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Population dynamics in fresh product markets with no posted prices

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Abstract

We introduce and mathematically study a conceptual model for the dynamics of buyers population in markets of perishable goods where prices are not posted. Buyers behaviours are driven partly by loyalty to previously visited merchants and partly by sensitivity to merchants intrinsic attractiveness. Moreover, attractiveness evolve in time depending on the relative volumes of buyers, assuming profit/competitiveness optimisation when favourable/unfavourable. While this negative feedback mechanism is a source of instability that promotes oscillatory behaviour, our analysis identifies those critical features that are responsible for the asymptotic stability of stationary states, both in their immediate neighbourhood and globally in phase space. In particular, we show that while full loss of clientele occurs (depending on the initial state) in case of a bounded reactivity rate, it cannot happen when this rate is unbounded and merchants resilience always prevails in this case. Altogether, our analysis provides mathematical insights into the consequences of introducing feedback into buyer-seller interactions and their diversified impacts on the long term levels of clientele in the markets.

November 6, 2023.

1 Introduction

In addition to their historical and cultural importance in rural societies [5], fresh product and foodstuff markets also play a central role in the supply chain of urban populations, including at the megalopolis scale. Some of the largest examples of such wholesale markets are the Rungis International Market near Paris, the Central de Abastos in Mexico City and Mercamadrid in Madrid, which each secure daily food supply for tens of millions of people [17].

Many such markets function based on pairwise buyer-seller interactions, without posted prices and/or include bargaining as a standard practice. In that case, buyer behaviours are known to be driven by some degree of loyalty to the previously visited merchants and, concurrently, by probing for best opportunities [2, 4, 10, 14, 15, 16]. Loyalty simply means the trend to return to the same sellers that were traded with at the previous market instance (e.g. previous week-day or previous week). On the other hand, best opportunity is a more convoluted notion which, in absence of public or solid prices, combines seller (stock-dependent) inclination towards negotiation with product quality and demand, and also sometimes, with accompanying services. In addition, opportunities may also vary in time depending on the merchant reaction to clientele changes [7].

As mathematical modelling is concerned, various attempts to capture the behavioural dynamics in fresh product markets have been developed in the literature. In particular, a fully analysable simple model has been proposed for the time evolution of the preference towards certain merchants

over others [16]. This dynamics includes dependence on profits and shows interesting parameter-dependent bifurcations in the phase portrait of the asymptotic functioning modes. However, the profits are assumed to be constant in time and the model does not include variable feedback from the merchants.

Besides, various agent-based detailed models have been introduced that incorporate several elements of the buyer-seller interactions, including feedback and bargaining. The accompanying numerical simulations have shown proved capacity to adequately reproduce the salient characteristics of the participants behaviour, such as persistent price dispersion and high loyalty [8, 11]. The numerics also have provided quantitative evaluations of various bargaining strategies [3, 12]. Yet, the proposed high level of detail in these models prevents one from a mathematically rigorous and complete analysis.

As a complement to previous studies, this paper aims to provide a mathematical analysis of the buyers population dynamics in presence of an adaptative feedback from the merchants. To that goal, a simple model is introduced (see Section 2.1), in which buyers are partly faithful and partly sensitive to merchants' current offers. Considering that the market venue is sufficiently compact so that all sellers can continuously assess the global buyers population, these offers, and more generally, the seller's attractiveness, vary in time depending on the (relative) volume of clientele. Postulating for simplicity that merchants reaction aims at optimising immediate profit, the resulting feedback is chosen to be of negative nature: a large volume induces a reduction of the attractiveness and conversely, the attractiveness increases when the clientele is scarce.

The presence of a negative feedback loop in the underlying interaction graph of a multidimensional dynamical system has been identified as a necessary condition for a stationary state/periodic orbit to be stable [6, 9, 13]. Here, we show that the local asymptotic stability of the stationary states, which correspond to constant functioning modes depend on the merchants' reactivity to clientele variations: stable when bounded (Proposition 2.1), unstable when the reactivity is arbitrary large (Proposition 2.2). Oscillatory behaviours may also take place in this model, especially when starting initially far away from the stationary points. However, under a suitable condition that ensures monotonic global behaviour of the merchants' attractiveness, these oscillations must asymptotically fade away and the system must approach a stationary state, unless the merchants behaviours tend to be indifferent (Theorem 2.3), which should remain an exceptional event. In addition, the asymptotic functioning mode, namely whether the clientele at every seller is asymptotically evanescent or maximal, may be sensitive to the initial condition (see Fig. 4), making it difficult to systematically predict the outcome of the buyer-seller interactions. Altogether, our analysis provides mathematical insights into the consequences of introducing feedback into buyer-seller interactions and their diversified impacts on the long term levels of clientele in the markets.

The proofs of the main results, which are all stated in Section 2.2, are given in Section 3. Finally, Section 4 concludes with a discussion about various limitations of the model and suggestions for improvements.

2 Population dynamics: definitions and main results

2.1 Definition of the model

We consider the dynamics of the buyers population in fresh product markets for which transaction prices remain private or are only based on declarations by the participants. Composed by $N \in \mathbb{Z}^+ = \{1, 2, \dots\}$ competing merchants, the market is assumed to be open at repetitive and regular events, typically every working day, which can be labelled by the discrete variable $t \in \mathbb{N} = \{0, 1, 2, \dots\}$.

In absence of public information about prices and other quantities such as volumes of transactions and stocks, we focus on the volumes of clientele of each merchant $i \in \{1, \dots, N\}$, which can be measured *a priori* by external observers. At day t , these volumes are represented by the fractions $p_i^t \in [0, 1]$ of the total buyer population present in the market that day. Importantly, a buyer may visit several competing merchants the same day,¹ so that we may have $\sum_{i=1}^N p_i^t > 1$. As specified below, the fractions evolve from one day to the next based on buyers' loyalty and merchant's attractiveness.

Merchant's attractiveness is a soft and relative notion that includes all aspects of economic competitiveness, as perceived by the buyers, namely price, product quality, adaptability, accompanying services, etc. Attractiveness is encapsulated in the positive coefficients $a_i^t \in \mathbb{R}_*^+$. When $a_i^t > 1$, merchant i is competitive and attracts prospecting customers; the higher a_i^t , the stronger the attraction. On the opposite, $a_i^t < 1$ means a non-competitive seller that repels their clientele, again as perceived by the buyers (NB: In the intermediate case $a_i^t = 1$, the fraction p_i^t remain unchanged.) The attractiveness coefficients a_i^t evolve in time depending on the volume of clientele, as explained below.

Altogether, the variable of the dynamical system is the pair

$$(\mathbf{p}, \mathbf{a}) \in M_N := [0, 1]^N \times (\mathbb{R}_*^+)^N,$$

where $\mathbf{p} = (p_1, \dots, p_N)$ and $\mathbf{a} = (a_1, \dots, a_N)$. Starting the first day $t = 0$ from a given initial condition $(\mathbf{p}^0, \mathbf{a}^0)$, the values $(\mathbf{p}^t, \mathbf{a}^t)$ are updated according to the following day-to-day iteration rule

$$\begin{cases} p_i^{t+1} = f_{\alpha, a_i^{t+1}}(p_i^t) \\ a_i^{t+1} = a_i^t g(p_i^t, \frac{1}{N} \sum_{i=1}^N p_i^t) \end{cases} \quad \text{for } i \in \{1, \dots, N\}, \quad (1)$$

where the parametrized one-dimensional maps $f_{\alpha, a}$ are defined by

$$f_{\alpha, a}(p) = \alpha p + (1 - \alpha) f_a(p),$$

and where f_a and g are discussed immediately below. For the sake of convenience, we may regard these iterations as the repeated action of some multidimensional map, *viz.* equation (1) implicitly defines a map $F : M_N \rightarrow M_N$ such that $(\mathbf{p}^{t+1}, \mathbf{a}^{t+1}) = F(\mathbf{p}^t, \mathbf{a}^t) = F^t(\mathbf{p}^0, \mathbf{a}^0)$ for all $t \in \mathbb{N}$.

In $f_{\alpha, a}$, the parameter $\alpha \in [0, 1)$ quantifies the amplitude of loyalty, namely the fraction of the population that is insensitive to merchants' attractiveness and that systematically returns to previously visited merchant(s).² Besides, that p_i^{t+1} in (1) is obtained using $f_{\alpha, a_i^{t+1}}$ (and not f_{α, a_i^t}) emphasizes on separating the contribution of loyalty to the one due to the merchant's attractiveness at the current day. Throughout the paper, α remains unchanged. For the sake of notation, we shall not mention any explicit dependence on this parameter.

Assumptions on the maps f_a . In order to comply with the desired buyer behaviours' dependence on merchants' attractiveness, the parametrized maps f_a in the contagion term of the first equation in (1) (which are all defined from $[0, 1]$ into itself) must be such that (see Fig. 1 for an illustration)

- $f_a(p) > p$ for every $p \in [0, 1)$ and $f_a(1) = 1$ when $a > 1$,

¹Visits to multiple merchants means buying, or merely bargaining about, different goods of the same nature (eg. fruits or vegetables) at different places the same day. In particular, this feature has been identified as a mechanism of progressive individual transfer from buying all goods of the same nature at a given seller to a different one [2].

²That α is independent of i reflects that loyalty is a uniform feature of the population which is assumed to be independent of the visited merchant.

- $f_a(p) < p$ for every $p \in (0, 1]$ and $f_a(0) = 0$ when $a < 1$ (and $f_1 = \text{Id}$).
- $f_{a_1}(p) < f_{a_2}(p)$ for every $a_1 < a_2 \in \mathbb{R}_*^+$ and $p \in (0, 1)$.

For the sake of analysis and simplicity, the following technical assumptions are also required.

- For every $a \in \mathbb{R}_*^+$, f_a is an increasing C^2 map on $[0, 1]$. Moreover, we have³

$$f'_a(0) = f'_{\frac{1}{a}}(1) = a, \quad \forall a \in (0, 1) \quad \text{and} \quad \sup_{a \in (0, 1)} \max\{|f''_a(0)|, |f''_{\frac{1}{a}}(1)|\} < +\infty.$$

- For every $x \in [0, 1]$, the map $a \mapsto f_a(x)$ is continuous in \mathbb{R}_*^+ . The map $a \mapsto \|f'_a\|_\infty := \max_{x \in [0, 1]} f'_a(x)$ is continuous in \mathbb{R}_*^+ .

In particular, all the conditions above hold for the following simple quadratic example (see fig. 1 for illustrations)⁴

$$f_a(x) = \begin{cases} ax + 0.9(1-a)x^2 & \text{if } a \leq 1 \\ 1 - \frac{1}{a}(1-x) - 0.9(1 - \frac{1}{a})(1-x)^2 & \text{if } a \geq 1 \end{cases} \quad (2)$$

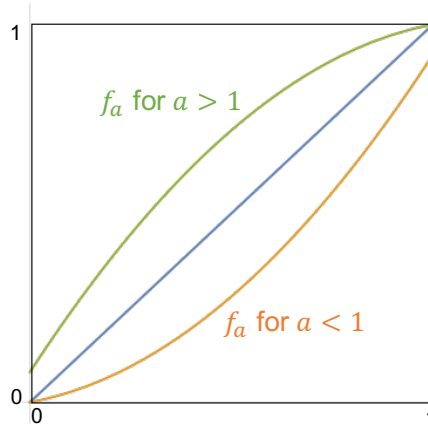


Figure 1: Illustration of graphs of the parametrized functions f_a defined by (2).

Basic assumptions on the map g . The second equation in (1) assumes that the merchants can evaluate the volume of clientele of their competitors in the market and that they react accordingly.⁵ Considering a negative feedback as indicated in the Introduction, attractiveness should decrease when the clientele is larger than the mean attendance, and conversely, it should increase in the case of clientele deficit. Accordingly, the map $g : [0, 1]^2 \rightarrow \mathbb{R}_*^+$ must be a continuous map such that $g(p, p) = 1$ for all $p \in [0, 1]$ and

$$(g(p, q) - 1)(q - p) > 0, \quad \forall p \neq q \in [0, 1]. \quad (3)$$

³Notice that this assumption on the value of the derivative is merely a matter of parametrization.

⁴Instead of 0.9 any constant in $(0, 1)$ could have been chosen in this example. We believe that the closer this constant is to 1, the more elaborated is the transient phenomenology of the system (1).

⁵In absence of any other public information such as sellers' background costs and current stocks, we assume that attractiveness is only influenced by the volumes of clientele. For simplicity, we also assume that all sellers react the same way to clientele changes.

Additional conditions on g will be imposed below in order to control the asymptotic dynamics of the system (1), in particular the fixed points' local stability and the global features of the dynamics.

Strictly speaking, we do not need to exclude that g vanishes on $[0, 1]^2$. It suffices that g remains positive in all orbits of (1), namely that we have

$$g(p_i, \frac{1}{N} \sum_{i=1}^N p_i) > 0, \quad \forall \mathbf{p} \in [0, 1]^N, \quad i \in \{1, \dots, N\} \text{ and } N \geq 2. \quad (4)$$

This weaker assumption allows one to consider the following linear example

$$g(p, q) = 1 + q - p. \quad (5)$$

2.2 Main features of the dynamics

We begin this section by listing some preliminary properties that provide a basis for the main results that follow.

Well-defined dynamics. The convex combination in the first equation of (1), together with the property $f_a([0, 1]) \subset [0, 1]$, and the inequality (4) imply that for every initial condition $(\mathbf{p}^0, \mathbf{a}^0) \in M_N$, the subsequent orbit $\{(\mathbf{p}^t, \mathbf{a}^t)\}_{t \in \mathbb{N}}$ is well-defined, and we have $(\mathbf{p}^t, \mathbf{a}^t) \in M_N$ for all $t \in \mathbb{N}$.

Anticipating a claim in Theorem 2.3, notice that the condition (9) on g in that statement ensures that the coefficients \mathbf{a}^t remain bounded in every orbit, which brings additional consistency to the modelling under consideration.

We aim at characterizing the asymptotic behaviour of orbits in this system. To that goal, we begin by characterizing the fixed points of F , namely the components of the stationary orbits.

Identification of the fixed points. Equation (1) implies that the \mathbf{p} -coordinates of every fixed point $F(\mathbf{p}, \mathbf{a}) = (\mathbf{p}, \mathbf{a})$ must satisfy

$$g(p_i, \frac{1}{N} \sum_{i=1}^N p_i) = 1, \quad \forall i \in \{1, \dots, N\}.$$

The condition (3) imposes that these coordinates must be all equal. Moreover, $\alpha \neq 1$ imposes that, individually, they must also be fixed points of the maps f_{a_i} themselves.

We conclude that the fixed points of F with $\mathbf{a} \neq \mathbf{1}$, where $\mathbf{1} = (1, \dots, 1)$, write either $(\mathbf{0}, \mathbf{a})$ for every $\mathbf{a} \in (0, 1]^N$ where $\mathbf{0} = (0, \dots, 0)$, or $(\mathbf{1}, \mathbf{a})$ for every $\mathbf{a} \in [1, +\infty)^N$. In particular, in absence of additional information other than clientele volumes, this means that the values of the attractiveness coefficients in stationary orbits are indifferent.

Synchronized dynamics. As a particular instance of coupled map system with mean-field coupling [1], the dynamics (1) commutes with the simultaneous permutations of buyers fractions and of attractiveness coefficients, *viz.* if the sequence $\{(\mathbf{p}^t, \mathbf{a}^t)\}_{t \in \mathbb{N}}$ is an orbit of (1), then for every permutation π of $\{1, \dots, N\}$, the sequence $\{(\pi \mathbf{p}^t, \pi \mathbf{a}^t)\}_{t \in \mathbb{N}}$ where

$$\pi \mathbf{p} = (p_{\pi(1)}, \dots, p_{\pi(N)}) \quad \text{and} \quad \pi \mathbf{a} = (a_{\pi(1)}, \dots, a_{\pi(N)}),$$

is also an orbit of this system.

This symmetry implies in particular that the subset of M_N of points where $\mathbf{p} = (p, \dots, p)$ and $\mathbf{a} = (a, \dots, a)$, is invariant under the action of F . Moreover, the dynamics in this set - the so-called

synchronized dynamics - reduces to the iterations of the one-dimensional map $f_{\alpha,a}$ (since a remains constant). The synchronized dynamics is very simple; for every $a \neq 1$, every orbit asymptotically converges to 0 (when $a < 1$) or to 1. In the original phase space, this corresponds to the convergence to one of the fixed points $(\mathbf{0}, \mathbf{a})$ or $(\mathbf{1}, \mathbf{a})$ where again $\mathbf{a} = (a, \dots, a)$. For $a = 1$, we have $f_{\alpha,1} = \text{Id}$, thus every orbit is stationary.

Clearly, the map F is invertible, ie. $(\mathbf{p}^t, \mathbf{a}^t)$ in (1) can be uniquely determined by $(\mathbf{p}^{t+1}, \mathbf{a}^{t+1})$. Moreover, its inverse also commutes with the simultaneous permutations of the \mathbf{p} - and \mathbf{a} -coordinates. As a consequence, no orbit can eventually become synchronized if its initial condition does not satisfy this property.⁶ Accordingly, the description of the synchronized dynamics above cannot apply to non-synchronized orbits. A proper analysis is required in order to determine asymptotic behaviours in this case. The rest of this section presents the main results of such an analysis.

Global symmetries. As a preliminary comment prior to the presentation of the results, notice the following symmetries of the sets of maps involved in (1).

- Given a parametrized map $f_a : [0, 1] \rightarrow [0, 1]$, let \bar{f}_a be defined by

$$\bar{f}_a(x) = 1 - f_a(1 - x), \quad \forall x \in [0, 1], a \in \mathbb{R}_*^+.$$

Clearly, when the family $\{f_a\}_{a \in \mathbb{R}_*^+}$ satisfies all assumptions in Section 2.1, the family $\{\bar{f}_a\}_{a \in \mathbb{R}_*^+}$ also satisfies these assumptions. Notice also that we have $\bar{\bar{f}}_a = f_a$ for the map defined by (2).

- Given a map $g : [0, 1]^2 \rightarrow \mathbb{R}_*^+$, let Sg be defined by

$$Sg(p, q) = \frac{1}{g(1-p, 1-q)}, \quad \forall (p, q) \in [0, 1]^2. \quad (6)$$

Again, if g satisfies all assumptions in Section 2.1, then so does the image Sg .

There is an obvious connection between the orbits of (1) and those of the dynamical system that results when substituting f_a by \bar{f}_a and g by Sg ; the sequence $\{(\mathbf{p}^t, \mathbf{a}^t)\}_{t \in \mathbb{N}}$ is an orbit of the former iff $\{(1 - \mathbf{p}^t, \frac{1}{\mathbf{a}^t})\}_{t \in \mathbb{N}}$, where

$$1 - \mathbf{p} = (1 - p_1, \dots, 1 - p_N) \quad \text{and} \quad \frac{1}{\mathbf{a}} = \left(\frac{1}{a_1}, \dots, \frac{1}{a_N}\right),$$

is an orbit of the latter. This property will be repeatedly employed below in order to shorten the analysis.

2.2.1 Local asymptotic stability

The first equation in (1) and the basic properties of the maps f_a imply that in any orbit for which there exists $T \in \mathbb{N}$ such that

$$\sup_{t > T} \max_{i \in \{1, \dots, N\}} a_i^t < 1 \quad \text{or} \quad \inf_{t > T} \min_{i \in \{1, \dots, N\}} a_i^t > 1, \quad (7)$$

all coordinates p_i^t must be monotone in t for $t \geq T$, and hence they must asymptotically converge (to 0 in the first case, to 1 otherwise). However, that such behaviours exist is not clear because

⁶Indeed, if, at any given $t \in \mathbb{N}$, the coordinates of \mathbf{p}^t and \mathbf{a}^t do not depend on i , then the same property must hold for $(\mathbf{p}^0, \mathbf{a}^0)$.

in every iteration, some of the \mathbf{a} -coordinates must increase and other(s) must decrease (unless the \mathbf{p} -coordinates are all equal). Therefore, the variations of these coordinates, and the displacements towards 1 especially, need to asymptotically vanish in order to make sure that (7) eventually holds.

This is the case provided that the merchants' reactivity to clientele changes, that is the amplitude of the attractiveness variations, remains bounded in this domain. More precisely, our first result claims asymptotic convergence to the continuum of fixed points with $\mathbf{p} = \mathbf{0}$ (or $\mathbf{p} = \mathbf{1}$) when starting initially close to that set. The asymptotic value of \mathbf{a} is itself hardly predictable from the initial condition (due to degeneracy inside the continuum), because no mechanism for attractiveness selection has been inserted into the dynamics.

Proposition 2.1. *Assume the existence of $K \in \mathbb{R}_*^+$ such that⁷*

$$|g(p, q) - 1| \leq K \max\{p, q\}, \quad \forall (p, q) \in (0, \frac{1}{2})^2. \quad (8)$$

Then for every $\mathbf{a}^0 \in (0, 1)^N$, there exists $\epsilon > 0$ such that for every $\mathbf{p}^0 \in [0, \epsilon]^N$, the subsequent orbit of (1) satisfies

$$\sup_{t \in \mathbb{N}} \max_{i \in \{1, \dots, N\}} a_i^t < 1 \quad \text{and} \quad \lim_{t \rightarrow +\infty} \mathbf{a}^t \text{ exists,}$$

and hence

$$\mathbf{p}^t \in [0, \epsilon]^N, \quad \forall t \in \mathbb{N} \quad \text{and} \quad \lim_{t \rightarrow +\infty} \mathbf{p}^t = \mathbf{0}.$$

Similarly, assume that the symmetric map Sg defined in (6) satisfies (8). Then for every $\mathbf{a}^0 \in (1, +\infty)^N$, there exists $\epsilon > 0$ such that for every $\mathbf{p}^0 \in (1 - \epsilon, 1]^N$, the subsequent orbit of (1) (still with $f_{\alpha, a}$ and g) satisfies

$$\inf_{t \in \mathbb{N}} \min_{i \in \{1, \dots, N\}} a_i^t > 1 \quad \text{and} \quad \lim_{t \rightarrow +\infty} \mathbf{a}^t \text{ exists,}$$

and hence

$$\mathbf{p}^t \in (1 - \epsilon, 1]^N, \quad \forall t \in \mathbb{N} \quad \text{and} \quad \lim_{t \rightarrow +\infty} \mathbf{p}^t = \mathbf{1}.$$

For the proof, see Section 3.1. As an example, notice that both the linear map g defined by (5) and its image Sg satisfy condition (8).

As market modelling is concerned, Proposition 2.1 can be interpreted as a robustness statement when close to vanishing or maximal clientele. In particular, when Sg satisfies (8), it shows that the introduction of an additional merchant into a market in which each seller already attracts almost all buyers, does not perturb the buyers' asymptotic behaviours, provided that the newcomer also initially attracts a large part of the population. An illustration of the dynamics for $N = 2$ when starting initially close to $\mathbf{p} = \mathbf{1}$ with all both coefficients above 1, is given in Fig. 2. Notice that in this example, the initial condition does not satisfy the conditions of Proposition 2.1 because $a_1^t < 1$ during some transient time interval. This suggests that convergence to $\mathbf{p} = \mathbf{1}$ may hold even if some merchants temporary become repulsive.

⁷The value $\frac{1}{2}$ here is arbitrary. Any value in $(0, 1)$ can be chosen.

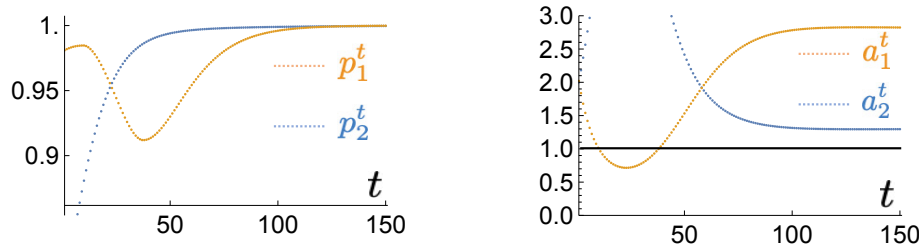


Figure 2: Illustration for $N = 2$ of the (local) asymptotic stability of the set of fixed points with $\mathbf{p} = \mathbf{1}$. The pictures represent the \mathbf{p}^t - and \mathbf{a}^t -coordinates time series of the orbit of (1) issued from $(\mathbf{p}^0, \mathbf{a}^0) = (0.981, 0.8, 2.02, 2)$, for f_a as in (2), $\alpha = 0.9$ and g defined by (5). Prompt convergence towards $(\mathbf{1}, \mathbf{a})$ (with both $a_i > 1$) is evident, despite temporary oscillations of p_1^t due a_1^t passing briefly below 1. This example can be regarded as representing the dynamics that results from introducing of a concurrent seller (indexed by 2) to an existing isolated one (seller 1) whose initial clientele fraction and attractiveness coefficients are similar, although more favourable. The initial strong increase of a_2^t reveals a rapid reaction from seller 2 in order to attract more customers and to bring the fraction p_2^t above p_1^t . When that happens, the roles and the variations of the a_i^t are exchanged and monotone convergence follows.

2.2.2 Role of condition (8) on fixed points stability

Condition (8) in Proposition 2.1 and its analogue for the map Sg are essential for the local stability of the continua of fixed points. Indeed, the following counter-example of the map $g : (0, 1] \times [0, 1] \rightarrow \mathbb{R}_*^+$ defined by

$$g(p, q) = \frac{q}{p},$$

shows that instability can result when this condition fails.⁸

Proposition 2.2. *For every $\mathbf{a}^0 \in (0, 1)^N$ and every $\mathbf{p} \in (0, 1)^N$ such that $(\mathbf{p}, \mathbf{a}^0)$ is not synchronized, there exist $T \in \mathbb{N}$ and $\Delta > 0$ such that for every $\delta \in (0, \Delta)$ we have*

$$\max_{i \in \{1, \dots, N\}} a_i^T > 1,$$

for the orbit issued from the initial condition $(\delta \mathbf{p}, \mathbf{a}^0)$ and generated by (1) with $g(p, q) = \frac{q}{p}$.

The proof is given in Section 3.2. As market modelling is concerned, Proposition 2.2 can be interpreted as merchants' resilience against clientele vanishing. Unlike when (8) holds, reactive changes in the attractiveness coefficients can become sufficiently large to counteract population decay and make the sellers become eventually attractive. When this happens, since the symmetric map Sg satisfies (8) in this example, asymptotic convergence to full clientele is expected from Proposition 2.1 and as illustrated on Fig. 3.

2.2.3 Asymptotic behaviour of orbits

Back to assuming that g satisfies (8), one may question about the main features of the dynamics (1) beyond fixed points' local stability. Because of the merchant's negative feedback, oscillatory

⁸Independently, on $(0, 1] \times [0, 1]$, this map satisfies all the assumptions imposed on g in Section 2.1. In this case, the dynamics of (1) is well-defined in $(0, 1]^N \times (\mathbb{R}_*^+)^N$, which is invariant.

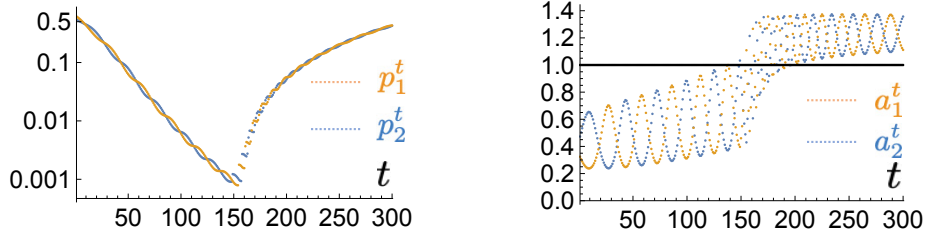


Figure 3: Illustration for $N = 2$ of the instability of the set of fixed points with $\mathbf{p} = \mathbf{0}$ for $g(p, q) = \frac{q}{p}$ which does not satisfies the condition (8) ((NB: Notice however that Sg satisfies (8)). The pictures represent the \mathbf{p}^t - and \mathbf{a}^t -coordinates time series of the orbit issued from $(\mathbf{p}^0, \mathbf{a}^0) = (0.546, 0.616, 0.473, 0.324)$, for f_a as in (2) and $\alpha = 0.9$. Notice the logarithmic scale on the left picture. The figure clearly indicates that a transient oscillating instability of the a_i^t occurs that eventually brings the two coefficients above 1. Accordingly, after a first phase of exponential decay to 0, both coordinates p_i^t start to grow and eventually converge to 1 (as expected when both $a_i^t > 1$).

behaviours may occur over arbitrary large time intervals if the coefficients a_i^t cross 1 arbitrarily many times.⁹ As illustrated in Fig. 4, this may be the case when close to the boundary of the fixed points' basins. In fact, we do not exclude that oscillations last forever (if in addition to infinitely many crossings of 1, the coefficients a_i approach 1). Yet, the central result of this paper states that, provided that g satisfies some weak form of concavity, more precisely that its mean value in phase space cannot exceed 1, permanent oscillations can happen only if the attractiveness coefficients do not remain away from 1.

Theorem 2.3. *In addition to satisfying condition (8), assume that g satisfies that following concavity inequality*

$$\frac{1}{N} \sum_{i=1}^N g(p_i, \frac{1}{N} \sum_{i=1}^N p_i) \leq 1, \quad \forall \mathbf{p} \in (0, 1)^N. \quad (9)$$

Then, for every orbit $\{(\mathbf{p}^t, \mathbf{a}^t)\}_{t \in \mathbb{N}}$ of (1), we have

$$\sup_{t \in \mathbb{N}} \max_{i \in \{1, \dots, N\}} a_i^t < +\infty.$$

Moreover, every orbit that satisfies

$$\liminf_{t \rightarrow +\infty} \min_{i \in \{1, \dots, N\}} |a_i^t - 1| > 0.$$

must asymptotically converge to a fixed point.

The fact that non-convergence to a fixed point requires that some attractiveness coefficient(s) must accumulate at 1 suggests that it can only occur for exceptional initial conditions. No such behaviour has ever been observed in simulations.

⁹In order to see that this is plausible, let $N = 2$ for simplicity and assume that we have $a_1^t < 1 < a_2^t$ for a number of consecutive values of t . Over this time interval, $p_1^t \searrow 0$ and $p_2^t \nearrow 1$. Depending on the interval length and on the values of p_i^t at the beginning of this interval, we may eventually have $p_1^t < p_2^t$. This in tun implies $a_1^t \searrow$ and $a_2^t \nearrow$. If this trend also lasts sufficiently long, we may eventually have $a_2^t < 1 < a_1^t$ and the whole process may repeat with the role of 1 and 2 exchanged.

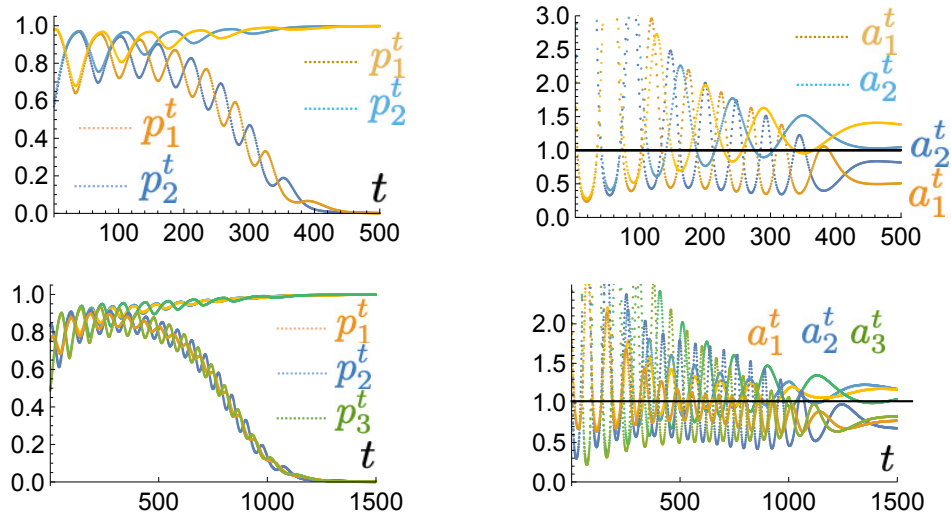


Figure 4: Illustration for $N = 2$ (top) and $N = 3$ (bottom) of the transient oscillatory behaviours when initially close to the boundary of the fixed points' basins. Similar time series as in previous figures but now for two orbits issued from close initial conditions across the boundary of the fixed points' basins: For $N = 2$, $a_2^0 = 0.57$ for the orbit that converges to $\mathbf{p} = \mathbf{0}$ (darker colours) and $a_2^0 = 0.6$ for the other orbit (lighter colours). The other initial coordinates and the parameters are as in Fig. 2. For $N = 3$, $p_3^0 = 0.487$ for the orbit that converges to $\mathbf{p} = \mathbf{0}$ and $p_3^0 = 0.497$ for the other orbits. All other coordinates and parameters are equal. As application to market modelling is concerned, the figure indicates that the outcome that results from introducing a new competitor in the market - either extinction of the buyers' population or convergence to maximal population - can be highly sensitive to small variations in the initial coordinates and may only emerge after a transient of substantial duration.

As application to modelling is concerned, Theorem 2.3 states that, possibly after a transient of variable duration, the market must eventually stabilize to a stationary functioning mode (that may include no clientele!), unless some sellers eventually become repeatedly indifferent, an event that we believe to be highly unlikely.

Theorem 2.3 follows from a more technical statement, namely Proposition 3.6, which is stated and proved in Section 3.3. For instance, the linear map defined (5) satisfies both conditions (8) and (9). Moreover, in the case $g(p, q) = \frac{q}{p}$, the map Sg also satisfies these two conditions (see in particular Section 3.2). Up to a minor adjustment (see footnote 12), the proof of Theorem 2.3 also applies in this example.

3 Stability analysis and proofs of the statements

3.1 Proof of Proposition 2.1

We only prove the first statement of the Proposition, since the one that involves Sg immediately follows by applying the global symmetries, using that the stability of the continuum of fixed points with $\mathbf{p} = \mathbf{1}$ for the original f_a and g is equivalent to the one with $\mathbf{p} = \mathbf{0}$ for the system with \tilde{f}_a and Sg .

The proof starts with the following crucial bootstrap statement that will also serve for future purposes.

Claim 3.1. *Given $a \in (0, 1)$ and $\beta \in (\alpha + a(1 - \alpha), 1)$, there exists $\zeta_{a,\beta} > 0$ such that for every $t \in \mathbb{N}$ such that*

$$\begin{cases} p_i^t \leq \frac{(1-\beta)\zeta}{K(a-\zeta)} \\ a_i^t \leq a - \zeta \end{cases} \quad \text{for } i \in \{1, \dots, N\},$$

where K is given by the condition (8) and $\zeta \in (0, \zeta_{a,\beta})$, we have

$$\begin{cases} p_i^{t+1} \leq \frac{(1-\beta)\beta\zeta}{K(a-\beta\zeta)} \\ a_i^{t+1} \leq a - \beta\zeta \end{cases} \quad \text{for } i \in \{1, \dots, N\}.$$

Proof of the Claim. Let $\zeta_{a,\beta} > 0$ be such that $\frac{(1-\beta)\zeta}{K(a-\zeta)} < \frac{1}{2}$ for all $\zeta \in (0, \zeta_{a,\beta})$. The definition of a_i^{t+1} and inequality (8) imply that we have

$$a_i^{t+1} \leq a_i^t \left(1 + \frac{(1-\beta)\zeta}{a-\zeta} \right) \leq a - \beta\zeta$$

Moreover, given the choice of β , let $\epsilon > 0$ be sufficiently small so that $\alpha + (1 - \alpha)(a + \epsilon) \leq (1 - \epsilon)\beta$. By continuity of the derivative f'_a and the fact that $f'_a(0) = a$, let $\zeta_{a,\beta} > 0$ be even smaller if necessary so that we have

$$f_{a_i^{t+1}}(x) \leq f_a(x) \leq (a + \epsilon)x,$$

when $x \leq \frac{(1-\beta)\zeta_{a,\beta}}{K(a-\zeta_{a,\beta})}$. Let again $\zeta_{a,\beta}$ be even smaller if necessary so that we also have

$$1 - \epsilon \leq \frac{a - \zeta}{a - \beta\zeta}, \quad \text{for all } \zeta \in (0, \zeta_{a,\beta}).$$

Altogether, we then have for every $p_i^t \leq \frac{(1-\beta)\zeta}{K(a-\zeta)}$ with $\zeta \in (0, \zeta_{a,\beta})$

$$p_i^{t+1} \leq \frac{(1-\epsilon)(1-\beta)\beta\zeta}{K(a-\zeta)} \leq \frac{(1-\beta)\beta\zeta}{K(a-\beta\zeta)},$$

as desired. □

Proof of Proposition 2.1. Given $\mathbf{a}^0 \in (0, 1)^N$, let $a \in (\max_i a_i^0, 1)$, $\beta \in (\alpha + a(1 - \alpha), 1)$ and

$$\zeta < \min \left\{ \zeta_{a,\beta}, a - \max_i a_i^0 \right\}.$$

Then for $\max_i p_i^0 < \frac{(1-\beta)\zeta}{K(a-\zeta)}$, Claim 3.1 implies that we have

$$\sup_{t \in \mathbb{N}} \max_{i \in \{1, \dots, N\}} a_i^t \leq a.$$

Thanks to the properties of the map f_a in the neighbourhood of 0, this inequality not only implies that all p_i^t tend to 0 but that this convergence is exponential, namely there exists $C \in \mathbb{R}^+$ and $b \in (0, 1)$ such that

$$\max_{i \in \{1, \dots, N\}} p_i^t \leq Cb^t, \quad \forall t \in \mathbb{N}.$$

According to the inequality (8), this yields the inequality

$$\max_{i \in \{1, \dots, N\}} |a_i^{t+1} - a_i^t| \leq aKCb^t \quad \forall t \in \mathbb{N},$$

which in turns implies that all sequences $\{a_i^t\}_{t \in \mathbb{N}}$ are Cauchy sequences, and hence ensures the existence of the limits $\lim_{t \rightarrow \infty} a_i^t$. \square

3.2 Proof of Proposition 2.2

The proof relies on the following preliminary statement.

Claim 3.2. *Assume that $\mathbf{p}^0 \in (0, 1)^N$ and that $(\mathbf{p}^0, \mathbf{a}^0)$ is not synchronized. Then, in the orbit issued from $(\mathbf{p}^0, \mathbf{a}^0)$, there are infinitely many instances of $t \in \mathbb{N}$ such that the coordinates of \mathbf{p}^t are not all equal.*

Proof of the Claim. Assume that all coordinates of \mathbf{p}^t are equal. Then we must have $\mathbf{a}^{t+1} = \mathbf{a}^t$. Moreover, the coordinates of \mathbf{a}^t cannot be all equal. Otherwise, $(\mathbf{p}^t, \mathbf{a}^t)$ would have to be synchronized, and hence $(\mathbf{p}^0, \mathbf{a}^0)$ would have to be synchronized too as we showed in Section 2.2.

In addition, that some of the coordinates of $\mathbf{a}^t = \mathbf{a}^{t+1}$ are distinct, strict monotonicity of $a \mapsto f_{\alpha, a}(p)$ for every $p \in (0, 1)$ and the fact that we must have $\mathbf{p}^t \notin \{\mathbf{0}, \mathbf{1}\}$,¹⁰ imply that some of the coordinates of \mathbf{p}^{t+1} must be distinct. The Claim then easily follows. \square

For the proof of the Proposition, consider the product $\pi^t := \prod_{i=1}^N a_i^t$. Expression (1) implies that we have

$$\frac{\pi^{t+1}}{\pi^t} = \frac{\left(\sum_{i=1}^N p_i^t\right)^N}{N^N \prod_{i=1}^N p_i^t}, \quad \forall t \in \mathbb{N}.$$

Moreover, the AM-GM inequality exactly claims that

$$\frac{\left(\sum_{i=1}^N p_i\right)^N}{N^N \prod_{i=1}^N p_i} \geq 1, \quad \forall \mathbf{p} \in (0, 1)^N,$$

with equality iff all \mathbf{p} -coordinates are equal, viz. iff $\frac{p_i}{p_1} = 1$ for all $i \in \{2, \dots, N\}$. Therefore, in every orbit, the sequence $\{\pi^t\}_{t \in \mathbb{N}}$ is non-decreasing. Actually, together with Claim 3.2, this argument shows that the sequence cannot remain constant in non-synchronized orbits. Moreover, assume that there exists $\epsilon > 0$ such that

$$\min_{i \in \{2, \dots, N\}} \left| \frac{p_i^t}{p_1^t} - 1 \right| \geq \epsilon, \quad (10)$$

for an arbitrary large number of instances of t . At these instances, we have $\frac{\pi^{t+1}}{\pi^t} \geq 1 + \epsilon'$ for some $\epsilon' > 0$, implying altogether that the sequence $\{\pi^t\}_{t \in \mathbb{N}}$ must grow with exponential rate. In particular, one can make sure that $\pi^t > 1$, and therefore $\max_i a_i^t > 1$ as desired, for t sufficiently large.

We are going to prove that the lower bound (10) holds for an arbitrary large number of instances of t , in the system for which the \mathbf{p} -coordinates are iterated using the linearized dynamics at $\mathbf{p} = \mathbf{0}$ for $\mathbf{a} \in (0, 1)^N$ and $\alpha = 0$. Then we shall use that under the assumption (otherwise there is nothing to prove)

$$\max_{i \in \{1, \dots, N\}} a_i^t < 1, \quad \forall t \in \mathbb{N},$$

¹⁰Indeed, by assumptions on the maps f_a and since $\alpha \in [0, 1)$, the only way that $\mathbf{p}^t \in \{\mathbf{0}, \mathbf{1}\}$ for $t > 0$ is that the same condition holds for $t = 0$, which is impossible given the current assumption on \mathbf{p}^0 .

an arbitrary large number of iterates under (1) must remain close to the corresponding ones under the linearization, provided that \mathbf{p}^0 lies in a sufficiently small neighbourhood of $\mathbf{0}$ (which depends on \mathbf{a}^0 and α).

Given that $f'_a(0) = a$ for $a < 1$, for $g(p, q) = \frac{q}{p}$, the system obtained by linearizing the first equation in (1) at $\mathbf{p}^t = \mathbf{0}$ and for $\mathbf{a}^{t+1} \in (0, 1)^N$ and $\alpha = 0$ is given by

$$\begin{cases} p_i^{t+1} = a_i^{t+1} p_i^t \\ a_i^{t+1} = a_i^t \frac{\sum_{i=1}^N p_i^t}{N p_i^t} \end{cases}, \quad \forall i \in \{1, \dots, N\}. \quad (11)$$

Letting $\rho_i^t = \frac{p_i^t}{p_1^t}$ and $\gamma_i^t = \frac{a_i^t}{a_1^t}$, (11) implies the following iteration rule for the variables $(\rho_2, \dots, \rho_N, \gamma_2, \dots, \gamma_N)$

$$\begin{cases} \rho_i^{t+1} = \gamma_i^t \\ \gamma_i^{t+1} = \frac{\gamma_i^t}{\rho_i^t} \end{cases} \quad \forall i \in \{2, \dots, N\}$$

Accordingly, we must have

$$\rho_i^{t+3} = \frac{1}{\rho_i^t} \quad \text{and} \quad \gamma_i^{t+3} = \frac{1}{\gamma_i^t} \quad \forall t \in \mathbb{N},$$

hence, the sequence $\{\rho_i^t\}_{t \in \mathbb{N}}$ must be periodic with period (at most) 6. Therefore, if the coordinates of \mathbf{p}^0 are not all equal, there must exist $i \in \{1, \dots, N\}$ such that

$$\rho_i^{6k} = \rho_i^0 \neq 1, \quad \forall k \in \mathbb{N},$$

proving that (10) holds for all $t \in 6\mathbb{N}$ as desired.

Consequently, for every $\mathbf{a}^0 \in (0, 1)^N$ and every $\mathbf{p}^0 \in (0, 1)^N$ whose coordinates are not all equal, there exist $t(\mathbf{p}^0, \mathbf{a}^0) \in \mathbb{N}$ such that

$$\max_{t \in \{0, \dots, t(\mathbf{p}^0, \mathbf{a}^0) - 1\}} \max_{i \in \{1, \dots, N\}} a_i^t \leq 1 < \max_{i \in \{1, \dots, N\}} a_i^{t(\mathbf{p}^0, \mathbf{a}^0)},$$

for the orbit generated by (11). Notice that $t(\epsilon \mathbf{p}, \mathbf{a})$ does not depend on $\epsilon \in \mathbb{R}_*^+$ because (11) commutes with the homogeneous scaling $\epsilon \mapsto \epsilon \mathbf{p}$ of the \mathbf{p} -coordinates.

Let F_{lin} be the multidimensional map associated with (11), ie. $(\mathbf{p}^{t+1}, \mathbf{a}^{t+1}) = F_{\text{lin}}(\mathbf{p}^t, \mathbf{a}^t)$. Given $\mathbf{x} = (x_1, \dots, x_N) \in \mathbb{R}^N$ let

$$\|\mathbf{x}\|_N := \max_{i \in \{1, \dots, N\}} |x_i|.$$

Claim 3.3. *Given $\mathbf{a} \in (0, 1)^N$, $t \in \mathbb{N}$ and $\epsilon > 0$, there exists $\delta > 0$ such that for every $\mathbf{p} \in (0, 1)^N$ satisfying $\|\mathbf{p}\|_N < \delta$, we have*

$$\max_{k \in \{1, \dots, t\}} \|F^k(\mathbf{p}, \mathbf{a}) - F_{\text{lin}}^k(\mathbf{p}, \mathbf{a})\|_{2N} < \epsilon,$$

provided that all \mathbf{a} -coordinates of each iterate $F_{\text{lin}}^k(\mathbf{p}, \mathbf{a})$ lie in $(0, 1)$.

This statement, whose proof will be given below, helps to complete the proof of the Proposition. Let $\mathbf{a}^0 \in (0, 1)^N$ and $\mathbf{p} \in (0, 1)^N$ be such that $(\mathbf{p}, \mathbf{a}^0)$ is not synchronized. Let $t_0 \in \mathbb{N}$ be the first time such that the coordinates of \mathbf{p}^{t_0} are not all equal. We may assume that $\mathbf{a}^{t_0} \in (0, 1)^N$, otherwise there is nothing to prove. For the sake of notation, we also assume $t_0 = 0$. Now, let

$$\epsilon_0 = \max_{i \in \{1, \dots, N\}} a_{\text{lin}, i}^{t(\mathbf{p}, \mathbf{a}^0)} - 1,$$

where $\mathbf{a}_{\text{lin}}^t$ are the \mathbf{a} -coordinates of $F_{\text{lin}}^t(\mathbf{p}, \mathbf{a}^0)$. Let also $\Delta > 0$ be the quantifier δ given by Claim 3.3 when $t = t(\mathbf{p}, \mathbf{a}^0)$ and $\epsilon = \epsilon_0$. For every $\delta \in (0, \Delta)$, the orbit of F issued from $(\delta\mathbf{p}, \mathbf{a}^0)$ certainly satisfies

$$\max_{t \in \{1, \dots, t(\mathbf{p}, \mathbf{a}^0)\}} \max_{i \in \{1, \dots, N\}} a_i^t > 1,$$

as desired, where \mathbf{a}^t are the \mathbf{a} -coordinates of $F^t(\delta\mathbf{p}, \mathbf{a}^0)$. The proof of Proposition 2.2 is complete. \square

Proof of Claim 3.3. The proof is standard and proceeds by induction based on the following properties.

- (i) For every $(\mathbf{p}, \mathbf{a}) \in (0, 1)^N \times (\mathbb{R}_*^+)^N$ and $\epsilon > 0$, there exists $\delta > 0$ such that for every $(\mathbf{p}', \mathbf{a}')$ satisfying $\|(\mathbf{p}', \mathbf{a}') - (\mathbf{p}, \mathbf{a})\|_{2N} < \delta$, we have

$$\|F(\mathbf{p}', \mathbf{a}') - F(\mathbf{p}, \mathbf{a})\|_{2N} < \epsilon,$$

as a direct consequence of the facts that $a \mapsto f_a(x)$ and $a \mapsto \|f'_a\|_\infty$ are continuous in \mathbb{R}_*^+ ,¹¹ together with continuity of the map g at every point of $(0, 1] \times [0, 1]$.

- (ii) For every $\mathbf{a} \in (0, 1)^N$ and $\epsilon > 0$, there exists $\delta > 0$ such that for every $\mathbf{p} \in (0, 1)^N$ satisfying $\|\mathbf{p}\|_N < \delta$, we have

$$\|F(\mathbf{p}, \mathbf{a}) - F_{\text{lin}}(\mathbf{p}, \mathbf{a})\|_{2N} < \epsilon,$$

as an immediate consequence of the fact that $f_a \in C^2([0, 1])$ for every $a \in \mathbb{R}_*^+$.

For the induction, consider the following decomposition

$$F^{t+1}(\mathbf{p}, \mathbf{a}) - F_{\text{lin}}^{t+1}(\mathbf{p}, \mathbf{a}) = F^{t+1}(\mathbf{p}, \mathbf{a}) - F \circ F_{\text{lin}}^t(\mathbf{p}, \mathbf{a}) + F \circ F_{\text{lin}}^t(\mathbf{p}, \mathbf{a}) - F_{\text{lin}}^{t+1}(\mathbf{p}, \mathbf{a}),$$

and use

- the induction hypothesis together with property (i) above in order to control the first difference,
- and the fact that all \mathbf{p} -coordinates of $F_{\text{lin}}^t(\mathbf{p}, \mathbf{a})$ remain small (also shown by induction) together with (ii) above (and $\mathbf{a}_{\text{lin}}^t \in (0, 1)^N$) in order to control the second difference. \square

3.3 Global analysis in phase space

This Section is devoted to the proof of Theorem 2.3 and begins by the presentation and proof of several auxiliary properties.

3.3.1 Preliminary statements

We begin with an easy preliminary result on the behaviour of the iterates

$$p_{t+1} = f_{\alpha, a_{t+1}}(p_t), \quad t \in \mathbb{N}, \tag{12}$$

given a sequence $\mathbf{a} = \{a_t\}_{t \in \mathbb{N}} \in (\mathbb{R}_*^+)^{\mathbb{N}}$. While the result is elementary in itself, it serves as a reference for the results about the collective system.

¹¹which implies that

$$|f_{a'}(p') - f_a(p)| \leq |f_{a'}(p') - f_{a'}(p)| + |f_{a'}(p) - f_a(p)|,$$

can be made arbitrarily small by taking (p', a') arbitrarily close to (p, a) .

Claim 3.4. *Assume that $\limsup_{t \rightarrow +\infty} a_t < 1$ (resp. $\liminf_{t \rightarrow +\infty} a_t > 1$). Then, under the iterations (12) above, the iterates have the following asymptotic behaviour*

$$\lim_{t \rightarrow +\infty} p_t = 0 \quad (\text{resp.} \quad \lim_{t \rightarrow +\infty} p_t = 1) \quad \text{for all } p_0 \in (0, 1).$$

Proof. We only prove the first case. The second one follows from symmetry. By assumption, let $T \in \mathbb{N}$ be such that $a_t < 1$ for all $t \geq T$. For $t \geq T - 1$, we have $f_{\alpha, a_{t+1}}(p) < p$ for all $p \in (0, 1)$; hence the sequence $\{p_t\}_{t \in \mathbb{N}}$ eventually decreases. Therefore, it must be convergent. Let us show that the limit $p_\infty \in [0, 1)$ must be equal to 0. Let $\{a_{t_n}\}_{n \in \mathbb{N}}$ be a convergent subsequence and let a_∞ be its limit. Writing

$$\begin{aligned} |f_{\alpha, a_{t_n}}(p_{t_n-1}) - f_{\alpha, a_\infty}(p_\infty)| &\leq |f_{\alpha, a_{t_n}}(p_{t_n-1}) - f_{\alpha, a_{t_n}}(p_\infty)| + |f_{\alpha, a_{t_n}}(p_\infty) - f_{\alpha, a_\infty}(p_\infty)| \\ &\leq \|f'_{\alpha, a_{t_n}}\|_\infty |p_{t_n-1} - p_\infty| + |f_{\alpha, a_{t_n}}(p_\infty) - f_{\alpha, a_\infty}(p_\infty)| \end{aligned}$$

and using the continuity of $a \mapsto \|f'_a\|_\infty$, we obtain

$$p_\infty = \lim_{n \rightarrow \infty} p_{t_n} = \lim_{n \rightarrow \infty} f_{\alpha, a_{t_n}}(p_{t_n-1}) = f_{\alpha, a_\infty}(p_\infty),$$

hence $p_\infty = 0$. □

The next statement collects some constraints on the asymptotic behaviours in the system (1).

Claim 3.5. *(i) It is impossible that*

$$\min_{i \in \{1, \dots, N\}} \limsup_{t \rightarrow +\infty} a_i^t < 1 < \max_{i \in \{1, \dots, N\}} \liminf_{t \rightarrow +\infty} a_i^t.$$

(ii) Assume that all the sequences $\{p_i^t\}_{t \in \mathbb{N}}$ converge. Then, their limits must be equal.

Proof. (i) By contradiction, assume the existence of $i_{\min}, i_{\max} \in \{1, \dots, N\}$ such that

$$\limsup_{t \rightarrow +\infty} a_{i_{\min}}^t < 1 < \liminf_{t \rightarrow +\infty} a_{i_{\max}}^t.$$

Claim 3.4 then implies the following limits

$$\lim_{t \rightarrow +\infty} p_{i_{\min}}^t = 0 \quad \text{and} \quad \lim_{t \rightarrow +\infty} p_{i_{\max}}^t = 1.$$

As a consequence, the continuity of g and the inequality (3) imply the existence of $\epsilon > 0$ and $T \in \mathbb{N}$ such that

$$g(p_{i_{\min}}^t, \frac{1}{N} \sum_{i=1}^N p_i^t) > 1 + \epsilon > 1 > 1 - \epsilon > g(p_{i_{\max}}^t, \frac{1}{N} \sum_{i=1}^N p_i^t), \quad \forall t > T,$$

from which it follows that

$$\lim_{t \rightarrow +\infty} a_{i_{\min}}^t = +\infty \quad \text{and} \quad \lim_{t \rightarrow +\infty} a_{i_{\max}}^t = 0, \tag{13}$$

in contradiction with the initial assumption.

(ii) Similarly, assume the existence of $i_{\min}, i_{\max} \in \{1, \dots, N\}$ such that

$$p_{i_{\min}}^\infty := \lim_{t \rightarrow +\infty} p_{i_{\min}}^t < p_{i_{\max}}^\infty := \lim_{t \rightarrow +\infty} p_{i_{\max}}^t.$$

Then, as before, the limits (13) must hold and therefore we must have $p_{i_{\min}}^\infty = 1$ and $p_{i_{\max}}^\infty = 0$, contradicting the inequality above. □

3.3.2 Proof of Theorem 2.3

The Theorem is an immediate corollary of the following more technical statement.

Proposition 3.6. *Assume that g satisfies the conditions (8) and (9). Then the following statements hold.*

(i) *In every orbit, the product π^t is non-increasing.*

(ii) *If $\lim_{t \rightarrow +\infty} \pi^t > 0$, then we have*

$$\lim_{t \rightarrow +\infty} \max_{i,j \in \{1, \dots, N\}} |p_i^t - p_j^t| = 0.$$

(iii) *The condition*

$$\liminf_{t \rightarrow +\infty} \min_{i \in \{1, \dots, N\}} |a_i^t - 1| > 0,$$

implies that the corresponding orbit must converge to a fixed point.

(iv) *For every orbit, we have*

$$\sup_{t \in \mathbb{N}} \max_{i \in \{1, \dots, N\}} a_i^t < +\infty.$$

Proof. (i) Applying the AM-GM inequality and then the inequality (9) yields

$$\prod_{i=1}^N g(p_i, \frac{1}{N} \sum_{i=1}^N p_i) \leq \left(\frac{1}{N} \sum_{i=1}^N g(p_i, \frac{1}{N} \sum_{i=1}^N p_i) \right)^N \leq 1, \quad \forall \mathbf{p} \in (0, 1)^N.$$

(ii) We have

$$\lim_{t \rightarrow +\infty} \frac{\pi^{t+1}}{\pi^t} = \lim_{t \rightarrow +\infty} \prod_{i=1}^N g(p_i^t, \frac{1}{N} \sum_{i=1}^N p_i^t) = 1,$$

which implies

$$\lim_{t \rightarrow +\infty} \frac{a_i^{t+1}}{a_i^t} = \lim_{t \rightarrow +\infty} g(p_i^t, \frac{1}{N} \sum_{i=1}^N p_i^t) = 1, \quad \forall i \in \{1, \dots, N\}. \quad (14)$$

Indeed, by compactness, let $\{t_k\}_{k \in \mathbb{N}}$ be an infinite subsequence such that

$$\lim_{k \rightarrow +\infty} g(p_i^{t_k}, \frac{1}{N} \sum_{i=1}^N p_i^{t_k}) = g_i^\infty \in [0, \gamma] \text{ exists for all } i \in \{1, \dots, N\}.$$

Then, we have

$$1 = \prod_{i=1}^N g_i^\infty \leq \left(\frac{1}{N} \sum_{i=1}^N g_i^\infty \right)^N = \lim_{k \rightarrow +\infty} \left(\frac{1}{N} \sum_{i=1}^N g(p_i^{t_k}, \frac{1}{N} \sum_{i=1}^N p_i^{t_k}) \right)^N \leq 1,$$

and optimality of the AM-GM inequality imposes that

$$g_i^\infty = 1, \quad \forall i \in \{1, \dots, N\},$$

as desired.

Furthermore, we claim that the second limits in (14) implies

$$\lim_{t \rightarrow +\infty} \left| p_i^t - \frac{1}{N} \sum_{i=1}^N p_i^t \right| = 0, \quad \forall i \in \{1, \dots, N\},$$

from where the desired conclusion immediately follows. Indeed, by contradiction, assume that $\{\mathbf{p}^t\}$ had an accumulation point \mathbf{p}^∞ such that

$$\min_{i \in \{1, \dots, N\}} \left| p_i^\infty - \frac{1}{N} \sum_{i=1}^N p_i^\infty \right| > 0.$$

The continuity of the map g implies that each sequence $\{g(p_i^t, \frac{1}{N} \sum_{i=1}^N p_i^t)\}$ must accumulate on $g(p_i^\infty, \frac{1}{N} \sum_{i=1}^N p_i^\infty)$. However, one of these values would have to differ from 1 because of the condition (3), which is impossible.

(iii) We separate the cases $\lim_{t \rightarrow +\infty} \pi^t = 0$ and $\lim_{t \rightarrow +\infty} \pi^t > 0$. In the first case, let $\{t_k\}_{k \in \mathbb{N}}$ be an infinite subsequence such that

$$\lim_{k \rightarrow +\infty} \min_{i \in \{1, \dots, N\}} |a_i^{t_k} - 1| > 0.$$

By passing to a further subsequence if necessary, the convergence $\pi^t \rightarrow 0$ implies the existence of $i_0 \in \{1, \dots, N\}$ such that

$$\lim_{k \rightarrow +\infty} a_{i_0}^{t_k} = 0.$$

By continuity and compactness, let¹²

$$\gamma = \max_{[0,1]^2} Sg < +\infty.$$

Together with the previous limit, that γ is finite implies that for every $t \in \mathbb{N}$, we have

$$\lim_{k \rightarrow +\infty} \max_{s \in \{0, \dots, t\}} a_{i_0}^{t_k - s} = 0.$$

Independently, the properties of the maps f_a ensure that

$$\lim_{t \rightarrow +\infty} \max_{p \in [0,1]} f_{\alpha, a}^t(p) = 0, \quad \forall a \in (0, 1).$$

These two limits imply the following one

$$\lim_{k \rightarrow +\infty} p_{i_0}^{t_k} = 0.$$

Besides, we may assume that $\{t_k\}_{k \in \mathbb{N}}$ is such that $a_{i_0}^{t_k - 1} \geq a_{i_0}^{t_k}$. Indeed, it is impossible that $\{a_{i_0}^t\}_{t \in \mathbb{N}}$ be eventually increasing. Moreover, its liminf is certainly attained in the set of values of t for which $a_{i_0}^{t-1} \geq a_{i_0}^t$. The inequality $a_{i_0}^{t_k - 1} \geq a_{i_0}^{t_k}$ forces the following one

$$p_{i_0}^{t_k} \geq \frac{1}{N} \sum_{i=1}^N p_i^{t_k}.$$

¹²For the example $g(p, q) = \frac{q}{p}$, we actually have

$$\sup_{\mathbf{p} \in (0,1)^N} Sg(p_i, \frac{1}{N} \sum_{i=1}^N p_i) < N,$$

which is all we need for our purpose.

Hence, we must have

$$\lim_{k \rightarrow +\infty} p_i^{t_k} = 0, \quad \forall i \in \{1, \dots, N\}.$$

Moreover, as shown next, this imposes in turn

$$\limsup_{k \rightarrow +\infty} a_i^{t_k} \leq 1, \quad \forall i \in \{1, \dots, N\},$$

and hence

$$\lim_{k \rightarrow +\infty} p_i^{t_k} = 0 \quad \text{and} \quad \limsup_{k \rightarrow +\infty} a_i^{t_k} < 1, \quad \forall i \in \{1, \dots, N\},$$

given the initial assumption in (iii). Therefore $(\mathbf{p}^{t_k}, \mathbf{a}^{t_k})$ must satisfy the conditions of Proposition 2.1 for k sufficiently large, viz. it must eventually belong to the basin of attraction of some fixed point, completing the proof in the case $\lim_{t \rightarrow +\infty} \pi^t = 0$. We prove the claimed inequality on the limsup above by contradiction. Given $i \in \{1, \dots, N\}$, assume the existence of $\epsilon > 0$ such that for every $k \in \mathbb{N}$, there exists $k' > k$ such that $a_i^{t_{k'}} > 1 + \epsilon$. Then

$$p_i^{t_{k'}} \geq f_{\alpha, 1+\epsilon}(0) > 0,$$

making it impossible that $\lim_{k \rightarrow +\infty} p_i^{t_k} = 0$.

In the case $\lim_{t \rightarrow +\infty} \pi^t > 0$, when combined with the assumption

$$\liminf_{t \rightarrow +\infty} \min_{i \in \{1, \dots, N\}} |a_i^t - 1| > 0,$$

and Claim 3.5 - (i), the fact that

$$\lim_{t \rightarrow +\infty} \frac{a_i^{t+1}}{a_i^t} = 1, \quad \forall i \in \{1, \dots, N\}$$

implies the existence of $T \in \mathbb{N}$ and $\epsilon > 0$ such that we have

$$\text{either } \sup_{t > T} \max_{i \in \{1, \dots, N\}} a_i^t \leq 1 - \epsilon, \quad \text{or } \inf_{t > T} \min_{i \in \{1, \dots, N\}} a_i^t \geq 1 - \epsilon.$$

Claim 3.4 then ensures convergence to a fixed point.

(iv) By contradiction, assume the existence of $i_1 \in \{1, \dots, N\}$ and an infinite subsequence $\{t_k\}_{k \in \mathbb{N}}$ such that

$$\lim_{k \rightarrow +\infty} a_{i_1}^{t_k} = +\infty.$$

Then similar arguments as in the proof of (iii) can be developed to show that we must have

$$\lim_{k \rightarrow +\infty} p_{i_1}^{t_k} = 1.$$

However, since π^t converges to a finite limit, there must exist $i_0 \in \{1, \dots, N\}$ such that

$$\lim_{k \rightarrow +\infty} a_{i_0}^{t_k} = 0,$$

for the same subsequence. Moreover, we showed in the proof of (iii) that this limit implies

$$\lim_{k \rightarrow +\infty} p_i^{t_k} = 0, \quad \forall i \in \{1, \dots, N\},$$

which contradicts the limit of $\{p_{i_1}^{t_k}\}$ above. □

4 Conclusion

In this paper, a macroscopic model for the dynamics of buyers population in markets of perishable goods where prices are not posted has been introduced and mathematically analyzed. In particular, a simple mechanism for merchants feedback has been included, based on assuming immediate profit/competitiveness optimisation and prompt estimation of the overall volumes of buyers. Such negative feedback implies oscillatory behaviours and makes (local) asymptotic stabilization dependent on the nature of the reactivity rate. A bounded rate cannot counteract intrinsic response to merchants attractiveness/repulsiveness. In such case of moderate reactivity, convergence towards stationary functioning modes takes place, with either full or absence of clientele at all merchants. On the other hand, an arbitrarily large reactivity can promote resilience against vanishing clientele, as shown with the example $g(p, q) = \frac{q}{p}$.

In addition to the control of local asymptotic stability of the fixed points, a concavity-type condition on the reactivity rate has been identified, that forces the oscillations to eventually cease in every orbit for which the merchants attractiveness remain bounded away from 1, a feature we believe to be generic and to fail only in exceptional cases. In particular, this may fail and oscillations may perdure forever when on the boundary of the fixed points basins of attraction, when neither convergence to a fixed point with $\mathbf{p} = \mathbf{1}$ or $\mathbf{p} = \mathbf{0}$ holds. Moreover, as evidenced in the numerics, to predict the asymptotic functioning mode, either full or empty clientele, is a delicate task when starting close to such boundary. In other words, while the basic ingredients of the dynamics are rather simple, the resulting temporal process that they generate can be rather involved and hardly predictable, especially in case of a major perturbation such as the introduction of a new seller in the market.

We conclude the paper with few additional comments about the modelling assumptions in the system (1) and some suggestions about possible improvements.

First, the modelling time scale has been chosen to be rather coarse for simplicity. It does not incorporate intraday variations of clientele or attractiveness. However, evidences have been given, especially at Rungis market [2], that sellers may react during sessions in order to adapt their attractiveness to initially scarce clienteles or to insufficient sale volumes given their stocks. Hence, a more elaborated modelling should incorporate more rapid changes such as hourly variations.

Likewise, the feedback term in (1) assumes instant evaluation by the merchants of the overall clientele volume in the market. While this hypothesis is justified when the market place is compact or the number of competitors is limited, in large markets with a high number of merchants selling goods of the same nature, it would be relevant to adopt, in the feedback evaluation, a local clientele estimation that is evaluated in a limited neighbourhood of the seller under consideration. In such case of local interaction terms, the dynamics is often richer than in globally coupled models, as for instance, spatial patterns and non-synchronous solutions emerge [1].

Equally important since real populations are of finite size, it is questionable to consider buyers fractions p_i^t that are extremely small or extremely large, and in particular smaller than the inverse of the total population size when the latter is given. A description of the dynamics at the level of individuals should be adopted when p_i^t is close to 0 or to 1, especially when experimenting the introduction of a new seller with very small initial clientele. In that setting, it would be particularly interesting to identify the analogues of the stability condition (8) at the level of individual dynamics.

In addition, a more detailed dynamics at the level of individuals gives the opportunity to take into account intrinsic heterogeneities in the individual features that may arise in such diversified populations of buyers that are present in these markets. Likewise, temporal non-systematic fluctuations of the individual behaviours could be integrated. In particular both these aspects could be taken into account through modelling of the dynamics using random process. All these considera-

tions can be the subject of future studies.

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