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Abstract

In this paper, we investigate the functional central limit theorem and the Marcinkiewicz strong law of large numbers for U-statistics having absolutely regular data and taking value in a separable Hilbert space. The novelty of our approach consists in using coupling in order to formulate a deviation inequality for original U-statistic, where the upper bound involves the mixing coefficient and the tail of several U-statistics of i.i.d. data. The presented results improve the known results in several directions: the case of metric space valued data is considered as well as Hilbert space valued, and the mixing rates are less restrictive in a wide range of parameters.

1 Main results

In all this paper, we will consider U-statistics of order two taking values in a separable Hilbert space $\mathbb{H}$ defined as follows: given a sequence of random variables $(X_i)_{i\geqslant 1}$ taking values in a separable metric space (S,d) and $h\colon S\times S\to\mathbb{H}$ is measurable,

$$U_n(h)=\sum_{1\leqslant i<j\leqslant n} h(X_i,X_j). \tag{1.1}$$

Halmos [19] and Hoeffding [20] showed that when $(X_i)_{i\geqslant 1}$ is i.i.d., $\mathbb{H}=\mathbb{R}$ and $\mathbb{E}[|h(X_1,X_2)|]<\infty$, $U_n(h)/\binom{n}{2}$ is a consistent estimator of $\mathbb{E}[h(X_1,X_2)]$. This results has been then extended to Hilbert valued case by Borovskikh in [7]. Asymptotic normality has also been established under the assumption $\mathbb{E}\left[h(X_1,X_2)^2\right]<\infty$ in [20]. Convergence rates in the central limit theorem for Hilbert space valued U-statistics were also considered in [26].

This paper is devoted to the obtention of a functional central central limit theorem and Marcinkiewicz strong law of large numbers for U-statistics whose data is a strictly stationary sequence and taking values in a separable Hilbert space. The motivation behind the consideration of vector-valued U-statistics is to consider spatial sign for robust tests (see [10, 30, 23]), or Wilcoxon-Mann-Whitney-type test [9]. Moreover, the assumption that the random variables X_i take values in a metric space instead of the real line is useful in order to consider function space analogue of Gini's mean difference (see Section 3.2 in [28]), or correlation dimension for metric space valued data (cf. [29]). Moreover, high dimensional or

functional data can be treated via the use of Hilbert space valued U-statistics, see for instance [8, 21]. Kendall's tau for functional data can be also treated via the use of Hilbert value U-statistics, see [22]. We refer the reader to the book [6] for a complete description of U-statistics taking values in Hilbert spaces.

The classical strong law of large numbers for U-statistics of strictly stationary data without dependence condition has been considered in [1, 4, 15]. A strong law of large numbers has been established in [17] for U-statistics of arbitrary order whose data comes from a Markov chain. Arcones [2] showed a law of large numbers for U-statistics of order m, where the normalisation is n^m. A similar result has been obtained for $2m$-wise independent sequences, that is, sequences $(X_i)_{i\geqslant 1}$ such that for each $i_1 < \cdots < i_{2m}$, the random variables $X_{i_1}, \ldots, X_{i_{2m}}$ are independent. Related results on V-functionals of α-mixing data were estalished in [32]. In [18], the law of large number for U-statistics of order two whose data comes from a function of an i.i.d. sequence has been investigated.

In order to study the asymptotic behavior of $U_n(h)$, we decompose the kernel as follows: take an independent copy X_1' of X_1 and let

$$h_{1,0}(x) = \mathbb{E}\left[h\left(x, X_1'\right)\right] - \mathbb{E}\left[h\left(X_1, X_1'\right)\right] \tag{1.2}$$

$$h_{0,1}(y) = \mathbb{E}\left[h\left(X_1, y\right)\right] - \mathbb{E}\left[h\left(X_1, X_1'\right)\right] \tag{1.3}$$

and

$$h_2(x,y) = h(x,y) - h_{1,0}(x) - h_{0,1}(y) - \mathbb{E}\left[h\left(X_1, X_1'\right)\right]. \tag{1.4}$$

In this way,

$$U_n(h) = \sum_{i=1}^{n-1} (n-i)\left(h_{1,0}(X_i) - \mathbb{E}\left[h_{1,0}(X_i)\right]\right) + \sum_{j=2}^{n} (j-1)\left(h_{0,1}(X_j) - \mathbb{E}\left[h_{0,1}(X_j)\right]\right) + U_n(h_2) \tag{1.5}$$

and when $(X_i)_{i\geqslant 1}$ is i.i.d. , one has

$$\mathbb{E}\left[h_2(X_i, X_j) \mid \sigma(X_k, k \leqslant j-1)\right] = 0 = \mathbb{E}\left[h_2(X_i, X_j) \mid \sigma(X_k, k \geqslant i+1)\right] \tag{1.6}$$

hence martingale and reversed martingale properties can be used in order to control moments of the sum over j and i respectively.

It is also worth pointing out that when h is symmetric, that is, $h(x,y) = h(y,x)$ for each $x, y \in S$, the functions $h_{1,0}$ and $h_{0,1}$ coincide hence (1.5) admits the simpler form

$$U_n(h) = n\sum_{k=1}^{n} \left(h_{1,0}(X_k) - \mathbb{E}\left[h_{1,0}(X_k)\right]\right) + U_n(h_2). \tag{1.7}$$

The dependence in our results will be quantified by the so-called mixing coefficients. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. The α-mixing and β-mixing coefficients between two sub-σ-algebras $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{F}$ are defined respectively by

$$\alpha(\mathcal{A}, \mathcal{B}) = \sup\left\{\left|\mathbb{P}(A \cap B) - \mathbb{P}(A)\,\mathbb{P}(B)\right|, A \in \mathcal{A}, B \in \mathcal{B}\right\}; \tag{1.8}$$

$$\beta(\mathcal{A}, \mathcal{B}) = \frac{1}{2}\sup\left\{\sum_{i=1}^{I}\sum_{j=1}^{J}\left|\mathbb{P}(A_i \cap B_j) - \mathbb{P}(A_i)\,\mathbb{P}(B_j)\right|\right\}, \tag{1.9}$$

where the supremum runs over all the partitions $(A_i)_{i=1}^I$ and $(B_j)_{j=1}^J$ of Ω of elements of $\mathcal{A}$ and $\mathcal{B}$ respectively. Given a strictly stationary sequence $(X_i)_{i\geqslant 1}$, we associate its sequences of α and β-mixing coefficients by letting

$$\alpha(k) := \sup_{\ell\geqslant 1} \alpha\left(\mathcal{F}_1^\ell, \mathcal{F}_{\ell+k}^\infty\right), \tag{1.10}$$

$$\beta(k) := \sup_{\ell\geqslant 1} \beta\left(\mathcal{F}_1^\ell, \mathcal{F}_{\ell+k}^\infty\right), \tag{1.11}$$

where $\mathcal{F}_u^v$, $1 \leqslant u \leqslant v \leqslant +\infty$ is the σ-algebra generated by the random variables X_i, $u \leqslant i \leqslant v$ ($u \leqslant i$ for $v = \infty$). A sequence $(X_i)_{i\geqslant 1}$ is said to be absolutely regular if $\lim_{k\to\infty} \beta(k) = 0$.

The paper is organized as follows: in Subsection 1.1, we will formulate a result for a partial sum process built on a U-statistic off absolutely regular data and in Subsection 1.2, results on the strong law of large numbers, distinguishing the degenerated and non-degenerated cases. The proofs are given in Section 2 and are a consequence of an inequality given in Subsection 2.1.

1.1 Functional central limit theorem

Define the process

$$\mathcal{U}_{n,h}(t) = \sum_{1\leqslant i<j\leqslant \lfloor nt\rfloor} h(X_i, X_j) + (nt - \lfloor nt\rfloor) \sum_{i=1}^{\lfloor nt\rfloor} h\left(X_i, X_{\lfloor nt\rfloor+1}\right). \tag{1.12}$$

Notice that $\mathcal{U}_{n,h}(k/n) = U_k(h)$ hence the process $\mathcal{U}_{n,h}$ contains the information of all the values of $U_k(h)$, $2 \leqslant k \leqslant n$. Moreover, for each $\omega \in \Omega$, the map $t \mapsto \mathcal{U}_{n,h}(t)$ belongs to $C_{\mathbb{H}}[0,1]$, the space of $\mathbb{H}$-valued continuous functions defined on the unit interval endowed with the norm $\|x\|_\infty = \sup_{t\in[0,1]} \|x(t)\|_{\mathbb{H}}$, because we interpolate linearly between the points $(k/n, U_k(h))$.

The limiting process will be described as follows.

Definition 1.1. *We say that a non-negative self-adjoint operator $\Gamma\colon \mathbb{H} \to \mathbb{H}$ is an $\mathcal{S}(\mathbb{H})$ operator if for some Hilbert basis $(e_i)_{i\geqslant 1}$, $\sum_{i=1}^\infty \langle \Gamma e_i, e_i\rangle_{\mathbb{H}} < \infty$.*

Definition 1.2. *Let $\pi_t\colon C_{\mathbb{H}}[0,1]$ be the projection map, that is, $\pi_t(x) = x(t)$. For $\Gamma \in \mathcal{S}(\mathbb{H})$, denote by W_Γ the process such that*

1. *$W_\Gamma(0) = 0$,*
2. *for all $0 \leqslant t_0 < t_1 < \cdots < t_N \leqslant 1$, the increments $(W(t_i) - W(t_{i-1}))_{i=0}^N$ are independent and $W(t_i) - W(t_{i-1})$ has a Gaussian distribution on $\mathbb{H}$ with mean zero and covariance operator $(t_i - t_{t-1})\Gamma$.*

We are now in position to state our result on functional central limit theorem for U-statistics of absolutely regular data taking values in a separable Hilbert space.

Theorem 1.3. *Let $(X_i)_{i\in\mathbb{Z}}$ be a strictly stationary sequence of random variables taking values in a separable metric space (S,d) and let $h\colon S\times S \to \mathbb{H}$ be a symmetric measurable function, where $S\times S$ is endowed with the product σ-algebra and the Hilbert space $(\mathbb{H}, \langle\cdot,\cdot\rangle_{\mathbb{H}})$ with the σ-algebra induced by the norm. Suppose that the following conditions are satisfied:*

(C.1) the following series is finite:

$$\sum_{k=1}^{\infty}\int_{0}^{\alpha(\sigma(X_i,i\leqslant 0),\sigma(X_k))} Q^2_{\|h_1(X_0)\|_{\mathbb{H}}}(u)\,du<\infty, \tag{1.13}$$

where $h_1(x)=\mathbb{E}[h(x,X_1)]-\mathbb{E}[h(X_1,X_1')]$ *and* X_1' *is an independent copy of* X_1,

(C.2) $\lim_{n\to\infty} n^2\beta(n)=0$,

(C.3) $\sup_{j\geqslant 2}\mathbb{E}\left[\|h(X_1,X_j)\|_{\mathbb{H}}\right]<\infty$.

Then the following convergence in distribution in $C_{\mathbb{H}}[0,1]$ *takes place:*

$$\frac{1}{n^{3/2}}\left(\mathcal{U}_{n,h}(t)-\mathbb{E}[\mathcal{U}_{n,h}(t)]\right)\to W_{\Gamma}, \tag{1.14}$$

where the operator Γ *is given by*

$$\langle \Gamma u,v\rangle_{\mathbb{H}}=\sum_{k\in\mathbb{Z}}\mathbb{E}\left[\langle h_1(X_0),u\rangle_{\mathbb{H}}\langle h_1(X_k),v\rangle_{\mathbb{H}}\right],\quad u,v\in\mathbb{H}. \tag{1.15}$$

Notice that our result also applies when $\mathbb{E}[h(X_1,X_1')\mid X_1]=\mathbb{E}[h(X_1,X_1')\mid X_1']=0$ almost surely, in which case the limiting process W_Γ is zero because so is Γ. In this case, the appropriated normalization is n^{-1} instead of $n^{-3/2}$ and completely different techniques have to be used, like in [24].

Let us compare this result with existing ones in the literature. Yoshihara [31] obtained a similar result for real-valued U-statistics, but at the cost of a more restrictive assumption on moments and the decay of mixing coefficients. Moreover, we address the Hilbert-valued case.

Dehling and Wendler [14] obtained a central limit theorem assuming the existence of a positive δ such that $\sup_{j\geqslant 2}\mathbb{E}\left[\|h(X_1,X_j)\|_{\mathbb{H}}^{2+\delta}\right]<\infty$ and that for some positive C,η, the β-mixing coefficient satisfies $\beta(n)\leqslant Cn^{-1-2/\delta-\eta}$. When $\delta<2$, our condition is less restrictive but our approach does not give better a weaker condition than (C.2) when we assume moments of order higher than two. The same authors obtained in [13] a central limit theorem for Hilbert-valued U-statistics whose date comes from functionals of an absolutely regular sequence. In [5], the more general case of functions of absolutely regular processes has been addressed but when restricted to β-mixing case, the obtained result is not better than ours.

Finally, let us mention that Theorem 1.3 may be applied even if there is no positive δ for which $\mathbb{E}\left[\|h_1(X_0)\|_{\mathbb{H}}^{2+\delta}\right]$ is finite. For instance, if we merely have $\mathbb{E}\left[\|h_1(X_0)\|_{\mathbb{H}}^2(\log(1+\|h_1(X_0)\|_{\mathbb{H}}))^{\gamma}\right]<\infty$, then (1.13) may be satisfied, but at the cost of the existence of constants C and $a\in(0,1)$ such that $\alpha(\sigma(X_i,i\leqslant 0),\sigma(X_k))\leqslant Ca^k$. We refer the reader to [27], pages 155-158.

1.2 Marcinkiewicz strong law of large numbers

In all the results on the Marcinkiewicz strong law of large numbers, a supplementary moment condition with respect to the i.i.d. case will be required. For $1<p<2$ and $\delta>0$, we will consider the assumption

$$\mathbb{E}\left[\left\|h(X_1,X_1')\right\|_{\mathbb{H}}^{p+\delta}\right]<\infty, \tag{1.16}$$

where X_1' is an independent copy of X_1. Since the derivation of moment inequalities was done via the use of moment inequalities for martingale, it will be natural to distinguish between the cases where the

exponent $p+\delta$ is bigger than 2 or not. An other reasonable assumption is boundedness in $\mathbb{L}^1$ of the summands which compose a U-statistic, namely,

$$\sup_{j\geqslant 2}\mathbb{E}\left[\left\|h\left(X_1,X_j\right)\right\|_{\mathbb{H}}\right]<\infty. \tag{1.17}$$

Theorem 1.4 (Law of large numbers, non-degenerated case, $p+\delta\geqslant 2$)**.** *Let $(X_i)_{i\in\mathbb{Z}}$ be a strictly stationary sequence of random variables taking values in a separable metric space (S,d) and let $h\colon S\times S\to\mathbb{H}$ be a measurable function, where $S\times S$ is endowed with the product σ-algebra and the Hilbert space $\mathbb{H}$ with the σ-algebra induced by the norm. Let $1<p<2$. Suppose that (1.16) and (1.17) hold with $p+\delta\geqslant 2$ and the following condition is satisfied: there exists a positive η such that*

$$\sum_{k=1}^{\infty}k^{\gamma(p,\delta,\eta)}<\infty, \tag{1.18}$$

where

$$\gamma\left(p,\delta,\eta\right)=\max\left\{p-1+\eta,p-2+\frac{p\left(p-1\right)}{\delta}\right\}. \tag{1.19}$$

Then the following convergence takes place:

$$\lim_{n\to\infty}\frac{1}{n^{1+1/p}}\left\|U_n\left(h\right)\right\|_{\mathbb{H}}=0\ \ a.s.. \tag{1.20}$$

Theorem 1.5 (Law of large numbers, non-degenerated case, $p+\delta<2$)**.** *Let $(X_i)_{i\in\mathbb{Z}}$ be a strictly stationary sequence of random variables taking values in a separable metric space (S,d) and let $h\colon S\times S\to\mathbb{H}$ be a measurable function, where $S\times S$ is endowed with the product σ-algebra and the Hilbert space $\mathbb{H}$ with the σ-algebra induced by the norm. Let $1<p<2$. Suppose that (1.16) and (1.17) hold with $p+\delta<2$ and that*

$$\sum_{k=1}^{\infty}k^{\gamma(p,\delta)}\beta\left(k\right)<\infty, \tag{1.21}$$

where

$$\gamma\left(p,\delta\right)=\max\left\{p-2+\frac{p\left(p-1\right)}{\delta},\frac{p\left(p-1\right)+\left(p-1\right)\delta}{p\left(p-1\right)+\left(p+1\right)\delta}\right\} \tag{1.22}$$

Then the following convergence takes place:

$$\lim_{n\to\infty}\frac{1}{n^{1+1/p}}\left\|U_n\left(h\right)\right\|_{\mathbb{H}}=0\ \ a.s.. \tag{1.23}$$

Dehling and Sharipov [16] obtained also the convergence (1.23), but with slightly different assumptions on the β-mixing coefficients, namely,

$$\sum_{k=1}^{\infty}k^{\gamma'(p,\delta)}\beta\left(k\right)<\infty, \tag{1.24}$$

where

$$\gamma'\left(p,\delta\right)=\max\left\{p-2+\frac{p\left(p-1\right)}{\delta},1\right\}. \tag{1.25}$$

Since $\gamma'\left(p,\delta\right)\geqslant\max\left\{\gamma\left(p,\delta\right),\gamma\left(p,\delta,\eta\right)\right\}$, our assumption is always equaly or less restrictive. It is also worth pointing out that we do not need symmetry of the kernel. Moreover, we can also treat Hilbert space valued kernels. These extensions allows us to consider kernels of the form

$$h\colon\mathbb{H}\times\mathbb{H}\to\mathbb{H},\quad h\left(x,y\right)=\begin{cases}\frac{x-y}{\left\|x-y\right\|_{\mathbb{H}}} & \text{if } x\neq y,\\ 0 & \text{if } x=y.,\end{cases} \tag{1.26}$$

which played an important role in [10, 30].

We now continue the presentation of the results in order to address the degenerated case. Like in the independent case, the appropriated normalisation is $n^{2/p}$.

Theorem 1.6 (Law of large numbers, degenerated case, $p+\delta=2$)**.** *Let $(X_i)_{i\in\mathbb{Z}}$ be a strictly stationary sequence of random variables taking values in a separable metric space (S,d) and let $h\colon S\times S\to\mathbb{H}$ be a measurable function, where $S\times S$ is endowed with the product σ-algebra and the Hilbert space $\mathbb{H}$ with the σ-algebra induced by the norm. Suppose that*

$$\mathbb{E}\left[h\left(X_1,X_1'\right)\mid X_1\right]=\mathbb{E}\left[h\left(X_1,X_1'\right)\mid X_1'\right]=0. \tag{1.27}$$

Let $1<p<2$. Assume that (1.16) and (1.17) hold with $p+\delta\geqslant 2$ and that there exists some positive η such that

$$\sum_{k=1}^{\infty}k^{\frac{2(p-1)}{2-p}+\eta}\beta\left(k\right)<\infty. \tag{1.28}$$

Then the following convergence takes place:

$$\lim_{n\to\infty}\frac{1}{n^{2/p}}\left\|U_n\left(h\right)\right\|_{\mathbb{H}}=0\ \ a.s.. \tag{1.29}$$

Note that in the previous result, we only assume the existence of moments of order two. In our approach, moments of higher order will not help to find a weaker condition on the decay of the β-mixing coefficients.

Theorem 1.7 (Law of large numbers, degenerated case, $p+\delta<2$)**.** *Let $(X_i)_{i\in\mathbb{Z}}$ be a strictly stationary sequence of random variables taking values in a separable metric space (S,d) and let $h\colon S\times S\to\mathbb{H}$ be a measurable function, where $S\times S$ is endowed with the product σ-algebra and the Hilbert space $\mathbb{H}$ with the σ-algebra induced by the norm. Suppose that*

$$\mathbb{E}\left[h\left(X_1,X_1'\right)\mid X_1\right]=\mathbb{E}\left[h\left(X_1,X_1'\right)\mid X_1'\right]=0. \tag{1.30}$$

Let $1<p<2$. Assume that (1.16) and (1.17) hold with $p+\delta<2$ and

$$\sum_{k=1}^{\infty}k^{p-1+\frac{p(p-1)}{\delta}}\beta\left(k\right)<\infty. \tag{1.31}$$

Then the following convergence takes place:

$$\lim_{n\to\infty}\frac{1}{n^{2/p}}\left\|U_n\left(h\right)\right\|_{\mathbb{H}}=0\ \ a.s.. \tag{1.32}$$

These two theorems complement the ones obtained in [16], where the degenerated case was considered, but only in the case of a bounded real valued kernel.

2 Proofs

2.1 A general deviation inequality

In this subsection, we give a bound for the maximum of a U-statistic of strictly stationary data in terms of U-statistics of i.i.d. data and partial sums of sequences of random variables.

Proposition 2.1. *Let $N > 2q \geqslant 1$ be integers, let $R, x > 0$, let $(X_i)_{i\geqslant 1}$ be a strictly stationary sequence of random variables taking values in a separable metric space (S, d) and let $h\colon S^2 \to \mathbb{H}$ be a measurable function, where $(\mathbb{H}, \langle , \rangle)$ is a separable Hilbert space, such that*

$$\mathbb{E}\left[h\left(X_1, X_1'\right) \mid X_1\right] = \mathbb{E}\left[h\left(X_1, X_1'\right) \mid X_1'\right] = 0, \tag{2.1}$$

where X_1' be an independent copy of X_1. Define $H := \|h(X_1, X_1')\|_{\mathbb{H}}$. For each $r \geqslant 2$, the following inequality takes place

$$\begin{aligned}\mathbb{P}\left(\max_{2\leqslant n\leqslant N}\left\|\sum_{1\leqslant i<j\leqslant n} h(X_i, X_j)\right\|_{\mathbb{H}} > x\right) &\leqslant C_r x^{-r} q^r N^r \mathbb{E}\left[H^r \mathbf{1}_{H\leqslant R}\right] \\ &\quad + C_r x^{-1} N^2 \mathbb{E}\left[H \mathbf{1}_{H>R}\right] + C_r x^{-1} q N \sup_{j\geqslant 2} \mathbb{E}\left[\|h(X_1, X_j)\|_{\mathbb{H}}\right] + 4N\beta(q), \end{aligned} \tag{2.2}$$

where the constant C_r depends only on r.

Note that when H has a finite moment of order r, one can simply let R going to infinity, which gives the simpler inequality

$$\begin{aligned}\mathbb{P}\left(\max_{2\leqslant n\leqslant N}\left\|\sum_{1\leqslant i<j\leqslant n} h(X_i, X_j)\right\|_{\mathbb{H}} > x\right) &\leqslant C_r x^{-r} q^r N^r \mathbb{E}\left[H^r\right] \\ &\quad + C_r x^{-1} q N \sup_{j\geqslant 2} \mathbb{E}\left[\|h(X_1, X_j)\|_{\mathbb{H}}\right] + 4N\beta(q). \end{aligned} \tag{2.3}$$

The contribution of each term is either increasing or decreasing in q hence q has to be chosen in a judicious way.

We start the proof of Proposition 2.3 by a Lemma which gives a bound of the maximum of a U-statistics in terms of several U-statistics whose data can be expressed in terms of blocks of vectors of elements of $(X_i)_{i\geqslant 1}$ having a gap of size q.

Lemma 2.2. *Let $N > 2q \geqslant 1$ be integers. Let $(X_i)_{i\geqslant 1}$ be a sequence of random variables taking values in a separable metric space (S, d) and let $h\colon S^2 \to \mathbb{H}$ be a measurable function, where $(\mathbb{H}, \langle , \rangle)$ is a separable Hilbert space. Define the S^{q-1}-valued vector $V_{k,u}$ by*

$$V_{k,u} = (X_{2qu+k}, \ldots, X_{2qu+k+q-1}). \tag{2.4}$$

Then the following inequality holds:

$$\max_{2\leqslant n\leqslant N}\left\|\sum_{1\leqslant i<j\leqslant n} h(X_i, X_j)\right\|_{\mathbb{H}} \leqslant \sum_{i=1}^{4} M_{N,q,i} + \sum_{i=1}^{5} R_{N,q,i}, \tag{2.5}$$

where

$$M_{N,q,1} = \sum_{\substack{\ell,\ell'=1\\ 0\leqslant \ell-\ell'\leqslant q-1}}^{2q} \max_{1\leqslant m\leqslant \left\lfloor \frac{N}{2q}\right\rfloor} \left\|\sum_{0\leqslant u<v\leqslant m} h_{\ell,\ell'}\left(V_{\ell',u}, V_{\ell',v}\right)\right\|_{\mathbb{H}}, \tag{2.6}$$

$$M_{N,q,2} = \sum_{\substack{\ell,\ell'=1\\ 0\leqslant \ell'-\ell\leqslant q-1}}^{2q} \max_{1\leqslant m\leqslant \left\lfloor \frac{N}{2q}\right\rfloor} \left\|\sum_{0\leqslant u<v\leqslant m} h_{\ell,\ell'}\left(V_{\ell,u}, V_{\ell,v}\right)\right\|_{\mathbb{H}}, \tag{2.7}$$

$$M_{N,q,3} = \sum_{\substack{\ell,\ell'=1 \\ q\leqslant \ell-\ell'\leqslant 2q-1}}^{2q} \max_{1\leqslant m\leqslant \left\lfloor \frac{N}{2q}\right\rfloor} \left\| \sum_{0\leqslant u<v\leqslant m} h_{\ell,\ell'}\left(V_{\ell-2q,u}, V_{\ell-2q,v}\right)\right\|_{\mathbb{H}}, \tag{2.8}$$

$$M_{N,q,4} = \sum_{\substack{\ell,\ell'=1 \\ q\leqslant \ell'-\ell\leqslant 2q-1}}^{2q} \max_{1\leqslant m\leqslant \left\lfloor \frac{N}{2q}\right\rfloor} \left\| \sum_{0\leqslant u<v\leqslant m} h_{\ell,\ell'}\left(V_{\ell',u}, V_{\ell',v}\right)\right\|_{\mathbb{H}}, \tag{2.9}$$

the functions $h_{\ell,\ell'}\colon S^q\times S^q\to\mathbb{H}$ *are defined by*

$$h_{\ell,\ell'}\left((x_a)_{a=1}^q,(y_b)_{b=1}^q\right) = \begin{cases} h\left(x_{\ell-\ell'+1},y_1\right) & \text{if } 0\leqslant \ell-\ell'\leqslant q-1,\\ h\left(x_1,y_{\ell'+1}\right) & \text{if } 0\leqslant \ell'-\ell\leqslant q-1,\\ h\left(x_1,y_{\ell'-\ell+2q+1}\right) & \text{if } q\leqslant \ell-\ell'\leqslant 2q-1,\\ h\left(x_{\ell'-\ell+2q+1},y_1\right) & \text{if } q\leqslant \ell'-\ell\leqslant 2q-1, \end{cases} \tag{2.10}$$

$$R_{N,q,1} = \sum_{\substack{\ell,\ell'=1 \\ q\leqslant \ell-\ell'\leqslant 2q-1}}^{2q} \max_{1\leqslant m\leqslant \left\lfloor \frac{N}{2q}\right\rfloor} \left\| \sum_{v=1}^{m} h\left(X_{2(v-1)q+\ell}, X_{2vq+\ell'}\right)\right\|_{\mathbb{H}}, \tag{2.11}$$

$$R_{N,q,2} = \sum_{\substack{\ell,\ell'=1 \\ q\leqslant \ell-\ell'\leqslant 2q-1}}^{2q} \max_{1\leqslant m\leqslant \left\lfloor \frac{N}{2q}\right\rfloor} \left\| \sum_{v=1}^{m} h\left(X_{\ell}, X_{2vq+\ell'}\right)\right\|_{\mathbb{H}}, \tag{2.12}$$

$$R_{N,q,3} = \sum_{\substack{\ell,\ell'=1 \\ q\leqslant \ell'-\ell\leqslant 2q-1}}^{2q} \max_{1\leqslant m\leqslant \left\lfloor \frac{N}{2q}\right\rfloor} \left\| \sum_{v=1}^{m} h\left(X_{\ell}, X_{2qv+\ell'}\right)\right\|_{\mathbb{H}}, \tag{2.13}$$

$$R_{N,q,4} = \sum_{\substack{\ell,\ell'=1 \\ q\leqslant \ell'-\ell\leqslant 2q-1}}^{2q} \max_{1\leqslant m\leqslant \left\lfloor \frac{N}{2q}\right\rfloor} \left\| \sum_{v=0}^{m} h\left(X_{2qv+\ell}, X_{2qv+\ell'}\right)\right\|_{\mathbb{H}}, \tag{2.14}$$

$$R_{N,q,5} = \sum_{1\leqslant \ell<\ell'\leqslant 2q} \max_{2\leqslant m\leqslant \left\lfloor \frac{N}{2q}\right\rfloor} \left\| \sum_{u=0}^{m} h\left(X_{2qu+\ell,2qv+\ell'}\right)\right\|_{\mathbb{H}}. \tag{2.15}$$

Let us explain the roled played by the different terms involved in (2.5). The four terms $M_{N,q,i}$, $i\in\{1,2,3,4\}$ are maxima of U-statistics whose data may be dependent, but their behavior is close to the one of U-statistics of ndependent data. The other terms $R_{N,q,i}$, $i\in\{1,2,3,4,5\}$, play a negligible role. Indeed, their contribution is of the same order as that of maximum of partial sums of a strictly stationary sequence, but with the normalization corresponding to a U-statistic, which is stronger than the one needed.

Proposition 2.3. *Let $N>2q\geqslant 1$ be integers. Let $(X_i)_{i\geqslant 1}$ be a strictly stationary sequence of random variables taking values in a separable metric space (S,d) and let $h\colon S^2\to\mathbb{H}$ be a measurable function, where $(\mathbb{H},\langle,\rangle)$ is a separable Hilbert space. Then the following inequality holds:*

$$\mathbb{P}\left(\max_{2\leqslant n\leqslant N}\left\|\sum_{1\leqslant i<j\leqslant n} h\left(X_i,X_j\right)\right\|_{\mathbb{H}}>8x\right)\leqslant \sum_{i=1}^{4}\mathbb{P}\left(M_{N,q,i}^{*}>x\right)+\sum_{i=1}^{5}\mathbb{P}\left(R_{N,q,i}>x\right)+4N\beta\left(q\right), \tag{2.16}$$

where the terms $R_{N,q,i}$ are defined as in (2.11), (2.12), (2.13), (2.14) and (2.15),

$$M^*_{N,q,1} = \sum_{\substack{\ell,\ell'=1 \\ 0\leqslant \ell-\ell'\leqslant q-1}}^{2q} \max_{1\leqslant m\leqslant \left\lfloor \frac{N}{2q}\right\rfloor} \left\| \sum_{0\leqslant u<v\leqslant m} h_{\ell,\ell'}\left(V^*_{\ell',u}, V^*_{\ell',v}\right)\right\|_{\mathbb{H}}, \tag{2.17}$$

$$M^*_{N,q,2} = \sum_{\substack{\ell,\ell'=1 \\ 0\leqslant \ell'-\ell\leqslant q-1}}^{2q} \max_{1\leqslant m\leqslant \left\lfloor \frac{N}{2q}\right\rfloor} \left\| \sum_{0\leqslant u<v\leqslant m} h_{\ell,\ell'}\left(V^*_{\ell,u}, V^*_{\ell,v}\right)\right\|_{\mathbb{H}}, \tag{2.18}$$

$$M^*_{N,q,3} = \sum_{\substack{\ell,\ell'=1 \\ q\leqslant \ell-\ell'\leqslant 2q-1}}^{2q} \max_{1\leqslant m\leqslant \left\lfloor \frac{N}{2q}\right\rfloor} \left\| \sum_{0\leqslant u<v\leqslant m} h_{\ell,\ell'}\left(V^*_{\ell-2q,u}, V^*_{\ell-2q,v}\right)\right\|_{\mathbb{H}}, \tag{2.19}$$

$$M^*_{N,q,4} = \sum_{\substack{\ell,\ell'=1 \\ q\leqslant \ell'-\ell\leqslant 2q-1}}^{2q} \max_{1\leqslant m\leqslant \left\lfloor \frac{N}{2q}\right\rfloor} \left\| \sum_{0\leqslant u<v\leqslant m} h_{\ell,\ell'}\left(V^*_{\ell',u}, V^*_{\ell',v}\right)\right\|_{\mathbb{H}}, \tag{2.20}$$

and $\left(V^*_{k,u}\right)_{\substack{k\in\{-q,\ldots,q\} \\ u\in\{0,\ldots,\lfloor N/(2q)\rfloor\}}}$ *satisfy the following:*

$$\text{for each } k\in\{-q,\ldots,q\}, \left(V^*_{k,u}\right)_{u=0}^{\lfloor N/2q\rfloor} \text{ is independent,} \tag{2.21}$$

$$\text{for each } k\in\{-q,\ldots,q\}, u\in\{0,\ldots,\lfloor N/(2q)\rfloor\}, V^*_{k,u} \text{ has the same distribution as } V_{k,u}. \tag{2.22}$$

It will turn out that we will not need the joint distribution of $M^*_{N,q,i}$ Only the fact that all there random variables are identically distributed and that the common distribution is the same as the maximum of norm of a U-statistic having kernel h and i.i.d. data (with the same distribution as X_1) will play a decisive role.

Proof of Lemma 2.2. We express for a fixed n the set $I := \left\{(i,j)\in\mathbb{N}^2, 1\leqslant i<j\leqslant n\right\}$ as

$$I = \bigcup_{\ell,\ell'=1}^{2q} \left\{\left(2qu+\ell, 2qv+\ell'\right), 0\leqslant u\leqslant v\leqslant \left\lfloor \left(n-\ell'\right)/(2q)\right\rfloor, 2qu+\ell<2qv+\ell'\right\}, \tag{2.23}$$

which can be splitted as the disjoint union $I = \bigcup_{a=1}^5 I_{n,a}$, where

$$I_{n,1} = \bigcup_{\substack{\ell,\ell'=1 \\ 0\leqslant \ell-\ell'\leqslant q-1}}^{2q} I_{n,1,\ell,\ell'}, \quad I_{n,1,\ell,\ell'} = \left\{\left(2qu+\ell, 2qv+\ell'\right), 0\leqslant u<v\leqslant \left\lfloor \left(n-\ell'\right)/(2q)\right\rfloor\right\}, \tag{2.24}$$

$$I_{n,2} = \bigcup_{\substack{\ell,\ell'=1 \\ 1\leqslant \ell'-\ell\leqslant q-1}}^{2q} I_{n,2,\ell,\ell'}, \quad I_{n,2,\ell,\ell'} = \left\{\left(2qu+\ell, 2qv+\ell'\right), 0\leqslant u<v\leqslant \left\lfloor \left(n-\ell'\right)/(2q)\right\rfloor\right\}, \tag{2.25}$$

$$I_{n,3} = \bigcup_{\substack{\ell,\ell'=1 \\ q\leqslant \ell-\ell'\leqslant 2q-1}}^{2q} I_{n,3,\ell,\ell'}, \quad I_{n,3,\ell,\ell'} := \left\{\left(2qu+\ell, 2qv+\ell'\right), 0\leqslant u<v\leqslant \left\lfloor \left(n-\ell'\right)/(2q)\right\rfloor\right\}, \tag{2.26}$$

$$I_{n,4} = \bigcup_{\substack{\ell,\ell'=1 \\ q\leqslant \ell'-\ell\leqslant 2q-1}}^{2q} I_{n,4,\ell,\ell'}, \quad I_{n,4,\ell,\ell'} = \left\{\left(2qu+\ell, 2qv+\ell'\right), 0\leqslant u<v\leqslant \left\lfloor \left(n-\ell'\right)/(2q)\right\rfloor\right\}, \tag{2.27}$$

$$I_{n,5}=\bigcup_{1\leqslant\ell<\ell'\leqslant 2q} I_{n,5,\ell,\ell'},\quad I_{n,5,\ell,\ell'}=\left\{\left(2qu+\ell,2qu+\ell'\right),0\leqslant u\leqslant\left\lfloor\left(n-\ell'\right)/\left(2q\right)\right\rfloor\right\}. \tag{2.28}$$

The first two sets I_1 and I_2 contain the indices of the form $(2qu+\ell,2qv+\ell')$ for which $|\ell-\ell'|\leqslant q-1$ and a distinction is made according to the order between ℓ and ℓ'. The sets I_3 and I_4 contain the indices of the form $(2qu+\ell,2qv+\ell')$ for which $|\ell-\ell'|>q-1$ (since $1\leqslant\ell,\ell'\leqslant 2q$, we necessarily have $|\ell-\ell'|\leqslant 2q-1$) and here again, a distinction is made according to the order between ℓ and ℓ'. Note that in the sets I_a, $1\leqslant a\leqslant 4$, one has $u<v$ which guarantees that $2qu+\ell<2qv+\ell'$. Finally, in the set I_5, the indexes corresponding to the case $u=v$ are collected.

As a consequence, the following inequality takes place:

$$\begin{aligned}\max_{2\leqslant n\leqslant N}\left\|\sum_{1\leqslant i<j\leqslant n}h\left(X_i,X_j\right)\right\|_{\mathbb{H}}\leqslant&\sum_{\substack{\ell,\ell'=1\\0\leqslant\ell-\ell'\leqslant q-1}}^{2q}\max_{2\leqslant n\leqslant N}\left\|\sum_{0\leqslant u<v\leqslant\left\lfloor\frac{n-\ell'}{2q}\right\rfloor}h\left(X_{2qu+\ell},X_{2qv+\ell'}\right)\right\|_{\mathbb{H}}\\&+\sum_{\substack{\ell,\ell'=1\\1\leqslant\ell'-\ell\leqslant q-1}}^{2q}\max_{2\leqslant n\leqslant N}\left\|\sum_{0\leqslant u<v\leqslant\left\lfloor\frac{n-b}{2q}\right\rfloor}h\left(X_{2qu+\ell},X_{2qv+\ell'}\right)\right\|_{\mathbb{H}}\\&+\sum_{\substack{\ell,\ell'=1\\q\leqslant\ell-\ell'\leqslant 2q-1}}^{2q}\max_{2\leqslant n\leqslant N}\left\|\sum_{0\leqslant u<v\leqslant\lfloor(n-\ell')/(2q)\rfloor}h\left(X_{2qu+\ell},X_{2qv+\ell'}\right)\right\|_{\mathbb{H}}\\&+\sum_{\substack{\ell,\ell'=1\\q\leqslant\ell'-\ell\leqslant 2q-1}}^{2q}\max_{2\leqslant n\leqslant N}\left\|\sum_{0\leqslant u<v\leqslant\lfloor(n-\ell')/(2q)\rfloor}h\left(X_{2qu+\ell},X_{2qv+\ell'}\right)\right\|_{\mathbb{H}}\\&+\sum_{1\leqslant\ell<\ell'\leqslant 2q}\max_{2\leqslant n\leqslant N}\left\|\sum_{u=0}^{\lfloor(n-\ell')/(2q)\rfloor}h\left(X_{2qu+\ell},X_{2qu+\ell'}\right)\right\|_{\mathbb{H}}.\end{aligned} \tag{2.29}$$

By the elementary inequality

$$\max_{2\leqslant n\leqslant N}a_{\lfloor(n-\ell')/(2q)\rfloor}\leqslant\max_{0\leqslant m\leqslant\lfloor N/(2q)\rfloor}a_m \tag{2.30}$$

valid for a non-negative sequence $(a_m)_{m\geqslant 0}$, we derive from (2.29) that

$$\begin{aligned}\max_{2\leqslant n\leqslant N}\left\|\sum_{1\leqslant i<j\leqslant n}h\left(X_i,X_j\right)\right\|_{\mathbb{H}}\leqslant&\sum_{\substack{\ell,\ell'=1\\0\leqslant\ell-\ell'\leqslant q-1}}^{2q}\max_{0\leqslant m\leqslant\left\lfloor\frac{N}{2q}\right\rfloor}\left\|\sum_{0\leqslant u<v\leqslant m}h\left(X_{2qu+\ell},X_{2qv+\ell'}\right)\right\|_{\mathbb{H}}\\&+\sum_{\substack{\ell,\ell'=1\\1\leqslant\ell'-\ell\leqslant q-1}}^{2q}\max_{0\leqslant m\leqslant\left\lfloor\frac{N}{2q}\right\rfloor}\left\|\sum_{0\leqslant u<v\leqslant m}h\left(X_{2qu+\ell},X_{2qv+\ell'}\right)\right\|_{\mathbb{H}}+\sum_{\substack{\ell,\ell'=1\\q\leqslant\ell-\ell'\leqslant 2q-1}}^{2q}\max_{0\leqslant m\leqslant\left\lfloor\frac{N}{2q}\right\rfloor}\left\|\sum_{0\leqslant u<v\leqslant m}h\left(X_{2qu+\ell},X_{2qv+\ell'}\right)\right\|_{\mathbb{H}}\\&+\sum_{\substack{\ell,\ell'=1\\q\leqslant\ell'-\ell\leqslant 2q-1}}^{2q}\max_{0\leqslant m\leqslant\left\lfloor\frac{N}{2q}\right\rfloor}\left\|\sum_{0\leqslant u<v\leqslant m}h\left(X_{2qu+\ell},X_{2qv+\ell'}\right)\right\|_{\mathbb{H}}\\&+\sum_{1\leqslant\ell<\ell'\leqslant 2q}\max_{0\leqslant m\leqslant\left\lfloor\frac{N}{2q}\right\rfloor}\left\|\sum_{u=0}^{m}h\left(X_{2qu+\ell},X_{2qu+\ell'}\right)\right\|_{\mathbb{H}}=:A_1+A_2+A_3+A_4+A_5.\end{aligned} \tag{2.31}$$

Using the expression of $h_{\ell,\ell'}$ and $V_{k,u}$ given by (2.10) and (2.4), we bound A_1 by $M_{N,q,1}$ and A_2 by $M_{N,q,2}$. Moreover, A_5 coincides with $R_{N,q,5}$. For A_3, we write

$$\sum_{0\leqslant u<v\leqslant m} h\left(X_{2qu+\ell}, X_{2qv+\ell'}\right) = \sum_{0\leqslant u<v\leqslant m} h\left(X_{2q(u-1)+\ell}, X_{2qv+\ell'}\right) + \sum_{v=1}^{m}\sum_{u=0}^{v-1}\left(h\left(X_{2qu+\ell}, X_{2qv+\ell'}\right) - h\left(X_{2q(u-1)+\ell}, X_{2qv+\ell'}\right)\right), \quad (2.32)$$

and since the sum over u is telescopic, the inequality

$$\left\|\sum_{0\leqslant u<v\leqslant m} h\left(X_{2qu+\ell}, X_{2qv+\ell'}\right)\right\|_{\mathbb{H}} \leqslant \left\|\sum_{0\leqslant u<v\leqslant m} h\left(X_{2q(u-1)+\ell}, X_{2qv+\ell'}\right)\right\|_{\mathbb{H}} + \left\|\sum_{v=1}^{m} h\left(X_{2qv+\ell}, X_{2qv+\ell'}\right)\right\|_{\mathbb{H}} + \left\|\sum_{v=1}^{m} h\left(X_{\ell}, X_{2qv+\ell'}\right)\right\|_{\mathbb{H}} \quad (2.33)$$

takes place. Then we use the expression of $h_{\ell,\ell'}$ in order to show that $A_3 \leqslant M_{N,q,3} + R_{N,q,1} + R_{N,q,3}$. The treatment of A_4 is similar, with the minor difference that we use the decomposition

$$\sum_{0\leqslant u<v\leqslant m} h\left(X_{2qu+\ell}, X_{2qv+\ell'}\right) = \sum_{0\leqslant u<v\leqslant m} h\left(X_{2q(u+1)+\ell}, X_{2qv+\ell'}\right) + \sum_{v=1}^{m}\sum_{u=0}^{v-1}\left(h\left(X_{2qu+\ell}, X_{2qv+\ell'}\right) - h\left(X_{2q(u+1)+\ell}, X_{2qv+\ell'}\right)\right). \quad (2.34)$$

This ends the proof of Lemma 2.2. □

In order to prove Proposition 2.3, we will need the following coupling lemma, due to Berbee [3].

Lemma 2.4. *Let X and Y be random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in a Polish space S. Let $\sigma(X)$ be the σ-algebra generated by X and let U be a random variable uniformly distributed on $[0,1]$ and independent of (X,Y). There exists a random variable Y^*, measurable with respect to $\sigma(X) \vee \sigma(Y) \vee \sigma(U)$, independent of X and distributed as Y, and such that $\mathbb{P}(Y \neq Y^*) = \beta(X,Y)$.*

Proof of Proposition 2.3. A consequence of Lemma 2.4 is the following. Given a sequence $(Y_u)_{u\geqslant 1}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in a Polish space S, we can find, on a richer probability space, an indepdent sequence $(Y_u^*)_{u\geqslant 1}$ of random variable, such that for each $u \geqslant 1$, Y_u have the same distribution as Y_u^* and $\mathbb{P}(Y_u^* \neq Y_u) \leqslant \beta(\sigma(Y_i, i \leqslant u-1), \sigma(Y_u))$.

Applying this result for each fixed $k \in \{-q, \ldots, q\}$ gives sequences $\left(V_{k,u}^*\right)_{u=0}^{\lfloor N/(2q)\rfloor}$ satisfying (2.21), (2.22) and $\mathbb{P}\left(V_{k,u}^* \neq V_{k,u}\right) \leqslant \beta(\sigma(V_{k,i}, i \leqslant u-1), \sigma(V_{k,i}))$. Define the events

$$A_{k,u} := \left\{V_{k,u}^* \neq V_{k,u}\right\}, \quad k \in \{-q, \ldots, q\}, u \in \{0, \ldots, \lfloor N/(2q)\rfloor\} \quad (2.35)$$

and

$$A = \bigcup_{k=-q}^{q} \bigcup_{u=0}^{\lfloor N/(2q)\rfloor} A_{k,u}. \quad (2.36)$$

In view of Lemma 2.2, one has

$$\mathbb{P}\left(\max_{2\leqslant n\leqslant N}\left\|\sum_{1\leqslant i<j\leqslant n}h\left(X_i,X_j\right)\right\|_{\mathbb{H}}>5x\right)\leqslant\mathbb{P}\left(\left\{M_{N,q,1}+M_{N,q,2}+M_{N,q,3}+M_{N,q,4}>x\right\}\cap A^c\right)$$
$$+\mathbb{P}\left(A\right)+\sum_{i=1}^{4}\mathbb{P}\left(M_{N,q,i}>x\right). \quad (2.37)$$

Since the vectors $V_{k,u}$ coincide with the vectors $V_{k,u}^*$, $k\in\{-q,\ldots,q\}$, $u\in\{0,\ldots,\lfloor N/(2q)\rfloor\}$ on A^c, the events $\{M_{N,q,1}+M_{N,q,2}+M_{N,q,3}+M_{N,q,4}>x\}\cap A^c$ and $\left\{M_{N,q,1}^*+M_{N,q,2}^*+M_{N,q,3}^*+M_{N,q,4}^*>x\right\}\cap A^c$ are equal, where $M_{N,q,i}^*$, $i\in\{1,2,3,4\}$, are defined by (2.17).

Moreover, since A is a union of $(2q+1)(\lfloor N/(2q)\rfloor+1)$ sets having probability smaller than $\beta(q)$, one derives that $\mathbb{P}(A)\leqslant(2q+1)(\lfloor N/(2q)\rfloor+1)\beta(q)\leqslant 4N\beta(q)$. This ends the proof of Proposition 2.3. □

Proof of Proposition 2.1. We apply Proposition 2.3 with x replaced by $2x$ and a use of Markov's inequality gives

$$\mathbb{P}\left(\max_{2\leqslant n\leqslant N}\left\|\sum_{1\leqslant i<j\leqslant n}h\left(X_i,X_j\right)\right\|_{\mathbb{H}}>16x\right)$$
$$\leqslant\sum_{i=1}^{4}\mathbb{P}\left(M_{N,q,i}^*>2x\right)+(2x)^{-1}\sum_{i=1}^{5}\mathbb{E}\left[R_{N,q,i}\right]+4N\beta(q). \quad (2.38)$$

One has

$$R_{N,q,i}\leqslant\sum_{\ell,\ell'=1}^{2q}\sum_{v=0}^{\left\lfloor\frac{N}{2q}\right\rfloor}\left\|Y_{\ell,\ell',v}\right\|_{\mathbb{H}}, \quad (2.39)$$

where $Y_{\ell,\ell',v}$ equal $h(X_i,X_j)$ for some indices i and j. Therefore, using $\lfloor N/(2q)\rfloor+1\leqslant N/q$, we find

$$\mathbb{E}\left[R_{N,q,i}\right]\leqslant 4qN\sup_{j\geqslant 2}\mathbb{E}\left[\left\|h\left(X_1,X_j\right)\right\|_{\mathbb{H}}\right]. \quad (2.40)$$

In order to control the terms $M_{N,q,i}$, we introduce truncated and degenerated kernels as follows For a kernel $h\colon S^2\to\mathbb{H}$ and a strictly stationary sequence $(X_i)_{i\geqslant 1}$, define

$$h^{\deg}(x,y)=h(x,y)-\mathbb{E}\left[h_1\left(X_1,y\right)\right]-\mathbb{E}\left[h_1\left(x,X_1\right)\right]+\mathbb{E}\left[h\left(X_1,X_1'\right)\right], \quad (2.41)$$

where X_1' is an independent copy of X_1. In this way, one has

$$\mathbb{E}\left[h^{\deg}\left(X_1,X_1'\right)\mid X_1\right]=\mathbb{E}\left[h^{\deg}\left(X_1,X_1'\right)\mid X_1'\right]=0 \quad (2.42)$$

hence if $(\xi_i)_{i\geqslant 1}$ is an i.i.d. sequence and ξ_1 has the same distribution as X_1, the U-statistic of data $(\xi_i)_{i\geqslant 1}$ and kernel $h^{\deg}$ is degenerated. Note that this property of degeneracy depends on the law of X_1, but since there will be no ambiguity, we will not write this dependence. Let $1<r\leqslant 2$, $N\geqslant 2q\geqslant 1$ and $R>0$. Let $h_{\leqslant}\colon S^2\to\mathbb{H}$ and $h_{>}\colon S^2\to\mathbb{H}$ be defined as

$$h_{\leqslant}(x,y)=h(x,y)\,\mathbf{1}_{\|h(x,y)\|_{\mathbb{H}}\leqslant R} \quad (2.43)$$

$$h_{>}(x,y)=h(x,y)\,\mathbf{1}_{\|h(x,y)\|_{\mathbb{H}}>R}. \quad (2.44)$$

In view of (2.1), one has $h(x,y) = (h_{\leqslant})^{\deg}(x,y) + (h_{>})^{\deg}(x,y)$. Therefore, defining $M^*_{N,q,i,\leqslant}$ and $M^*_{N,q,i,>}$ as in (2.17) with h replaced respectively by $(h_{\leqslant})^{\deg}$ and $(h_{>})^{\deg}$ the equality $M_{N,q,i} \leqslant M^*_{N,q,i,\leqslant} + M^*_{N,q,i,>}$ holds hence by (2.38) and Markov's inequality, it follows that

$$\begin{aligned}
&\mathbb{P}\left(\max_{2\leqslant n\leqslant N}\left\|\sum_{1\leqslant i<j\leqslant n} h(X_i,X_j)\right\|_{\mathbb{H}} > 16x\right) \\
&\quad\leqslant \sum_{i=1}^{4}\mathbb{P}\left(M^*_{N,q,i,\leqslant} > x\right) + \sum_{i=1}^{4}\mathbb{P}\left(M^*_{N,q,i,>} > x\right) + \frac{10}{x}qN\sup_{j\geqslant 2}\mathbb{E}\left[\|h(X_1,X_j)\|_{\mathbb{H}}\right] + 4N\beta(q) \\
&\quad\leqslant x^{-r}\sum_{i=1}^{4}\mathbb{E}\left[\left(M^*_{N,q,i,\leqslant}\right)^r\right] + x^{-1}\sum_{i=1}^{4}\mathbb{E}\left[M^*_{N,q,i>}\right] + \frac{10}{x}qN\sup_{j\geqslant 2}\mathbb{E}\left[\|h(X_1,X_j)\|_{\mathbb{H}}\right] + 4N\beta(q). \qquad (2.45)
\end{aligned}$$

We now control the moment of order r of $M^*_{N,q,1,\leqslant}$ in the following way:

$$\begin{aligned}
\mathbb{E}\left[\left(M^*_{N,q,1,\leqslant}\right)^r\right] &= \left\|\sum_{\substack{\ell,\ell'=1\\0\leqslant\ell-\ell'\leqslant q-1}}^{2q}\max_{1\leqslant m\leqslant\left\lfloor\frac{N}{2q}\right\rfloor}\left\|\sum_{0\leqslant u<v\leqslant m} h_{\ell,\ell'}\left(V^*_{\ell',u},V^*_{\ell',v}\right)\right\|_{\mathbb{H}}\right\|_r^r && (2.46)\\
&\leqslant \left(\sum_{\substack{\ell,\ell'=1\\0\leqslant\ell-\ell'\leqslant q-1}}^{2q}\left\|\max_{1\leqslant m\leqslant\left\lfloor\frac{N}{2q}\right\rfloor}\left\|\sum_{0\leqslant u<v\leqslant m} h_{\ell,\ell'}\left(V^*_{\ell',u},V^*_{\ell',v}\right)\right\|_{\mathbb{H}}\right\|_r\right)^r && (2.47)\\
&\leqslant 4\left(\sum_{\substack{\ell,\ell'=1\\0\leqslant\ell-\ell'\leqslant q-1}}^{2q}\left\|\left\|\sum_{0\leqslant u<v\leqslant\left\lfloor\frac{N}{2q}\right\rfloor} h_{\ell,\ell'}\left(V^*_{\ell',u},V^*_{\ell',v}\right)\right\|_{\mathbb{H}}\right\|_r\right)^r && (2.48)\\
&\leqslant C_{1,r}\left((2q)^2\frac{N}{2q}\right)^r\mathbb{E}\left[\left\|(h_{\leqslant})^{\deg}\left(X_1,X_1'\right)\right\|_{\mathbb{H}}^r\right] && (2.49)\\
&\leqslant C_{2,r}q^rN^r\mathbb{E}\left[\left\|h\left(X_1,X_1'\right)\right\|_{\mathbb{H}}^r\mathbf{1}_{\|h(X_1,X_1')\|_{\mathbb{H}}\leqslant R}\right], && (2.50)
\end{aligned}$$

where $C_{1,r}$ and $C_{2,r}$ depend only on r, the step from (2.46) to (2.47) is justified by the triangle inequality, the one from (2.47) to (2.48) from Doob's inequality, (2.49) follows by the use of degeneracy of $(h_{\leqslant})^{\deg}$, which gives a martingale and reversed martingale property for the summation over j and i respectively and a double use of Theorem 4.1 in [25]. Finally, (2.50) is a consequence of the fact that

$$(h_{\leqslant})^{\deg}\left(X_1,X_1'\right) = h_{\leqslant}\left(X_1,X_1'\right) - \mathbb{E}\left[h_{\leqslant}\left(X_1,X_1'\right)\mid X_1\right] - \mathbb{E}\left[h_{\leqslant}\left(X_1,X_1'\right)\mid X_1'\right] + \mathbb{E}\left[h_{\leqslant}\left(X_1,X_1'\right)\right] \qquad (2.51)$$

and an appplication of the triangle inequality. A similar bound holds for $\mathbb{P}\left(M^*_{N,q,i} > x\right)$, $i\in\{2,3,4\}$.

The control of the tail of the U-statistic associated to $(h_{>})^{\deg}$ is much simpler and follows from

$$\max_{2\leqslant n\leqslant N}\left\|\sum_{1\leqslant i<j\leqslant n}(h_{>})^{\deg}(X_i,X_j)\right\|_{\mathbb{H}} \leqslant \sum_{1\leqslant i<j\leqslant N}\left\|(h_{>})^{\deg}(X_i,X_j)\right\|_{\mathbb{H}}, \qquad (2.52)$$

Markov's inequality and

$$(h_{>})^{\deg}\left(X_1,X_1'\right) = h_{>}\left(X_1,X_1'\right) - \mathbb{E}\left[h_{>}\left(X_1,X_1'\right)\mid X_1\right] - \mathbb{E}\left[h_{>}\left(X_1,X_1'\right)\mid X_1'\right] + \mathbb{E}\left[h_{>}\left(X_1,X_1'\right)\right].$$

This ens the proof of Proposition 2.1. □

2.2 Proof of Theorem 1.3

We start from decomposition 1.5. Theorem 2 in [12] applied with X_i replaced by $h_1(X_i)$ gives the convergence in distribution in $C_{\mathbb{H}}[0,1]$

$$\frac{1}{\sqrt{n}}\left(\sum_{j=1}^{\lfloor nt\rfloor} h_1(X_i) + (nt - \lfloor nt\rfloor)\, h_1\left(X_{\lfloor nt\rfloor+1}\right)\right) \to W_\Gamma, \tag{2.53}$$

where

$$\langle \Gamma u, v\rangle_{\mathbb{H}} = \lim_{n\to\infty} \mathbb{E}\left[\left\langle n^{-1/2}S_n, u\right\rangle_{\mathbb{H}} \left\langle n^{-1/2}S_n, v\right\rangle_{\mathbb{H}} \mid \sigma\left(X_k, k\leqslant 0\right)\right], \quad u, v\in\mathbb{H}, \tag{2.54}$$

and $S_n = \sum_{i=1}^n h_1(X_i)$. As Remark 1 in [12] says, when the sequence $(X_i)_{i\geqslant 1}$ is ergodic, the operator Γ is non random. Ergodicity is ensured by the absolute regularity of $(X_i)_{i\geqslant 1}$. Then a computation combined with the absolute convergence of the series in (1.15) shows that Γ has the expression given in that equation.

By (1.5), the convergence (1.14) will follows from

$$\frac{1}{N^{3/2}} \max_{2\leqslant n\leqslant N} \left\| \sum_{1\leqslant i<j\leqslant n} h_2(X_i, X_j)\right\|_{\mathbb{H}} \to 0 \text{ in probability}, \tag{2.55}$$

where h_2 is defined as in (1.4), or in other words, that for each positive ε,

$$\lim_{N\to\infty} \mathbb{P}\left(\frac{1}{N^{3/2}} \max_{2\leqslant n\leqslant N} \left\| \sum_{1\leqslant i<j\leqslant n} h_2(X_i, X_j)\right\|_{\mathbb{H}} > \varepsilon\right) = 0. \tag{2.56}$$

To do so, let us fix $\eta > 0$. We use (2.3) with $r = 2$, $q = \left\lfloor \eta\sqrt{N}\right\rfloor$ and $x = N^{3/2}$ and get that

$$\begin{aligned}\mathbb{P}\left(\frac{1}{N^{3/2}} \max_{2\leqslant n\leqslant N} \left\| \sum_{1\leqslant i<j\leqslant n} h_2(X_i, X_j)\right\|_{\mathbb{H}} > \varepsilon\right) &\leqslant C_2\varepsilon^{-2}N^{-3}\left\lfloor \eta\sqrt{N}\right\rfloor^{2u} N^2\mathbb{E}\left[H^2\right] \\ &\quad + C_2\varepsilon^{-1}N^{-3/2}\left\lfloor \eta\sqrt{N}\right\rfloor N \sup_{j\geqslant 2}\mathbb{E}\left[\left\|h(X_1, X_j)\right\|_{\mathbb{H}}\right] + 4N\beta\left(\left\lfloor \eta\sqrt{N}\right\rfloor\right). \end{aligned}\tag{2.57}$$

Since the assumption $n^2\beta(n)\to 0$ implies that for each fixed $\eta > 0$, $4N\beta\left(\left\lfloor \eta\sqrt{N}\right\rfloor\right)\to 0$, we find that

$$\limsup_{N\to\infty} \mathbb{P}\left(\frac{1}{N^{3/2}} \max_{2\leqslant n\leqslant N} \left\| \sum_{1\leqslant i<j\leqslant n} h_2(X_i, X_j)\right\|_{\mathbb{H}} > 9\varepsilon\right) \leqslant C_2\varepsilon^{-2}\mathbb{E}\left[H^2\right]\eta^2 + C_2\varepsilon^{-1}\eta \sup_{j\geqslant 2}\mathbb{E}\left[\left\|h(X_1, X_j)\right\|_{\mathbb{H}}\right].$$

Since η is arbitrary, the proof of Theorem 1.3 is complete.

2.3 Proof of Theorem 1.4

In view of the decomposition (1.5), it suffices to prove that

$$\lim_{n\to\infty} \frac{1}{n^{1+1/p}} \sum_{i=1}^{n-1} (i-1)\left(h_{1,0}(X_i) - \mathbb{E}\left[h_{1,0}(X_i)\right]\right) = 0 \text{ a.s.}, \tag{2.58}$$

$$\lim_{n\to\infty}\frac{1}{n^{1+1/p}}\sum_{j=2}^{n}(n-j)\left(h_{0,1}(X_j)-\mathbb{E}\left[h_{0,1}(X_j)\right]\right)=0 \text{ a.s. and} \tag{2.59}$$

$$\lim_{n\to\infty}\frac{1}{n^{1+1/p}}\left\|U_n(h_2)\right\|_{\mathbb{H}}=0, \tag{2.60}$$

where the functions $h_{1,0}$, $h_{0,1}$ and h_2 are defined respectively by (1.2), (1.3) and (1.4). Define $S_i=\sum_{k=1}^{i}\left(h_{1,0}(X_k)-\mathbb{E}\left[h_{1,0}(X_k)\right]\right)$ and $S'_j=\sum_{k=1}^{j}\left(h_{0,1}(X_k)-\mathbb{E}\left[h_{0,1}(X_k)\right]\right)$. Then

$$\sum_{i=1}^{n-1}(i-1)\left(h_{1,0}(X_i)-\mathbb{E}\left[h_{1,0}(X_i)\right]\right)=\sum_{j=2}^{n}(S_n-S_j) \tag{2.61}$$

hence

$$\frac{1}{n^{1+1/p}}\left\|\sum_{i=1}^{n-1}(i-1)\left(h_{1,0}(X_i)-\mathbb{E}\left[h_{1,0}(X_i)\right]\right)\right\|_{\mathbb{H}}\leqslant\frac{2}{n^{1+1/p}}\max_{1\leqslant i\leqslant n}\left\|S_i\right\|_{\mathbb{H}} \tag{2.62}$$

and similarly,

$$\frac{1}{n^{1+1/p}}\left\|\sum_{j=2}^{n}(n-j)\left(h_{0,1}(X_i)-\mathbb{E}\left[h_{0,1}(X_i)\right]\right)\right\|_{\mathbb{H}}\leqslant\frac{2}{n^{1+1/p}}\max_{1\leqslant j\leqslant n}\left\|S'_j\right\|_{\mathbb{H}}. \tag{2.63}$$

By Corollary 3 in [11], condition 1.18 implies that $n^{-1/p}\left\|S_n\right\|_{\mathbb{H}}+n^{-1/p}\left\|S'_n\right\|_{\mathbb{H}}\to 0$ a.s. Therefore, it suffices to prove (2.60), which reduces, by the Borel-Cantelli lemma, to prove that for each positive ε,

$$\sum_{M=0}^{\infty}\mathbb{P}\left(2^{-M\left(1+\frac{1}{p}\right)}\max_{2\leqslant n\leqslant 2^M}\left\|\sum_{1\leqslant i<j\leqslant n}h_2(X_i,X_j)\right\|_{\mathbb{H}}>\varepsilon\right)<\infty. \tag{2.64}$$

By (2.3) applied with $R=p+\delta$, N replaced by 2^M, x by $\varepsilon 2^{M(1+1/p)}$ and $q=\left\lfloor 2^{Ma}\right\rfloor$ where $a=1/(p+\eta)$, we infer that

$$\begin{aligned}\mathbb{P}\left(2^{-M\left(1+\frac{1}{p}\right)}\max_{2\leqslant n\leqslant 2^M}\left\|\sum_{1\leqslant i<j\leqslant n}h_2(X_i,X_j)\right\|_{\mathbb{H}}>9\varepsilon\right)&\leqslant C_{p+\delta}\varepsilon^{-(p+\delta)}2^{-2M\left(1+\frac{1}{p}\right)}2^{\frac{2}{p+\eta}M}2^{2M}\mathbb{E}\left[H^{p+\delta}\right]\\&+C_r\varepsilon^{-1}2^{-M\left(1+\frac{1}{p}\right)}2^{\frac{M}{p+\eta}}2^M\sup_{j\geqslant 2}\mathbb{E}\left[\left\|h(X_1,X_j)\right\|_{\mathbb{H}}\right]+4\cdot 2^M\beta\left(\left\lfloor 2^{\frac{M}{p+\eta}}\right\rfloor\right).\end{aligned} \tag{2.65}$$

Since $a<1/p$, the convergence of $\sum_{M=0}^{\infty}\mathbb{P}\left(2^{-M\left(1+\frac{1}{p}\right)}\max_{2\leqslant n\leqslant 2^M}\left\|\sum_{1\leqslant i<j\leqslant n}h_2(X_i,X_j)\right\|_{\mathbb{H}}>9\varepsilon\right)$ is guaranteed by the convergence of $\sum_{M=0}^{\infty}2^M\beta\left(\left\lfloor 2^{\frac{M}{p+\eta}}\right\rfloor\right)$. By dividing the sum over sets of the form $\left\{\left\lfloor 2^{Ma}\right\rfloor+1,\ldots,\left\lfloor 2^{(M+1)a}\right\rfloor\right\}$ and the fact that $(\beta(k))_{k\geqslant 1}$ is non-increasing, the convergence of $\sum_{M=0}^{\infty}2^M\beta\left(\left\lfloor 2^{Ma}\right\rfloor\right)$ is equivalent to that of $\sum_{k=1}^{\infty}k^{\frac{1}{a}-1}\beta(k)$, which is guaranteed by 1.18.

This ends the proof of Theorem 1.4.

2.4 Proof of Theorem 1.5

By the same arguments as in the proof of Theorem 1.4, it suffices to prove that (2.64) holds for each positive ε. To do so, we apply Proposition 2.3 with $x=2^{M(1+1/p)}\varepsilon$, N replaced by 2^M, $q=\left\lfloor 2^{Ma}\right\rfloor$ and $R=2^{Mb}$, where $a\in(0,1/p)$ and $b>0$ will be specified later and in such a way that

$$\sum_{M=0}^{\infty}2^{-2M/p}2^{2Ma}\mathbb{E}\left[H^2\mathbf{1}_{H\leqslant 2^{Mb}}\right]<\infty, \tag{2.66}$$

$$\sum_{M=0}^{\infty} 2^{-M(1+1/p)} 2^{2M} \mathbb{E}\left[H \mathbf{1}_{H>2^{Mb}}\right] < \infty, \tag{2.67}$$

$$\sum_{M=0}^{\infty} 2^{M} \beta\left(\left\lfloor 2^{Ma} \right\rfloor\right) < \infty. \tag{2.68}$$

Notice that for each positive b and c and each non-negative random variable Y,

$$\sum_{M=0}^{\infty} 2^{-Mc} \mathbb{E}\left[Y^2 \mathbf{1}_{Y \leqslant 2^{Mb}}\right] \leqslant K(b,c)\, \mathbb{E}\left[Y^{2-\frac{c}{b}}\right], \tag{2.69}$$

$$\sum_{M=0}^{\infty} 2^{Mc} \mathbb{E}\left[Y \mathbf{1}_{Y>2^{Mb}}\right] \leqslant K(b,c)\, \mathbb{E}\left[Y^{1+\frac{c}{b}}\right]. \tag{2.70}$$

Therefore, applying (2.70) with $c = 1 - 1/p$ imposes the choice $b = (p-1)/(p(p+\delta-1))$. Now, applying (2.69) with $c = 2(1+1/p) - 2a$ shows that a must satisfy

$$2 - \frac{2/p - 2a}{b} = p + \delta, \tag{2.71}$$

hence

$$a = \frac{p(p-1) + \delta(p+1)}{2p(p+\delta-1)}. \tag{2.72}$$

Notice that (2.68) is equivalent to the convergence of $\sum_{k=1}^{\infty} k^{1/a-1} \beta(k)$. Therefore, (2.64) is a consequence of (1.21).

2.5 Proof of Theorem 1.6

By the Borel-Cantelli lemma, it suffices to prove that

$$\sum_{M=0}^{\infty} \mathbb{P}\left(2^{-2M/p} \max_{2 \leqslant n \leqslant 2^M} \left\| \sum_{1 \leqslant i < j \leqslant n} h(X_i, X_j) \right\|_{\mathbb{H}} > \varepsilon \right) < \infty. \tag{2.73}$$

To do so, let M and $\varepsilon > 0$ and let us apply Proposition 2.3 with $r = 2$, $x = 2^{2M/p}\varepsilon$, N replaced by 2^M and $q = \left\lfloor 2^{Ma} \right\rfloor$, where $a = 1/(\eta + p/(2-p))$. The convergence of the series $\sum_{M=0}^{\infty} 2^M \beta\left(\left\lfloor 2^{Ma} \right\rfloor\right)$ is equivalent to that of $\sum_{k=1}^{\infty} k^{-1+1/a} \beta(k)$, which is guaranteed by (1.31).

2.6 Proof of Theorem 1.7

Here again, it suffices to check (2.73). Letting $a = \delta/(p(p+\delta-1))$ and $b := p(p+\delta-1)/(2(p-1))$, then applying Proposition 2.1 with N replaced by 2^M, $r = 2$, $x = 2^{2M/p}\varepsilon$, $q = \left\lfloor 2^{Ma} \right\rfloor$ and $R = 2^{Mb}$ gives the convergence of $\sum_{M=0}^{\infty} \mathbb{P}\left(2^{-2M/p} \max_{2 \leqslant n \leqslant 2^M} \left\| \sum_{1 \leqslant i < j \leqslant n} h(X_i, X_j) \right\|_{\mathbb{H}} > \varepsilon\right)$ thanks to (1.31).

References

[1] J. Aaronson, R. Burton, H. Dehling, D. Gilat, T. Hill, and B. Weiss. Strong laws for L- and U-statistics. *Trans. Amer. Math. Soc.*, 348(7):2845–2866, 1996.

[2] M. A. Arcones. The law of large numbers for U-statistics under absolute regularity. *Electron. Comm. Probab.*, 3:13–19, 1998.

[3] H. C. P. Berbee. *Random walks with stationary increments and renewal theory*, volume 112 of *Mathematical Centre Tracts*. Mathematisch Centrum, Amsterdam, 1979.

[4] S. Borovkova, R. Burton, and H. Dehling. Consistency of the Takens estimator for the correlation dimension. *Ann. Appl. Probab.*, 9(2):376–390, 1999.

[5] S. Borovkova, R. Burton, and H. Dehling. Limit theorems for functionals of mixing processes with applications to U-statistics and dimension estimation. *Trans. Amer. Math. Soc.*, 353(11):4261–4318, 2001.

[6] Yu. V. Borovskikh. The theory of U-statistics in a Hilbert space. *Akad. Nauk Ukrain. SSR Inst. Mat. Preprint*, (78):56, 1986.

[7] Yu. V. Borovskikh. The law of large numbers for UH-statistics. *Dokl. Akad. Nauk Ukrain. SSR Ser. A*, (7):6–9, 85, 1988.

[8] S. Bouzebda, A. Nezzal, and T. Zari. Uniform consistency for functional conditional U-statistics using delta-sequences. *Mathematics*, 11(1), 2023.

[9] A. Chakraborty and P. Chaudhuri. A Wilcoxon-Mann-Whitney-type test for infinite-dimensional data. *Biometrika*, 102(1):239–246, 2015.

[10] A. Chakraborty and P. Chaudhuri. Tests for high-dimensional data based on means, spatial signs and spatial ranks. *Ann. Statist.*, 45(2):771–799, 2017.

[11] J. Dedecker and F. Merlevède. Inequalities for partial sums of Hilbert-valued dependent sequences and applications. *Math. Methods Statist.*, 15(2):176–206, 2006.

[12] J. Dedecker and F. Merlevède. The conditional central limit theorem in Hilbert spaces. *Stochastic Process. Appl.*, 108(2):229–262, 2003.

[13] H. Dehling, O. Sh. Sharipov, and M. Wendler. Bootstrap for dependent Hilbert space-valued random variables with application to von Mises statistics. *J. Multivariate Anal.*, 133:200–215, 2015.

[14] H. Dehling and M. Wendler. Central limit theorem and the bootstrap for U-statistics of strongly mixing data. *J. Multivariate Anal.*, 101(1):126–137, 2010.

[15] H. G. Dehling, D. Giraudo, and D. Volny. Some remarks on the ergodic theorem for U-statistics, 2023.

[16] H. G. Dehling and O. Sh. Sharipov. Marcinkiewicz-Zygmund strong laws for U-statistics of weakly dependent observations. *Statist. Probab. Lett.*, 79(19):2028–2036, 2009.

[17] G. Fort, E. Moulines, P. Priouret, and P. Vandekerkhove. A simple variance inequality for U-statistics of a Markov chain with applications. *Statist. Probab. Lett.*, 82(6):1193–1201, 2012.

[18] D. Giraudo. Limit theorems for U-statistics of Bernoulli data. *ALEA Lat. Am. J. Probab. Math. Stat.*, 18(1):793–828, 2021.

[19] P. R. Halmos. The theory of unbiased estimation. *Ann. Math. Statistics*, 17:34–43, 1946.

[20] W. Hoeffding. A class of statistics with asymptotically normal distribution. *Ann. Math. Statistics*, 19:293–325, 1948.

[21] W. J. Hu, L. Wang, B. X. Zhang, and G. C. Wang. A two sample test based on U-statistic for functional data. *Acta Math. Sin. (Engl. Ser.)*, 39(3):533–552, 2023.

[22] S. Jadhav and S. Ma. An association test for functional data based on Kendall's tau. *J. Multivariate Anal.*, 184:Paper No. 104740, 9, 2021.

[23] F. Jiang, R. Wang, and X. Shao. Robust inference for change points in high dimension. *J. Multivariate Anal.*, 193:Paper No. 105114, 23, 2023.

[24] A. Leucht. Degenerate U- and V-statistics under weak dependence: asymptotic theory and bootstrap consistency. *Bernoulli*, 18(2):552–585, 2012.

[25] I. Pinelis. Optimum bounds for the distributions of martingales in Banach spaces. *Ann. Probab.*, 22(4):1679–1706, 1994.

[26] M. L. Puri and V. V. Sazonov. On the central limit theorem in Hilbert space with application to U-statistics. In *Statistical sciences and data analysis (Tokyo, 1991)*, pages 407–413. VSP, Utrecht, 1993.

[27] E. Rio. *Théorie asymptotique des processus aléatoires faiblement dépendants*, volume 31 of *Mathématiques & Applications (Berlin) [Mathematics & Applications]*. Springer-Verlag, Berlin, 2000.

[28] O. Sh. Sharipov, J. Tewes, and M. Wendler. Bootstrap for U-statistics: a new approach. *J. Nonparametr. Stat.*, 28(3):576–594, 2016.

[29] A. van Delft and A. J. Blumberg. A statistical framework for analyzing shape in a time series of random geometric objects, 2023.

[30] L. Wegner and M. Wendler. Robust change-point detection for functional time series based on U-statistics and dependent wild bootstrap, 2023.

[31] K. Yoshihara. Limiting behavior of U-statistics for stationary, absolutely regular processes. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 35(3):237–252, 1976.

[32] H. Zähle. Marcinkiewicz-Zygmund and ordinary strong laws for empirical distribution functions and plug-in estimators. *Statistics*, 48(5):951–964, 2014.