{(5.2)}{\leq} \sqrt{C'} A^{d/2} e^{-r_0 p} \mathbb{E}[\#B_{S(r_0)}(u, Ae^{-r_0}) \cap \text{RST}]^{1/2}. \end{aligned} \quad (5.3)$$

Let us show that $\mathbb{E}[\#B_{S(r_0)}(u, Ae^{-r_0}) \cap \text{RST}] \leq CA^d$ for some $C > 0$ independent of A, r_0 . We use the covering of $S(r_0)$ by balls of radius e^{-r_0} introduced by Lemma 4.2 in Section 4. For any $1 \leq i \leq N(r_0)$, by Proposition 4.4 applied with $p = 1$, $\mathbb{E}[\#\text{RST} \cap B_{S(r_0)}(z_i, e^{-r_0})] \leq C$ for C independent of r_0, z_i . By Lemma 4.2, the number of balls intersecting $B_{S(r_0)}(u, Ae^{-r_0})$ is bounded by $C_{\text{ball}}A^d$. It follows that $\mathbb{E}[\#B_{S(r_0)}(u, Ae^{-r_0}) \cap \text{RST}] \leq CA^d$.

Thus, resuming to (5.3),

$$\mathbb{E} \left[\sum_{z \in B_{S(r_0)}(u, Ae^{-r_0}) \cap \text{RST}} \left(\text{MBD}_{r_0}^{r_0+n\delta}(z) \right)^p \right] \leq C e^{-r_0 p}. \quad (5.4)$$

Since $r \mapsto \text{MBD}_{r_0}^r(z)$ is non-decreasing for any $z \in S(r)$,

$$\sum_{z \in B_{S(r_0)}(u, Ae^{-r_0}) \cap \text{RST}} (\text{MBD}_{r_0}^\infty(z))^p = \lim_{n \rightarrow \infty} \uparrow \sum_{z \in B_{S(r_0)}(u, Ae^{-r_0}) \cap \text{RST}} \left(\text{MBD}_{r_0}^{r_0+n\delta}(z) \right)^p. \quad (5.5)$$

Proposition 2.6 follows by (5.4) and by monotone convergence theorem.

Step 2: We build the random variables $Y_n^M(A, r_0)$ and $Y^M(A, r_0)$, as presented in the beginning of Step 1. **Let us define**

$$Z_n(A, r_0) := \sum_{z' \in B_{S(r_0+n\delta)}(u, Ae^{-r_0(M^d/(2p)+1)}) \cap \text{RST}} \left(\text{MBD}_{r_0+n\delta}^{r_0+(n+1)\delta}(z') \right)^{2p}. \quad (5.6)$$

We will show that the following choice works: we set $Y_0^M(A, r_0) := 0$, and for any $n \geq 0$,

$$Y_{n+1}^M(A, r_0) := \left(1 - \frac{1}{n^2} \right)^{1-2p} Y_n^M(A, r_0) + n^{4p-2} A^{-d} e^{2r_0 p} Z_n(A, r_0). \quad (5.7)$$

Let us also define $Y^M(A, r_0) := \lim_{n \rightarrow \infty} \uparrow Y_n^M(A, r_0)$ (this is well-defined since $n \rightarrow Y_n^M(A, r_0)$ is non-decreasing).

To understand the choice of (5.6) and (5.7), let us establish a formula where $S_{n+1}(A, r_0)$ is upper bounded by $S_n(A, r_0)$ and $Z_n(A, r_0)$.

Let $A, r_0 > 0$, let $n \in \mathbb{N}$. Fix $z \in B_{S(r_0)}(u, Ae^{-r_0})$. The quantity $\text{MBD}_{r_0}^{r_0+n\delta}$ takes into account finite backward paths that stop before level $r_0 + n\delta$ and those (potentially infinite) that continue after level $r_0 + n\delta$. Let us define the random set $\text{Stop}(z)$ as the set of ending points (in the backward direction) of finite paths from z stopping before level $r_0 + n\delta$:

$$\text{Stop}(z) := \{z' = (r'; u') \in \mathcal{N} \cap \mathcal{D}(z), r_0 \leq r' \leq r_0 + n\delta, A^{-1}(z') = \emptyset\} \subset \mathcal{N}.$$

By definition of $\text{MBD}_{r_0}^{r_0+n\delta}(z)$ (resp. $\text{MBD}_{r_0}^{r_0+(n+1)\delta}(z)$),

$$\text{MBD}_{r_0}^{r_0+n\delta}(z) = \max_{z'=(r';u') \in \text{Stop}(z)} \text{CFD}_{r_0}^{r'}(z') \vee \max_{z' \in \mathcal{D}_{r_0}^{r_0+n\delta}(z)} \text{CFD}_{r_0}^{r_0+n\delta}(z') \quad (5.8)$$

and

$$\begin{aligned} & \text{MBD}_{r_0}^{r_0+(n+1)\delta}(z) \\ &= \max_{z'=(r';u') \in \text{Stop}(z)} \text{CFD}_{r_0}^{r'}(z') \vee \max_{z' \in \mathcal{D}_{r_0}^{r_0+n\delta}(z)} \left(\text{CFD}_{r_0}^{r_0+n\delta}(z') + \text{MBD}_{r_0+n\delta}^{r_0+(n+1)\delta}(z') \right). \end{aligned} \quad (5.9)$$

For any $p \geq 1$, $a, b \geq 0$ and $t \in [0, 1]$, Jensen inequality gives,

$$(a+b)^p = \left(t \frac{a}{t} + (1-t) \frac{b}{1-t} \right)^p \leq t \left(\frac{a}{t} \right)^p + (1-t) \left(\frac{b}{1-t} \right)^p = t^{1-p} a^p + (1-t)^{1-p} b^p. \quad (5.10)$$

Applying (5.10) with $t = 1/n^2$ leads to:

$$\begin{aligned} & \left(\text{MBD}_{r_0}^{r_0+(n+1)\delta}(z) \right)^{2p} \\ & \stackrel{(5.9)}{=} \max_{z'=(r';u') \in \text{Stop}(z)} \text{CFD}_{r_0}^{r'}(z')^{2p} \vee \max_{z' \in \mathcal{D}_{r_0}^{r_0+n\delta}(z)} \left(\text{CFD}_{r_0}^{r_0+n\delta}(z') + \text{MBD}_{r_0+n\delta}^{r_0+(n+1)\delta}(z') \right)^{2p} \\ & \leq \max_{z'=(r';u') \in \text{Stop}(z)} \text{CFD}_{r_0}^{r'}(z')^{2p} \vee \\ & \max_{z' \in \mathcal{D}_{r_0}^{r_0+n\delta}(z)} \left[\left(1 - \frac{1}{n^2} \right)^{1-2p} \left(\text{CFD}_{r_0}^{r_0+n\delta}(z') \right)^{2p} + n^{4p-2} \left(\text{MBD}_{r_0+n\delta}^{r_0+(n+1)\delta}(z') \right)^{2p} \right]. \end{aligned}$$

To simplify future expressions, set

$$p(n) := \left(1 - \frac{1}{n^2}\right)^{1-2p}. \quad (5.11)$$

Since $1 - 2p < 0$ and $1 - 1/n^2 \in (0, 1)$, $p(n) > 1$ and

$$\begin{aligned} (\text{MBD}_{r_0}^{r_0+(n+1)\delta}(z))^{2p} &\leq p(n) \left[\max_{z'=(r';u') \in \text{Stop}(z)} (\text{CFD}_{r_0}^{r'}(z'))^{2p} \vee \max_{z' \in \mathcal{D}_{r_0}^{r_0+n\delta}(z)} (\text{CFD}_{r_0}^{r_0+n\delta}(z'))^{2p} \right] \\ &\quad + n^{4p-2} \max_{z' \in \mathcal{D}_{r_0}^{r_0+n\delta}(z)} (\text{MBD}_{r_0+n\delta}^{r_0+(n+1)\delta}(z'))^{2p} \\ &\stackrel{(5.8)}{=} p(n) (\text{MBD}_{r_0}^{r_0+n\delta}(z))^{2p} + n^{4p-2} \max_{z' \in \mathcal{D}_{r_0}^{r_0+n\delta}(z)} (\text{MBD}_{r_0+n\delta}^{r_0+(n+1)\delta}(z'))^{2p}. \end{aligned} \quad (5.12)$$

Summing (5.12) over all $z \in B_{S(r_0)}(u, Ae^{-r_0})$ leads to:

$$S_{n+1}(A, r_0) \leq p(n) S_n^S(A, r_0) + n^{4p-2} \sum_{\substack{z \in B_{S(r_0)}(u, Ae^{-r_0}) \\ z' \in \mathcal{D}_{r_0}^{r_0+n\delta}(z)}} (\text{MBD}_{r_0+n\delta}^{r_0+(n+1)\delta}(z'))^{2p}. \quad (5.13)$$

Let us condition on the event $\{S_n(A, r_0) \leq MA^d e^{-2r_0p}\}$. Then, for any $z \in B_{S(r_0)}(u, Ae^{-r_0})$,

$$(\text{MBD}_{r_0}^{r_0+n\delta}(z))^{2p} \leq S_n(A, r_0) \leq MA^d e^{-2r_0p},$$

so, for any $z' \in \mathcal{D}_{r_0}^{r_0+n\delta}(z)$, $\widehat{z'0z} \leq M^{1/2p} A^{d/(2p)} e^{-r_0}$. Denoting by $z_\infty \in \partial\mathbb{H}^{d+1}$ the point of direction u , we have

$$\begin{aligned} \widehat{z'0z_\infty} &\leq \widehat{z'0z} + \widehat{z0z_\infty} \\ &\leq M^{\frac{d}{2p}} A^{\frac{d}{2p}} e^{-r_0} + Ae^{-r_0}, \text{ since } z \in B_{S(r_0)}(u, Ae^{-r_0}) \\ &\leq Ae^{-r_0} \left(M^{\frac{d}{2p}} + 1 \right), \text{ since } \frac{d}{2p} \leq 1. \end{aligned}$$

Therefore, for any $z \in B_{S(r_0)}(u, Ae^{-r_0})$,

$$\mathcal{D}_{r_0}^{r_0+n\delta}(z) \subset B_{S(r_0+n\delta)} \left(u, Ae^{-r_0} \left(M^{\frac{d}{2p}} + 1 \right) \right). \quad (5.14)$$

By (5.6) and (5.14),

$$\sum_{\substack{z \in B_{S(r_0)}(u, Ae^{-r_0}) \\ z' \in \mathcal{D}_{r_0}^{r_0+n\delta}(z)}} (\text{MBD}_{r_0+n\delta}^{r_0+(n+1)\delta}(z'))^{2p} \leq Z_n(A, r_0), \quad (5.15)$$

thus, combining (5.13) and (5.15), on the event $\{S_n \leq MA^d e^{-2r_0p}\}$,

$$S_{n+1}(A, r_0) \leq p(n) S_n(A, r_0) + n^{4p-2} Z_n(A, r_0). \quad (5.16)$$

This upper-bound of $S_{n+1}(A, r_0)$ suggests the definition of the random variables $Y_n^M(A, r_0)$ given in (5.7).

Step 3: We now show that the random variables $Y_n^M(A, r_0)$ verify the items (1)-(3) of Step 1. Let us start with item (3): for $n \in \mathbb{N}$, on the event $\{S_n(A, r_0) \leq (M \wedge Y_n^M(A, r_0)) A^d e^{-2r_0p}\}$, by (5.16),

$$\begin{aligned} S_{n+1}(A, r_0) &\leq p(n) S_n(A, r_0) + n^{4p-2} Z_n(A, r_0) \\ &\leq p(n) Y_n^M(A, r_0) A^d e^{-2r_0p} + n^{4p-2} Z_n(A, r_0) \\ &= A^d e^{-2r_0p} [p(n) Y_n^M(A, r_0) + n^{4p-2} A^{-d} e^{2r_0p} Z_n(A, r_0)] \\ &= A^d e^{-2r_0p} Y_{n+1}^M(A, r_0). \end{aligned}$$

Thus the random variables $Y_n^M(A, r_0)$ verify (3).

We move on to show that $(Y_n^M(A, r_0))_{n, M, A, r_0}$ and $(Y^M(A, r_0))_{M, A, r_0}$ also verify (2) of Step 1. To proceed, we upper-bound $\mathbb{E}[Y_n^M(A, r_0)]$ by induction on n .

For any M, A, r_0, n , Proposition 2.5 applied for $\theta = Ae^{-r_0} (M^{d/(2p)} + 1)$ gives,

$$\begin{aligned} \mathbb{E}[Z_n(A, r_0)] &\leq C \left(Ae^{-r_0} \left(M^{\frac{d}{2p}} + 1 \right) \right)^d e^{(d-2p)(r_0+n\delta)} \\ &= CA^d \left(M^{\frac{d}{2p}} + 1 \right)^d e^{-2pr_0+n(d-2p)\delta}. \end{aligned} \quad (5.17)$$

Let us define, for any $n \in \mathbb{N}$,

$$q(n) := n^{4p-2} e^{n(d-2p)\delta}, \quad \text{and} \quad P(n) := \prod_{k=0}^{n-1} p(k), \quad Q(n) := \sum_{k=0}^{n-1} q(k), \quad (5.18)$$

with the convention $P(0) = 1$ and $Q(0) = 0$. It can be noticed that

$$\lim_{n \rightarrow \infty} P(n) < \infty, \quad \lim_{n \rightarrow \infty} Q(n) < \infty \text{ since } d - 2p < 0. \quad (5.19)$$

Let us show by induction on n that $\mathbb{E}[Y_n^M(A, r_0)] \leq C(M^{d/(2p)} + 1)^d P(n)Q(n)$ for any $n \in \mathbb{N}$. The assertion is clear for $n = 0$ and, for $n \geq 0$,

$$\begin{aligned} \mathbb{E}[Y_{n+1}^M(A, r_0)] &= p(n) \mathbb{E}[Y_n^M(A, r_0)] + n^{4p-2} A^{-d} e^{2r_0 p} \mathbb{E}[Z_n(A, r_0)] \\ &= p(n) \mathbb{E}[Y_n^M(A, r_0)] + A^{-d} e^{2r_0 p - n(d-2p)\delta} q(n) \mathbb{E}[Z_n(A, r_0)] \\ &\stackrel{(5.17)}{\leq} p(n) \mathbb{E}[Y_n^M(A, r_0)] + C \left(M^{\frac{d}{2p}} + 1 \right)^d q(n) \\ &\leq C \left(M^{\frac{d}{2p}} + 1 \right)^d [p(n)P(n)Q(n) + q(n)] \text{ by induction hypothesis} \\ &= C \left(M^{\frac{d}{2p}} + 1 \right)^d [P(n+1)Q(n) + q(n)] \\ &\leq C \left(M^{\frac{d}{2p}} + 1 \right)^d P(n+1)Q(n+1) \text{ since } P(n+1) \leq 1, \end{aligned} \quad (5.20)$$

which achieves the induction. Thus, by (5.19), there exists some constant $C > 0$ such that, for any $M, A, r_0 \geq 0$, for any $n \in \mathbb{N}$, $\mathbb{E}[Y_n^M(A, r_0)] \leq C(M^{d/(2p)} + 1)^d$. By monotone convergence,

$$\mathbb{E}[Y^M(A, r_0)] \leq C \left(M^{1/(2p)} + 1 \right)^d.$$

Thus, for any $M, A, r_0 \geq 0$, Markov inequality gives,

$$\mathbb{P}[Y^M(A, r_0) \geq M] \leq \frac{C(M^{1/(2p)} + 1)^d}{M} = O(M^{-2/3})$$

since $2p > 3d$. Thus the family of random variables $Y^M(A, r_0)$ verifies (2). This achieves the proof.

Proof of Proposition 2.7. This is a direct consequence of Proposition 2.6. Let $\varepsilon > 0$ and let us choose p such that $d/p < \varepsilon$. Applying Proposition 2.6 with $A = \pi e^{r_0}$ gives that, for any $r_0 \geq 0$,

$$\mathbb{E} \left[\sum_{z \in S(r_0) \cap \text{RST}} (\text{MBD}_{r_0}^\infty(z))^p \right] \leq C e^{r_0(d-p)}. \quad (5.21)$$

Thus:

$$\mathbb{E} \left[\sum_{n \in \mathbb{N}} \left(\max_{z \in \mathcal{L}_n} \text{MBD}_n^\infty(z) e^{(1-\varepsilon)n} \right)^p \right] \leq \sum_{n \in \mathbb{N}} C e^{n((d-p)+(1-\varepsilon)p)} < \infty$$

since $\varepsilon > d/p$. Therefore, a.s.,

$$\lim_{n \rightarrow \infty} e^{(1-\varepsilon)n} \max_{z \in \mathcal{L}_n} \text{MBD}_n^\infty(z) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Moreover, $r_0 \mapsto \max_{z \in \mathcal{L}_{r_0}} \text{MBD}_{r_0}^\infty$ is non-increasing, so for any $n \leq r_0 < n+1$,

$$e^{(1-\varepsilon)r_0} \max_{z \in \mathcal{L}_{r_0}} \text{MBD}_{r_0}^\infty(z) \leq \max_{z \in \mathcal{L}_n} \text{MBD}_n^\infty(z) e^{(1-\varepsilon)r_0} \leq e^{1-\varepsilon} \max_{z \in \mathcal{L}_n} \text{MBD}_n^\infty(z) e^{(1-\varepsilon)n}, \quad (5.22)$$

thus

$$\lim_{r_0 \rightarrow \infty} e^{(1-\varepsilon)r_0} \max_{z \in \mathcal{L}_{r_0}} \text{MBD}_{r_0}^\infty(z) \rightarrow 0 \text{ as } r_0 \rightarrow \infty.$$

Define R_0 such that, for any $r_0 \geq R_0$, $e^{(1-\varepsilon)r_0} \max_{z \in \mathcal{L}_{r_0}} \text{MBD}_{r_0}^\infty(z) \leq 1/2$. For any $r_0 \geq R_0$, $z \in \mathcal{L}_{r_0}$, $z_1, z_2 \in \mathcal{D}(z)$, defining $r_1 := d(0, z_1)$ and $r_2 := d(0, z_2)$,

$$\begin{aligned} \widehat{z_1 0 z_2} &\leq \widehat{z_1 0 z} + \widehat{z 0 z_2} \leq \text{CFD}_{r_0}^{r_1}(z_1) + \text{CFD}_{r_0}^{r_2}(z_2) \leq \text{MBD}_{r_0}^{r_1}(z) + \text{MBD}_{r_0}^{r_2}(z) \\ &\leq 2 \text{MBD}_{r_0}^\infty(z) \leq 2 \max_{z' \in \mathcal{L}_{r_0}} \text{MBD}_{r_0}^\infty(z') \leq e^{-(1-\varepsilon)r_0}. \end{aligned} \quad (5.23)$$

This achieves the proof of Proposition 2.7.

A. Proof of Proposition 2.2

We first show that the RST is a tree. If the RST contains some loop z_0, \dots, z_n , then the furthest vertex to the origin in the loop, say z_i , must have two parents, which contradicts the definition of the RST. Moreover, for some given vertex $z \in \mathcal{N}$, the sequence $(d(A^{(k)}(z), 0))_k$ is decreasing. In addition, since $\mathcal{N} \cap B(r)$ is finite for any $r \geq 0$, there is no infinite decreasing sequence $(d(A^{(k)}(z), 0))_k$. Thus $A^{(k)}(z) = 0$ for some finite $k \geq 0$. Therefore, the RST is a connected graph, so it is a tree.

We move on to show that the RST is locally finite. Let us assume for the moment that, for any $z = (r; u) \in \mathbb{H}^{d+1}$ and $\rho > 0$,

$$\text{Vol}(B^+(z, \rho)) \geq c e^{d(\rho \wedge r)/2}. \quad (\text{A.1})$$

for some c independent of z, ρ .

For $z_0 = (r_0; u_0) \in \mathcal{N} \cup \{0\}$, $z = (r; u) \in \mathbb{H}^d$, let us define

$$a(z, z_0) = \mathbf{1}_{r > r_0} \mathbf{1}_{B^+(z, d(z, z_0)) \cap \mathcal{N} = \emptyset}.$$

For any $z \in \mathcal{N}$, $z_0 = A(z)$ if and only if $a(z, z_0) = 1$. By Campbell formula [13],

$$\begin{aligned} \mathbb{E} \left[\#\{z_0 \in \mathcal{N}, \#A^{(-1)}(z_0) = \infty\} \right] &= \mathbb{E} \left[\sum_{z_0 \in \mathcal{N}} \mathbf{1}_{\sum_{z \in \mathcal{N}} a(z, z_0) = \infty} \right] \\ &= \lambda \int_{\mathbb{H}^{d+1}} \mathbb{P}_{z_0} \left[\sum_{z \in \mathcal{N}} a(z, z_0) = \infty \right] d\text{Vol}(z_0), \end{aligned}$$

where we recall that \mathbb{P}_{z_0} is the Palm measure conditioned on having an atom at z_0 . Thus it suffices to show that $\mathbb{P}_{z_0} [\sum_{z \in \mathcal{N}} a(z, z_0) = \infty] = 0$ for any $z_0 \in \mathbb{H}^{d+1}$. Let $z_0 = (r_0; u_0) \in \mathbb{H}^{d+1}$. Note that, if $d(z, z_0) \geq r_0$, then $0 \in B^+(z, d(z, z_0))$ so $a(z, z_0) = 0$. Thus,

$$\begin{aligned} \mathbb{E} \left[\sum_{z \in \mathcal{N}} a(z, z_0) \right] &= \lambda \int_{\mathbb{H}^{d+1}} \mathbb{E} [a(z, z_0)] d\text{Vol}(z) \leq \lambda \int_{\mathbb{H}^{d+1}} \mathbf{1}_{d(z, z_0) < r_0} \mathbb{P}[B^+(z, d(z, z_0)) \cap \mathcal{N} = \emptyset] d\text{Vol}(z) \\ &= \lambda \int_{\mathbb{H}^{d+1}} \mathbf{1}_{d(z, z_0) < r_0} \exp(-\lambda \text{Vol}(B^+(z, d(z, z_0)))) d\text{Vol}(z) \\ &\stackrel{(\text{A.1})}{\leq} \lambda \int_{\mathbb{H}^{d+1}} \exp(-\lambda c e^{-d/2} (d(z, z_0) \wedge d(0, z))) d\text{Vol}(z) \\ &\stackrel{(2.4)}{=} \lambda \nu(\mathbb{S}^d) \int_0^\infty \exp(-\lambda c e^{-r/2}) \sinh(r)^d dr < +\infty, \end{aligned}$$

where $\nu(\mathbb{S}^d)$ is the surface area of the Euclidean unit ball \mathbb{S}^d . Thus $\mathbb{P}[\sum_{z \in \mathcal{N}} a(z, z_0) = \infty] = 0$.

It remains to show (A.1). Recall that $z = (r; u)$ and that $\rho \geq 0$. Let us introduce $z' = (r - r \wedge \rho; u)$ and $z'' = (r - \frac{r \wedge \rho}{2}; u)$. The latter point is the center of the geodesic $[z', z]$ (which is here also the segment $[z', z]$ as these points are aligned with 0). We have that

$$B(z'', \frac{r \wedge \rho}{2}) \subset B^+(z, r \wedge \rho) \subset B^+(z, \rho). \quad (\text{A.2})$$

The second inclusion is obvious as $r \wedge \rho \leq \rho$. For the first inclusion, let us consider $x \in B(z'', \frac{r \wedge \rho}{2})$. Since

$$d'(x, z) \leq d(x, z'') + d(z'', z) \leq \frac{r \wedge \rho}{2} + \frac{r \wedge \rho}{2} = r \wedge \rho,$$

we have $B(z'', \frac{r \wedge \rho}{2}) \subset B(z, r \wedge \rho)$. Moreover,

$$d(x, 0) \leq d(x, z'') + d(z'', 0) \leq \frac{r \wedge \rho}{2} + (r - \frac{r \wedge \rho}{2}) = r,$$

so $B(z'', \frac{r \wedge \rho}{2}) \subset B(r)$. From these two inclusions, we deduce the first inclusion in (A.2). As a consequence,

$$\text{Vol}(B^+(z, \rho)) \geq \text{Vol}(B(z'', \frac{r \wedge \rho}{2})) = \text{Vol}(B(\frac{r \wedge \rho}{2})) = \nu(\mathbb{S}^d) \int_0^{\frac{r \wedge \rho}{2}} \sinh^d(t) dt \geq c(d) e^{d \frac{r \wedge \rho}{2}},$$

where the last inequality holds when r and ρ are sufficiently far from 0.

It remains to show that the geodesics $[z, A(z)]$ for $z \in \mathcal{N}$ do not cross a.s. in the bi-dimensional case ($d = 1$). Let us suppose that there are no two points z_1, z_2 with $d(0, z_1) = d(0, z_2)$ (this happens with probability 0). Let $z_1 = (r_1; u_1), z_2 = (r_2; u_2) \in \mathcal{N}$ and let us set $A(z_1) := (r'_1; u'_1), A(z_2) := (r'_2; u'_2)$. Suppose that $[z_1, A(z_1)]$ and $[z_2, A(z_2)]$ meet at some point $P_{hyp} := (r_{hyp}; u_{hyp})$. We have $r'_1 < r_{hyp} < r_2$, thus by definition of the parent, $d(z_2, A(z_2)) < d(z_2, A(z_1))$. Then

$$\begin{aligned} d(z_2, P_{hyp}) + d(P_{hyp}, A(z_2)) &= d(z_2, A(z_2)) < d(z_2, A(z_1)) \\ &\leq d(z_2, P_{hyp}) + d(P_{hyp}, A(z_1)), \end{aligned} \quad (\text{A.3})$$

so $d(P_{hyp}, A(z_2)) < d(P_{hyp}, A(z_1))$. On the other hand, interchanging z_1 and z_2 in the previous calculation leads to $d(P_{hyp}, A(z_1)) < d(P_{hyp}, A(z_2))$. This is a contradiction. Therefore $[z_1, A(z_1)] \cap [z_2, A(z_2)] = \emptyset$. This achieves the proof of Proposition 2.2.

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