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Coverings of the plane by self-avoiding curves which satisfy the local isomorphism property

Francis OGER

Abstract. A self-avoiding plane-filling curve cannot be periodic, but we show that it can satisfy the local isomorphism property. We investigate three families of coverings of the plane by finite sets of nonoverlapping self-avoiding curves which satisfy that property in a strong form. These curves are respectively inductive limits of: 1) n-folding square curves such as the dragon curve, obtained by folding n times a strip of paper in 2 and unfolding it with $\pi/2$ angles; 2) n-folding triangular curves such as the terdragon curve, obtained by folding n times a strip of paper in 3 and unfolding it with $\pi/3$ angles; 3) generalizations of Peano-Gosper curves. In each family, the coverings consist of a small number of curves (at most 6), and in many examples only 1 curve. We do not know presently if similar examples exist in $\mathbb{R}^n$ for $n \geq 3$.

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We denote by $\mathbb{N}^*$ the set of strictly positive integers, and $\mathbb{R}^+$ the set of positive real numbers.

For each $n \in \mathbb{N}^*$, we consider $\mathbb{R}^n$ equipped with a norm $x \to \|x\|$. For any $x, y \in \mathbb{R}^n$, we write $d(x, y) = \|y - x\|$. For each $x \in \mathbb{R}^n$ and each $r \in \mathbb{R}^+$, we denote by $B(x, r)$ the closed ball of center x and radius r.

For any $E, F \subset \mathbb{R}^n$, an *isomorphism* from E to F is a translation τ such that $\tau(E) = F$. They are *locally isomorphic* if, for each $x \in \mathbb{R}^n$ (resp. $y \in \mathbb{R}^n$) and each $r \in \mathbb{R}^+$, there exists $y \in \mathbb{R}^n$ (resp. $x \in \mathbb{R}^n$) such that $(B(x, r) \cap E, x) \cong (B(y, r) \cap F, y)$.

$E \subset \mathbb{R}^n$ satisfies the *local isomorphism property* if, for each $x \in \mathbb{R}^n$ and each $r \in \mathbb{R}^+$, there exist $s \in \mathbb{R}^+$ such that each $B(y, s)$ contains some z with $(B(z, r) \cap E, z) \cong (B(x, r) \cap E, x)$. We say that E satisfies the *strong local isomorphism property* if there exist $s, t \in \mathbb{R}^+$ such that, for each $x \in \mathbb{R}^n$ and each $r \in \mathbb{R}^+$, each $B(y, rs + t)$ contains some z with $(B(z, r) \cap E, z) \cong (B(x, r) \cap E, x)$. These definitions are similar to those for aperiodic tilings.

For $n \geq 2$, we say that a closed subset $C \subset \mathbb{R}^n$ is a *self-avoiding curve* if there exists a bicontinuous map from a closed interval $I \subset \mathbb{R}$ to C. If $I = [a, b]$ with $a, b \in \mathbb{R}$ (resp. $I = [a, +\infty[$ with $a \in \mathbb{R}$, $I = \mathbb{R}$), then C is a *bounded curve* (resp. a *half curve*, a *complete curve*).

We say that $A \subset \mathbb{R}^n$ is *space-filling* if there exists $\alpha \in \mathbb{R}^+$ such that $\sup_{x\in\mathbb{R}^n} \inf_{y\in A} d(x,y) \leq \alpha$. It follows from the proposition below that a self-avoiding space-filling complete curve cannot be invariant through a nontrivial translation:

Proposition. For each integer $n \geq 2$, each self-avoiding complete curve $C \subset \mathbb{R}^n$ and each $w \in \mathbb{R}^n$ such that $C = w + C$, there exists $B \subset C$ bounded such that C is the disjoint union of the subsets $kw + B$ for $k \in \mathbb{Z}$.

Proof. Consider $f : \mathbb{R} \to C$ bicontinuous. Then $\tau_w : C \to C : y \to w + y$ and $\varphi : \mathbb{R} \to \mathbb{R} : s \to f^{-1}(w + f(s))$ satisfy $\varphi = f^{-1} \circ \tau_w \circ f$. The map φ is bicontinuous because f, τ_w and f^{-1} are bicontinuous. In particular, φ is strictly increasing or strictly decreasing. Actually, φ is strictly increasing since we have $\varphi(s) \neq s$ for each $s \in \mathbb{R}$.

Now consider $x \in C$ and write $t_k = f^{-1}(kw+x)$ for each $k \in \mathbb{Z}$. The equalities $\varphi(t_k) = t_{k+1}$ with $t_k < t_{k+1}$ or $t_{k+1} < t_k$ for $k \in \mathbb{Z}$ imply $\varphi([t_k, t_{k+1}[) = [t_{k+1}, t_{k+2}[$ and $w + f([t_k, t_{k+1}[) = f([t_{k+1}, t_{k+2}[)$ for each $k \in \mathbb{Z}$. Consequently, C is the union of the disjoint subsets $kw + f([t_0, t_1[) = f([t_k, t_{k+1}[)$ for $k \in \mathbb{Z}$. ■

On the other hand, the results of the next three sections give various examples of self-avoiding plane-filling complete curves which satisfy the strong local isomorphism property.

In each of them, we start with a set $\mathcal{B}$ of self-avoiding bounded curves. We consider the set $\mathcal{C}$ of complete curves obtained as inductive limits of curves in $\mathcal{B}$. We show that any curve in $\mathcal{C}$ can be extended into a generally unique plane-filling set of at least 1 and at most 6 nonoverlapping such curves which satisfies the local isomorphism property. In each case, the bounded self-avoiding curves that we consider are described in B. Mandelbrot's book [3].

We do not know presently if similar examples exist in $\mathbb{R}^n$ for $n \geq 3$.

1. Dragons and other square folding curves

We consider a regular tiling of the plane by squares. For each $n \in \mathbb{N}$, an *n-folding curve* is obtained by folding n times a strip of paper in two, each time possibly to the left or to the right, then unfolding it with $\pi/2$ angles so that the support of each segment of the curve is a side of one of the squares. If all the foldings are done in the same direction, then we obtain the dragon curve of order n.

For each n-folding curve, each side of each square is the support of at most 1 segment, so that we make the curve self-avoiding by rounding the angles.

The *complete folding curves* are the complete curves obtained as inductive limits of n-folding curves for $n \in \mathbb{N}$. We say that a set of such curves is a

covering of the plane if each side of each square is the support of exactly 1 segment of 1 curve. Then the curves are disjoint.

Here, we consider nonoriented curves. The following results were proved in [4] and [5]:

Theorem 1.1. Each complete folding curve has a unique extension into a covering of the plane by such curves which satisfies the local isomorphism property, except in one case where there are 2 such extensions. Each covering actually satisfies the strong local isomorphism property.

Theorem 1.2. Each such covering consists of 1, 2, 3, 4 or 6 curves.

Theorem 1.3. There exist 2^ω local isomorphism classes of such coverings.

We note that, for any such coverings $\mathcal{C}, \mathcal{D}$ and any curves $C \in \mathcal{C}$, $D \in \mathcal{D}$, the coverings $\mathcal{C}, \mathcal{D}$ are locally isomorphic if and only if, for each $n \in \mathbb{N}$, the isomorphism classes of n-folding subcurves of C and D are the same.

Theorem 1.4. Each local isomorphism class is the union of 2^ω isomorphism classes, including one with a covering by 1 curve and one with a covering by 2 curves.

In [5], the local isomorphism classes which contain a covering, and actually two nonisomorphic coverings, by 6 curves are characterized; an example is given by Figure 1.1 below. Some examples of classes with coverings by 3 (resp. 4) curves are also given. It would be intersting to determine for each local isomorphism class the number of isomorphism classes of coverings by 1, 2, 3 or 4 curves, especially since similar results are obtained in the two next sections for triangular folding curves and for Peano-Gosper curves.

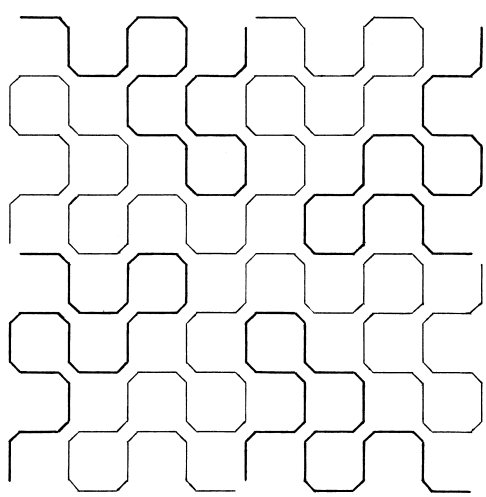

Figure 1.1

2. Terdragons and other triangular folding curves

For each $n \in \mathbb{N}$, we consider n-folding triangular curves such as the terdragon curve, which are obtained by folding n times a strip of paper in 3, each time possibly left then right or right then left, and unfolding it with $\pi/3$ angles. We prove that they are self-avoiding.

The complete folding triangular curves are obtained as inductive limits of n-folding triangular curves for $n \in \mathbb{N}$. We show that each nonoriented such curve can be extended into a unique covering of the plane by disjoint such curves and that this covering satisfies the strong local isomorphism property. We also prove that there exist 2^ω local isomorphism classes of such coverings and that each of them contains 2^ω (resp. 2^ω, 2 or 5, 0) isomorphism classes of coverings by 1 (resp. 2, 3, ≥ 4) curves.

2.1. Definitions and main results.

Here, we consider a regular tiling $\mathcal{P}$ of the plane by equilateral triangles. We define *oriented triangular curves* which we call *t-curves*. A *bounded t-curve* (resp. *half t-curve*, *complete t-curve*) is a sequence $(A_k)_{0\leq k\leq n}$ (resp. $(A_k)_{k\in\mathbb{N}}$, $(A_k)_{k\in\mathbb{Z}}$) of segments which are oriented sides of triangles of $\mathcal{P}$ such that, for any A_k, A_{k+1}, the terminal point of A_k is the initial point of A_{k+1} and A_k, A_{k+1} form a $\mp\pi/3$ angle. We associate to each such curve the sequence $(a_k)_{1\leq k\leq n}$ (resp. $(a_k)_{k\in\mathbb{N}^*}$, $(a_k)_{k\in\mathbb{Z}}$) with $a_k = +1$ (resp. $a_k = -1$) for each k such that we turn left (resp. right) when we pass from A_{k-1} to A_k.

We say that a t-curve C is *self-avoiding* if each side of a triangle is the support of at most one segment of C. A set $\mathcal{C}$ of t-curves is a *covering of the plane* if each side of a triangle is the support of exactly 1 segment of 1 curve of $\mathcal{C}$. We represent the curves with slightly rounded angles, so that they do not pass through the vertices of triangles. Then each self-avoiding t-curve passes at most once through each point of the plane and the curves in a covering are disjoint.

We define by induction on $n \in \mathbb{N}$ the sequences $T_{\lambda_1\cdots\lambda_n}$ for $\lambda_1, \ldots, \lambda_n \in \{-1, +1\}$. We denote by T the empty sequence. For each $n \in \mathbb{N}$ and any $\lambda_1, \ldots, \lambda_{n+1}$, we write $T_{\lambda_1\cdots\lambda_{n+1}} = (T_{\lambda_1\cdots\lambda_n}, \lambda_{n+1}, T_{\lambda_1\cdots\lambda_n}, -\lambda_{n+1}, T_{\lambda_1\cdots\lambda_n})$.

For each $n \in \mathbb{N}$, an *n-folding t-curve* associated to $T_{\lambda_1\cdots\lambda_n}$ is realized as follows: We successively fold n times in three a strip of paper. For each $k \in \{1, \ldots, n\}$, the k-th folding is done left then right if $\lambda_{n+1-k} = +1$ and right then left if $\lambda_{n+1-k} = -1$. Then we unfold the strip, keeping a $\pi/3$ angle for each fold. The *terdragon* curve of order n, which was first considered in [1], is the curve $T_{\lambda_1\cdots\lambda_n}$ with $\lambda_1 = \cdots = \lambda_n = +1$.

The *∞-folding t-curves* (resp. *complete folding t-curves*) are the half t-curves (resp. complete t-curves) obtained as inductive limits of n-folding t-curves C_n for $n \in \mathbb{N}$. We are going to see that, for each complete folding

t-curve C, there exists a unique sequence $\Lambda = (\lambda_n)_{n\in\mathbb{N}^*} \in \{-1,+1\}^{\mathbb{N}^*}$, not depending on the orientation of C, such that each bounded subcurve of C is countained in a t-curve associated to some $T_{\lambda_1\cdots\lambda_n}$.

In the second part of the present section, we show that folding t-curves are self-avoiding.

We denote by (P) the following property of a set E of oriented sides of triangles of $\mathcal{P}$: If $A, B \in E$ are sides of the same triangle, then they define the same direction of rotation around its center.

There are 2 opposite sets E_1, E_2 which satisfy (P) and such that each nonoriented side of a triangle of $\mathcal{P}$ is the support of 1 element of E_1 and 1 element of E_2. The segments of any t-curve are all contained in E_1, or all contained in E_2, and therefore satisfy (P).

We say that a covering $\mathcal{C}$ of the plane by oriented complete t-curves satisfies (P) if the set of segments of curves of $\mathcal{C}$ satisfies (P), or equivalently if it is equal to E_1 or E_2. Each covering by nonoriented complete t-curves induces 2 covering by oriented complete t-curves which satisfy (P) and have opposite orientations.

Theorem 2.1. Each nonoriented (resp. oriented) complete folding t-curve can be extended in a unique way into a covering of the plane by such curves (resp. a covering of the plane by such curves which satisfies (P)). This covering satisfies the strong local isomorphism property and consists of curves associated to the same sequence Λ.

Theorem 2.2. Two coverings of the plane by nonoriented complete folding t-curves are locally isomorphic if and only if they are associated to the same sequence Λ.

Theorem 2.3. For each covering of the plane by oriented complete folding t-curves, the local isomorphism property implies (P) and (P) implies the strong local isomorphism property. Any coverings with these properties are locally isomorphic if and only if they are associated to the same sequence Λ and have the same orientation.

Theorem 2.4. For each $\Lambda = (\lambda_n)_{n\in\mathbb{N}^*} \in \{-1,+1\}^{\mathbb{N}^*}$, the class of coverings of the plane by nonoriented complete folding t-curves associated to Λ is the union of:
1) 2^ω isomorphism classes of coverings by 1 curve;
2) 2^ω isomorphism classes of coverings by 2 curves;
3) 2 isomorphism classes of coverings by 3 curves having 1 vertex in common, where each of the 3 curves is the union of 2 ∞-folding t-curves starting from that vertex;

4) 3 isomorphism classes of coverings by 3 curves with 1 curve separating the 2 others, if there exists $k \in \mathbb{N}^*$ such that $\lambda_{n+1} = -\lambda_n$ for each $n \geq k$ (Figure 2.1 gives an example with $k = 1$ and $\lambda_1 = +1$); no such class otherwise.
In 3) and in 4), all the coverings are equivalent up to isometry.

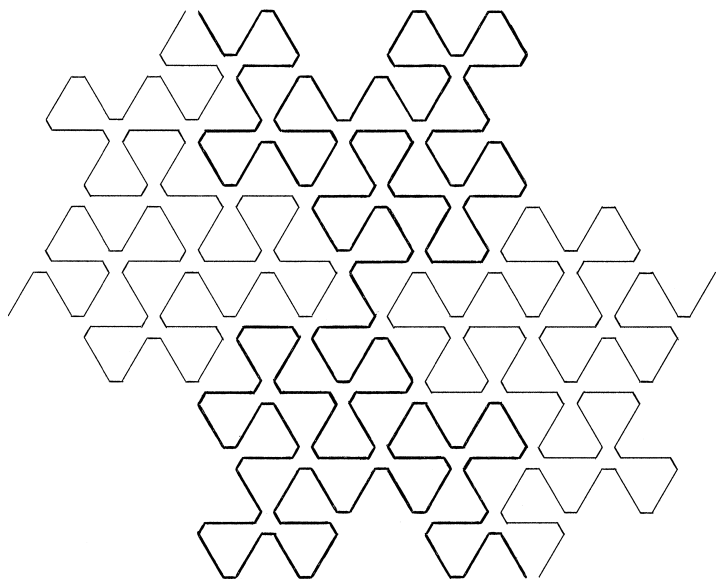

Figure 2.1

2.2. Detailed results and proofs.

Unless otherwise specified, all the curves that we consider are oriented. The property (P) and the sets E_1, E_2 are defined as above.

For any t-curves C, D, if the terminal point of C is the initial point of D and if the terminal segment of C and the initial segment of D form a $\mp\pi/3$ angle, then we denote by CD the t-curve obtained by connecting them.

For each $n \in \mathbb{N}^*$, any $\lambda_1, \ldots, \lambda_n \in \{-1, +1\}$ and each n-folding t-curve C associated to $T_{\lambda_1 \cdots \lambda_n}$, we consider the 3 $(n-1)$-folding t-curves $C^{\mathrm{I}}, C^{\mathrm{M}}, C^{\mathrm{S}}$ associated to $T_{\lambda_1 \cdots \lambda_{n-1}}$ such that $C = C^{\mathrm{I}} C^{\mathrm{M}} C^{\mathrm{S}}$. Here, $\mathrm{I}, \mathrm{M}, \mathrm{S}$ can be viewed as abbreviations for "inferior", "middle", "superior".

For $k \in \mathbb{N}$ and $S = (\alpha_1, \ldots, \alpha_k) \in \{-1, +1\}^k$, we write $\overline{S} = (-\alpha_k, \ldots, -\alpha_1)$. The reverse of a curve associated to S is associated to $\overline{S}$. For $S = (\alpha_n)_{n \in \mathbb{N}^*} \in \{-1, +1\}^{\mathbb{N}^*}$, we write $\overline{S} = (-\alpha_{-n})_{n \in -\mathbb{N}^*}$.

For $n \in \mathbb{N}$ and $\lambda_1, \ldots, \lambda_n \in \{-1, +1\}$, we have $T_{\lambda_1 \cdots \lambda_n} = \overline{T_{\lambda_1 \cdots \lambda_n}}$. Consequently, the reverse of a curve associated to $T_{\lambda_1 \cdots \lambda_n}$ is also associated to $T_{\lambda_1 \cdots \lambda_n}$.

For each $\Lambda = (\lambda_k)_{k \in \mathbb{N}^*} \in \{-1, +1\}^{\mathbb{N}^*}$, we denote by T_Λ the inductive limit of the sequences $T_{\lambda_1 \cdots \lambda_n}$ with $T_{\lambda_1 \cdots \lambda_n}$ initial segment of $T_{\lambda_1 \cdots \lambda_{n+1}}$ for each $n \in \mathbb{N}$.

Proposition 2.5. Let C be a complete t-curve associated to a sequence $S = (s_h)_{h\in\mathbb{Z}} \in \{-1,+1\}^{\mathbb{Z}}$. Then the properties 1), 2), 3) below are equivalent:
1) C is a complete folding t-curve.
2) For each $k \in \mathbb{N}$, there exists $h \in \mathbb{Z}$ such that $s_{h+3^k+3^{k+1}i} = -s_{h+3^k2+3^{k+1}j}$ for any $i, j \in \mathbb{Z}$.
3) There exists a unique sequence $\Lambda = (\lambda_n)_{n\in\mathbb{N}^*} \in \{-1,+1\}^{\mathbb{N}^*}$ such that exactly one of the two following properties is true:
a) S is equivalent to $(\overline{T_\Lambda}, +1, T_\Lambda)$ or $(\overline{T_\Lambda}, -1, T_\Lambda)$ modulo a translation of $\mathbb{Z}$;
b) $C = \cup_{n\in\mathbb{N}} C_n$ for a sequence $(C_n)_{n\in\mathbb{N}}$ such that, for each $n \in \mathbb{N}$, C_n is an n-folding t-curve associated to $T_{\lambda_1\cdots\lambda_n}$ and $C_n \in \left\{C^I_{n+1}, C^M_{n+1}, C^S_{n+1}\right\}$.

Proof. 1) or 3) implies 2) because we have $s_{3^k+3^{k+1}i} = -s_{3^k2+3^{k+1}j}$ for each $n \in \mathbb{N}$, each sequence $(s_h)_{1\le h\le 3^n-1}$ associated to an n-folding t-curve, each $k \in \{0,\dots,n-1\}$ and any integers $0 \le i, j \le 3^{n-k-1}-1$. If the case a) of 3) is realized, then, in 2), we can take the same h for each $k \in \mathbb{N}$.

Now we show that 2) implies 1) and 3). For each $g \in \mathbb{Z}$, we consider the point z_g of the plane which is associated to s_g in the correspondance between S and C. For $g \in \mathbb{Z}$ and $n \in \mathbb{N}$, we denote by $\alpha_n(g)$ the largest integer $h \le g$ such that $s_{h+3^k+3^{k+1}i} = -s_{h+3^k2+3^{k+1}j}$ for each $k \in \{0,\dots,n-1\}$ and any $i, j \in \mathbb{Z}$. We have $g < \beta_n(g)$ for $\beta_n(g) = \alpha_n(g) + 3^n$. The part $C_n(g)$ of C between $z_{\alpha_n(g)}$ and $z_{\beta_n(g)}$ is an n-folding t-curve. We have $C_n(g) \in \{C_{n+1}(g)^I, C_{n+1}(g)^M, C_{n+1}(g)^S\}$.

There exists a unique sequence $\Lambda = (\lambda_n)_{n\in\mathbb{N}^*} \in \{-1,+1\}^{\mathbb{N}^*}$ such that, for each $n \in \mathbb{N}$ and each $g \in \mathbb{Z}$, $C_n(g)$ is associated to $T_{\lambda_1\cdots\lambda_n}$. For any $g, h \in \mathbb{Z}$ such that $g < h$, we have $C_n(g) = C_n(h)$ for n large enough, or $\beta_n(g) = \alpha_n(h)$ for n large enough.

If there exists $g \in \mathbb{Z}$ such that $\cup_{n\in\mathbb{N}} C_n(g) = C$, then the case b) of 3) is realized and C is clearly a complete folding t-curve.

Otherwise, there exist $m \in \mathbb{N}^*$ and $g, h \in \mathbb{Z}$ with $g < h$ such that $\beta_n(g) = \alpha_n(h)$ for each $n \ge m$. Then the case a) of 3) is realized. Moreover, for each $n \ge m$, $C_n(g)^M C_n(g)^S C_n(h)^I$ or $C_n(g)^S C_n(h)^I C_n(h)^M$ is an n-folding t-curve D_n contained in C. We have $D_n \subset D_{n+1}$ for each $n \ge m$ and $C = \cup_{n\ge m} D_n$, whence 1). ■

Similarly to the case of folding curves in [4] and [5], and Peano-Gosper curves in Section 3 below, we define a *derivation* Δ on folding t-curves.

For each $n \in \mathbb{N}^*$, we divide each n-folding t-curve C into sequences of 3 consecutive segments; $\Delta(C)$ is obtained by replacing each such sequence with a unique segment whose initial and terminal points are the initial point of the first segment and the terminal point of the third segment; if C is associated to a sequence $(\alpha_i)_{1\le i\le 3^n-1} \in \{-1,+1\}^{3^n-1}$, then $\Delta(C)$ is associated to $(\alpha_{3i})_{1\le i\le 3^{n-1}-1}$.

The definition of Δ naturally extends to ∞-folding t-curves. Now we extend it to complete folding t-curves.

Consider any such curve $C = (A_k)_{k\in\mathbb{Z}}$, and the associated sequence $(s_k)_{k\in\mathbb{Z}} \in \{-1,+1\}^{\mathbb{Z}}$. Then, by Proposition 2.5, there exist $\varepsilon \in \{-1,+1\}$ and $h \in \mathbb{Z}$, whose remainder modulo 3 is completely determined by C, such that $s_{h+3k+1} = \varepsilon$ and $s_{h+3k+2} = -\varepsilon$ for each $k \in \mathbb{Z}$. The curve $\Delta(C)$ is obtained by replacing each $A_{h+3k}A_{h+3k+1}A_{h+3k+2}$ with a unique segment; it is associated to the sequence $(s_{h+3k})_{k\in\mathbb{Z}}$.

If an ∞-folding or a complete folding t-curve C is the inductive limit for $n \in \mathbb{N}$ of some n-folding t-curves $C_n \subset C$, then $\Delta(C)$ is the inductive limit of the curves $\Delta(C_n)$ for $n \in \mathbb{N}^*$.

For each $k \in \mathbb{N}^*$ and each t-curve C which is n-folding for some $n \geq k$, ∞-folding or complete folding, $\Delta^k(C)$ is obtained in the same way by replacing sequences of 3^k consecutive segments with 1 segment.

Now we begin to show the self-avoiding and plane-filling properties of folding t-curves. We denote by U the set of vertices of triangles of $\mathcal{P}$ and we fix $i \in \{1,2\}$.

We prove by induction on n that, for each $n \in \mathbb{N}$, each $\Lambda \in \{-1,+1\}^n$ and each $x \in U$, there exists a unique covering $\mathcal{C}(\Lambda, x)$ of the plane by n-folding t-curves associated to T_Λ and with segments in E_i such that x is an endpoint of some of the curves.

If $n = 0$, then Λ is the empty sequence and we write $\mathcal{C}(\Lambda, x) = E_i$. It remains to be proved that, if the property is true for an integer n, then it is true for $n+1$.

If X is the set of vertices of equilateral triangles which form a regular tiling of the plane, then, for each $x \in X$, there exists a unique partition $X = G(X,x) \cup H(X,x)$ such that:
1) $H(X,x)$ is the set of vertices of hexagons which form a regular tiling of the plane;
2) $x \in G(X,x)$ and the elements of $G(X,x)$ are the centers of the hexagons. Moreover, $G(X,x)$ is also the set of vertices of equilateral triangles which form a regular tiling of the plane.

For each $x \in U$, we write $V_0(x) = U$ and, for each $k \in \mathbb{N}$, $V_{k+1}(x) = G(V_k(x), x)$ and $W_{k+1}(x) = H(V_k(x), x)$.

For each $\Lambda \in \{-1,+1\}^n$ and each $\lambda \in \{-1,+1\}$, supposing that $\mathcal{C}(\Lambda, x)$ is already defined, we define $\mathcal{C}((\Lambda,\lambda), x)$ as follows: If $\lambda = +1$ (resp. -1), then, for each $y \in V_{n+1}(x)$ and each curve $A \in \mathcal{C}(\Lambda, x)$ starting from y, we put in $\mathcal{C}((\Lambda,\lambda), x)$ the curve ABC, where:
1) B is the curve of $\mathcal{C}(\Lambda, x)$ starting from the endpoint of A and such that its first segment is just at the left (resp. right) of the last segment of A;
2) C is the curve of $\mathcal{C}(\Lambda, x)$ starting from the endpoint of B and such that its first segment is just at the right (resp. left) of the last segment of B.

The following properties are true for each $n \in \mathbb{N}$, each $\Lambda \in \{-1, +1\}^n$ and each $x \in U$. For $n \geq 2$, they are proved by using Δ^{n-1}.

Each curve of $\mathcal{C}(\Lambda, x)$ connects a pair of points of $V_n(x)$ with minimal distance. Each such pair (y, z) is connected by a unique curve of $\mathcal{C}(\Lambda, x)$. If $n \geq 1$, then this curve contains the 2 points of $W_n(x)$ which are between y and z.

Using the derivations, we see that, if C is a curve associated to some T_Λ, then C or its reverse belongs to $\mathcal{C}(\Lambda, x) = \mathcal{C}(\Lambda, y)$, where x and y are the endpoints of C. It follows that C is self-avoiding.

For each $n \in \mathbb{N}^*$, we call an *n-triangle* any equilateral triangle such that each side consists of n sides of triangles of $\mathcal{P}$. We say that a set of curves $\mathcal{C}$ *covers* an n-triangle $\mathcal{T}$ if, in each 1-triangle of $\mathcal{P}$ contained in $\mathcal{T}$, each side is the support of a segment of a curve of $\mathcal{C}$.

Proposition 2.6. For each integer $n \geq 2$, each $(2n)$-folding t-curve covers a $(3^{n-1} + 3)/2$-triangle.

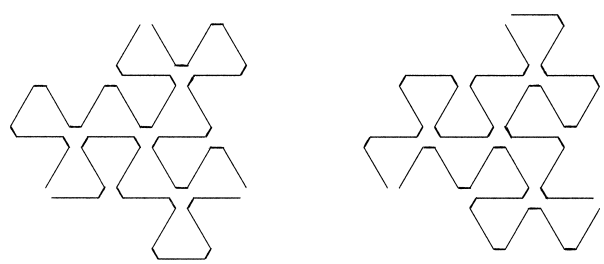

Figure 2.2

Proof. For each $n \geq 2$, we denote by k_n the largest integer k such that each $(2n)$-folding t-curve covers a k-triangle. We simultaneously prove that $k_2 \geq 3$ and $k_n \geq 3k_{n-1} - 3$ for each $n \geq 3$.

For each $(2n)$-folding t-curve C, we consider the tiling $\mathcal{Q}$ of the plane by equilateral triangles which is associated to $\Delta^2(C)$. Each triangle of $\mathcal{Q}$ is the union of 9 triangles of $\mathcal{P}$.

For any segments A_1, A_2, A_3 of $\Delta^2(C)$, if their supports are the sides of a 1-triangle $\mathcal{T}$ of $\mathcal{Q}$, then their orientations define the same direction of rotation around the center of $\mathcal{T}$ since $\Delta^2(C)$ satisfies (P). Figure 2.2 above shows two possible configurations for $\Delta^{-2}(A_1)$, $\Delta^{-2}(A_2)$, $\Delta^{-2}(A_3)$. Any other configuration is equivalent to one of them up to isometry.

We see from Figure 2.2 that $\mathcal{E} = \{\Delta^{-2}(A_1), \Delta^{-2}(A_2), \Delta^{-2}(A_3)\}$ necessarily covers a 3-triangle of $\mathcal{P}$. If $n = 2$, then C covers a 3-triangle of $\mathcal{P}$ since the 2-folding t-curve $\Delta^2(C)$ has such segments A_1, A_2, A_3.

We also see from Figure 2.2 that each side of a 1-triangle of $\mathcal{P}$ contained in $\mathcal{T}$ is the support of a segment of $\mathcal{E}$, except possibly if one of its endpoints is a vertex of $\mathcal{T}$.

In order to prove that $k_n \geq 3k_{n-1} - 3$ for each $n \geq 3$, it suffices to show that, for each integer $k \geq 2$, if $\Delta^2(C)$ covers a k-triangle $\mathcal{W}$ of $\mathcal{Q}$, then C covers the $(3k-3)$-triangle $\mathcal{V}$ of $\mathcal{P}$ contained in the interior of $\mathcal{W}$.

This property is a consequence of the two following facts: First, by the argument above, for each 1-triangle $\mathcal{T}$ of $\mathcal{Q}$, if $\Delta^2(C)$ covers $\mathcal{T}$, then, in each 1-triangle of $\mathcal{P}$ contained in $\mathcal{T}$, each side is the support of a segment of C if neither of its endpoints is a vertex of $\mathcal{T}$. Second, for each vertex z of a 1-triangle of $\mathcal{Q}$, if z belongs to $\mathcal{V}$, then z is an endpoint of 6 segments of C since it is an endpoint of 6 segments of $\Delta^2(C)$; as C is self-avoiding, it follows that each side with endpoint z of a 1-triangle of $\mathcal{P}$ is the support of a segment of C. ■

Corollary 2.7. For each covering of the plane by complete folding t-curves, the local isomorphism property implies (P).

Proof. Let $\mathcal{C}$ be such a covering. By Proposition 2.6, each $C \in \mathcal{C}$ covers arbitrarily large triangles. The covering $\mathcal{C}$ satisfies (P) on any such triangle, and therefore on the whole plane by the local isomorphism property. ■

For each nontrivial t-curve C, we say that $x \in U$ belongs to C and we write $x \in C$ if x is an endpoint of at least one segment of C. For any $x, y \in U$ with $d(x, y) = 1$, we say that the nonoriented segment $[x, y]$ belongs to C and we write $[x, y] \in C$ if $[x, y]$ is the support of a segment of C.

For each folding t-curve C, we denote by $\mathrm{F_L}(C)$ (resp. $\mathrm{F_R}(C)$) the union of the sides $[x, y]$ of triangles of $\mathcal{P}$ such that $x, y \in C$, such that C contains exactly 1 of the 2 points which form equilateral triangles with x and y, and such that the second point is at the left (resp. right) of C.

For each integer $n \geq 1$ and each n-folding t-curve C with initial point u and terminal point v, there exists a unique sequence $(y_i)_{1 \leq i \leq h} \subset U$ with $y_1 = u$, $y_h = v$, $d(y_{i-1}, y_i) = 1$ and $[y_{i-1}, y_i] \subset \mathrm{F_L}(C)$ for $2 \leq i \leq h$. There is also a unique sequence $(z_j)_{1 \leq j \leq k} \subset U$ with $z_1 = u$, $z_k = v$, $d(z_{j-1}, z_j) = 1$ and $[z_{j-1}, z_j] \subset \mathrm{F_R}(C)$ for $2 \leq j \leq k$.

We denote by $\mathrm{F^0_L}(C)$ (resp. $\mathrm{F^0_R}(C)$) the union of the segments $[y_{i-1}, y_i]$ (resp. $[z_{j-1}, z_j]$). We have $\mathrm{F^0_L}(C) \cap \mathrm{F^0_R}(C) = \{u, v\}$ and no point of C is outside $\mathrm{F^0_L}(C) \cup \mathrm{F^0_R}(C)$.

It follows that C starts at y_1, then successively passes through $y_2, \ldots, y_{h-1}$ in that order, possibly once or twice for each of them, then ends at y_h. The same property is true for $z_1, \ldots, z_k$.

Now we show that C contains the segments $[w_1, w_2]$ with $w_1, w_2 \in U$ and $d(w_1, w_2) = 1$ which are inside $\mathrm{F^0_L}(C) \cup \mathrm{F^0_R}(C)$. It follows $\mathrm{F_L}(C) = \mathrm{F^0_L}(C)$ and $\mathrm{F_R}(C) = \mathrm{F^0_R}(C)$.

For the proof, we use the existence of a covering $\mathcal{C}(\Lambda, u)$ of the plane which contains C. If a segment $[w_1, w_2]$ as above does not belong to C, then

it belongs to a curve $D \in \mathcal{C}(\Lambda, u)$ with $D \neq C$.

The segment $[w_1, w_2]$ is between $\mathrm{F}^0_{\mathrm{L}}(C)$ and C, or between C and $\mathrm{F}^0_{\mathrm{R}}(C)$. We can suppose that the first case is true, since the second one can be treated in the same way.

Then there exists i such that $[w_1, w_2]$ is inside the loop formed by C between two occurrences of x_i, or between an occurrence of x_i and an occurrence of x_{i+1}. We observe that, in the first case, no curve of $\mathcal{C}(\Lambda, u)$ crosses the limit of the loop, and in the second case at most one curve of $\mathcal{C}(\Lambda, u)$ crosses that limit and only once, since the part of the curve which crosses the limit must contain $[x_i, x_{i+1}]$.

In both cases, it follows that at least one endpoint of D is inside the loop. Then another curve of $\mathcal{C}(\Lambda, u)$ having that endpoint is completely inside the loop, whence a contradiction since C and the other curves of $\mathcal{C}(\Lambda, u)$ are equivalent up to isometry.

The following properties are true since, if an ∞-folding t-curve or a complete folding t-curve C is the inductive limit of some n-folding t-curves C_n, then any segment $[x, y]$ is contained in $\mathrm{F}_{\mathrm{L}}(C)$ (resp. $\mathrm{F}_{\mathrm{R}}(C)$) if and only if it is contained in $\mathrm{F}_{\mathrm{L}}(C_n)$ (resp. $\mathrm{F}_{\mathrm{R}}(C_n)$) for n large enough.

If C is an ∞-folding t-curve with initial point x, then we have $\mathrm{F}_{\mathrm{L}}(C) \cap \mathrm{F}_{\mathrm{R}}(C) = \{x\}$ and $\mathrm{F}_{\mathrm{L}}(C)$, $\mathrm{F}_{\mathrm{R}}(C)$ are half curves with endpoint x. If C is a complete folding t-curve, then $\mathrm{F}_{\mathrm{L}}(C)$, $\mathrm{F}_{\mathrm{R}}(C)$ are disjoint and each of them is a complete curve or empty.

For each folding t-curve C, we write $\mathrm{F}(C) = \mathrm{F}_{\mathrm{L}}(C) \cup \mathrm{F}_{\mathrm{R}}(C)$. We denote by $\mathrm{Dom}(C)$ the closed part of the plane limited by $\mathrm{F}(C)$ and containing C. The interior of $\mathrm{Dom}(C)$ is connected and C contains the sides of triangles of $\mathcal{P}$ which are in $\mathrm{Dom}(C)$ and not in $\mathrm{F}(C)$.

For any Λ, x, the sets $\mathrm{Dom}(C)$ for $C \in \mathcal{C}(\Lambda, x)$ are nonoverlapping and cover the plane. If $(C_i)_{i \in I}$ is a covering of the plane by complete folding t-curves, then the sets $\mathrm{Dom}(C_i)$ are nonoverlapping and cover the plane, except possibly the 1-triangles (u, v, w) such that $[u, v]$, $[u, w]$, $[v, w]$ belong to three different curves. We shall see later that, actually, no such triangle exists.

Now let $n \geq 1$ be an integer and let C be an n-folding t-curve associated to a sequence $T_{\lambda_1 \cdots \lambda_n}$. Consider the initial points w, x of $C^{\mathrm{I}}, C^{\mathrm{M}}$ and the terminal points y, z of $C^{\mathrm{M}}, C^{\mathrm{S}}$.

If $\lambda_n = +1$ (resp. -1), then $y \in \mathrm{F}_{\mathrm{L}}(C)$ and $x \in \mathrm{F}_{\mathrm{R}}(C)$ (resp. $x \in \mathrm{F}_{\mathrm{L}}(C)$ and $y \in \mathrm{F}_{\mathrm{R}}(C)$). We denote by $\mathrm{F}_{\mathrm{LI}}(C)$ the part of $\mathrm{F}_{\mathrm{L}}(C)$ between w and y (resp. x), $\mathrm{F}_{\mathrm{LS}}(C)$ the part of $\mathrm{F}_{\mathrm{L}}(C)$ between y (resp. x) and z, $\mathrm{F}_{\mathrm{RI}}(C)$ the part of $\mathrm{F}_{\mathrm{R}}(C)$ between w and x (resp. y), $\mathrm{F}_{\mathrm{RS}}(C)$ the part of $\mathrm{F}_{\mathrm{R}}(C)$ between x (resp. y) and z.

Now suppose $n \geq 2$. If $\lambda_n = +1$, then we have $\mathrm{F}_{\mathrm{LI}}(C) = \mathrm{F}_{\mathrm{LI}}(C^{\mathrm{I}})\mathrm{F}_{\mathrm{LS}}(C^{\mathrm{M}})$, $\mathrm{F}_{\mathrm{LS}}(C) = \mathrm{F}_{\mathrm{L}}(C^{\mathrm{S}})$, $\mathrm{F}_{\mathrm{RI}}(C) = \mathrm{F}_{\mathrm{R}}(C^{\mathrm{I}})$, $\mathrm{F}_{\mathrm{RS}}(C) = \mathrm{F}_{\mathrm{RI}}(C^{\mathrm{M}})\mathrm{F}_{\mathrm{RS}}(C^{\mathrm{S}})$. If $\lambda_n = -1$, then we have $\mathrm{F}_{\mathrm{LI}}(C) = \mathrm{F}_{\mathrm{L}}(C^{\mathrm{I}})$, $\mathrm{F}_{\mathrm{LS}}(C) = \mathrm{F}_{\mathrm{LI}}(C^{\mathrm{M}})\mathrm{F}_{\mathrm{LS}}(C^{\mathrm{S}})$, $\mathrm{F}_{\mathrm{RI}}(C) =$

$\mathrm{F}_{\mathrm{RI}}(C^{\mathrm{I}})\mathrm{F}_{\mathrm{RS}}(C^{\mathrm{M}})$, $\mathrm{F}_{\mathrm{RS}}(C) = \mathrm{F}_{\mathrm{R}}(C^{\mathrm{S}})$.

Lemma 2.8. Consider $n \in \mathbb{N}^*$ and $\lambda_1, \ldots, \lambda_n \in \{-1, +1\}$. Let C be an n-folding t-curve associated to $T_{\lambda_1 \cdots \lambda_n}$. Then there exist some sequences $(x_i)_{0 \leq i \leq 2^n}$ and $(y_i)_{0 \leq i \leq 2^n}$, with $x_0 = y_0$ initial point of C and $x_{2^n} = y_{2^n}$ terminal point of C, such that:
1) each segment $[x_i, x_{i+1}]$ or $[y_i, y_{i+1}]$ is a side of a triangle of $\mathcal{P}$;
2) each angle $\widehat{x_{i-1}x_ix_{i+1}}$ or $\widehat{y_{i-1}y_iy_{i+1}}$ is equal to $\mp 2\pi/3$;
3) $\mathrm{F}_{\mathrm{LI}}(C) = \cup_{1 \leq i \leq 2^{n-1}} [x_{i-1}, x_i]$, $\mathrm{F}_{\mathrm{LS}}(C) = \cup_{2^{n-1}+1 \leq i \leq 2^n} [x_{i-1}, x_i]$, $\mathrm{F}_{\mathrm{RI}}(C) = \cup_{1 \leq i \leq 2^{n-1}} [y_{i-1}, y_i]$, $\mathrm{F}_{\mathrm{RS}}(C) = \cup_{2^{n-1}+1 \leq i \leq 2^n} [y_{i-1}, y_i]$;
4) for $0 \leq i \leq 2^n$ and $1 \leq k \leq n$, each point x_i, y_i belongs to $V_k(C)$ if and only if i is divisible by 2^k.

Now associate to C the sequences $(\alpha_i)_{1 \leq i \leq 2^n - 1}$ and $(\beta_i)_{1 \leq i \leq 2^n - 1}$ with $\alpha_i = +1$ (resp. -1) if $\widehat{x_{i-1}x_ix_{i+1}} = +2\pi/3$ (resp. $-2\pi/3$), and $\beta_i = +1$ (resp. -1) if $\widehat{y_{i-1}y_iy_{i+1}} = +2\pi/3$ (resp. $-2\pi/3$). Then we have $\alpha_{2^{n-1}} = -1$, $\beta_{2^{n-1}} = +1$ and $\alpha_{2^k + 2^{k+1}i} = \beta_{2^k + 2^{k+1}i} = (-1)^i \lambda_{k+2}$ for $0 \leq k \leq n-2$ and $0 \leq i \leq 2^{n-k-1} - 1$.

Proof. Lemma 2.8 is clearly true for $n = 1$. Now suppose that it is true for an integer $n \geq 1$ and consider an $(n+1)$-folding t-curve C associated to a sequence $T_{\lambda_1 \cdots \lambda_{n+1}}$. We can assume $\lambda_{n+1} = +1$ since the case $\lambda_{n+1} = -1$ is similar. We only show the results for $\mathrm{F}_{\mathrm{L}}(C)$ since the proof for $\mathrm{F}_{\mathrm{R}}(C)$ is similar.

The curves $C^{\mathrm{I}}, C^{\mathrm{M}}, C^{\mathrm{S}}$ are associated to $T_{\lambda_1 \cdots \lambda_n}$. We have $\mathrm{F}_{\mathrm{LI}}(C) = \mathrm{F}_{\mathrm{LI}}(C^{\mathrm{I}})\mathrm{F}_{\mathrm{LS}}(C^{\mathrm{M}})$ and $\mathrm{F}_{\mathrm{LS}}(C) = \mathrm{F}_{\mathrm{L}}(C^{\mathrm{S}}) = \mathrm{F}_{\mathrm{LI}}(C^{\mathrm{S}})\mathrm{F}_{\mathrm{LS}}(C^{\mathrm{S}})$. The sequences associated to $\mathrm{F}_{\mathrm{LI}}(C^{\mathrm{I}})$ and $\mathrm{F}_{\mathrm{LI}}(C^{\mathrm{S}})$ are equal. The sequences associated to $\mathrm{F}_{\mathrm{LS}}(C^{\mathrm{M}})$ and $\mathrm{F}_{\mathrm{LS}}(C^{\mathrm{S}})$ are also equal. We turn with a $+2\pi/3$ (resp. $-2\pi/3$, $-2\pi/3$) angle when we pass from $\mathrm{F}_{\mathrm{LI}}(C^{\mathrm{I}})$ to $\mathrm{F}_{\mathrm{LS}}(C^{\mathrm{M}})$ (resp. from $\mathrm{F}_{\mathrm{LS}}(C^{\mathrm{M}})$ to $\mathrm{F}_{\mathrm{LI}}(C^{\mathrm{S}})$, from $\mathrm{F}_{\mathrm{LI}}(C^{\mathrm{S}})$ to $\mathrm{F}_{\mathrm{LS}}(C^{\mathrm{S}})$) because C passes twice (resp. once, once) through the terminal point of $\mathrm{F}_{\mathrm{LI}}(C^{\mathrm{I}})$ (resp. $\mathrm{F}_{\mathrm{LS}}(C^{\mathrm{M}})$, $\mathrm{F}_{\mathrm{LI}}(C^{\mathrm{S}})$). The results follow from these facts and from the induction hypothesis applied to $C^{\mathrm{I}}, C^{\mathrm{M}}, C^{\mathrm{S}}$. ■

Corollary 2.9. Let C be a complete folding t-curve such that $\mathrm{F}_{\mathrm{L}}(C)$ is nonempty. Consider the sequence $(x_n)_{n \in \mathbb{Z}}$ of endpoints of segments of $\mathrm{F}_{\mathrm{L}}(C)$ and the sequence $(\alpha_n)_{n \in \mathbb{Z}}$, with $\alpha_n = +1$ if $\widehat{x_{n-1}x_nx_{n+1}} = +2\pi/3$ and $\alpha_n = -1$ if $\widehat{x_{n-1}x_nx_{n+1}} = -2\pi/3$. Then there exists a sequence $(n_k)_{k \in \mathbb{N}^*}$ such that $x_{n_k + 2^k i} \in W_k(C)$ and $\alpha_{n_k + 2^k i} = (-1)^i \alpha_{n_k}$ for each $k \in \mathbb{N}^*$ and each $i \in \mathbb{Z}$. The same property is true for $\mathrm{F}_{\mathrm{R}}(C)$ if it is nonempty.

Remark. If C is obtained from an ∞-folding t-curve, then there exists a unique $n \in \mathbb{Z} - \cup_{k \in \mathbb{N}^*}(n_k + 2^k\mathbb{Z})$; we have $x_n \in \cap_{k \in \mathbb{N}^*} V_k(C)$. Otherwise, we have $\mathbb{Z} = \cup_{k \in \mathbb{N}^*}(n_k + 2^k\mathbb{Z})$.

Lemma 2.10. Any set of disjoint complete folding t-curves is finite.

Proof. We assume that the sides of triangles of $\mathcal{P}$ have length 1 and we show by induction on $n \in \mathbb{N}^*$ that $d(x,y) \leq \rho_n = \left[(\sqrt{3})^{n-1}(4-\sqrt{3})-1\right]/(\sqrt{3}-1)$ for any points x, y of an n-folding t-curve. We have $\rho_1 = \sqrt{3}$ and the property is clearly true for $n = 1$.

We prove that, if the property is true for an integer $n \geq 1$, then it is also true for $n+1$. We consider an $(n+1)$-folding t-curve C. We represent $D = \Delta(C)$ in such a way that C and D have the same endpoints, which gives the length $\sqrt{3}$ to the segments of D.

By the induction hypothesis, for D represented in that way, we have $d(x,y) \leq \rho_n\sqrt{3}$ for any $x, y \in D$. Moreover, for each $z \in C$, there exists $y \in D$ such that $d(y,z) \leq 1/2$. It follows that, for any $v, w \in C$, we have $d(v,w) \leq \rho_n\sqrt{3}+1 = \rho_{n+1}$.

Now we see that, for $x \in \mathbb{R}^2$ and $k \in \mathbb{N}$ large enough, there exists no $r \in \mathbb{R}^+$ such that $B(x,r)$ contains points of k disjoint n-folding t-curves for each $n \in \mathbb{N}^*$. This follows since each n-folding t-curve contains 3^n sides of triangles of $\mathcal{P}$, each such curve containing a point of $B(x,r)$ is necessarily contained in $B(x, r+\rho_n)$ and, for n large compared to r, the number of sides of triangles of $\mathcal{P}$ contained in $B(x, r+\rho_n)$ is smaller than $3^n k$ for an integer k not depending on n and r. ■

Lemma 2.11. Let $\mathcal{C}$ be a covering of the plane by complete folding t-curves. Then the sets $\mathrm{Dom}(C)$ for $C \in \mathcal{C}$ are nonoverlapping and cover the plane, we have $V_k(C) = V_k(D)$ for each $k \in \mathbb{N}^*$ and any $C, D \in \mathcal{C}$, and all the curves of $\mathcal{C}$ are associated to the same sequence $(\lambda_n)_{n\in\mathbb{N}^*}$.

Proof. We saw that the sets $\mathrm{Dom}(C)$ for $C \in \mathcal{C}$ are nonoverlapping and cover the plane, except possibly the 1-triangles (u,v,w) such that $[u,v]$, $[u,w]$, $[v,w]$ belong to three different curves. Suppose that such a triangle exists and consider the curves $C, D \in \mathcal{C}$ such that $[u,v] \in C$ and $[u,w] \in D$.

Then we have $[u,v] \subset F(C)$ because w cannot belong to C and $[u,w] \subset F(D)$ because v cannot belong to D. The value of the angle covered by $\mathrm{Dom}(C)$ (resp. $\mathrm{Dom}(D)$) in u is $2\pi/3$ if C (resp. D) passes once through u, and $4\pi/3$ if C (resp. D) passes twice through u. If C and D pass only once through u, then another curve $E \in \mathcal{C}$ passes through u and the value of the angle covered by $\mathrm{Dom}(E)$ in u is $2\pi/3$. In each case, we obtain a contradiction since $\widehat{vuw}$ and these angles are all nonoverlapping.

Now we prove the other parts of Lemma 2.11. For each covering $\mathcal{C}$ and each $C \in \mathcal{C}$, we consider the sequence $\Lambda(C) = (\lambda_n(C))_{n\in\mathbb{N}^*}$ associated to C. It suffices to show that $V_1(C) = V_1(D)$ and $\lambda_1(C) = \lambda_1(D)$ for any $C, D \in \mathcal{C}$, since these properties imply that $\{\Delta(C) \mid C \in \mathcal{C}\}$ is also a covering of the plane by complete folding t-curves. As C is finite by Lemma 2.10, it is

enough to prove these properties for the pairs (C, D) such that $\mathrm{F}(C) \cap \mathrm{F}(D)$ is unbounded.

For each such pair, we consider some consecutive segments $[x_0, x_1], \ldots, [x_8, x_9]$ of $\mathrm{F}(C) \cap \mathrm{F}(D)$, and the integers $\alpha_1, \ldots, \alpha_8 \in \{-1, +1\}$ associated to the vertices $x_1, \ldots, x_8$. According to Corollary 2.9, we have $\alpha_1 = -\alpha_3 = \alpha_5 = -\alpha_7$ or $\alpha_2 = -\alpha_4 = \alpha_6 = -\alpha_8$, but the 2 properties cannot be simultaneously true since the first one implies $\alpha_2 = -\alpha_6$ or $\alpha_4 = -\alpha_8$.

Let us suppose for instance that $\alpha_1 = -\alpha_3 = \alpha_5 = -\alpha_7$. Then we have $x_2 \in V_1(C)$ and $x_2 \in V_1(D)$, whence $V_1(C) = V_1(D)$. It follows $\lambda_1(C) = \lambda_1(D)$ because the 6 sequences of 3 segments of curves of $\mathcal{C}$ with endpoint x_2 are necessarily equivalent up to rotation. ■

Lemma 2.12. Here, for each $n \in \mathbb{N}$ and each complete folding t-curve C, we represent $\Delta^n(C)$ with segments of length 1. For each $x \in \mathbb{R}^2$, we denote by $\Delta^n(x)$ the image of x in this representation. Then, for each $x \in \mathbb{R}^2$, any complete folding t-curves C, D and each $n \in \mathbb{N}$ large enough, we have $d(\Delta^n(x), \Delta^n(C)) < 3/4$ and $d(\Delta^n(C), \Delta^n(D)) < 3/2$.

Proof. For each complete folding t-curve C and each $n \in \mathbb{N}$, we obtain $\Delta^{n+1}(C)$ from $\Delta^n(C)$ by replacing each sequence of 3 consecutive segments with 1 segment, then scaling down the curve by $\sqrt{3}$. Before the second operation, for each $x \in \Delta^n(C)$, there exists $y \in \Delta^{n+1}(C)$ such that $d(x, y) \leq 1/2$.

Consequently, for each $x \in \mathbb{R}^2$ and each $n \in \mathbb{N}$, we have
$d(\Delta^{n+1}(x), \Delta^{n+1}(C)) \leq (d(\Delta^n(x), \Delta^n(C)) + 1/2)/\sqrt{3}$, and therefore
$d(\Delta^{n+1}(x), \Delta^{n+1}(C)) \leq (5\sqrt{3}/9) d(\Delta^n(x), \Delta^n(C))$ if $d(\Delta^n(x), \Delta^n(C)) \geq 3/4$.
It follows $d(\Delta^n(x), \Delta^n(C)) < 3/4$ for n large enough.

The statement for the curves C, D is an immediate consequence. ■

Now we consider a sequence $X = (x_n)_{n \in \mathbb{N}} \subset U$ with $x_{n+1} \in V_n(x_n)$ for each $n \in \mathbb{N}$, a sequence $\Lambda = (\lambda_n)_{n \in \mathbb{N}^*} \in \{-1, +1\}^{\mathbb{N}^*}$ and the sequences $\Lambda_n = (\lambda_1, \ldots, \lambda_n)$ for $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, each curve of $\mathcal{C}(\Lambda_{n+1}, x_{n+1})$ is obtained by concatenation of 3 curves of $\mathcal{C}(\Lambda_n, x_n)$. We denote by $\mathcal{C}(\Lambda, X)$ the set of inductive limits of curves $C_n \in \mathcal{C}(\Lambda_n, x_n)$.

If there exists $x \in U$ such that $\cap_{n \in \mathbb{N}} V_n(x_n) = \{x\}$, then $\mathcal{C}(\Lambda, X)$ contains 3 half curves starting at x and 3 reversed such curves ending at x. We denote by $\mathcal{C}^+(\Lambda, X)$ (resp. $\mathcal{C}^-(\Lambda, X)$) the set of curves obtained from $\mathcal{C}(\Lambda, X)$ by connecting each terminal segment of reversed half curve with the initial segment of half curve just at its left (resp. right).

Otherwise, we have $\cap_{n \in \mathbb{N}} V_n(x_n) = \emptyset$. Then we write $\mathcal{C}^0(\Lambda, X) = \mathcal{C}(\Lambda, X)$. By Proposition 2.5, each $\mathcal{C}^\alpha(\Lambda, X)$ is a covering of the plane by complete folding t-curves.

Now, for each $y \in U$ and each $k \in \mathbb{N}^*$, we denote by $H(y,k)$ the regular hexagon of center y such that each of its sides is the union of k sides of triangles of $\mathcal{P}$, and $H^*(y,k)$ its interior.

Proposition 2.13. Consider a sequence $\Lambda = (\lambda_k)_{k\in\mathbb{N}^*} \in \{-1,+1\}^{\mathbb{N}^*}$ and two sequences $X = (x_k)_{k\in\mathbb{N}}$, $Y = (y_k)_{k\in\mathbb{N}}$ such that $x_{k+1} \in V_k(x_k)$ and $y_{k+1} \in V_k(y_k)$ for each $k \in \mathbb{N}$. Then, for any $\alpha, \beta \in \{0,-,+\}$ such that $\mathcal{C}^\alpha(\Lambda, X)$ and $\mathcal{C}^\beta(\Lambda, Y)$ exist, each $n \in \mathbb{N}^*$ and any $x, y \in U$, there exists $z \in U \cap H(y, 5.3^n)$ such that $\mathcal{C}^\alpha(\Lambda, X) \restriction H^*(x, 3^n) \cong \mathcal{C}^\beta(\Lambda, Y) \restriction H^*(z, 3^n)$.

Proof. There exists $t \in V_{2n+2}(x_{2n+2})$ such that $x \in H(t, 2.3^n)$, and therefore $H^*(x, 3^n) \subset H^*(t, 3^{n+1})$. For each $u \in V_{2n+2}(y_{2n+2})$, the sets $\mathcal{C}^\alpha(\Lambda, X) \restriction H^*(t, 3^{n+1})$ and $\mathcal{C}^\beta(\Lambda, Y) \restriction H^*(u, 3^{n+1})$ are isomorphic, except possibly concerning the way to connect the 6 segments with endpoint t and the 6 segments with endpoint u.

There exist $u, v, w \in V_{2n+2}(y_{2n+2})$ which form a 3^{n+1}-triangle containing y, which implies $u, v, w \in H(y, 3^{n+1})$. One of these points, say w, belongs to $V_{2n+3}(y_{2n+3})$, while the 2 others belong to $W_{2n+3}(y_{2n+3})$. Then the connexions of the 6 segments of $\mathcal{C}^\beta(\Lambda, Y)$ in u and v are different.

Suppose for instance that the connexions of $\mathcal{C}^\alpha(\Lambda, X)$ in t are the same as the connexions of $\mathcal{C}^\beta(\Lambda, Y)$ in u. Then we have $\mathcal{C}^\alpha(\Lambda, X) \restriction H^*(t, 3^{n+1}) \cong \mathcal{C}^\beta(\Lambda, Y) \restriction H^*(u, 3^{n+1})$. Consequently, there exists $z \in U \cap H(u, 2.3^n)$ such that $\mathcal{C}^\alpha(\Lambda, X) \restriction H^*(x, 3^n) \cong \mathcal{C}^\beta(\Lambda, Y) \restriction H^*(z, 3^n)$. We have $z \in U \cap H(y, 5.3^n)$. ■

Proof of Theorems 2.1, 2.2, 2.3. It suffices to prove the results for oriented curves, since they imply the results for nonoriented curves.

For the proof of Theorem 2.1, it is enough to consider a curve C with segments in E_i. Then it follows from Proposition 2.5 that C belongs to a covering $\mathcal{C}^\alpha(\Lambda, X)$. Lemma 2.11 implies that $\mathcal{C}^\alpha(\Lambda, X)$ is the only covering of the plane by oriented complete folding t-curves which contains C and satisfies (P). It satisfies the strong local isomorphism property by Proposition 2.13. Conversely, by Corollary 2.7, for each covering of the plane by oriented complete folding t-curves, the local isomorphism property implies (P).

If $\mathcal{C}$ is a covering of the plane by oriented complete folding t-curves which satisfies (P), then, for any $x, y \in U$, the orientation of the 6 segments with endpoint x and the orientation of the 6 segments with endpoint y are equivalent up to translation. Consequently, if two such coverings are locally isomorphic, then they have the same orientation. We see from 3) of Proposition 2.5 that they are associated to the same sequence Λ. Conversely, Proposition 2.13 implies that any two such coverings are locally isomorphic if they have the same orientation and if they are associated to the same sequence Λ. ■

Proof of Theorem 2.4. Throughout the proof, we use the notations and the properties introduced just before Lemma 2.8.

First we show that each covering $\mathcal{C}$ contains no more than 3 curves. By Lemma 2.11, there exist $\alpha \in \{0, -, +\}$ and $X = (x_n)_{n\in\mathbb{N}} \subset U$ such that $\mathcal{C}$ is obtained from $\mathcal{C}^\alpha(\Lambda, X)$ by forgetting its orientation.

If $\cap_{n\in\mathbb{N}} V_n(x_n) = \{x\}$ with $x \in U$, then $\mathcal{C}^+(\Lambda, X)$ (resp. $\mathcal{C}^-(\Lambda, X)$) only contains the 3 curves C_1, C_2, C_3 passing through x: For each $n \in \mathbb{N}$, all the segments with points in the interior of $B(\Delta^n(x), 1)$ belong to the 6 1-folding subcurves of $\Delta^n(C_1)$, $\Delta^n(C_2)$, $\Delta^n(C_3)$ which have x as an endpoint. It follows that any other curve D in $\mathcal{C}^+(\Lambda, X)$ (resp. $\mathcal{C}^-(\Lambda, X)$) satisfies $d(\Delta^n(x), \Delta^n(D)) \geq 1$ for each $n \in \mathbb{N}$, contrary to Lemma 2.12.

If $\cap_{n\in\mathbb{N}} V_n(x_n) = \varnothing$ and if $\mathcal{C}^0(\Lambda, X)$ contains more than 3 curves, then it contains 4 curves C, D, E, F with D separating C, E and E separating D, F. For each $n \in \mathbb{N}$, $\Delta^n(D)$ is separating $\Delta^n(C)$, $\Delta^n(E)$ and $\Delta^n(E)$ is separating $\Delta^n(D)$, $\Delta^n(F)$. It follows $d(\Delta^n(C), \Delta^n(F)) \geq \sqrt{3}$ since, for any consecutive segments $r, s, t \in \Delta^n(C) \cup \Delta^n(D) \cup \Delta^n(E) \cup \Delta^n(F)$, if $r \in \Delta^n(C)$, then $s \in \Delta^n(C) \cup \Delta^n(D)$, $t \in \Delta^n(C) \cup \Delta^n(D) \cup \Delta^n(E)$, and therefore $s, t \notin \Delta^n(F)$. This contradicts Lemma 2.12 for n large.

Now we consider the coverings $\mathcal{C}$ associated to Λ which contain 3 curves. For each such $\mathcal{C}$, there exists $C, D, E \in \mathcal{C}$ distinct such that $F(C) \cap D \neq \varnothing$ and $F(C) \cap E \neq \varnothing$.

If $F(C)$ consists of 1 complete curve, then there exists $x \in F(C)$ which belongs to C, D, E. We have $x \in \cap_{n\in\mathbb{N}} V_n(\mathcal{C})$.

Conversely, 3) is realized by any covering $\mathcal{C}$ associated to Λ such that there exists $x \in \cap_{n\in\mathbb{N}} V_n(\mathcal{C})$. Then $\mathcal{C}$ contains 3 curves, the isomorphism class of $\mathcal{C}$ does not depend on x and $\mathcal{C}$ is invariant through a rotation of center x and angle $2\pi/3$. We obtain a representative of the other isomorphism class of coverings of that type by changing the connections in x.

If $F(C)$ consists of 2 complete curves, then we can orient C, D, E in such a way that $F_L(C) = F_L(D)$ and $F_R(C) = F_R(E)$. We consider 2 segments $s \subset F_L(C)$, $t \subset F_R(C)$ of length 1, and 2 sequences $(C_n)_{n\in\mathbb{N}}$, $(P_n)_{n\in\mathbb{N}}$ such that $C = \cup_{n\in\mathbb{N}} C_n$ and such that, for each $n \in \mathbb{N}$, C_n is an n-folding t-curve, $P_n \in \{\mathrm{I}, \mathrm{M}, \mathrm{S}\}$ and $C_n = C_{n+1}^{P_n}$. There exists an integer h such that $s \subset \mathrm{F_L}(C_n)$ and $t \subset \mathrm{F_R}(C_n)$ for each $n \geq h$. We consider the sequences $(\alpha_n)_{n\geq h}$ and $(\beta_n)_{n\geq h}$ such that $\alpha_n, \beta_n \in \{\mathrm{I}, \mathrm{S}\}$, $s \subset \mathrm{F}_{\mathrm{L}\alpha_n}(C_n)$ and $t \subset \mathrm{F}_{\mathrm{R}\beta_n}(C_n)$ for each n.

For $n \geq h$, the properties $s \subset \mathrm{F}_{\mathrm{L}\alpha_n}(C_n) \cap \mathrm{F}_{\mathrm{L}\alpha_{n+1}}(C_{n+1})$ and $t \subset \mathrm{F}_{\mathrm{R}\beta_n}(C_n) \cap \mathrm{F}_{\mathrm{R}\beta_{n+1}}(C_{n+1})$ respectively imply $\mathrm{F}_{\mathrm{L}\alpha_n}(C_{n+1}^{P_n}) = \mathrm{F}_{\mathrm{L}\alpha_n}(C_n) \subset \mathrm{F}_{\mathrm{L}\alpha_{n+1}}(C_{n+1})$ and $\mathrm{F}_{\mathrm{R}\beta_n}(C_{n+1}^{P_n}) = \mathrm{F}_{\mathrm{R}\beta_n}(C_n) \subset \mathrm{F}_{\mathrm{R}\beta_{n+1}}(C_{n+1})$. Considering C_{n+1}, we see that one of the 4 cases below is realized:

$P_n = \mathrm{I}$, $(\alpha_n, \beta_n) \in \{(\mathrm{I}, \mathrm{I}), (\mathrm{I}, \mathrm{S}), (\mathrm{S}, \mathrm{I})\}$, $(\alpha_{n+1}, \beta_{n+1}) = (\mathrm{I}, \mathrm{I})$;
$P_n = \mathrm{S}$, $(\alpha_n, \beta_n) \in \{(\mathrm{I}, \mathrm{S}), (\mathrm{S}, \mathrm{I}), (\mathrm{S}, \mathrm{S})\}$, $(\alpha_{n+1}, \beta_{n+1}) = (\mathrm{S}, \mathrm{S})$;
$P_n = \mathrm{M}$, $(\alpha_n, \beta_n) = (\mathrm{S}, \mathrm{I})$, $(\alpha_{n+1}, \beta_{n+1}) = (\mathrm{I}, \mathrm{S})$, $\lambda_{n+1} = +1$;

$P_n = \mathrm{M}$, $(\alpha_n, \beta_n) = (\mathrm{I}, \mathrm{S})$, $(\alpha_{n+1}, \beta_{n+1}) = (\mathrm{S}, \mathrm{I})$, $\lambda_{n+1} = -1$.

It follows that there exists $m \geq h$ such that one of the properties below is true for each $n \geq m$:
a) $P_n = \mathrm{I}$ and $(\alpha_n, \beta_n) = (\mathrm{I}, \mathrm{I})$;
b) $P_n = \mathrm{S}$ and $(\alpha_n, \beta_n) = (\mathrm{S}, \mathrm{S})$;
c) $P_n = \mathrm{M}$; $(\alpha_n, \beta_n) = (\mathrm{I}, \mathrm{S})$ and $\lambda_n = +1$ for n even; $(\alpha_n, \beta_n) = (\mathrm{S}, \mathrm{I})$ and $\lambda_n = -1$ for n odd;
d) $P_n = \mathrm{M}$; $(\alpha_n, \beta_n) = (\mathrm{I}, \mathrm{S})$ and $\lambda_n = +1$ for n odd; $(\alpha_n, \beta_n) = (\mathrm{S}, \mathrm{I})$ and $\lambda_n = -1$ for n even.

The cases a) and b) imply $\cup_{n\in\mathbb{N}} C_n \neq C$, contrary to our hypotheses.

On the other hand, each of the cases c), d), for each of the values of Λ which realize it, gives a covering of the plane by complete folding t-curves, with 1 curve separating 2 others. This covering is determined by Λ modulo a positive isometry. We obtain representatives of the 2 other isomorphism classes of coverings by applying a rotation of angle $\mp 2\pi/3$.

It remains to be proved that there exist 2^ω isomorphism classes of coverings of the plane by 1 (resp. 2) complete folding t-curves associated to Λ. We consider the complete folding t-curves C for which there exists a pair $((C_n)_{n\in\mathbb{N}}, (P_n)_{n\in\mathbb{N}})$ such that $C = \cup_{n\in\mathbb{N}} C_n$ and such that, for each $n \in \mathbb{N}$, C_n is an n-folding t-curve, $P_n \in \{\mathrm{I}, \mathrm{M}, \mathrm{S}\}$ and $C_n = C_{n+1}^{P_n}$. Then C is determined modulo a positive isometry by $(P_n)_{n\in\mathbb{N}}$ and there are countably many choices of $(P_n)_{n\in\mathbb{N}}$ which give C, since any 2 such sequences only differ by a finite number of terms. Consequently, it suffices to show that 2^ω choices of $(P_n)_{n\in\mathbb{N}}$ give coverings by 1 (resp. 2) complete folding t-curves.

In order to obtain a curve C which covers the plane, it suffices to have $\mathrm{F_L}(C_{5n}) \cap \mathrm{F_L}(C_{5n+2}) = \varnothing$ and $\mathrm{F_R}(C_{5n+2}) \cap \mathrm{F_R}(C_{5n+4}) = \varnothing$ for each $n \in \mathbb{N}$. The first property is realized for $P_{5n} = \mathrm{S}$, $P_{5n+1} = \mathrm{I}$ if $\lambda_{5n+2} = +1$, and $P_{5n} = \mathrm{I}$, $P_{5n+1} = \mathrm{S}$ if $\lambda_{5n+2} = -1$. The second property is realized for $P_{5n+2} = \mathrm{I}$, $P_{5n+3} = \mathrm{S}$ if $\lambda_{5n+4} = +1$, and $P_{5n+2} = \mathrm{S}$, $P_{5n+3} = \mathrm{I}$ if $\lambda_{5n+4} = -1$. For each $n \in \mathbb{N}$, P_{5n+4} can be chosen arbitrarily. It follows that 2^ω choices give coverings by 1 curve.

Now we prove that 2^ω choices give coverings by 2 curves. As only countably many choices give coverings by 3 curves, it suffices to show that 2^ω choices give a curve C with $\mathrm{F_L}(C) \neq \varnothing$.

For each $n \in \mathbb{N}^*$, we consider the sequences $(P_1, \ldots, P_{n-1})$, $(C_1, \ldots, C_n)$, $(\alpha_1, \ldots, \alpha_n)$, with C_i an i-folding t-curve for $1 \leq i \leq n$, $C_i = C_{i+1}^{P_i}$ and $\mathrm{F}_{\mathrm{L}\alpha_i}(C_i) \subset \mathrm{F}_{\mathrm{L}\alpha_{i+1}}(C_{i+1})$ for $1 \leq i \leq n-1$. We observe that, for any such sequences, there are 2 different ways to choose P_n so that the $(n+1)$-folding t-curve C_{n+1} defined by $C_n = C_{n+1}^{P_n}$ satisfies $\mathrm{F}_{\mathrm{L}\alpha_n}(C_n) \subset \mathrm{F_L}(C_{n+1})$, and therefore $\mathrm{F}_{\mathrm{L}\alpha_n}(C_n) \subset \mathrm{F}_{\mathrm{L}\alpha_{n+1}}(C_{n+1})$ for the appropriate α_{n+1}. ■

Remark. Each covering of the plane by 3 nonoriented complete folding t-

curves is invariant through a central symmetry.

3. Peano-Gosper curves

In the present section, we obtain a set Ω of coverings of $\mathbb{R}^2$ by sets of disjoint self-avoiding nonoriented curves, generalizing the Peano-Gosper curves, such that:
1) each $\mathcal{C} \in \Omega$ satisfies the strong local isomorphism property; any set of curves locally isomorphic to $\mathcal{C}$ belongs to Ω;
2) Ω is the union of 2^ω equivalence classes for the relation "$\mathcal{C}$ locally isomorphic to $\mathcal{D}$"; each of them contains 2^ω (resp. 2^ω, 1, 0) isomorphism classes of coverings by 1 (resp. 2, 3, ≥ 4) curves.
Each $\mathcal{C} \in \Omega$ gives exactly 2 coverings of $\mathbb{R}^2$ by sets of oriented curves which satisfy the local isomorphism property. They have opposite orientations.

For each regular tiling $\mathcal{P}$ of $\mathbb{R}^2$ by hexagons and each center x of a tile of $\mathcal{P}$, we construct some tilings $\mathcal{P}_{x\lambda_1\cdots\lambda_n}$ of $\mathbb{R}^2$ by isomorphic tiles such that, for each $n \in \mathbb{N}$ and any $\lambda_1, \ldots, \lambda_n \in \{-, +\}$, x is the center of a tile of $\mathcal{P}_{x\lambda_1\cdots\lambda_n}$ and, for $n \geq 1$, each tile of $\mathcal{P}_{x\lambda_1\cdots\lambda_n}$ is the union of 7 nonoverlapping tiles of $\mathcal{P}_{x\lambda_1\cdots\lambda_{n-1}}$ with one of them surrounded by the 6 others.

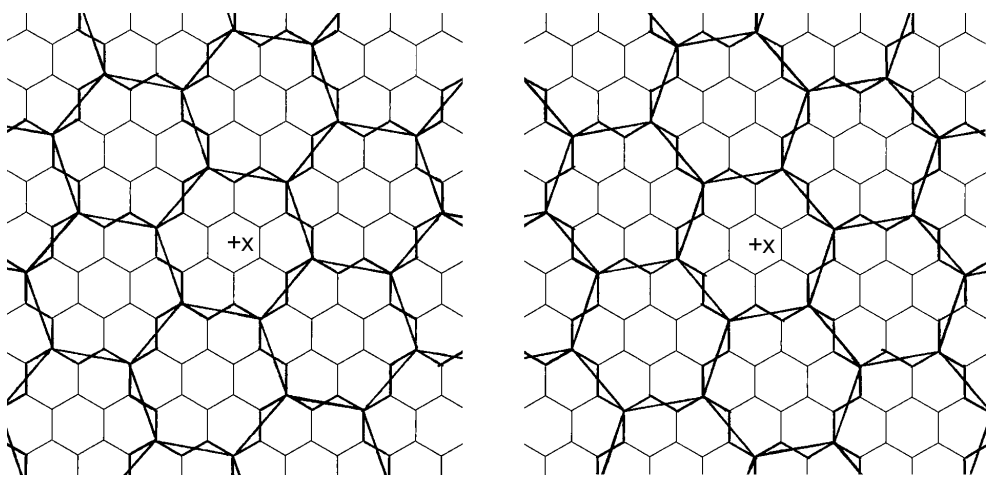

Figure 3.1

We write $\mathcal{P}_x = \mathcal{P}$. We consider the tilings $\mathcal{P}_{x+}$ and $\mathcal{P}_{x-}$ respctively given by the first and the second part of Figure 3.1. The points of $\mathbb{R}^2$ which are common to 3 tiles of $\mathcal{P}_{x+}$ (resp. $\mathcal{P}_{x-}$) determine a regular tiling by hexagons $\mathcal{Q}_{x+}$ (resp. $\mathcal{Q}_{x-}$).

We denote by Δ_{x+} (resp. Δ_{x-}) the bijection which associates to each tile of $\mathcal{P}_{x+}$ (resp. $\mathcal{P}_{x-}$) the tile of $\mathcal{Q}_{x+}$ (resp. $\mathcal{Q}_{x-}$) with the same center. We see from Figure 3.1 that, for each tile Q of $\mathcal{Q}_{x+}$ (resp. $\mathcal{Q}_{x-}$), $\Delta_{x+}^{-1}(Q)$ (resp. $\Delta_{x-}^{-1}(Q)$) is obtained by replacing each side S of Q with three consecutive

sides of tiles of $\mathcal{P}$. The first side is at the right (resp. left) of S and the third side is at its left (resp. right) for the direction given by anticlockwise rotation around the center of Q.

For $n \geq 2$ and $\lambda_1, \ldots, \lambda_n \in \{-, +\}$ we write $\mathcal{P}_{x\lambda_1\cdots\lambda_n} = \Delta_{x\lambda_1}^{-1}((\mathcal{Q}_{x\lambda_1})_{x\lambda_2\cdots\lambda_n})$. Each $\mathcal{P}_{x\lambda_1\cdots\lambda_n}$ consists of isomorphic tiles with connected interior and connected exterior. It is regular and invariant through a rotation of center x and angle $\pi/3$.

In each $\mathcal{P}_{x\lambda_1\cdots\lambda_n}$, each tile P has nonempty frontiers with 6 others. We call *sides* of P these frontiers, which are unions of 3^n sides of tiles of $\mathcal{P}$. The *vertices* of P are the endpoints of its sides.

For each $n \in \mathbb{N}$, any $\lambda_1, \ldots, \lambda_n \in \{-, +\}$ and any centers x, y of tiles of $\mathcal{P}$, each tile of $\mathcal{P}_{x\lambda_1\cdots\lambda_n-}$ and each tile of $\mathcal{P}_{y\lambda_1\cdots\lambda_n+}$ are isomorphic.

For any integers $n \geq m \geq 1$, we define some *derivations* $\Delta_{x\lambda_1\cdots\lambda_m}$ on the sets $\mathcal{P}_{x\lambda_1\cdots\lambda_n}$. We use the definitions of Δ_{x+} and Δ_{x-} given above and we write $\Delta_{x\lambda_1\cdots\lambda_m}(P) = \Delta_{x\lambda_m}(\cdots(\Delta_{x\lambda_1}(P))\cdots)$ for any $\lambda_1, \ldots, \lambda_n \in \{-, +\}$ and each $P \in \mathcal{P}_{x\lambda_1\cdots\lambda_n}$. We have
$\Delta_{x\lambda_1\cdots\lambda_m}(\mathcal{P}_{x\lambda_1\cdots\lambda_n}) = (\Delta_{x\lambda_1\cdots\lambda_m}(\mathcal{P}_{x\lambda_1\cdots\lambda_m}))_{x\lambda_{m+1}\cdots\lambda_n}$.

Remark. For each $n \in \mathbb{N}$, denote by P_n the tile of center x in $\mathcal{P}_{x\lambda_1\cdots\lambda_n}$ with $\lambda_1 = \cdots = \lambda_n = +$, and Q_n the tile with the same vertices as P_0 which is the image of P_n under a similarity. Then the limit of the tiles Q_n is the Peano-Gosper island considered in [3, p. 46]. It is the union of 7 isomorphic nonoverlapping tiles which are similar to it, with one of them surrounded by the 6 others.

Now, for each $\Lambda = (\lambda_n)_{n\in\mathbb{N}^*} \in \{-, +\}^{\mathbb{N}^*}$, we consider the sequences $X = (x_n)_{n\in\mathbb{N}}$ of centers of tiles of $\mathcal{P}$ with $x_{n+1} = x_n$ or x_n, x_{n+1} centers of adjacent tiles of $\mathcal{P}_{x_n\lambda_1\cdots\lambda_n}$ for each $n \in \mathbb{N}$. For each such X and each $n \in \mathbb{N}$, we write $\mathcal{P}_{X\Lambda}^n = \mathcal{P}_{x_n\lambda_1\cdots\lambda_n}$ and we denote by $P_{X\Lambda}^n$ the tile of $\mathcal{P}_{X\Lambda}^n$ which contains x_n. For $n \in \mathbb{N}$, we have $P_{X\Lambda}^n \subset P_{X\Lambda}^{n+1}$ and each tile of $\mathcal{P}_{X\Lambda}^{n+1}$ is the union of 7 nonoverlapping tiles of $\mathcal{P}_{X\Lambda}^n$. We write $\mathcal{P}_{X\Lambda} = \cup_{n\in\mathbb{N}}\mathcal{P}_{X\Lambda}^n$.

For each center u of a tile of $\mathcal{P}$, there exists a unique sequence $S_u = (u_n)_{n\in\mathbb{N}}$ with the properties above such that $u_0 = u$ and $\mathcal{P}_{S_u\Lambda} = \mathcal{P}_{X\Lambda}$. We write $R_{uX\Lambda} = \cup_{n\in\mathbb{N}}P_{S_u\Lambda}^n$.

We call a *region* of $\mathcal{P}_{X\Lambda}$ any minimal nonempty closed subset $R \subset \mathbb{R}^2$ such that any tile of $\mathcal{P}_{X\Lambda}$ is completely inside R or completely outside. Each $R_{uX\Lambda}$ is a region of $\mathcal{P}_{X\Lambda}$ and each region of $\mathcal{P}_{X\Lambda}$ is obtained in that way.

For each $n \in \mathbb{N}$ and any centers u, v of tiles of $\mathcal{P}$, $P_{S_u\Lambda}^n$ and $P_{S_v\Lambda}^n$ are disjoint, or they have one common side, or they are equal. If the third possibility is realized for some n, then we have $R_{uX\Lambda} = R_{vX\Lambda}$. Otherwise, the second possibility is realized for n large enough and $R_{uX\Lambda}, R_{vX\Lambda}$ are adjacent.

For any centers u, v, w of tiles of $\mathcal{P}$, if $P_{S_u\Lambda}^n, P_{S_v\Lambda}^n, P_{S_w\Lambda}^n$ are distinct for each $n \in \mathbb{N}$, then, for n large enough, they have 1 common point and any 2 of them

have a common side containing that point. Consequently, $R_{uX\Lambda}, R_{vX\Lambda}, R_{wX\Lambda}$ are the three regions of $\mathcal{P}_{X\Lambda}$.

It follows that one of the three following properties is true for each $\mathcal{P}_{X\Lambda}$:
1) There is 1 region.
2) There are 2 regions; their frontier is a complete curve which consists of sides of hexagons of $\mathcal{P}$.
3) There are 3 regions with 1 common point which is a vertex of tiles of $\mathcal{P}^n_{X\Lambda}$ for each $n \in \mathbb{N}$; the frontier of 2 regions is a half curve starting from that point which consists of sides of hexagons of $\mathcal{P}$.

Proposition 3.1. For each Λ, the isomorphism classes of sets $\mathcal{P}_{X\Lambda}$ are countable; there exist 2^ω classes of sets with 1 region, 2^ω classes of sets with 2 regions, and 2 classes of sets with 3 regions, obtained from each other by a rotation of angle $\pi/3$.

Proof. For each sequence $X = (x_n)_{n\in\mathbb{N}}$ such that $\mathcal{P}_{X\Lambda}$ exists, we write $R_{X\Lambda} = R_{x_0X\Lambda}$. For any such sequences X, Y, we write $X \sim_1 Y$ if $\mathcal{P}_{X\Lambda} \cong \mathcal{P}_{Y\Lambda}$, $X \sim_2 Y$ if $\mathcal{P}_{X\Lambda} = \mathcal{P}_{Y\Lambda}$, and $X \sim_3 Y$ if $\mathcal{P}_{X\Lambda} = \mathcal{P}_{Y\Lambda}$ and $R_{X\Lambda} = R_{Y\Lambda}$.

Each $\sim_3$-class is countable since any two sequences in such a class are ultimately equal. As each $\mathcal{P}_{X\Lambda}$ only has finitely many regions, each $\sim_2$-class is also countable.

For any sequences X, Y such that $X \sim_1 Y$, there exist a sequence Z and a translation τ such that $X \sim_2 Z$ and $\tau(Z) = Y$; we have $\tau(\mathcal{P}) = \mathcal{P}$. As only countably many translations satisfy that property, each $\sim_1$-class is the union of countably many $\sim_2$-classes, and therefore countable.

Now we prove the second part of the proposition.

First we consider a vertex w of a tile of $\mathcal{P}$ and the sequences Y such that w is a vertex of $P^n_{Y\Lambda}$ for each $n \in \mathbb{N}$. This property is true for exactly 6 sequences $Y_1, \ldots, Y_6$ with $R_{Y_1\Lambda}, R_{Y_2\Lambda}, R_{Y_3\Lambda}$ distinct regions of $\mathcal{P}_{Y_1\Lambda} = \mathcal{P}_{Y_2\Lambda} = \mathcal{P}_{Y_3\Lambda}$, $R_{Y_1\Lambda} \cap R_{Y_2\Lambda} \cap R_{Y_3\Lambda} = \{w\}$ and Y_4, Y_5, Y_6 obtained from Y_1, Y_2, Y_3 by a rotation of center w and angle $\pi/3$.

The statement concerning the classes of sets with 3 regions follows since, for each sequence X, if $\mathcal{P}_{X\Lambda}$ has 3 regions with the common point w, then X is ultimately equal to some Y_i.

Now, it suffices to prove that there exist 2^ω sequences $X = (x_n)_{n\in\mathbb{N}}$ such that $\mathcal{P}_{X\Lambda}$ has 1 region and 2^ω sequences with 2 or 3 regions.

In order to obtain a sequence X with 1 region, it suffices to choose successively the elements x_k so that $P^{k-1}_{X\Lambda}$ is contained in the interior of $P^k_{X\Lambda}$ for each k even. Consequently, there are 2^ω such sequences.

For each side S of a tile of $\mathcal{P}$, in order to obtain a sequence X with 2 or 3 regions, it suffices to choose successively the elements x_k so that S is contained in a side of $P^k_{X\Lambda}$. There are 3 possible choices for each $k \geq 1$, and therefore 2^ω such sequences. ■

Now we define the curves associated to $\mathcal{P}$. The set V of vertices of hexagons of $\mathcal{P}$ is the disjoint union of 2 subsets which contain no consecutive vertices of an hexagon. We choose one of them and we denote it by W.

An *oriented bounded curve* (resp. *half curve*, *complete curve*) is a sequence $C = (A_k)_{0\leq k\leq n}$ (resp. $(A_k)_{k\in\mathbb{N}}$, $(A_k)_{k\in\mathbb{Z}}$) of consecutive oriented segments, each of them joining 2 vertices of an hexagon of $\mathcal{P}$ which belong to W, such that each hexagon contains at most 1 segment and each endpoint of C, if it exists, only belongs to 1 segment. With this definition, C is self-avoiding.

For each oriented bounded curve $(A_0, \ldots, A_n)$, we consider the sequence $(a_1, \ldots, a_n)$ with each a_k equal to $+2, +1, 0, -1, -2$ if the angle between A_{k-1} and A_k is $+2\pi/3, +\pi/3, 0, -\pi/3, -2\pi/3$. For each sequence $S = (a_1, \ldots, a_n)$, we write $\overline{S} = (-a_n, \ldots, -a_1)$. We have $\overline{\overline{S}} = S$. The curves associated to $\overline{S}$ are obtained by changing the orientation of the curves associated to S.

For each $n \in \mathbb{N}$, any $\lambda_1, \ldots, \lambda_n \in \{-, +\}$, each center w of a tile of $\mathcal{P}$ and each $P \in \mathcal{P}_{w\lambda_1\cdots\lambda_n}$, we say that a curve C is a *covering* of P if it is contained in P and if, for each $m \in \{0, \ldots, n\}$ and each $Q \in \mathcal{P}_{w\lambda_1\cdots\lambda_m}$ contained in P, the segments of C contained in Q form a subcurve whose endpoints are vertices of Q.

For any integers $n \geq m \geq 1$ and each covering C of $P \in \mathcal{P}_{w\lambda_1\cdots\lambda_n}$, we obtain a covering $\Delta_{w\lambda_1\cdots\lambda_m}(C)$ of $\Delta_{w\lambda_1\cdots\lambda_m}(P)$ by replacing each $C \restriction Q$ for $Q \in \mathcal{P}_{w\lambda_1\cdots\lambda_m}$ contained in P with the segment from its initial point to its terminal point. We have $\Delta_{w\lambda_1\cdots\lambda_m}(C) = \Delta_{w\lambda_m}(\cdots(\Delta_{w\lambda_1}(C))\cdots)$.

Now we define by induction on $n \in \mathbb{N}^*$ some sequences
$S_{\lambda_1\cdots\lambda_n} \in \{-2, -1, 0, +1, +2\}^{7^n-1}$ for $\lambda_1, \ldots, \lambda_n \in \{-, +\}$. We write
$S_+ = S = (+1, +2, -1, -2, 0, -1)$, $S_- = -S$,
$S_{++} = (S, +1, \overline{S}, +1, \overline{S}, -1, S, -1, S, +1, S, -1, \overline{S})$, $S_{--} = -S_{++}$,
$S_{+-} = (\overline{S}, -1, S, -1, S, +1, \overline{S}, +1, \overline{S}, -1, \overline{S}, +1, S)$, $S_{-+} = -S_{+-}$.
For $n \geq 3$, $S_{\lambda_1\cdots\lambda_n}$ is obtained from $S_{\lambda_2\cdots\lambda_n} = (a_1, \ldots, a_{7^{n-1}-1})$ by replacing each subsequence $(a_{7k+1}, \ldots, a_{7k+6})$ equal to S_{λ_2} (resp. $\overline{S_{\lambda_2}}$) with $S_{\lambda_1\lambda_2}$ (resp. $\overline{S_{\lambda_1\lambda_2}}$).

The first (resp. second) part of Figure 3.2 below shows a covering of a tile of $\mathcal{P}_{w+}$ (resp. $\mathcal{P}_{w-}$) by a curve associated to S_+ (resp. S_-).

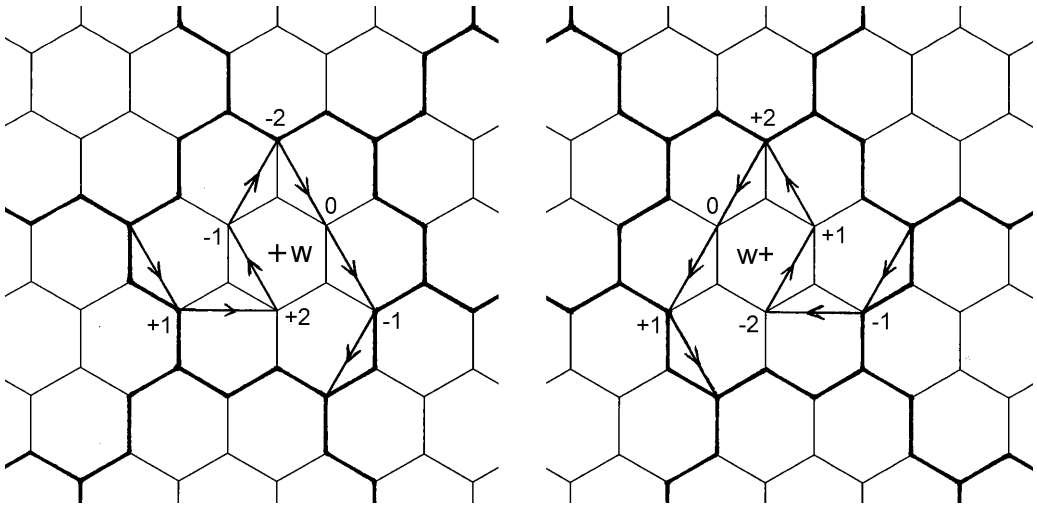

Figure 3.2

The first (resp. second) part of Figure 3.3 below shows a covering of a tile of $\mathcal{P}_{w++}$ (resp. $\mathcal{P}_{w+-}$) by a curve associated to S_{++} (resp. S_{+-}).

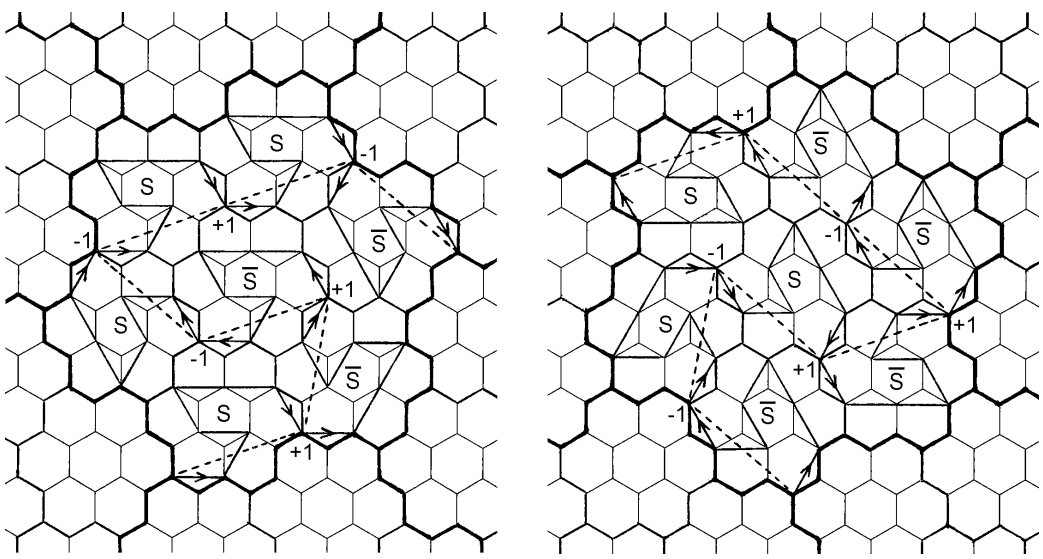

Figure 3.3

Proposition 3.2. Consider $w \in \mathbb{R}^2$, $n \in \mathbb{N}^*$, $\lambda_1, \ldots, \lambda_n \in \{-,+\}$ and $P \in \mathcal{P}_{w\lambda_1\cdots\lambda_n}$. Then P has 6 coverings by oriented curves, each of them determined by its initial and terminal points. Each covering of P is associated to $S_{\lambda_1\cdots\lambda_n}$ or $\overline{S_{\lambda_1\cdots\lambda_n}}$. The 3 coverings associated to $S_{\lambda_1\cdots\lambda_n}$ are obtained from one of them by rotations of angles $2k\pi/3$, and the 3 others by changing their orientations. For $n \geq 2$, if C is a covering of P associated to $S_{\lambda_1\cdots\lambda_n}$ (resp. $\overline{S_{\lambda_1\cdots\lambda_n}}$), then $\Delta_{w\lambda_1}(C)$ is a covering of $\Delta_{w\lambda_1}(C)$ associated to $S_{\lambda_2\cdots\lambda_n}$ (resp. $\overline{S_{\lambda_2\cdots\lambda_n}}$).

Proof. We see from Figure 3.2 that Proposition 3.2 is true for $n = 1$. Now we show that it is true for $n \geq 2$ if it is true for $n - 1$.

By the induction hypothesis, there exists a covering D of $\Delta_{w\lambda_1}(P)$ associated to $S_{\lambda_2 \cdots \lambda_n}$. Each segment A of D is a covering of an hexagon $H_A \in \Delta_{w\lambda_1}(\mathcal{P}_{w\lambda_1 \cdots \lambda_n})$ which is contained in $\Delta_{w\lambda_1}(P)$. We obtain a covering C of P with $\Delta_{w\lambda_1}(C) = D$ by replacing each such A with a covering of $\Delta^{-1}_{w\lambda_1}(H_A)$ which has the corresponding initial and terminal points.

As P is invariant through a rotation of angle $2\pi/3$, it has 3 coverings obtained from C by rotations of angles $0, 2\pi/3, 4\pi/3$, and 3 others obtained by changing their orientations. These coverings are associated to the 6 possible pairs of initial and terminal points.

Any other covering B of P has the same initial and terminal points as one of the 6 coverings above, say C'. Then $\Delta_{w\lambda_1}(B)$ and $\Delta_{w\lambda_1}(C')$ are coverings of $\Delta_{w\lambda_1}(P)$ with the same initial and terminal points. By the induction hypothesis, we have $\Delta_{w\lambda_1}(B) = \Delta_{w\lambda_1}(C')$. Consequently, for each $Q \in \mathcal{P}_{w\lambda_1}$ contained in P, $B \restriction Q$ and $C' \restriction Q$ are coverings of Q with the same initial and terminal points, which implies $B \restriction Q = C' \restriction Q$. It follows $B = C'$.

Now it suffices to show that C is associated to $S_{\lambda_1 \cdots \lambda_n}$. We see from Figure 3.3 that it is true for $n = 2$. For $n \geq 3$, we write $C = (C_h)_{0 \leq h \leq 7^n - 1}$ and $D = (D_h)_{0 \leq h \leq 7^{n-1} - 1}$. We consider the associated sequences
$(c_h)_{1 \leq h \leq 7^n - 1} \in \{-2, -1, 0, +1, +2\}^{7^n - 1}$ and
$(d_h)_{1 \leq h \leq 7^{n-1} - 1} \in \{-2, -1, 0, +1, +2\}^{7^{n-1} - 1}$.

For $0 \leq k \leq 7^{n-2} - 1$, we write $U_k = (c_h)_{49k+1 \leq h \leq 49k+48}$ and $V_k = (d_h)_{7k+1 \leq h \leq 7k+6}$. There exists $Q_k \in \mathcal{P}_{w\lambda_1\lambda_2}$ contained in P such that U_k is associated to $C \restriction Q_k$ and V_k is associated to $D \restriction \Delta_{w\lambda_1}(Q_k) = \Delta_{w\lambda_1}(C \restriction Q_k)$. Consequently, we have $U_k = S_{\lambda_1\lambda_2}$ (resp. $\overline{S_{\lambda_1\lambda_2}}$) if and only if $V_k = S_{\lambda_2}$ (resp. $\overline{S_{\lambda_2}}$).

It remains to be proved that $c_{49k} = d_{7k}$ for $1 \leq k \leq 7^{n-2} - 1$. As $U_{k-1}, U_k \in \{S_{\lambda_1\lambda_2}, \overline{S_{\lambda_1\lambda_2}}\}$, there exists $T \in \{S_{\lambda_1}, \overline{S_{\lambda_1}}\}$ such that U_{k-1} is ending with T and U_k is beginning with $\overline{T}$. It follows that the angle between C_{49k} and C_{49k+1} is equal to the angle between D_{7k} and D_{7k+1}, which implies $c_{49k} = d_{7k}$. ■

Example. The Peano-Gosper curves considered in [3, p. 46] and [2, p. 63] are associated to the sequences $T_n = S_{\lambda_1 \cdots \lambda_n}$ with $\lambda_1 = \cdots = \lambda_n = +$. The case $n = 2$ is shown in the first part of Figure 3.3. We have
$T_{n+1} = (T_n, +1, \overline{T_n}, +1, \overline{T_n}, -1, T_n, -1, T_n, +1, T_n, -1, \overline{T_n})$
for each $n \in \mathbb{N}^*$. In [2] and several papers mentioned among its references, W. Kuhirun, D.H. Werner and P.L. Werner prove that an antenna with the shape of a Peano-Gosper curve has particular electromagnetic properties. We can suppose that similar properties exist for the other values of $\lambda_1, \ldots, \lambda_n$.

Corollary 3.3. For each $w \in \mathbb{R}^2$, each $n \in \mathbb{N}$, any $\lambda_1, \ldots, \lambda_{n+1} \in \{-, +\}$ (resp. $\lambda_1, \ldots, \lambda_{n+2} \in \{-, +\}$) and each $P \in \mathcal{P}_{w\lambda_1 \cdots \lambda_n}$, each covering of some

$Q \in \mathcal{P}_{w\lambda_1\cdots\lambda_{n+1}}$ (resp. $\mathcal{P}_{w\lambda_1\cdots\lambda_{n+2}}$) by a nonoriented (resp. oriented) curve contains copies of the 3 (resp. 6) coverings of P by nonoriented (resp. oriented) curves.

Proof. By Proposition 3.2, it suffices to show that each covering of some $Q \in \mathcal{P}_{w\lambda_1\cdots\lambda_{n+1}}$ (resp. $\mathcal{P}_{w\lambda_1\cdots\lambda_{n+2}}$) by a nonoriented (resp. oriented) curve contains 3 (resp. 6) nonisomorphic coverings of tiles $P \in \mathcal{P}_{w\lambda_1\cdots\lambda_n}$ by nonoriented (resp. oriented) curves. We see from Figure 3.2 (resp. 3.3) that the statement for nonoriented (resp. oriented) curves is true for $n = 0$.

For each $n \geq 1$ and each covering C of some $Q \in \mathcal{P}_{w\lambda_1\cdots\lambda_{n+1}}$ (resp. $\mathcal{P}_{w\lambda_1\cdots\lambda_{n+2}}$) by a nonoriented (resp. oriented) curve, we consider some hexagonal tiles Q_1, Q_2, Q_3 (resp. $Q_1, \ldots, Q_6$) contained in $\Delta_{w\lambda_1\cdots\lambda_n}(Q)$ such that the nonoriented (resp. oriented) segments $\Delta_{w\lambda_1\cdots\lambda_n}(C) \restriction Q_i$ are nonisomorphic. Then the nonoriented (resp. oriented) curves $C \restriction \Delta^{-1}_{w\lambda_1\cdots\lambda_n}(Q_i) = \Delta^{-1}_{w\lambda_1\cdots\lambda_n}(\Delta_{w\lambda_1\cdots\lambda_n}(C) \restriction Q_i)$ are nonisomorphic. ■

Lemma 3.4. Here the curves are nonoriented. Consider $w \in \mathbb{R}^2$, $n \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_{n+1} \in \{-, +\}$, $P \in \mathcal{P}_{w\lambda_1\cdots\lambda_{n+1}}$, $Q_1, \ldots, Q_7 \in \mathcal{P}_{w\lambda_1\cdots\lambda_n}$ such that $P = Q_1 \cup \cdots \cup Q_7$, and $Q \in \{Q_1, \ldots, Q_7\}$. If Q contains the center of P, then each of the 3 coverings of Q extends into a covering of P. Otherwise, denote by x_1 (resp. x_2, x_3) the vertex of Q belonging to W which is a vertex of 1 (resp. 2, 3) tiles Q_i. If x_1 is not a vertex of P, then the covering of Q which joins x_2 and x_3 extends into 3 coverings of P and the 2 other coverings of Q do not extend. If x_1 is a vertex of P, then the covering of Q which joins x_1 and x_2 (resp. x_2 and x_3, x_1 and x_3) extends into 2 (resp. 1, 0) coverings of P.

Proof. We see from Figure 3.2 that Lemma 3.4 is true for $n = 0$. Then, as in the proof of Corollary 3.3, we use the derivations $\Delta_{w\lambda_1\cdots\lambda_n}$ to show that it is also true for $n \geq 1$. ■

For each $\mathcal{P}_{X\Lambda}$, we say that a curve C is a *covering* of a region R of $\mathcal{P}_{X\Lambda}$ if $C \subset R$ and if C contains a covering of each $P \in \mathcal{P}_{X\Lambda}$ contained in R. We say that a set $\mathcal{C}$ of disjoint complete curves is a *covering* of $\mathcal{P}_{X\Lambda}$ if each $P \in \mathcal{P}_{X\Lambda}$ has a covering by a subcurve of a curve of $\mathcal{C}$. Then each region of $\mathcal{P}_{X\Lambda}$ also has a covering by a subcurve of a curve of $\mathcal{C}$.

Proposition 3.5. We consider nonoriented curves. Any region R of some $\mathcal{P}_{X\Lambda}$ has a covering by a complete curve. This covering is unique if $R \subsetneq \mathbb{R}^2$. If R is the union of an increasing sequence of tiles with 1 common vertex $y \in W$, then R also has a unique covering by a half curve and y is the endpoint of that curve. Otherwise, no such covering exists.

Proof. We have $R = \cup_{n\in\mathbb{N}} P_n$ for an increasing sequence $(P_n)_{n\in\mathbb{N}} \in \times_{n\in\mathbb{N}} \mathcal{P}^n_{X\Lambda}$.

For each $m \in \mathbb{N}$, consider the nonempty set E_m which consists of the coverings of P_m with endpoints different from the common vertex of the tiles P_n if it exists. Then, for $m < n$, each element of E_n gives by restriction an element of E_m. By König's lemma, there exists an increasing sequence $(C_n)_{n\in\mathbb{N}} \in \times_{n\in\mathbb{N}} E_n$; $\cup_{n\in\mathbb{N}} C_n$ is a covering of R by a complete curve.

Now we can suppose $R \subsetneq \mathbb{R}^2$ for the remainder of the proof. Then there exists $h \in \mathbb{N}$ such that, for each $n \geq h$, P_n does not contain the center of P_{n+1}. For $n \geq h$, denote by y_n^1 (resp. y_n^2, y_n^3) the vertex of P_n belonging to W which is a vertex of 1 (resp. 2, 3) tiles of $\mathcal{P}_{X\Lambda}^n$ contained in P_{n+1}.

If R is the union of an increasing sequence of tiles with a common vertex belonging to W, then there exists $k \geq h$ such that $y_n^1 = y_k^1$ for each $n \geq k$. For $n \geq k$, denote by F_n the nonempty set which consists of the coverings of P_n with endpoint y_k^1. Then, for $k \leq m \leq n$, each element of F_n gives by restriction an element of F_m. By König's lemma, there exists an increasing sequence $(D_n)_{n\geq k} \in \times_{n\geq k} F_n$; $\cup_{n\in\mathbb{N}} D_n$ is a covering of R by a half curve with endpoint y_k^1.

Conversely, suppose that R has 2 coverings C, D. For each $n \in \mathbb{N}$, write $C_n = C \restriction P_n$ and $D_n = D \restriction P_n$. Consider $k \geq h$ such that $C_k \neq D_k$.

For $n \geq k$, we have $C_n \neq D_n$. As $C_n = C_{n+1} \restriction P_n$ and $D_n = D_{n+1} \restriction P_n$, it follows from Lemma 3.4 applied to P_n and P_{n+1} that y_n^1 is a vertex of P_{n+1} and that one of the curves C_n, D_n connects y_n^1 and y_n^2, while the other one connects y_n^2 and y_n^3.

Consequently, we have $y_n^1 = y_k^1$ for $n \geq k$, one of the curves C, D is a complete curve, the other one is a half curve with endpoint y_k^1 and there is only one possibility for each of them. ■

Now we state and prove our main results. As above, we only consider curves with segments in W.

Theorem 3.6. For each Λ, we consider the coverings of the sets $\mathcal{P}_{X\Lambda}$ by sets of nonoriented complete curves:
1) Suppose that $\mathcal{P}_{X\Lambda}$ has 1 region. Then each covering of $\mathcal{P}_{X\Lambda}$ consists of 1 curve. If X is ultimately constant, then $\mathcal{P}_{X\Lambda}$ has 3 coverings, obtained from one another by rotations of angles $\pm 2\pi/3$. Otherwise, $\mathcal{P}_{X\Lambda}$ has 1 or 2 coverings, and each case is realized for 2^ω values of X.
2) If $\mathcal{P}_{X\Lambda}$ has 2 regions, then $\mathcal{P}_{X\Lambda}$ has 1 covering. It consists of 2 curves. This case is realized for 2^ω values of X.
3) If $\mathcal{P}_{X\Lambda}$ has 3 regions and if their common point y does not belong to W, then $\mathcal{P}_{X\Lambda}$ has 1 covering. It consists of 3 curves obtained from one another by rotations of center y and angles $\pm 2\pi/3$.
4) If $\mathcal{P}_{X\Lambda}$ has 3 regions and if their common point y belongs to W, then $\mathcal{P}_{X\Lambda}$ has 3 coverings obtained from one another by rotations of center y and angles $\pm 2\pi/3$. Each covering of $\mathcal{P}_{X\Lambda}$ consists of 2 curves. One of them is a covering

of a region. The other one is the union of 2 half curves with endpoint y, which are equivalent modulo a rotation of center y and angle $2\pi/3$; each half curve is a covering of a region.

Theorem 3.6 will be proved after the remarks and the example below, which concern nonoriented curves:

Remark. It follows from Theorem 3.6 that, for each region $R \subsetneq \mathbb{R}^2$ of some $\mathcal{P}_{X\Lambda}$, the covering of R by a complete curve can be extended into 1 covering of $\mathcal{P}_{X\Lambda}$. The covering of R by a half curve, if it exists, can be extended into 2 coverings of $\mathcal{P}_{X\Lambda}$ by complete curves, which are equivalent modulo a rotation of angle $2\pi/3$.

Remark. For each Λ, as each $\mathcal{P}_{X\Lambda}$ has finitely many coverings, Proposition 3.1 implies that each isomorphism class of coverings of sets $\mathcal{P}_{X\Lambda}$ is countable. Consequently, it follows from Theorem 3.6 that we have 2^ω isomorphism classes of coverings for case 2) and 2^ω classes for case 1) with X not ultimately constant. On the other hand, Proposition 3.1 and Theorem 3.6 imply that we have 1 isomorphism class for case 3), 3 classes for case 4) and 3 classes for case 1) with X ultimately constant.

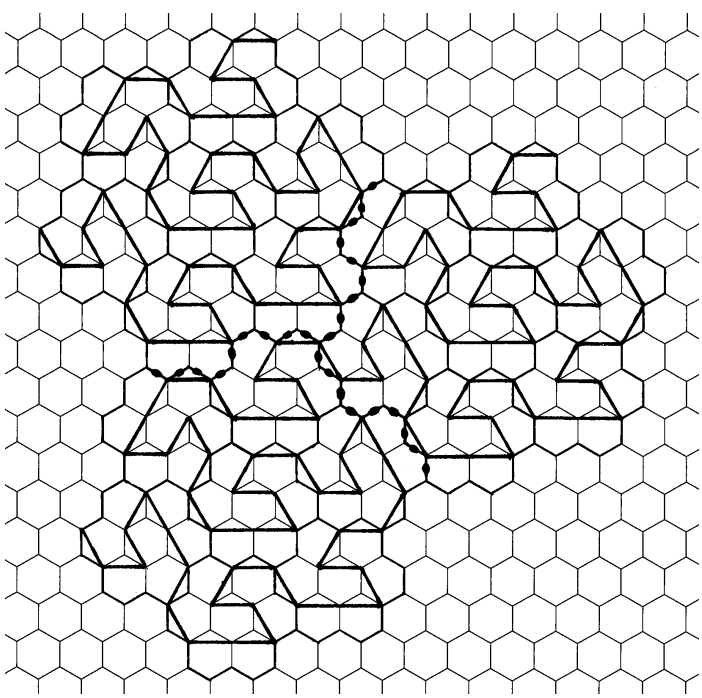

Figure 3.4

Example. For each $n \in \mathbb{N}^*$, denote by C_n the Peano-Gosper curve associated to the sequence T_n. Consider the inductive limit B (resp. C, D), defined up

to rotation, of the curves C_n with each C_n embedded in C_{n+1} by identifying T_n with its fourth (resp. third, first) copy in
$T_{n+1} = (T_n, +1, \overline{T_n}, +1, \overline{T_n}, -1, T_n, -1, T_n, +1, T_n, -1, \overline{T_n})$.
Then there exist: 1) a covering of $\mathbb{R}^2$ by 3 copies of the complete curve B; 2) a covering of $\mathbb{R}^2$ by a copy of the complete curve C, and 2 copies of the half curve D which form a complete curve (see Figure 3.4 above).

Proof of Theorem 3.6. We write $X = (x_n)_{n\in\mathbb{N}}$. For each $n \in \mathbb{N}$, we consider the tile $P_n \in \mathcal{P}^n_{X\Lambda}$ which contains x_n.
Proof of 1). We have $\mathbb{R}^2 = \cup_{n\in\mathbb{N}} P_n$. Each covering of $\mathcal{P}_{X\Lambda}$ consists of 1 complete curve by Proposition 3.5.

First we suppose that there exists h such that $x_n = x_h$ for each $n \geq h$. Then, by Proposition 3.2, for $m \geq h$, the 3 coverings of P_m are obtained from one of them by rotations of center x_h and angles $2k\pi/3$, and they are the restrictions of the 3 coverings of P_n for each $n \geq m$. It follows that $\mathcal{P}_{X\Lambda}$ has exactly 3 coverings, which are the inductive limits of the coverings of the tiles P_n for $n \geq h$, and therefore obtained from one of them by rotations of center x_h and angles $2k\pi/3$.

From now on, we suppose X not ultimately constant. First we observe that $\mathcal{P}_{X\Lambda}$ cannot have 3 coverings. Otherwise, for n large enough, these 3 coverings would give by restriction 3 distinct coverings of P_n, which is not possible when $x_{n+1} \neq x_n$ since, by Lemma 3.4, at most 2 coverings of P_n extend into coverings of P_{n+1}.

By Lemma 3.4, for each $n \in \mathbb{N}$ and any coverings $A \neq B$ of P_n, we have 2 coverings of P_{n+1} which extend A and B in 2 cases: first if P_n contains the center of P_{n+1}, second if P_n does not contain the center of P_{n+1}, if the common endpoint of A and B belongs to exactly 2 tiles of $\mathcal{P}^n_{X\Lambda}$ contained in P_{n+1}, and if the other endpoint of A or the other endpoint of B is a vertex of P_{n+1}. The first case is realized for $x_{n+1} = x_n$ and the second one for another choice of x_{n+1}.

It follows that, for any coverings $C \neq D$ of some $P \in \mathcal{P}$ and each $K \subset \mathbb{N}$ with K and $\mathbb{N} - K$ infinite, we can choose $X = (x_n)_{n\in\mathbb{N}}$, with x_0 center of P and $x_{n+1} = x_n$ if and only if $n \in K$, so that there exist some increasing sequences $(C_n)_{n\in\mathbb{N}}$, $(D_n)_{n\in\mathbb{N}}$ with $C_0 = C$, $D_0 = D$ and C_n, D_n coverings of P_n for each $n \in \mathbb{N}$. For each $n \in \mathbb{N} - K$, supposing that x_n, C_n, D_n are already defined, we choose x_{n+1} in such a way that the second case above is realized for $A = C_n$ and $B = D_n$. Then $\mathcal{P}_{X\Lambda}$ only has 1 region and $\cup_{n\in\mathbb{N}} C_n$, $\cup_{n\in\mathbb{N}} D_n$ are the only coverings of $\mathcal{P}_{X\Lambda}$.

By Lemma 3.4, for each $n \in \mathbb{N}$, if $x_{n+1} \neq x_n$ and if the point of P_n which is a vertex of P_{n+1} does not belong to W, then the 3 coverings of P_{n+1} give by restriction the same covering of P_n, so that only 1 covering of P_n is the restriction of a covering of $\mathcal{P}_{X\Lambda}$. There are 2^ω different ways to choose X so

that this property is true for infinitely many integers n, and so that $x_{n+1} = x_n$ is also true for infinitely many integers n. Then $\mathcal{P}_{X\Lambda}$ only has 1 covering and 1 region.

Proof of 2), 3), 4). Suppose that $\mathcal{P}_{X\Lambda}$ has 2 or 3 regions. Write $R = \cup_{n\in\mathbb{N}} P_n$.

If R is not the union of an increasing sequence of tiles with 1 common vertex, then $\mathcal{P}_{X\Lambda}$ has 2 regions and the second region satisfies the same property. By Proposition 3.5, each region has a unique covering and it is a complete curve. The union of these coverings is the unique covering of $\mathcal{P}_{X\Lambda}$.

This case is realized if:

a) there exists a sequence $(\Sigma_n)_{n\in\mathbb{N}}$ with Σ_n side of P_n and $\Sigma_n \subset \Sigma_{n+1}$ for each $n \in \mathbb{N}$;

b) there is no common vertex of the tiles P_n for n large.

For each $n \in \mathbb{N}$ and each choice of $x_1, \ldots, x_n$ compatible with a), there are 3 choices of x_{n+1} compatible with a). Consequently, there are 2^ω sequences X which satisfy a), and also 2^ω sequences which satisfy a) and b) since countably many sequences do not satisfy b).

If R is the union of an increasing sequence of tiles with one common vertex y, then $\mathcal{P}_{X\Lambda}$ has 3 regions R_1, R_2, R_3 obtained from R by rotations of center y and angles $2k\pi/3$.

If y does not belong to W, then, by Proposition 3.5, each R_i has a unique covering. These coverings are complete curves obtained from one of them by rotations of center y and angles $2k\pi/3$. Their union is the unique covering of $\mathcal{P}_{X\Lambda}$.

If y belongs to W, then, by Proposition 3.5, each R_i has 1 covering by a complete curve and 1 covering by a half curve with endpoint y. Each of the 3 coverings of $\mathcal{P}_{X\Lambda}$ is obtained by taking a covering by a complete curve for 1 region and 2 coverings by half curves for the 2 other regions. The 2 half curves are equivalent modulo a rotation of center y and angle $2\pi/3$. They form a complete curve since y is their common endpoint. ■

Now, for each set $\mathcal{C}$ of oriented curves, we consider the following property:

(P) If 2 segments of curves of $\mathcal{C}$ are opposite sides of a rhombus, then they have opposite orientations.

We observe that (P) is satisfied if $\mathcal{C}$ consists of 1 complete curve or 1 curve which is a covering of a tile of some $\mathcal{P}_{X\Lambda}$.

Theorem 3.7. The local isomorphism property implies (P) for coverings by sets of oriented curves. The strong local isomorphism property is true for the coverings of sets $\mathcal{P}_{X\Lambda}$ by sets of nonoriented curves or by sets of oriented curves which satisfy (P).

Remark. Theorem 3.7 implies that each covering of some $\mathcal{P}_{X\Lambda}$ by a set of

oriented curves satisfies the local isomorphism property or can be transformed into a covering which satisfies that property by changing the orientation of one of the curves.

Theorem 3.8. For each Λ and any X, Y, any coverings of $\mathcal{P}_{X\Lambda}$ and $\mathcal{P}_{Y\Lambda}$ by sets of nonoriented curves are locally isomorphic. Any coverings of $\mathcal{P}_{X\Lambda}$ and $\mathcal{P}_{Y\Lambda}$ by sets of oriented curves are locally isomorphic if they satisfy (P).

Remark. It follows from Theorem 3.6 that each local isomorphism class of coverings by sets of nonoriented curves associated to a sequence Λ contains 2^ω (resp. 2^ω, 1, 0) isomorphism classes of coverings by 1 (resp. 2, 3, ≥ 4) curves.

Proof of Theorem 3.7 and Theorem 3.8. It suffices to prove the results for oriented curves since they imply the results for nonoriented curves.

First suppose that a covering $\mathcal{C}$ of some $\mathcal{P}_{X\Lambda}$ satisfies the local isomorphism property. Consider 2 segments of curves of $\mathcal{C}$ which are opposite sides of a rhombus A. Then there exist a translation τ and a tile T of $\mathcal{P}_{X\Lambda}$ such that $\tau(A)$ is contained in T and $\tau(A \cap \mathcal{C}) = \tau(A) \cap C$, where C is the curve of $\mathcal{C}$ which contains a coverig of T. The 2 segments have opposite orientations because their images through τ, which belong to the same curve, necessarily have opposite orientations.

Now, for each X such that $\mathcal{P}_{X\Lambda}$ exists, each $n \in \mathbb{N}$ and each $x \in \mathbb{R}^2$, we consider the tile $P^n_{Xx} \in \mathcal{P}^n_{X\Lambda}$ with center x if it exists, and the union Q^n_{Xx} of the 3 tiles with common vertex x belonging to $\mathcal{P}^n_{X\Lambda}$ if they exist. We are going to prove the following property, which implies the other statements of Theorem 3.7 and Theorem 3.8:
For any sets X, Y such that $\mathcal{P}_{X\Lambda}, \mathcal{P}_{Y\Lambda}$ exist, for any coverings $\mathcal{C}, \mathcal{D}$ of $\mathcal{P}_{X\Lambda}, \mathcal{P}_{Y\Lambda}$ by sets of curves which satisfy (P), for each $n \in \mathbb{N}$ and for each $x \in \mathbb{R}^2$ such that Q^n_{Xx} exists, each P^{n+4}_{Yy} contains some Q^n_{Yz} with $\mathcal{D} \restriction Q^n_{Yz} \cong \mathcal{C} \restriction Q^n_{Xx}$.

By Corollary 3.3, for any $y, z \in \mathbb{R}^2$ such that P^{n+2}_{Yy} and P^{n+4}_{Yz} exist, each covering of P^{n+4}_{Yz} contains copies of the 6 coverings of P^{n+2}_{Yy}. Consequently, it suffices to show that, for each $x \in \mathbb{R}^2$ such that Q^n_{Xx} exists, each P^{n+2}_{Yy} contains some Q^n_{Yz} such that $\mathcal{C} \restriction Q^n_{Xx}$ and $\mathcal{D} \restriction Q^n_{Yz}$ are equivalent modulo a rotation of angle $2k\pi/3$ and/or changing the orientation of all the curves.

We write $\Lambda = (\lambda_k)_{k\in\mathbb{N}^*}$, $X = (x_k)_{k\in\mathbb{N}}$ and $Y = (y_k)_{k\in\mathbb{N}}$. We have $\Delta_{x_n\lambda_1\cdots\lambda_n}(\mathcal{C}) \restriction \Delta_{x_n\lambda_1\cdots\lambda_n}(P) = \Delta_{x_n\lambda_1\cdots\lambda_n}(\mathcal{C} \restriction P)$ for each $P \in \mathcal{P}^n_{X\Lambda}$ and $\Delta_{y_n\lambda_1\cdots\lambda_n}(\mathcal{D}) \restriction \Delta_{y_n\lambda_1\cdots\lambda_n}(P) = \Delta_{y_n\lambda_1\cdots\lambda_n}(\mathcal{D} \restriction P)$ for each $P \in \mathcal{P}^n_{Y\Lambda}$.

For any $x, y \in \mathbb{R}^2$, $\mathcal{C} \restriction Q^n_{Xx}$ and $\mathcal{D} \restriction Q^n_{Yy}$ are equivalent modulo a translation, or a rotation of angle $2k\pi/3$, or changing the orientation of the curves, or a combination of these operations, if and only if the same property is true for $\Delta_{x_n\lambda_1\cdots\lambda_n}(\mathcal{C}) \restriction \Delta_{x_n\lambda_1\cdots\lambda_n}(Q^n_{Xx})$ and $\Delta_{y_n\lambda_1\cdots\lambda_n}(\mathcal{D}) \restriction \Delta_{y_n\lambda_1\cdots\lambda_n}(Q^n_{Yy})$. Consequently, it suffices to prove the statement for $n = 0$.

Figure 3.5 below shows representatives of the 9 equivalence classes of possible configurations for $\mathcal{C} \restriction Q^0_{Xx}$, modulo the operations mentioned just above. We see from Figure 3.3 that, for each curve C which is a covering of some P^2_{Yy}, each of these classes is realized by $C \restriction Q^0_{Yz}$ for some $Q^0_{Yz} \subset P^2_{Yy}$. ■

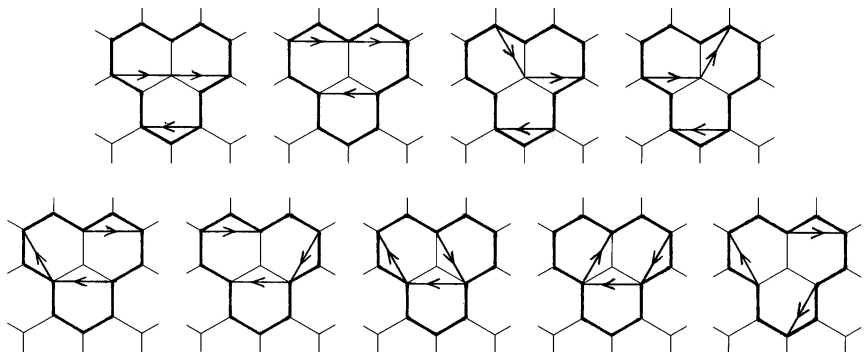

Figure 3.5

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