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# Enveloping space of a globally hyperbolic conformally flat spacetime

Rym SMAÏ

**Abstract.** We prove that any simply-connected globally hyperbolic conformally flat spacetime  $V$  can be conformally embedded in a bigger conformally flat spacetime, called **enveloping space** of  $V$ , containing all the conformally flat Cauchy-extensions of  $V$ , in particular its  $\mathcal{C}_0$ -maximal extension. As a result, we establish a new proof of the existence and the uniqueness of the  $\mathcal{C}_0$ -maximal extension of a globally hyperbolic conformally flat spacetime. Furthermore, this approach allows us to prove that  $\mathcal{C}_0$ -maximal extensions respect inclusion.

## 1 Introduction

The notion of maximal extension of a globally hyperbolic spacetime arises from the resolution of Einstein equations in general relativity. This physical theory suggests that our universe is modeled by a Lorentzian manifold  $(M, g)$  of dimension 4 where the metric  $g$  satisfies some PDEs, the so-called *Einstein equations*. A way to solve them is to require that  $M$  is homeomorphic to  $S \times \mathbb{R}$  where  $S$  is a Riemannian manifold. This allows to define a Cauchy problem where the initial data is the Riemannian manifold  $(S, h)$  equipped with a  $(2, 0)$ -tensor  $\text{II}$ . A solution is a Lorentzian metric  $g$  on  $S \times \mathbb{R}$  such that the restriction of  $g$  to  $S \times \{0\}$  is  $h$  and  $\text{II}$  is the shape operator of this hypersurface. It turns out that a necessary condition to have such a solution is that  $h$  and  $\text{II}$  satisfy *the constraint equations*. Conversely, Choquet-Bruhat and Geroch [12, Section 2] proved that when the constraint equations are satisfied, a local solution exists. Two natural questions arise: is it possible to extend this solution to a maximal one? If yes, is it unique up to isometry? Choquet-Bruhat and Geroch answered positively to both questions (see [12, Theorem 3]).

The solutions of the Cauchy problem associated to Einstein equations turn out to be *globally hyperbolic* (abbrev. GH) (see Definition 2). More generally, Geroch [6] proved that any GH spacetime admits an embedded Riemannian hypersurface which intersects every inextendible causal curve exactly once, called *Cauchy hypersurface*. It turns out that all smooth Cauchy hypersurfaces of a GH spacetime are diffeomorphic.

There is a natural partial ordering on GH spacetimes: given two GH spacetimes,  $M$  and  $N$ , we say that  $N$  is a Cauchy-extension of  $M$  if there exists an isometric embedding from  $M$  to  $N$  sending every Cauchy hypersurface of  $M$  on a Cauchy hypersurface of  $N$ . Such an embedding is called *an isometric Cauchy-embedding*. We can ask again in this general setting the questions of the existence and the uniqueness, up to isometry, of a maximal extension. The answer to both questions is yes within a *rigid category*<sup>1</sup> of spacetimes. Actually, the spacetimes which are solution of a Cauchy problem associated

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<sup>1</sup>See e.g. [10, Definitions 2 and 4]. Spacetimes of constant curvature are examples of rigid categories.

to Einstein equations constitute a rigid category and it turns out that the arguments of Choquet-Bruhat and Geroch could be adapted to any other rigid category.

In this paper, we are interested in the notion of maximality in the setting of *conformally flat* spacetimes of dimension  $n \geq 3$ . The morphisms preserving these structures are the *conformal* diffeomorphisms. In [10, Section 3.1], C. Rossi adapted to this setting the definition of the ordering relation on GH spacetimes by considering conformal Cauchy-embeddings instead of isometric Cauchy-embeddings. A GH conformally flat spacetime  $M$  is then said to be  $\mathcal{C}_0$ -*maximal* if any conformal Cauchy-embedding from  $M$  to any GH conformally flat spacetime is surjective. C. Rossi proved that any GH conformally flat spacetime admits a  $\mathcal{C}_0$ -maximal extension, unique up to conformal diffeomorphism (see [10, Sections 3.2 & 3.3]). Her proof is mainly based on Zorn lemma and so does not give any description of the  $\mathcal{C}_0$ -maximal extension. In this paper, we propose a new approach which allows us to give a constructive proof of the existence and the uniqueness of the  $\mathcal{C}_0$ -maximal extension. Indeed, given a simply-connected GH conformally flat spacetime  $M$ , we construct a bigger conformally flat spacetime  $E(M)$  in which  $M$  and all its conformally flat Cauchy-extensions embeds conformally. The  $\mathcal{C}_0$ -maximal extension of  $M$  turns out to be the Cauchy development of a Cauchy hypersurface of  $M$  in  $E(M)$ . The images in  $E(M)$  of the previous embeddings satisfy the nice property of being *causally convex*. A subset  $U$  of a spacetime is causally convex if any causal curve joining two points of  $U$  is contained in  $U$ . While convexity is a metric notion, causal convexity is a conformal notion. Let us add that causal convexity is a strong property in a GH spacetime: it is a classical fact that any causally convex open subset of a GH spacetime is GH.

**Theorem 1.** *Let  $M$  be a simply-connected globally hyperbolic conformally flat spacetime. There exists a conformally flat spacetime  $E(M)$  with the following properties:*

1.  $E(M)$  fibers trivially over a conformally flat Riemannian manifold  $\mathcal{B}$  diffeomorphic to any Cauchy hypersurface of  $M$ ;
2.  $M$  embeds conformally in  $E(M)$  as a causally convex open subset;
3. all the conformally flat Cauchy-extensions of  $M$  embeds conformally in  $E(M)$  as causally convex open subsets. In particular, the  $\mathcal{C}_0$ -maximal extension of  $M$  is the Cauchy development of a Cauchy hypersurface of  $M$  in  $E(M)$ .

Such a spacetime  $E(M)$  is called **an enveloping space** of  $M$ .

This result still holds for the larger class of *developable* GH conformally flat spacetimes (see Definition 11). In Section 4.3, we describe causally convex open subsets of an enveloping space  $E(M)$  then, in Section 7, we characterize those which are  $\mathcal{C}_0$ -maximal.

A consequence of Theorem 1 is the following result:

**Corollary 1.** *Any globally hyperbolic conformally flat spacetime admits a  $\mathcal{C}_0$ -maximal extension, unique up to conformal diffeomorphism.*

Now we ask the following question. Let  $V$  be a globally hyperbolic conformally flat spacetime and let  $U$  be a causally convex open subset of  $V$ . Does the  $\mathcal{C}_0$ -maximal extension of  $U$  embed conformally in the  $\mathcal{C}_0$ -maximal extension of  $V$ ?

The  $\mathcal{C}_0$ -maximal extensions of  $U$  and  $V$  are *a priori* abstract objects which depends on the Cauchy hypersurfaces of  $U$  and  $V$  respectively. These last ones are completely independant so the question above is not tautological. We prove in Section 6 that the answer is yes, in other words, that  $\mathcal{C}_0$ -maximal extensions preserve inclusion.

**Theorem 2.** *Let  $V$  be a globally hyperbolic conformally flat spacetime and let  $U$  be a causally convex open subset of  $V$ . Then, the  $\mathcal{C}_0$ -maximal extension of  $U$  is conformally equivalent to a causally convex open subset of the  $\mathcal{C}_0$ -maximal extension of  $V$ .*

## Overview of the paper

In Section 2, we introduce the preliminary material on causality of spacetimes. We focus in particular on *globally hyperbolic* spacetimes and we recall some of their main properties. Section 3 deals with the model space of conformally flat Lorentzian structure, the so-called *Einstein universe*. After a quick description of its geometry, we characterize *causally convex open subsets* of its universal cover (see Sections 3.5 and 3.6). We devote Section 4 to the proof of Theorem 1: we construct an enveloping space (see Section 4.2) and we describe its causally convex globally hyperbolic open subsets (see Section 4.3). In Section 5, we propose a new proof of the existence and the uniqueness of the maximal extension of a globally hyperbolic conformally flat spacetime, using the notion of enveloping space. Section 6 is devoted to the proof of Theorem 2. Lastly, we establish a link between the notion of  $\mathcal{C}_0$ -maximality and the notion of *eikonal functions* in Section 7.

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## 2 Globally hyperbolic spacetimes

### 2.1 Preliminaries on spacetimes

The aim of this preliminary section is to introduce the concept of *causality* in a Lorentzian manifold and briefly recall some basic causal notions as *causal curves*, *future and past* of points, *lightcones*, *achronal and acausal* subsets, etc.

Throughout this paper, we denote by  $(p, q)$  the signature of a non-degenerate quadratic form  $Q$  where  $p$  and  $q$  are respectively the number of negative and positive coefficients in the polar decomposition of  $Q$ .

**Spacetimes.** A *Lorentzian metric* on a manifold of dimension  $n$  is a non-degenerate symmetric 2-tensor  $g$  of signature  $(1, n - 1)$ . A manifold equipped with a Lorentzian is called *Lorentzian*.

In a Lorentzian manifold  $(M, g)$ , we say that a non-zero tangent vector  $v$  is *timelike*, *lightlike*, *spacelike* if  $g(v, v)$  is respectively negative, zero, positive. The set of timelike vectors is the union of two convex open cones. When it is possible to make a continuous choice of a connected component in each tangent space, the manifold  $M$  is said *time-orientable*. The timelike vectors in the chosen component are said *future-directed* while those in the other component are said *past-directed*. A *spacetime* is an oriented and time-oriented Lorentzian manifold.

**Future, past.** In a spacetime  $M$ , a differential curve is *timelike*, *lightlike*, *spacelike* if its tangent vectors are timelike, lightlike, spacelike. It is *causal* if its tangent vectors are either timelike or lightlike.

Given a point  $p$  in  $M$ , the *future* (resp. *chronological future*) of  $p$ , denoted  $J^+(p)$  (resp.  $I^+(p)$ ), is the set of endpoints of future-directed causal (resp. timelike) curves starting from  $p$ . More generally, the future (resp. chronological future) of a subset  $A$  of  $M$ , denoted  $J^+(A)$  (resp.  $I^+(A)$ ), is the union of  $J^+(a)$  (resp.  $I^+(a)$ ) where  $a \in A$ .

An open subset  $U$  of  $M$  is a spacetime and the intrinsic causality relations of  $U$  imply the corresponding ones in  $M$ . We denote  $J^+(A, U)$  (resp.  $I^+(A, U)$ ) the future (resp. chronological future) in the manifold  $U$  of a set  $A \subset U$ . Then,  $I^+(A, U) \subset I^+(A) \cap U$ .

Dual to the preceding definitions are corresponding *past* versions. In general, *past* definitions and proofs follows from future versions (and vice versa) by reversing time-orientation.

**Diamonds.** We call *diamond* of  $M$  any intersection  $J^-(p) \cap J^+(q)$ , where  $p, q \in M$  such that  $p \in J^+(q)$ . We denote it  $J(p, q)$ . Given two points  $p, q \in M$  such that  $p \in I^+(q)$ , the interior of the diamond  $J(p, q)$  is the intersection  $I^-(p) \cap I^+(q)$  and is denoted  $I(p, q)$  (see [8, Lemma 6, p.404]).

**Achronal, acausal subsets.** A subset  $A$  of a spacetime  $M$  is called *achronal* (resp. *acausal*) if no timelike (resp. causal) curve intersects  $A$  more than once.

**Causal convexity.** In Riemannian geometry, it is often useful to consider open neighborhoods which are *geodesically convex*. In Lorentzian geometry, there is, in addition, a *causal convexity* notion. A subset  $U$  of  $M$  is said *causally convex* if for every  $p, q \in U$ , any causal curve of  $M$  joining  $p$  to  $q$  is contained in  $U$ . Equivalently, if every diamond  $J(p, q)$  of  $M$  with  $p, q \in U$  is contained in  $U$ . It is easy to check that the intersection of two causally convex subsets is causally convex.

**Cauchy developments.** Let  $A$  be an achronal subset of  $M$ . The future (resp. past) Cauchy development of  $A$ , denoted  $\mathcal{C}^+(A)$  (resp.  $\mathcal{C}^-(A)$ ), is the set of points  $p$  of  $M$  such that every past-inextendible (resp. future-inextendible) causal curve through  $p$  meets  $A$ . The Cauchy development of  $A$  is the union of  $\mathcal{C}^+(A)$  and  $\mathcal{C}^-(A)$ , denoted  $\mathcal{C}(A)$ .

## 2.2 Global hyperbolicity

**Definition 1.** A spacetime  $M$  is said strongly causal if for every point  $p \in M$  and every neighborhood  $U$  of  $p$ , there exists a neighborhood  $V$  of  $p$  contained in  $U$ , which is causally convex in  $M$ .

**Definition 2.** A spacetime  $M$  is said globally hyperbolic (abbrev. GH) if the two following conditions hold:

1.  $M$  is strongly causal.
2. all diamonds of  $M$  are compact.

It was proved by Sanchez in [3] that the first condition can be weakened to  $M$  is causal, that is  $M$  contains no causal loop.

A classical result of Geroch [6], later improved by Bernal and Sanchez [2], gives a characterization of global hyperbolicity involving the notion of Cauchy hypersurface.

**Definition 3.** A topological (resp. smooth) Cauchy hypersurface is an achronal topological hypersurface (resp. an embedded Riemannian hypersurface) that is met by exactly once by every inextendible causal curve of  $M$ .

**Theorem 3** ([6]). A spacetime  $M$  is globally hyperbolic if and only if it contains a topological Cauchy hypersurface.

Bernal and Sanchez [2] improved this result by proving the existence of a smooth Cauchy hypersurface.

Topological (resp. smooth) Cauchy hypersurfaces of a globally hyperbolic spacetime are homeomorphic (resp. diffeomorphic). Therefore, one can set the following definition.

**Definition 4.** A globally hyperbolic spacetime is said Cauchy-compact (or spatially compact) if it admits a compact Cauchy hypersurface.

A remarkable property of globally hyperbolic spacetimes is that causal convexity implies global hyperbolicity.

**Proposition 1.** Let  $M$  be a globally hyperbolic spacetime. Then, any causally convex open subset of  $M$  is globally hyperbolic.

*Proof.* Since  $M$  is globally hyperbolic, there is no causal loop in  $U$ . Since  $U$  is causally convex, the diamonds of  $U$  are exactly the diamonds of  $M$  contained in  $U$ . Thus, they are compact.  $\square$

## 2.3 Shadows

In this section, we show that the causal structure of globally hyperbolic spacetimes is encoded by compact subsets of a Cauchy hypersurface, called *shadows*.<sup>2</sup>

Let  $M$  be a globally hyperbolic spacetime and let  $S \subset M$  be a Riemannian Cauchy hypersurface.

**Definition 5.** Let  $p \in M$ . We call shadow of  $p$  on  $S$ , denoted by  $O(p, S)$ , the set of points in  $S$  which are causally related to  $p$ . When there is no confusion about the Cauchy hypersurface  $S$ , we will simply write  $O(p)$  instead of  $O(p, S)$ .

If  $p \in I^\pm(S)$ , then  $O(p, S) = J^\mp(p) \cap S$ ; if  $p \in S$ ,  $O(p, S)$  is reduced to  $\{p\}$ . Thus, by [8, Lemma 40, p.423], shadows are *compact*.

The main interest of the notion *shadows* is given by the following proposition proved by C. Rossi in her thesis (see [9, Prop. 2.6, chap. 4]).

**Proposition 2.** Suppose  $S$  is not compact. Then, two points  $p$  and  $q$  of  $M$  in the chronological future of  $S$  coincide if and only if their shadows on  $S$  are equal.  $\square$

By Proposition 2, the shadows on  $S$  characterize completely the points of the globally hyperbolic spacetime  $M$ . This allows to reduce, in some situations, the study of the spacetime to the study of compact subsets of a Riemannian manifold.

## 3 Geometry of Einstein universe

In this section, we introduce the model space of conformally flat Lorentzian structures, the so-called *Einstein universe*, and we describe its causal structure.

### 3.1 The Klein model

Let  $\mathbb{R}^{2,n}$  be the vector space  $\mathbb{R}^{n+2}$  of dimension  $(n+2)$  equipped with the nondegenerate quadratic form  $q_{2,n}$  of signature  $(2, n)$  given by

$$q_{2,n}(u, v, x_1, \dots, x_n) = -u^2 - v^2 + x_1^2 + \dots + x_n^2$$

in the coordinate system  $(u, v, x_1, \dots, x_n)$  associated to the canonical basis of  $\mathbb{R}^{n+2}$ .

**Definition 6.** Einstein universe of dimension  $n$ , denoted by  $\text{Ein}_{1,n-1}$ , is the space of isotropic lines of  $\mathbb{R}^{2,n}$  with respect to the quadratic form  $q_{2,n}$ , namely

$$\text{Ein}_{1,n-1} = \{[x] \in \mathbb{P}(\mathbb{R}^{2,n}) : q_{2,n}(x) = 0\}.$$

In practice, it is more convenient to work with the double cover of the Einstein universe, denoted by  $\text{Ein}_{1,n-1}$ :

$$\text{Ein}_{1,n-1} = \{[x] \in \mathbb{S}(\mathbb{R}^{2,n}) : q_{2,n}(x) = 0\}$$

where  $\mathbb{S}(\mathbb{R}^{2,n})$  is the sphere of rays, namely the quotient of  $\mathbb{R}^{2,n} \setminus \{0\}$  by positive homotheties.

<sup>2</sup>This terminology has been introduced by C. Rossi in her thesis [9, Chapitre 4].

### 3.2 Spatio-temporal decomposition of Einstein universe

The choice of a timelike plane of  $\mathbb{R}^{2,n}$ , i.e. a plane on which the restriction of  $q_{2,n}$  is negative definite, defines a spatio-temporal decomposition of Einstein universe:

**Lemma 1.** *Any timelike plane  $P \subset \mathbb{R}^{2,n}$  defines a diffeomorphism between  $\mathbb{S}^{n-1} \times \mathbb{S}^1$  and  $Ein_{1,n-1}$ .*

*Proof.* Consider the orthogonal splitting  $\mathbb{R}^{2,n} = P^\perp \oplus P$  and call  $q_{P^\perp}$  and  $q_P$  the positive definite quadratic form induced by  $\pm q_{2,n}$  on  $P^\perp$  and  $P$  respectively. The restriction of the canonical projection  $\mathbb{R}^{2,n} \setminus \{0\}$  on  $\mathbb{S}(\mathbb{R}^{2,n})$  to the set of points  $(x, y) \in P^\perp \oplus P$  such that  $q_{P^\perp}(x) = q_P(y) = 1$  defines a map from  $\mathbb{S}^{n-1} \times \mathbb{S}^1$  to  $Ein_{1,n-1}$ . It is easy to check that this map is a diffeomorphism.  $\square$

For every timelike plane  $P \subset \mathbb{R}^{2,n}$ , the quadratic form  $q_{2,n}$  induces a Lorentzian metric  $g_P$  on  $\mathbb{S}^{n-1} \times \mathbb{S}^1$  given by

$$g_P = d\sigma^2(P) - d\theta^2(P)$$

where  $d\sigma^2(P)$  is the round metric on  $\mathbb{S}^{n-1} \subset (P^\perp, q_{P^\perp})$  induced by  $q_{P^\perp}$  and  $d\theta^2(P)$  is the round metric on  $\mathbb{S}^1 \subset (P, q_P)$  induced by  $q_P$ .

An easy computation shows that if  $P' \subset \mathbb{R}^{2,n}$  is another timelike plane, the Lorentzian metric  $g_{P'}$  is conformally equivalent to  $g_P$ , i.e.  $q_P$  and  $q_{P'}$  are proportional by a positive smooth function on  $\mathbb{S}^{n-1} \times \mathbb{S}^1$ . As a result, Einstein universe is naturally equipped with a conformal class of Lorentzian metrics. This Lorentzian conformal structure induces causality on Einstein universe. Indeed, changing the metric in the conformal class consists in multiplying by a positive function and so does not change the sign of the norm of a tangent vector. The causal structure of Einstein universe is trivial: any point is causally related to any other one (see e.g. [9, Cor. 2.10, Chap. 2]).

Let us point out that in general geodesics are not well-defined in a conformal space-time. Indeed, a computation of the Levi Civita connexion shows that geodesics are not preserved by conformal changes of metrics. Nevertheless, lightlike geodesics are preserved as *non-parametrized* curves (see e.g. [4, Théorème 3]).

### 3.3 Causal structure of the universal cover

Let  $\widetilde{Ein}_{1,n-1}$  be the universal cover of  $Ein_{1,n-1}$ . When  $n \geq 3$ , every diffeomorphism between  $Ein_{1,n-1}$  and  $\mathbb{S}^{n-1} \times \mathbb{S}^1$  lifts to a diffeomorphism between  $\widetilde{Ein}_{1,n-1}$  and  $\mathbb{S}^{n-1} \times \mathbb{R}$ . The pull-back by the projection  $\mathbb{S}^{n-1} \times \mathbb{R} \rightarrow \mathbb{S}^{n-1} \times \mathbb{S}^1$  of the conformal class  $[d\sigma^2 - d\theta^2]$  on  $\mathbb{S}^{n-1} \times \mathbb{S}^1$  defined previously is the conformal class of the Lorentzian metric  $d\sigma^2 - dt^2$  where  $dt^2$  is the usual metric on  $\mathbb{R}$ . This induces a natural conformally flat Lorentzian structure on  $\widetilde{Ein}_{1,n-1}$ .

**Definition 7.** *We call spatio-temporal decomposition of  $\widetilde{Ein}_{1,n-1}$  any conformal diffeomorphism between  $\widetilde{Ein}_{1,n-1}$  and  $\mathbb{S}^{n-1} \times \mathbb{R}$ .*



In what follows, we fix a spatio-temporal decomposition and we identify  $\widetilde{Ein}_{1,n-1}$  to  $\mathbb{S}^{n-1} \times \mathbb{R}$ .

The fundamental group of  $Ein_{1,n-1}$  is isomorphic to  $\mathbb{Z}$ , generated by the transformation  $\delta : \widetilde{Ein}_{1,n-1} \rightarrow \widetilde{Ein}_{1,n-1}$  defined by  $\delta(x, t) = (x, t + 2\pi)$ . That of  $Ein_{1,n-1}$  is generated by the transformation  $\sigma : \widetilde{Ein}_{1,n-1} \rightarrow \widetilde{Ein}_{1,n-1}$  such that  $\sigma^2 = \delta$ , i.e. the map defined by  $\sigma(x, t) = (-x, t + \pi)$ .

**Definition 8.** *Two points  $p$  and  $q$  of  $\widetilde{Ein}_{1,n-1}$  are said to be conjugate if one is the image under  $\sigma$  of the other.*

While the causal structure of  $Ein_{1,n-1}$  is trivial, the causal structure of  $\widetilde{Ein}_{1,n-1}$  is rich. We give a brief description below. We direct to [9, Chap. 2] for more details.

Lightlike geodesics of  $\widetilde{Ein}_{1,n-1}$  are the curves which can be written, up to reparametrization, as  $(x(t), t)$  where  $x : I \rightarrow \mathbb{S}^{n-1}$  is a geodesic of  $\mathbb{S}^{n-1}$  defined on an interval  $I$  of  $\mathbb{R}$ . The inextensible ones are those for which  $x$  is defined on  $\mathbb{R}$ .

It turns out that the photons going through a point  $(x_0, t_0)$  have common intersections at the points  $\sigma^k(x_0, t_0)$ , for  $k \in \mathbb{Z}$ ; and are pairwise disjoint outside these points. The lightcone of a point  $(x_0, t_0)$  is the set of points  $(x, t)$  such that  $d(x, x_0) = |t - t_0|$  where  $d$  is the distance on the sphere  $\mathbb{S}^{n-1}$  induced by the round metric. It disconnects  $\widetilde{Ein}_{1,n-1}$  in three connected components:

- The *chronological future* of  $(x_0, t_0)$ : this is the set of points  $(x, t)$  of  $\mathbb{S}^{n-1} \times \mathbb{R}$  such that  $d(x, x_0) < t - t_0$ .
- The *chronological past* of  $(x_0, t_0)$ : this is the set of points  $(x, t)$  of  $\mathbb{S}^{n-1} \times \mathbb{R}$  such that  $d(x, x_0) < t_0 - t$ .
- The set of points non-causally related to  $(x_0, t_0)$ , i.e. the set of points  $(x, t)$  of  $\mathbb{S}^{n-1} \times \mathbb{R}$  such that  $d(x, x_0) > |t - t_0|$ . This is exactly the interior of the diamond of vertices  $\sigma(x_0, t_0)$  and  $\sigma^{-1}(x_0, t_0)$ . It is conformally diffeomorphic to Minkowski spacetime (see e.g. [11, Lemma 2.38 and Corollary 2.43]) and is called *affine chart*. We denote it  $\text{Mink}_0(x_0, t_0)$ .

There are two other affine charts associated to the point  $(x_0, t_0)$ , namely:

- the set of points non-causally related to  $\sigma(x_0, t_0)$ , contained in the chronological future of  $(x_0, t_0)$ , denoted  $\text{Mink}_+(x_0, t_0)$ ;
- the set of points non-causally related to  $\sigma^{-1}(x_0, t_0)$ , contained in the chronological past of  $(x_0, t_0)$ , denoted  $\text{Mink}_-(x_0, t_0)$ .

The universal cover  $\widetilde{Ein}_{1,n-1}$  is globally hyperbolic: any sphere  $\mathbb{S}^{n-1} \times \{t\}$ , where  $t \in \mathbb{R}$ , is a Cauchy hypersurface.

### 3.4 Conformal group

The subgroup  $O(2, n) \subset Gl_{n+2}(\mathbb{R})$  preserving  $q_{2,n}$ , acts conformally on  $Ein_{1,n-1}$ . When  $n \geq 3$ , the conformal group of  $Ein_{1,n-1}$  is *exactly*  $O(2, n)$ . This is a consequence of the following result, which is an extension to Einstein universe, of a classical theorem of Liouville in Euclidean conformal geometry (see e.g. [4]):

**Theorem 4.** *Let  $n \geq 3$ . Any conformal transformation between two open subsets of  $Ein_{1,n-1}$  is the restriction of an element of  $O(2, n)$ .*

It is a classical fact that every conformal diffeomorphism of  $Ein_{1,n-1}$  lifts to a conformal diffeomorphism of  $\widetilde{Ein}_{1,n-1}$ . Conversely, by Theorem 4, every conformal transformation of  $\widetilde{Ein}_{1,n-1}$  defines a unique conformal transformation of the quotient space  $Ein_{1,n-1} = \widetilde{Ein}_{1,n-1} / \langle \delta \rangle$ .

Let  $\text{Conf}(\widetilde{Ein}_{1,n-1})$  denote the group of conformal transformations of  $\widetilde{Ein}_{1,n-1}$ . Let  $j : \text{Conf}(\widetilde{Ein}_{1,n-1}) \rightarrow O(2, n)$  be the natural projection. This is a surjective group morphism whose kernel is generated by  $\delta$ .

### 3.5 Causally convex open subsets of Einstein universe

In this section, we characterize causally convex open subsets of  $\widetilde{Ein}_{1,n-1}$  in a spatio-temporal decomposition  $\mathbb{S}^{n-1} \times \mathbb{R}$ . We denote by  $d$  the distance on  $\mathbb{S}^{n-1}$  induced by the round metric.

**Proposition 3.** *Let  $\Omega$  be a causally convex open subset of  $\widetilde{Ein}_{1,n-1}$ . Then, there exist two 1-Lipschitz functions  $f^+$  and  $f^-$  from an open subset  $U$  of  $\mathbb{S}^{n-1}$  to  $\overline{\mathbb{R}}$  such that the following hold:*

- $f^- < f^+$  on  $U$ ;
- the extensions of  $f^+$  and  $f^-$  to  $\partial U$  coincide.
- $\Omega$  is the set of points  $(x, t)$  of  $\widetilde{Ein}_{1,n-1}$  such that  $f^-(x) < t < f^+(x)$ .

The proof of Proposition 3 uses the following lemma.

**Lemma 2.** *For every points  $p$  and  $q$  in the closure of  $\Omega$  such that  $p \in I^+(q)$ , the intersection  $I^-(p) \cap I^+(q)$  is contained in  $\Omega$ .*

*Proof.* Let  $p, q \in \overline{\Omega}$  such that  $p \in I^+(q)$ . There exist two sequences  $\{p_i\}$  and  $\{q_i\}$  of elements of  $\Omega$  such that  $\lim p_i = p$  and  $\lim q_i = q$ . Let  $r \in I(p, q)$ . Then,  $I^+(r)$  is an open neighborhood of  $p$  and  $I^-(r)$  is an open neighborhood of  $q$ . As a result, there exists an integer  $i_0$  such that  $p_{i_0} \in I^+(r)$  and  $q_{i_0} \in I^-(r)$ . It follows that  $r \in I(p_{i_0}, q_{i_0}) \subset \Omega$ .  $\square$

*Proof of Proposition 3.* Let  $U$  be the projection of  $\Omega$  on the sphere  $\mathbb{S}^{n-1}$ . Since  $\Omega$  is causally convex, the intersection of  $\Omega$  with any timelike line  $\{x\} \times \mathbb{R}$ , where  $x \in U$ , is connected, i.e. it is a segment  $\{x\} \times ]f^-(x), f^+(x)[$ . This defines two functions  $f^+$  and  $f^-$  from  $U$  to  $\overline{\mathbb{R}}$  such that  $\Omega$  is the set of points  $(x, t)$  such that  $f^-(x) < t < f^+(x)$ .

**Fact 1.** *If there exists  $x \in U$  such that  $f^+(x) = +\infty$  then  $f^+ \equiv +\infty$ .*

This is equivalent to prove that  $\Omega$  is future-complete, i.e.  $I^+(\Omega) \subset \Omega$ , as soon as it contains a timelike half-line  $\alpha = \{x\} \times [t, +\infty[$ . Let  $p \in \Omega$  and let  $q \in I^+(p)$ . Since  $\alpha$  is future-inextendible, it intersects  $I^+(q)$ . Let  $q'$  be a point in this intersection. Then,  $q \in J(q', p')$  where  $p' \in \Omega \cap I^-(p)$ . Thus,  $q \in \Omega$ .

**Fact 2.** *If  $f^+$  is finite, it is 1-Lipschitz.*

This is equivalent to prove that the graph of  $f^+$  is achronal. Suppose there exist two distinct points  $p, q$  in the graph of  $f^+$  such that  $p \in I^+(q)$ . Since  $p \in \partial\Omega$ , we have  $I^+(q) \cap \Omega \neq \emptyset$ . Then,  $q \in I(p', q')$  where  $p' \in I^+(q) \cap \Omega$  and  $q' \in \Omega \cap I^-(q)$ . Hence,  $q \in \Omega$ . Contradiction.

**Fact 3.** *If  $f^+$  and  $f^-$  are finite and  $\partial U$  is non-empty, then the extensions of  $f^+$  and  $f^-$  to  $\partial U$  are equal.*

Let  $\overline{f^+}$  (resp.  $\overline{f^-}$ ) be the extension of  $f^+$  (resp.  $f^-$ ) to  $\overline{U}$ . Suppose there exists  $x \in U$  such that  $\overline{f^-}(x) < \overline{f^+}(x)$ . Then, the timelike segment  $\{x\} \times ]\overline{f^-}(x), \overline{f^+}(x)[$  is contained in the boundary of  $\Omega$ . This contradicts Lemma 2.

□

Conversely:

**Proposition 4.** *Let  $f^+, f^- : U \subset \mathbb{S}^{n-1} \rightarrow \overline{\mathbb{R}}$  be two 1-Lipschitz functions defined on an open subset  $U$  of  $\mathbb{S}^{n-1}$  such that:*

- $f^- < f^+$  on  $U$ ;
- the extensions of  $f^+$  and  $f^-$  to  $\partial U$  coincide.

*Then, the set of points  $(x, t)$  of  $\widetilde{Ein}_{1, n-1}$  such that  $f^-(x) < t < f^+(x)$ , named  $\Omega$ , is causally convex in  $\widetilde{Ein}_{1, n-1}$ .*

The proof of Proposition 4 uses the following lemma.

**Lemma 3.** *Let  $\overline{f^+}$  be the extension of  $f^+$  to  $\overline{U}$ . Then, for every point  $p$  in the graph of  $\overline{f^+}$ , the causal future of  $p$  is disjoint from  $\Omega$ .*

*Proof.* Let  $x \in U$  and set  $p = (x, f^+(x))$ . Suppose there exists  $(y, s) \in \Omega \cap J^+(p)$ . Then,  $d(x, y) \leq s - f^+(x) < f^+(y) - f^+(x) \leq d(x, y)$ . Contradiction. Then,  $J^+(p) \cap \Omega = \emptyset$ .

Now, let  $x \in \partial U$  and let  $\{x_i\}$  be a sequence of elements of  $U$  such that  $x = \lim x_i$ . Set  $p_i = (x_i, f^+(x_i))$ . Suppose there exists  $q \in J^+(p) \cap \Omega$ . Since  $\Omega$  is open, there exists  $q' \in I^+(q) \cap \Omega$ . By transitivity,  $q' \in I^+(p)$ . Then,  $I^-(q')$  is an open neighborhood of  $p$ . Since  $\lim p_i = p$ , we deduce that  $p_i \in I^-(q')$  for  $i$  big enough. Equivalently,  $q' \in I^+(p_i)$ . Thus,  $I^+(p_i) \cap \Omega \neq \emptyset$ . Contradiction. □

There is an analogue statement for the extension of  $f^-$  to  $\overline{U}$ , denoted  $\overline{f^-}$ , with the reverse time-orientation.

*Proof of Proposition 4.* Since  $f^\pm$  is 1-Lipschitz, if  $f^\pm$  is infinite in a point of  $U$ , it is infinite on  $U$ . If  $f^+ \equiv +\infty$  and  $f^- \equiv -\infty$ , we have  $U = \mathbb{S}^{n-1}$  and  $\Omega = \widetilde{Ein}_{1,n-1}$ .

Suppose  $f^+ < +\infty$  and  $f^- \equiv -\infty$ . Let  $p, q \in \Omega$  such that  $q \in J^+(p)$ . Let  $\gamma$  be a future causal curve of  $\widetilde{Ein}_{1,n-1}$  joining  $p$  to  $q$ . Suppose that  $\gamma \not\subseteq \Omega$ . Then,  $\gamma$  intersects the boundary of  $\Omega$ , reduced in this case to the graph of  $f^+$ , in a point  $r = (x, f^+(x))$  where  $x \in U$ . By Lemma 3,  $J^+(r)$  is disjoint from  $\Omega$ . Then, the segment of  $\gamma$  joining  $r$  to  $q$  is contained in  $J^+(r)$ . Thus,  $q \notin \Omega$ . Contradiction.

Suppose now that  $f^+$  and  $f^-$  are finite. If  $\partial U$  is empty, the proof is similar to the previous case. Otherwise, we call  $f$  the common extension of  $f^+$  and  $f^-$  to  $\partial U$ . In this case, the boundary of  $\Omega$  is the union of the graphs of  $f^+, f^-$  and  $f$ . By Lemma 3, the points of  $\Omega$  are not causally related to any point in the graph of  $f$ . Therefore, the previous arguments still hold.  $\square$

Now, we describe Cauchy hypersurfaces of causally convex open subsets of  $\widetilde{Ein}_{1,n-1}$ . Let

$$\Omega := \{(x, t) \in U \times \mathbb{R} : f^-(x) < t < f^+(x)\}$$

be a causally convex open subset of  $\widetilde{Ein}_{1,n-1}$  where  $U$  is an open subset of  $\mathbb{S}^{n-1}$  and  $f^+, f^-$  are the functions from  $U$  to  $\overline{\mathbb{R}}$  given by Proposition 3.

**Proposition 5.** *Let  $h$  be a 1-Lipschitz real-valued function defined on  $U$  such that its extension to  $\partial U$  coincide with that of  $f^+$  and  $f^-$  and  $f^- < h < f^+$  on  $U$ . Then, the graph of  $h$  is a topological Cauchy hypersurface of  $\Omega$ .*

Proposition 5 is a consequence of the following lemma.

**Lemma 4.** *Suppose  $f^+$  and  $f^-$  are finite. Then, every inextensible timelike curve of  $\widetilde{Ein}_{1,n-1}$  that intersects  $\Omega$  meet each of the graphs of  $f^+$  and  $f^-$ .*

*Proof.* Let  $\gamma$  be an inextensible timelike curve of  $\widetilde{Ein}_{1,n-1}$  that intersects  $\Omega$ . Then,  $\gamma$  intersects the boundary of  $\Omega$ . If  $\partial U$  is empty,  $\partial\Omega$  is the union of the graphs of  $f^+$  and  $f^-$ . Otherwise,  $\partial\Omega$  is the union of the graphs of  $f^+, f^-$  and  $f$  where  $f$  is the common extension of  $f^+$  and  $f^-$  to  $\partial U$ . Since the points of  $\Omega$  are not causally related to any point of the graph of  $f$  (see Lemma 3), we deduce that in both cases,  $\gamma$  intersects the graph of  $f^+$  or the graph of  $f^-$ .

Suppose  $\gamma$  meets the graph of  $f^+$ . Since  $\Omega$  is not past-complete,  $\gamma$  leaves  $\Omega$  and so intersects again the its boundary. Since the graph of  $f^+$  is achronal,  $\gamma$  could not intersects the graph of  $f^+$  a second time. Thus,  $\gamma$  intersects the graph of  $f^-$ .  $\square$

*Proof of Proposition 5.* Let  $\gamma$  be an inextensible timelike curve of  $\Omega$ . Since  $\Omega$  is causally convex,  $\gamma$  is the intersection of  $\Omega$  with an inextensible timelike curve  $\tilde{\gamma}$  of  $\widetilde{Ein}_{1,n-1}$ . By Lemma 4,  $\tilde{\gamma}$  meets the graph of  $h$ . Moreover, since the graph of  $h$  is achronal,  $\tilde{\gamma}$  intersects it exactly once. The proposition follows from [8, Definition 28 and Lemma 29, p. 415].  $\square$

### 3.6 Duality

In this section, we highlight a particular class of causally convex open subsets of  $\widetilde{Ein}_{1,n-1}$  involving a notion of *duality* in Einstein universe.

**Duality in the Klein model.** Recall that a subset of  $\mathbb{S}(\mathbb{R}^{2,n})$  is said to be *convex* if it is the projectivization of a convex subset of  $\mathbb{R}^{2,n}$ . The convex hull of a subset  $A$  of  $\mathbb{S}(\mathbb{R}^{2,n})$  is the smallest convex containing  $A$ .

Let  $\Lambda \subset Ein_{1,n-1}$ . Let us denote  $\text{Conv}(\Lambda)$  the convex hull of  $\Lambda$  in  $\mathbb{S}(\mathbb{R}^{2,n})$ . The *dual convex cone* of  $\Lambda$  in  $\mathbb{S}(\mathbb{R}^{n+2})$  is

$$\text{Conv}^*(\Lambda) = \{x \in \mathbb{S}(\mathbb{R}^{2,n}) : \langle x, y \rangle_{2,n} < 0 \ \forall y \in \text{Conv}(\Lambda)\}.$$

**Definition 9.** We call dual of  $\Lambda$  the intersection of  $Ein_{1,n-1}$  with the dual cone  $\text{Conv}^*(\Lambda)$ .

Notice that

$$\text{Conv}^*(\Lambda) \cap Ein_{1,n-1} = \{x \in Ein_{1,n-1} : \langle x, y \rangle_{2,n} < 0 \ \forall y \in \Lambda\}.$$

**Duality in the universal cover.** Let  $\pi : \widetilde{Ein}_{1,n-1} \rightarrow Ein_{1,n-1}$  be the universal covering map.

**Lemma 5.** Let  $\Lambda \subset \widetilde{Ein}_{1,n-1}$ . The restriction of the projection  $\pi$  to the set of points which are non-causally related to any point of  $\Lambda$  is injective. Furthermore, its image is contained in the dual of the projection of  $\Lambda$  in  $Ein_{1,n-1}$ . If in addition,  $\Lambda$  is acausal, we have equality.

*Proof.* Set  $\Omega := \widetilde{Ein}_{1,n-1} \setminus (J^+(\Lambda) \cup J^-(\Lambda))$ . By definition,  $\Omega$  is the intersection of the affine charts  $\text{Mink}_0(p)$  where  $p \in \Lambda$ . As a result, the restriction of  $\pi$  to  $\Omega$  is injective and its image is contained in the dual of  $\pi(\Lambda)$  (see [11, Corollary 2.43]).

Suppose  $\Lambda$  is acausal. Then,  $\pi(\Lambda)$  is *negative*, i.e. for every  $x, y \in \pi(\Lambda)$ , we have  $\langle x, y \rangle_{2,n} < 0$  for every representant  $x, y \in \mathbb{R}^{2,n}$  of  $x, y$  (see [1, Lemma 10.13]).

Let  $x \in Ein_{1,n-1} \cap \text{Conv}^*(\pi(\Lambda))$ . Set  $\widehat{\Lambda}_0 := \{x\} \cup \pi(\Lambda)$ . By definition,  $\widehat{\Lambda}_0$  is a negative subset of  $Ein_{1,n-1}$ . By [11, Proposition 2.47], there exists an acausal subset  $\Lambda_0$  of  $\widetilde{Ein}_{1,n-1}$  which projects on  $\widehat{\Lambda}_0$ . Furthermore, the proof of [11, Proposition 2.47] shows that we can choose such a  $\Lambda_0$  such that it contains  $\Lambda$ . As a consequence,  $\Lambda_0$  is the union of a lift  $p$  of  $x$  and  $\Lambda$ . Since  $\Lambda_0$  is acausal,  $p$  is non-causally related to any point of  $\Lambda$ . The lemma follows.  $\square$

Lemma 5 motivates the following definition.

**Definition 10.** Let  $\Lambda$  be a subset of  $\widetilde{Ein}_{1,n-1}$ . We call dual of  $\Lambda$ , denoted by  $\Lambda^\circ$ , the set of points which are non-causally related to any point of  $\Lambda$ .

**Lemma 6.** Let  $\Lambda$  be a subset of  $\widetilde{Ein}_{1,n-1}$  such that its dual is non-empty. Then, the dual of  $\Lambda$  is causally convex. If in addition  $\Lambda$  is closed, its dual is open.

*Proof.* Let  $p$  and  $q$  be two points in the dual  $\Lambda^\circ$ , joined by a causal curve  $\gamma : I \subset \mathbb{R} \rightarrow M$ . Suppose there is  $t \in I$  such that  $\gamma(t) \notin \Lambda^\circ$ , in other words  $\gamma(t)$  is causally related to a point  $\lambda \in \Lambda$ . By transitivity, it follows that  $p$  or  $q$  is causally related to  $\lambda$ . Contradiction.

Suppose  $\Lambda$  is closed. If  $\Lambda$  is not compact, it would contain a causal curve inextensible in the future or in the past. Then,  $J^+(\Lambda) \cup J^-(\Lambda)$  would be the whole space  $\widetilde{Ein}_{1,n-1}$  and  $\Lambda^\circ$  would be empty. Therefore,  $\Lambda$  is compact. It follows that  $J^\pm(\Lambda)$  is closed. Hence,  $\Lambda^\circ$  is open.  $\square$

Now, we characterize duals of achronal closed subsets of  $\widetilde{Ein}_{1,n-1}$  in a spatio-temporal decomposition  $\mathbb{S}^{n-1} \times \mathbb{R}$ .

Let  $\Lambda$  be a closed achronal subset of  $\widetilde{Ein}_{1,n-1}$ . It is the graph of a 1-Lipschitz real-valued function  $f$  defined on a closed subset  $\Lambda_0$  of the sphere  $\mathbb{S}^{n-1}$ . Let  $f^+, f^-$  be the real-valued functions defined for every  $x \in \mathbb{S}^{n-1}$  by:

$$f^+(x) = \inf_{x_0 \in \Lambda_0} \{f(x_0) + d(x, x_0)\}$$

$$f^-(x) = \sup_{x_0 \in \Lambda_0} \{f(x_0) - d(x, x_0)\}.$$

Notice that  $f^-(x) < f^+(x)$  for every  $x \in \mathbb{S}^{n-1} \setminus \Lambda_0$  and that  $f^+$  and  $f^-$  are equal to  $f$  on  $\Lambda_0$ .

**Proposition 6.** *The dual of  $\Lambda$  is the set of points  $(x, t)$  of  $\widetilde{Ein}_{1,n-1}$  such that  $f^-(x) < t < f^+(x)$ .*

*Proof.* Let  $(x, t) \in \mathbb{S}^{n-1} \times \mathbb{R}$  be a point in the dual of  $\Lambda$ . By definition,  $(x, t)$  is non-causally related to any point  $(x_0, f(x_0))$  where  $x_0 \in \Lambda_0$ . In other words, for every  $x_0 \in \Lambda_0$ , we have  $d(x, x_0) > |t - f(x_0)|$ , i.e.  $f(x_0) - t < d(x, x_0) < f(x_0) + t$ . Hence,

$$\sup_{x_0 \in \Lambda_0} \{f(x_0) - d(x, x_0)\} \leq t \leq \inf_{x_0 \in \Lambda_0} \{f(x_0) + d(x, x_0)\}.$$

Since  $\Lambda_0$  is compact, the supremum and infimum above are attained; the previous inequalities are then strict. Thus, we obtain  $f^-(x) < t < f^+(x)$ . The converse inclusion is clear.  $\square$

## 4 Enveloping space of a simply connected GH conformally flat spacetime

### 4.1 Conformally flat spacetimes

A spacetime is called *conformally flat* if it is locally conformal to Minkowski spacetime. Einstein universe is conformally flat since any point of Einstein universe admits a neighborhood conformally equivalent to Minkowski spacetime (see Section 3.3). It follows that any spacetime locally modeled on Einstein universe, i.e. equipped with a

$(O(2, n), \widetilde{Ein}_{1, n-1})$ -structure, is conformally flat. Conversely, by Theorem 4, any conformally flat spacetime of dimension  $n \geq 3$  admits a  $(O(2, n), \widetilde{Ein}_{1, n-1})$ -structure. We deduce the following statement.

**Proposition 7.** *A conformally flat Lorentzian structure on a manifold  $M$  of dimension  $n \geq 3$  is equivalent to a  $(O_0(2, n), \widetilde{Ein}_{1, n-1})$ -structure.  $\square$*

From the causal point of view, it is more relevant to consider as model space the universal cover of Einstein universe with its group of conformal diffeomorphisms. As a result, a conformally flat Lorentzian structure on a manifold  $M$  of dimension  $n \geq 3$  is encoded by the data of a development pair  $(D, \rho)$  where  $D : \tilde{M} \rightarrow \widetilde{Ein}_{1, n-1}$  is a developing map and  $\rho : \pi_1(M) \rightarrow \text{Conf}(\widetilde{Ein}_{1, n-1})$  is the associated holonomy morphism<sup>3</sup>.

**Lemma 7.** *The restriction of the developing map  $D$  to a causal curve of  $\tilde{M}$  is injective.*

*Proof.* Let  $\gamma : I \subset \mathbb{R} \rightarrow \tilde{M}$  be a causal curve. Since  $D$  is conformal, it follows that  $D \circ \gamma : I \rightarrow \widetilde{Ein}_{1, n-1}$  is a causal curve. Then, if there exist  $t_0, t_1 \in I$  such that  $t_0 \neq t_1$  and  $D(\gamma(t_0)) = D(\gamma(t_1))$ , the curve  $D \circ \gamma$  would be a causal loop. Contradiction.  $\square$

**Definition 11.** *A conformally flat spacetime  $M$  is said developable if any developing map descends to the quotient, giving a local diffeomorphism from  $M$  to  $\widetilde{Ein}_{1, n-1}$ , called again developing map.*

**Lemma 8.** *Any developable conformally flat spacetime  $M$  is strongly causal.*

*Proof.* Let  $D : M \rightarrow \widetilde{Ein}_{1, n-1}$  be a developing map. Let  $p \in M$  and let  $U$  be a neighborhood of  $p$ . Without loss of generality, we suppose that the restriction of  $D$  to  $U$  is a diffeomorphism on its image. Then,  $D(U)$  is a neighborhood of  $D(p)$ . Since  $\widetilde{Ein}_{1, n-1}$  is GH, it is in particular strongly causal. Thus, there exists a neighborhood  $V'$  of  $D(p)$  contained in  $D(U)$  and causally convex in  $\widetilde{Ein}_{1, n-1}$ . Let  $V$  be the preimage of  $V'$  under  $D|_U$ . By definition,  $V$  is a neighborhood of  $p$  contained in  $U$ . Moreover,  $V$  is causally convex in  $M$ . Indeed, let  $\gamma$  be a causal curve of  $M$  joining two points  $q, q' \in V$ . By Lemma 7, the image under  $D$  of  $\gamma$  is a causal curve of  $\widetilde{Ein}_{1, n-1}$  joining  $D(q)$  to  $D(q')$ . Since  $V'$  is causally convex,  $D(\gamma)$  is contained in  $V'$ . If  $\gamma$  is not contained in  $V$ , there exists  $r \in \gamma \cap (U \setminus V)$ . Hence,  $D(r) \in D(\gamma) \setminus V'$ . Contradiction.  $\square$

## 4.2 Construction of an enveloping space

Let  $V$  be a simply-connected globally hyperbolic conformally flat spacetime of dimension  $n \geq 3$ . In this section, we prove Theorem 1. Our proof still hold if we weaken the assumption *simply-connected* by *developable*.

Let  $(D, \varphi)$  be a pair where

- $D : V \rightarrow \widetilde{Ein}_{1, n-1}$  is a developing map;

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<sup>3</sup>We direct the reader not familiar with  $(G, X)$ -structures to [7, Chapter 5].

- $\varphi : \widetilde{Ein}_{1,n-1} \rightarrow \mathbb{S}^{n-1} \times \mathbb{R}$  is a spatio-temporal decomposition of  $\widetilde{Ein}_{1,n-1}$ .

Throughout this section, we call  $\pi$  the projection of  $\widetilde{Ein}_{1,n-1}$  on  $\mathbb{S}^{n-1}$  defined as  $\pi_0 \circ \varphi$  where  $\pi_0 : \mathbb{S}^{n-1} \times \mathbb{R} \rightarrow \mathbb{S}^{n-1}$  is the projection on the first factor.

**Timelike foliation on  $V$ .** Consider the vector field  $T$  on  $V$  defined as the pull-back by  $D$  of  $\partial_t$ . The flow of  $T$  defines a foliation of  $V$  by smooth timelike curves. Let  $\mathcal{B}$  be the leaf space, namely the quotient space of  $V$  by the equivalence relation that identifies two points if they are on the same leaf. We denote by  $\psi : V \rightarrow \mathcal{B}$  the canonical projection.

**Fact 1.** *The leaf space  $\mathcal{B}$  is homeomorphic to a Cauchy hypersurface  $S$  of  $V$ .*

*Proof.* Every leaf is a timelike curve of  $V$  and so meets  $S$  in a unique point. Therefore, the restriction of  $\psi$  to  $S$  is a continuous bijection on  $\mathcal{B}$ . The restriction of  $\psi$  to  $S$  is open. Indeed, any open subset  $U$  of  $S$  coincides with  $\psi|_S^{-1}(\psi|_S(U))$ , so  $\psi|_S(U)$  is open in  $\mathcal{B}$ . Then, the restriction of  $\psi$  to  $S$  is a homeomorphism on  $\mathcal{B}$ .  $\square$

By definition, the map  $\pi \circ D$  is constant on each leaf, therefore it induces a map  $d$  from  $\mathcal{B}$  to  $\mathbb{S}^{n-1}$  such that the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{D} & \widetilde{Ein}_{1,n-1} \\ \psi \downarrow & & \downarrow \pi \\ \mathcal{B} & \xrightarrow{d} & \mathbb{S}^{n-1} \end{array}$$

that is  $d \circ \psi = \pi \circ D$ . Since  $\pi \circ D$  and  $\psi$  are submersions,  $d$  is a local homeomorphism.

**Fiber bundle over the leaf space.** Let  $E(V)$  be the fiber bundle over  $\mathcal{B}$  defined as the pullback by  $d : \mathcal{B} \rightarrow \mathbb{S}^{n-1}$  of the trivial bundle  $\pi : \widetilde{Ein}_{1,n-1} \rightarrow \mathbb{S}^{n-1}$ , in other words:

$$E(V) := \{(p, b) \in \widetilde{Ein}_{1,n-1} \times \mathcal{B} : \pi(p) = d(b)\}.$$

We denote by  $\hat{\pi} : E(V) \rightarrow \mathcal{B}$  the projection on the second factor.

**Fact 2.** *The fiber bundle  $E(V)$  is trivial.*

*Proof.* Let  $f$  be the continuous map from  $\mathcal{B} \times \mathbb{R}$  in  $E(V)$  that sends  $(b, t)$  on  $(p, b)$  where  $p$  is the point of  $\widetilde{Ein}_{1,n-1}$  with coordinates  $(d(b), t)$  in the decomposition  $\mathbb{S}^{n-1} \times \mathbb{R}$ . It is easy to see that  $f$  is bijective. Indeed, the inverse is the continuous map that sends  $(p, b) \in E(V)$  on  $(b, t) \in \mathcal{B} \times \mathbb{R}$  where  $t$  is the projection of  $p \in \widetilde{Ein}_{1,n-1} \simeq \mathbb{S}^{n-1} \times \mathbb{R}$  on  $\mathbb{R}$ . Therefore,  $f$  is a homeomorphism. Clearly, the following diagram commutes

$$\begin{array}{ccc} \mathbb{R} \times \mathcal{B} & \xrightarrow{f} & E(V) \\ \hat{\pi}_0 \searrow & & \swarrow \hat{\pi} \\ & \mathcal{B} & \end{array}$$

i.e.  $\hat{\pi} \circ f = \hat{\pi}_0$ . In other words,  $f$  is an isomorphism of fiber bundles. The lemma follows.  $\square$



The projection on the first factor  $\hat{D} : E(V) \rightarrow \widetilde{Ein}_{1,n-1}$  is a local homeomorphism inducing a structure of conformally flat spacetime on  $E(V)$ . In particular,  $E(V)$  is strongly causal (see Lemma 8).

**Fact 3.** *The fibers of  $E(V)$  are inextendible timelike curves.*

*Proof.* The fiber  $E_b$  of  $E(V)$  over a point  $b \in \mathcal{B}$  is the set of points  $(p, b)$  such that  $\pi(p) = d(b)$ . It is then easy to see that the restriction of  $\hat{D}$  to  $E_b$  is a homeomorphism on the fiber  $\pi^{-1}(d(b))$  of  $\pi : \widetilde{Ein}_{1,n-1} \rightarrow \mathbb{S}^{n-1}$ . The lemma follows.  $\square$

**Conformal embedding of  $V$  in  $E(V)$ .** Let  $i$  be the map from  $V$  to  $E(V)$  defined by

$$i(p) := (D(p), \psi(p))$$

for every  $p \in V$ .

**Fact 4.** *The map  $i$  is a conformal embedding of  $V$  into  $E(V)$ .*

*Proof.* Since the maps  $D$  and  $\psi$  are continuous, open and conformal, so does the map  $i$ . All we need to check is that  $i$  is injective. Let  $p, q \in V$  such that  $i(p) = i(q)$ . Then,  $D(p) = D(q)$  and  $\psi(p) = \psi(q)$ . This last inequality implies that  $p$  and  $q$  belongs to the same timelike leaf. Since the restriction of  $D$  to a leaf is injective (see Lemma 7), it follows from  $D(p) = D(q)$  that  $p = q$ .  $\square$

**Remark 1.** *The restriction of  $\hat{D}$  to  $i(M)$  coincide with  $D$ , more precisely  $\hat{D} \circ i = D$ .*

Now we prove that the image of  $i$  is causally convex in  $E(M)$ . The proof uses the following lemma.

**Lemma 9.** *The image under  $i$  of a Cauchy hypersurface  $S$  of  $V$  is a spacelike hypersurface of  $E(V)$  which disconnects  $E(V)$ .*

*Proof.* Since  $i$  is a conformal embedding,  $i(S)$  is a spacelike embedded hypersurface of  $E(V)$ . Let  $\psi : V \rightarrow \mathcal{B}$  be the canonical projection of  $V$  on the leaf space  $\mathcal{B}$ . Recall that the restriction of  $\psi$  to  $S$  is a homeomorphism on  $\mathcal{B}$ . Clearly, the map  $i|_S \circ \psi|_S^{-1} : \mathcal{B} \rightarrow E(V)$  is a section of  $\hat{\pi}$ . Hence,  $i(S)$  is a global section of  $E(V)$ . The lemma follows.  $\square$

**Fact 5.** *The image  $i(V)$  is causally convex in  $E(V)$ .*

*Proof.* Let  $p_0, p_1 \in V$  such that there is a causal curve  $\gamma$  of  $E(V)$  joining  $p_0$  to  $p_1$ . Let  $\hat{\gamma}$  be an inextendible causal curve of  $E(V)$  containing  $\gamma$ . Each connected component of the intersection of  $\hat{\gamma}$  with  $i(V)$  is an inextendible causal curve of  $i(V)$ . We call  $\gamma_0$  and  $\gamma_1$  the connected components containing  $p_0$  and  $p_1$  respectively. To prove that  $\gamma$  is contained in  $i(V)$  is equivalent to prove that  $\gamma_0 = \gamma_1$ . Suppose that  $\gamma_0$  and  $\gamma_1$  are disjoint. Then, each one of them meets  $i(S)$  in a single point,  $x_0$  and  $x_1$  respectively, which are distinct. Therefore, the curve  $\hat{\gamma}$  intersects  $i(S)$  in at least two distinct points. But,  $i(S)$  is acausal in  $E(M)$  (see Lemma 9 and [8, chap. 14, Lemma 45 and Lemma 42]). Contradiction.  $\square$

**Remark 2.** *All the results of this section stated until now are based on the existence of a developing map, and so are still valid if  $V$  is not-simply connected but developable.*

**Embedding of the conformally flat Cauchy extensions of  $V$  in  $E(V)$ .**

**Proposition 8.** *Let  $W$  be a conformally flat Cauchy-extension of  $V$ . Then, there is a conformal embedding  $i'$  of  $W$  into  $E(V)$  such that the following assertions hold:*

- *The image  $i'(W)$  is causally convex in  $E(V)$  and contains  $i(V)$ ;*
- *Every Cauchy hypersurface of  $i(V)$  is a Cauchy hypersurface of  $i'(W)$ .*

The proof of Proposition 8 uses the following lemma.

**Lemma 10.** *Let  $W$  be a conformally flat Cauchy-extension of  $V$ . Then, the enveloping spaces  $E(V)$  and  $E(W)$  are isomorphic.*

*Proof.* Let  $f : V \rightarrow W$  be a Cauchy embedding and let  $D' : W \rightarrow \widetilde{Ein}_{1,n-1}$  be a developing map such that  $D' \circ f = D$ . Consider the foliation of  $W$  by inextendible timelike curves induced by the pull-back by  $D'$  of the vector field  $\partial_t$  on  $\widetilde{Ein}_{1,n-1} \simeq \mathbb{S}^{n-1} \times \mathbb{R}$  and let  $\psi' : W \rightarrow \mathcal{B}'$  be the canonical projection on the leaf space. Since  $f$  is a Cauchy-embedding,  $f(V)$  is causally convex in  $W$  (see [10, Lemma 8]). Then, the image under  $f$  of every leaf of  $V$  is the intersection of a unique leaf of  $W$  with  $f(V)$ . It follows that the map  $\psi' \circ f$  descends to the quotient in a diffeomorphism  $\bar{f} : \mathcal{B} \rightarrow \mathcal{B}'$ . Let  $d' : \mathcal{B}' \rightarrow \mathbb{S}^{n-1}$  be the developing map induced by  $D'$ . It is clear that  $d := d' \circ \bar{f}$  is the developing map induced by  $D$ . Therefore, the map  $F : E(V) \rightarrow E(W)$  defined by  $F(p, b) = (p, \bar{f}(p))$  is a conformal diffeomorphism that sends every fiber of  $E(V)$  on a fiber of  $E(W)$ . The lemma follows.  $\square$

*Proof of Proposition 8.* Let  $j : W \hookrightarrow E(W)$  be the conformal embedding of  $W$  into  $E(W)$  and let  $F : E(V) \rightarrow E(W)$  be the isomorphism defined in the proof of Lemma 10. The map  $i' := F^{-1} \circ j$  defines a conformal embedding of  $W$  into  $E(V)$ . By Lemma 5,  $i(V)$  and  $i'(W)$  are causally convex in  $E(V)$ . Moreover, according to the proof of Lemma 10, the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{i} & E(V) \\ f \downarrow & & \downarrow F \\ W & \xrightarrow{j} & E(W) \end{array}$$

that is  $F \circ i = j \circ f$ , i.e.  $i = F^{-1} \circ j \circ f = i' \circ f$ . It follows that:

- $i(V) = i'(f(V)) \subset i'(W)$ .
- Every Cauchy hypersurface of  $i(V)$  is a Cauchy hypersurface of  $i'(W)$ . Indeed, since  $f$  is a Cauchy-embedding, if  $S$  is a Cauchy hypersurface of  $V$ ,  $f(S)$  is a Cauchy hypersurface of  $W$ . Then,  $i(S) = i'(f(S))$  is a Cauchy hypersurface of  $i'(W)$ .

The proposition follows.  $\square$

**Remark 3.** *If  $V$  is developable, it is easy to see that any conformally flat Cauchy extension of  $W$  is also developable. As a result, Proposition 8 is still true in this setting.*

**The  $\mathcal{C}_0$ -maximal extension of  $V$ .** From now on, we identify  $V$  and the conformally flat Cauchy-extensions of  $V$  with their images in  $E(V)$ . Let  $S$  be a Cauchy hypersurface of  $V$ .

**Lemma 11.** *The Cauchy development of  $S$  in  $E(V)$  contains all the Cauchy-extensions of  $V$ . In particular, it contains  $V$ .*

*Proof.* Let  $W$  be a Cauchy extension of  $V$ . Let  $x \in W$  and let  $\hat{\gamma}$  be an inextendible causal curve of  $E(V)$  going through  $x$ . Since  $W$  is causally convex in  $E(V)$  (see Proposition 8), the intersection of  $\hat{\gamma}$  with  $W$  is an inextendible causal curve  $\gamma$  of  $W$ . By Proposition 8,  $S$  is a Cauchy hypersurface of  $W$ , then  $\gamma$  intersects  $S$  in a single point. It follows that  $x$  belongs to the Cauchy development of  $S$  in  $E(V)$ .  $\square$

**Proposition 9.** *The Cauchy development  $\mathcal{C}(S)$  of  $S$  in  $E(V)$  is a  $\mathcal{C}_0$ -maximal extension of  $V$ .*

*Proof.* By [8, Theorem 38, p. 421],  $\mathcal{C}(S)$  is a globally hyperbolic spacetime for which  $S$  is a Cauchy hypersurface. According to Lemma 11, it is a Cauchy extension of  $V$ . Let  $W$  be a conformally flat Cauchy-extension of  $\mathcal{C}(S)$ . In particular,  $W$  is a Cauchy-extension of  $V$ . Then,  $W$  embeds conformally in  $E(V)$  and the image is a causally convex open subset of  $E(V)$  containing  $\mathcal{C}(S)$ . By Lemma 11,  $\mathcal{C}(S)$  is exactly  $W$  (seen in  $E(V)$ ). Hence,  $\mathcal{C}(S)$  is  $\mathcal{C}_0$ -maximal.  $\square$

**Corollary 2.** *The  $\mathcal{C}_0$ -maximal extension  $\mathcal{C}(S)$  is unique up to conformal diffeomorphism.*

*Proof.* Let  $\hat{V}$  another  $\mathcal{C}_0$ -maximal extension of  $V$ . By Lemma 11,  $\hat{V}$ , seen in  $E(V)$ , is contained in  $\mathcal{C}(S)$ . If this inclusion is strict,  $\mathcal{C}(S)$  would be a Cauchy extension of  $\hat{V}$ . This contradicts the  $\mathcal{C}_0$ -maximality of  $\hat{V}$ .  $\square$

We proved again the existence and the uniqueness of the  $\mathcal{C}_0$ -maximal extension for *simply-connected* conformally flat globally hyperbolic flat spacetimes.

**Corollary 3.** *If  $V$  is Cauchy-compact then the  $\mathcal{C}_0$ -maximal extension of  $V$  is conformally equivalent to  $\widetilde{Ein}_{1,n-1}$ .*

*Proof.* By Lemma 1, the leaf space  $\mathcal{B}$  is compact. Since  $d$  is a local homeomorphism, it follows that  $d$  is a covering. But,  $\mathbb{S}^{n-1}$  is simply connected, so  $d$  is a homeomorphism. As a result,  $\hat{D} : E(V) \rightarrow \widetilde{Ein}_{1,n-1}$  is a conformal diffeomorphism. Indeed, let  $p \in \widetilde{Ein}_{1,n-1}$ . There exists a unique  $b \in \mathcal{B}$  such that  $d(b) = \pi(p)$ . Thus,  $(p, b)$  is the unique point of  $E(V)$  such that  $\hat{D}(p, b) = p$ . The corollary follows.  $\square$

**Definition 12.** *The trivial fiber bundle  $\hat{\pi} : E(V) \rightarrow \mathcal{B}$  is called **an enveloping space** of  $V$ .*

The construction of this fiber bundle depends on the choice of a pair  $(D, \varphi)$  where

- $D : V \rightarrow \widetilde{Ein}_{1,n-1}$  is a developing map ;

- $\varphi : \mathbb{S}^{n-1} \times \mathbb{R} \rightarrow \widetilde{Ein}_{1,n-1}$  is a spatio-temporal decomposition of  $\widetilde{Ein}_{1,n-1}$ .

Given two such pairs  $(D, \varphi)$ ,  $(D', \varphi')$ , there exists a conformal transformation  $\phi$  of  $\widetilde{Ein}_{1,n-1}$  such that  $D' = \phi \circ D$ . We say that  $(D, \varphi)$  and  $(D', \varphi')$  are *equivalent* if  $\varphi' = \varphi \circ \phi^{-1}$ .

**Lemma 12.** *If  $(D, \varphi)$  and  $(D', \varphi')$  are equivalent, the enveloping spaces  $E(V)$  and  $E'(V)$  defined by  $(D, \varphi)$  and  $(D', \varphi')$  are isomorphic, i.e. there exists a conformal diffeomorphism from  $E(V)$  to  $E'(V)$  which sends fiber on fiber.*  $\square$

**Example 1** (Enveloping space of Minkowski spacetime). *Minkowski spacetime  $\mathbb{R}^{1,n-1}$  is conformally equivalent to the set of points of  $\widetilde{Ein}_{1,n-1}$  which are not causally related to a point  $p \in \widetilde{Ein}_{1,n-1}$ . Given a spatio-temporal decomposition  $\varphi : \widetilde{Ein}_{1,n-1} \rightarrow \mathbb{S}^{n-1} \times \mathbb{R}$ , the enveloping space of  $\mathbb{R}^{1,n-1}$  is the complement in  $\widetilde{Ein}_{1,n-1}$  of the fiber going through  $p$  of the trivial bundle  $\pi_0 \circ \varphi : \widetilde{Ein}_{1,n-1} \rightarrow \mathbb{S}^{n-1}$ . This is a trivial fiber bundle over  $\mathbb{S}^{n-1} \setminus \{\pi_0 \circ \varphi(p)\} \simeq \mathbb{R}^{n-1}$ .*

**Example 2** (Enveloping space of De-Sitter spacetime). *Given a spatio-temporal decomposition  $\varphi : \widetilde{Ein}_{1,n-1} \rightarrow \mathbb{S}^{n-1} \times \mathbb{R}$ , De-Sitter spacetime is conformally equivalent to  $\mathbb{S}^{n-1} \times ]0, \pi[$ . Therefore, the enveloping space of De-Sitter spacetime relatively to this decomposition is  $\pi_0 \circ \varphi : \widetilde{Ein}_{1,n-1} \rightarrow \mathbb{S}^{n-1}$  where  $\pi_0 : \mathbb{S}^{n-1} \times \mathbb{R} \rightarrow \mathbb{S}^{n-1}$  is the projection on the first factor.*

### 4.3 Causally convex GH open subsets of the enveloping space

In this section, we describe causally convex open subsets of the enveloping space  $E(V)$  - defined in the previous section - which are globally hyperbolic. Notice that since  $E(V)$  is, *a priori*, not globally hyperbolic (see Example 1), causal convexity does not imply global hyperbolicity anymore.

By [10, Theorem 10], causally convex GH open subsets of  $E(V)$  which contain *conjugate points*, i.e. points whose image under  $\hat{D}$  are conjugate in  $\widetilde{Ein}_{1,n-1}$ , are conformally equivalent to causally convex open subsets of  $\widetilde{Ein}_{1,n-1}$ . These are described in Section 3.5. This is why we only deal here with causally convex open subsets of  $E(V)$  without conjugate points. We basically generalise the description of causally convex open subsets of  $\widetilde{Ein}_{1,n-1}$ .

The following definition introduces a class of sections of the enveloping space  $E(V) \rightarrow \mathcal{B}$ , expressed in a global trivialization.

**Definition 13.** *A real-valued function  $f$  defined on an open subset  $U$  of  $\mathcal{B}$  is said 1-Lipschitz if for every  $x \in U$ , there exists an open neighborhood  $U_x$  of  $x$  contained in  $U$  such that the following hold:*

1. *the restriction of  $d$  to  $U_x$  is injective;*
2. *the map  $f \circ d_{|U_x}^{-1} : d(U_x) \subset \mathbb{S}^{n-1} \rightarrow \mathbb{R}$  is 1-Lipschitz.*

This definition generalizes the (local) notion of 1-Lipschitz functions defined on an open subset of the sphere  $\mathbb{S}^{n-1}$ .

**Remark 4.** *The function  $f$  is 1-Lipschitz if and only if its graph is locally achronal.*

**Lemma 13.** *Any 1-Lipschitz real-valued function  $f$  defined on an open subset  $U \subsetneq \mathcal{B}$  admits a unique extension to  $\overline{U}$ .  $\square$*

**Proposition 10.** *Any causally convex GH open subset  $\Omega$  of  $E(V)$  is the domain bounded by the graphs of two 1-Lipschitz real-valued functions  $f^+$  and  $f^-$  defined on an open subset  $U$  of  $\mathcal{B}$  such that:*

1.  $f^- < f^+$  on  $U$ ;
2. the extensions of  $f^+$  and  $f^-$  to  $\partial U$  coincide.

*Proof.* Let  $\hat{\pi}$  the natural projection of  $E(V)$  on  $\mathcal{B}$ . We call  $U$  the projection of  $\Omega$  on  $\mathcal{B}$ . Since  $\Omega$  is causally convex, the intersection of  $\Omega$  with any fiber  $\hat{\pi}^{-1}(x)$ , where  $x \in U$ , is connected, i.e. it is a segment  $\{x\} \times ]f^-(x), f^+(x)[$ . Notice that  $-\infty < f^-(x)$  and  $f^+(x) < +\infty$ , otherwise  $\Omega$  would contain conjugate points. Contradiction. This defines two real-valued functions  $f^+, f^-$  defined on  $U$  such that:

$$\Omega = \{(x, t) \in U \times \mathbb{R} : f^-(x) < t < f^+(x)\}.$$

Let  $x \in U$ . Set  $p_+ = (x, f^+(x))$ . Let  $p \in I^-(p_+) \cap \Omega$ . Since  $\Omega$  is GH, the restriction of  $\hat{D}$  to  $I^+(p, \Omega)$  is injective and its image is causally convex in  $\widehat{Ein}_{1, n-1}$  (see [9, Prop. 2.7 and Cor. 2.8, p.151]). It follows that:

- $\hat{D}(I^+(p, \Omega))$  is the domain bounded by the graphs of two 1-Lipschitz real-valued functions  $g^- < g^+$  defined on an open subset of  $\mathbb{S}^{n-1}$  (see Proposition 3);
- $d$  is injective on  $U_x := \hat{\pi}(I^+(p, \Omega))$  and  $g^+ = f^+ \circ d_{|U_x}^{-1}$ .

Thus,  $f^+$  is 1-Lipschitz. The proof is similar for  $f^-$  with the reverse time-orientation. Lemma 2 is still valid for causally convex open subsets of  $E(V)$ . Hence, the same arguments used in the proof of Proposition 3 shows that the extensions of  $f^+$  and  $f^-$  to  $\partial U$  coincide.  $\square$

**Corollary 4.** *The graphs of  $f^+$  and  $f^-$  are achronal.*

*Proof.* The proof is similar than that of Fact 2 in the proof of Proposition 3.  $\square$

## 5 $\mathcal{C}_0$ -maximal extensions of conformally flat globally hyperbolic spacetimes

In this section, we give a new proof of the existence and the uniqueness of the  $\mathcal{C}_0$ -maximal extension of a globally hyperbolic conformally flat spacetime  $V$  of dimension  $n \geq 3$ , using the notion of enveloping space introduced in Section 4.

When  $V$  is simply-connected, the proof is given in Section 4.2 (see Proposition 9 and Corollary 2). We deal here with the case where  $V$  is not simply-connected. The proof consists to extend the action of  $\pi_1 V$  on  $\tilde{V}$  to a proper action on the  $\mathcal{C}_0$ -maximal extension of  $\tilde{V}$ . We prove then that the  $\mathcal{C}_0$ -maximal extension of  $V$  is the quotient of the  $\mathcal{C}_0$ -maximal extension of  $\tilde{V}$  by  $\pi_1 V$ .

**Notations.** Let  $D : \tilde{V} \rightarrow \widetilde{Ein}_{1,n-1}$  be a developing map and let  $\rho : \Gamma \rightarrow \text{Conf}(\widetilde{Ein}_{1,n-1})$  be the associated holonomy representation, where  $\Gamma := \pi_1(V)$ .

Fix a decomposition  $\varphi : \widetilde{Ein}_{1,n-1} \rightarrow \mathbb{S}^{n-1} \times \mathbb{R}$ . Let  $E(\tilde{V})$  be the enveloping space related to the pair  $(D, \varphi)$  (see Section 4.2). We denote by  $\hat{\pi} : E(\tilde{V}) \rightarrow \mathcal{B}$  the projection on the second factor and by  $\hat{D} : E(\tilde{V}) \rightarrow \widetilde{Ein}_{1,n-1}$  the projection on the first factor.

**Fact 6.** *If  $\tilde{V}$  is Cauchy-compact, the  $\mathcal{C}_0$ -maximal extension of  $V$  is a finite quotient of  $\widetilde{Ein}_{1,n-1}$ .*

*Proof.* This is an immediate consequence of Corollary 3. □

Suppose now that  $\tilde{V}$  is not Cauchy-compact. Let  $\tilde{S}$  be a non-compact Cauchy hypersurface of  $\tilde{V}$ . The  $\mathcal{C}_0$ -maximal extension of  $\tilde{V}$  is the Cauchy development  $\mathcal{C}(\tilde{S})$  of  $\tilde{S}$  in  $E(\tilde{V})$  (see Proposition 9). In what follows, we extend the action of  $\Gamma$  to  $\mathcal{C}(\tilde{S})$ .

**Action of  $\Gamma$  on  $\mathcal{C}(\tilde{S})$ .** The points of  $\mathcal{C}(\tilde{S})$  are characterized by their shadows on  $\tilde{S}$  (see Proposition 2). As a result, we show that the action of  $\Gamma$  on  $\tilde{S}$  induces naturally an action of  $\Gamma$  on  $\mathcal{C}(\tilde{S})$ :

**Proposition 11.** *Let  $p \in \mathcal{C}(\tilde{S})$  and let  $\gamma \in \Gamma$ . There exists a unique point  $\gamma.p$  in  $\mathcal{C}(\tilde{S})$  such that its shadow on  $\tilde{S}$  is exactly the image under  $\gamma$  of the shadow of  $p$  on  $\tilde{S}$ . This defines an action of  $\Gamma$  on  $\mathcal{C}(\tilde{S})$  which satisfies the following properties:*

- *the restriction of the action of  $\Gamma$  on  $\mathcal{C}(\tilde{S})$  to  $\tilde{V}$  coincide with the usual action of  $\Gamma$  on  $\tilde{V}$ ;*
- *the action of  $\Gamma$  on  $\mathcal{C}(\tilde{S})$  preserves the causality relations, i.e. for every  $p \in \mathcal{C}(\tilde{S})$  and for every  $\gamma \in \Gamma$ , we have:*

$$\begin{aligned} p \in J^-(q) &\Leftrightarrow \gamma.p \in J^-(\gamma.q) \\ p \in I^-(q) &\Leftrightarrow \gamma.p \in I^-(\gamma.q); \end{aligned}$$

- *the restriction of  $\hat{D}$  to  $\mathcal{C}(\tilde{S})$  is  $\rho$ -equivariant, i.e. for every  $p \in \mathcal{C}(\tilde{S})$  and for every  $\gamma \in \Gamma$ , we have:*

$$\hat{D}(\gamma.p) = \rho(\gamma)\hat{D}(p).$$

We prove Proposition 11 by an analysis-synthesis reasoning. In the analysis, we suppose that  $\gamma.p$  exists and is unique, and we look for properties satisfied by  $\gamma.p$  that will characterize it. In the synthesis, we use the criteria found in the analysis to determine the point  $\gamma.p$ . In what follows, the shadow of a point  $p \in \mathcal{C}(\tilde{S})$  on  $\tilde{S}$  is denoted by  $O(p)$ .

**Analysis.** Suppose that for every  $\gamma \in \Gamma$  and every  $p \in \mathcal{C}(\tilde{S})$ , the point  $\gamma.p$  exists and is unique. For every  $\gamma \in \Gamma$  and every  $p \in \tilde{V}$ , we denote by  $\gamma p$  (without the dot between  $\gamma$  and  $p$ ) the usual action of the deck transformation  $\gamma$  on  $p$ . Let us start with this easy remark.

**Remark 5.** Let  $p \in \mathcal{C}(\tilde{S})$  and let  $\gamma \in \Gamma$ . If  $p \in \tilde{V}$ , then  $\gamma.p = \gamma p$ . Indeed, since the action of  $\Gamma$  on  $\tilde{V}$  respect the causality relations, we have  $J^-(\gamma p) = \gamma J^-(p)$ . Therefore,

$$O(\gamma p) := J^-(\gamma p) \cap \tilde{S} = \gamma J^-(p) \cap \tilde{S} = \gamma(J^-(p) \cap \gamma^{-1}\tilde{S}).$$

Since  $\tilde{S}$  is  $\Gamma$ -invariant, we deduce that  $\gamma(J^-(p) \cap \gamma^{-1}\tilde{S}) = \gamma O(p)$ . Hence,  $O(\gamma p) = \gamma O(p)$ , i.e.  $\gamma.p = \gamma p$ .

**Lemma 14.** The map which associates to every  $(\gamma, p) \in \Gamma \times \mathcal{C}(\tilde{S})$  the point  $\gamma.p \in \mathcal{C}(\tilde{S})$  is a group action. Moreover, this action respect causality relations.

*Proof.* The fact that the map  $(\gamma, p) \in \Gamma \times \mathcal{C}(\tilde{S}) \mapsto \gamma.p$  is a group action follows easily from the fact that the restriction to  $\Gamma \times \tilde{S}$  is the usual group action of  $\Gamma$  on  $\tilde{S}$ .

Let  $\gamma \in \Gamma$  and let  $p, q \in \mathcal{C}(\tilde{S})$ . Suppose that  $p, q \in J^+(S)$ . Then,

$$p \in J^-(q) \Leftrightarrow O(p) \subset O(q) \Leftrightarrow O(\gamma.p) := \gamma O(p) \subset \gamma O(q) =: O(\gamma.q) \Leftrightarrow \gamma.p \in J^-(\gamma.q).$$

By symmetry, the same arguments still hold if  $p, q \in J^-(\tilde{S})$ . It remains the case where  $q \in I^+(\tilde{S})$  and  $p \in I^-(\tilde{S})$ . In this case, we have

$$p \in J^-(q) \Leftrightarrow O(p) \cap O(q) \neq \emptyset \Leftrightarrow \gamma O(p) \cap \gamma O(q) \neq \emptyset \Leftrightarrow O(\gamma.p) \cap O(\gamma.q) \neq \emptyset.$$

Since  $\Gamma$  preserves  $\tilde{S}$ , it preserves the chronological future/past of  $\tilde{S}$  in  $\mathcal{C}(\tilde{S})$ . Thus,  $O(\gamma.p) \cap O(\gamma.q) \neq \emptyset \Leftrightarrow \gamma.p \in J^-(\gamma.q)$ . The lemma follows.  $\square$

**Lemma 15.** The restriction of the developing map  $\hat{D} : E(\tilde{V}) \rightarrow \widetilde{Ein}_{1,n-1}$  to  $\mathcal{C}(\tilde{S})$  is  $\rho$ -equivariant, i.e. for every  $p \in \mathcal{C}(\tilde{S})$  and for every  $\gamma \in \Gamma$ , we have  $\hat{D}(\gamma.p) = \rho(\gamma)\hat{D}(p)$ .

*Proof.* Let  $\gamma \in \Gamma$  and let  $p \in \mathcal{C}(\tilde{S})$ . Suppose that  $p \in I^+(\tilde{S})$ . Let  $q \in \mathcal{C}(\tilde{S}) \cap I^+(p)$ . Since  $\mathcal{C}(\tilde{S})$  is globally hyperbolic, the restriction of  $\hat{D}$  to  $I^-(q, \mathcal{C}(\tilde{S}))$  is injective and its image is causally convex in  $\widetilde{Ein}_{1,n-1}$  (see [9, Prop. 2.7 and Cor. 2.8, p. 151]). It follows that  $D(I^-(q) \cap \tilde{S})$  is an achronal hypersurface of  $\hat{D}(I^-(q, \mathcal{C}(\tilde{S})))$ .

Set  $\Sigma := D(I^-(q) \cap \tilde{S})$ . Since  $O(p) \subset I^-(q, \mathcal{C}(\tilde{S}))$ , the restriction of  $D$  to  $O(p)$  is injective and its image is exactly  $O(\hat{D}(p), \Sigma)$ . Thus

$$D(\gamma O(p)) = \rho(\gamma)D(O(p)) = \rho(\gamma)O(\hat{D}(p), \Sigma) = O(\rho(\gamma)\hat{D}(p), \rho(\gamma)\Sigma). \quad (1)$$

By definition,  $\gamma O(p) = O(\gamma.p)$ . From Lemma 14, we get  $O(\gamma.p) \subset I^-(\gamma.q, \mathcal{C}(\tilde{S}))$ . As above, we deduce that the restriction of  $D$  to  $O(\gamma.p)$  is injective and its image is exactly  $O(\hat{D}(\gamma.p), D(I^-(\gamma.p) \cap \tilde{S}))$ . But,  $D(I^-(\gamma.p) \cap \tilde{S}) = \rho(\gamma)\Sigma$ . Indeed, since  $O(\gamma.q) = \gamma O(q)$  we have  $I^-(\gamma.q) \cap \tilde{S} = \gamma(I^-(q) \cap \tilde{S})$ . Thus,  $D(I^-(\gamma.q) \cap \tilde{S}) = \rho(\gamma)\Sigma$ . Hence,

$$D(\gamma O(p)) = D(O(\gamma.p)) = O(\hat{D}(\gamma.p), \rho(\gamma)\Sigma). \quad (2)$$

It follows from (1) and (2) that  $O(\rho(\gamma)\hat{D}(p), \rho(\gamma)\Sigma) = O(\hat{D}(\gamma.p), \rho(\gamma)\Sigma)$ . By Proposition 2, we deduce that  $\hat{D}(\gamma.p) = \rho(\gamma)\hat{D}(p)$ .

If  $p \in I^-(\tilde{S})$ , the same arguments hold with the reverse time-orientation. Lastly, if  $p \in \tilde{S}$ , the lemma follows from Remark 5.  $\square$

**Remark 6.** *In the proof of Lemma 15, we can replace  $I^-(q, \mathcal{C}(\tilde{S}))$  by any other causally convex open neighborhood  $U$  of  $p$  in  $\mathcal{C}(\tilde{S})$  such that:*

- $U$  intersects  $\tilde{S}$ ;
- the restriction of  $\hat{D}$  to  $U$  is injective;
- the image  $\hat{D}(U)$  is causally convex in  $\widetilde{Ein}_{1,n-1}$ .

**Synthesis.** Let  $p \in \mathcal{C}(\tilde{S})$  and let  $\gamma \in \Gamma$ . Without loss of generality, we suppose that  $p \in I^+(\tilde{S})$ . The idea is to reconstruct the proof of Lemma 15 to determine  $\gamma.p$ . More precisely, we choose a relevant causally convex open neighborhood  $U$  of  $p$  in  $\mathcal{C}(\tilde{S})$  satisfying the properties stated in Remark 6 and we construct the open subset that will turn out to be  $\gamma.U$  and will therefore contain  $\gamma.p$  (see Figure 1).

Fix  $q \in \mathcal{C}(\tilde{S}) \cap I^+(p)$ . Let  $U$  be the Cauchy development of  $I^-(q) \cap \tilde{S}$  in  $E(\tilde{V})$ .

**Lemma 16.** *The image under  $D$  of  $I^-(q) \cap \tilde{S}$  is achronal in  $\widetilde{Ein}_{1,n-1}$ .*

*Proof.* We have  $I^-(q) \cap \tilde{S} \subset I^-(q, \mathcal{C}(\tilde{S}))$ . Since  $\mathcal{C}(\tilde{S})$  is GH, the restriction of  $\hat{D}$  to  $I^-(q, \mathcal{C}(\tilde{S}))$  is injective (see [9, Prop. 2.7, p. 151]). Then,  $\underline{D}(I^-(q) \cap \tilde{S})$  is achronal in  $\hat{D}(I^-(q, \mathcal{C}(\tilde{S})))$ . This last one being causally convex in  $\widetilde{Ein}_{1,n-1}$  (see [9, Cor. 2.8, p. 151]), we deduce easily that  $D(I^-(q) \cap \tilde{S})$  is achronal in  $\widetilde{Ein}_{1,n-1}$ .  $\square$

**Fact 7.** *The restriction of  $\hat{D}$  to  $U$  is injective. Moreover, the image under  $\hat{D}$  of  $U$  is equal to the Cauchy development of  $D(I^-(q) \cap \tilde{S})$  in  $\widetilde{Ein}_{1,n-1}$ .*

*Proof.* Clearly,  $U \subset \mathcal{C}(\tilde{S}) \cap I^-(q)$ . Since  $\mathcal{C}(\tilde{S})$  is causally convex in  $E(\tilde{V})$ , we have  $\mathcal{C}(\tilde{S}) \cap I^-(q) = I^-(q, \mathcal{C}(\tilde{S}))$ . Since the restriction of  $\hat{D}$  to  $I^-(q, \mathcal{C}(\tilde{S}))$  is injective and its image is causally convex in  $\widetilde{Ein}_{1,n-1}$  (see [9, Prop. 2.7 and Cor. 2.8, p. 151]), the lemma follows.  $\square$

Let  $W$  be the Cauchy development of  $\gamma(I^-(q) \cap \tilde{S})$  in  $E(\tilde{V})$ .

**Remark 7.** *The image under  $D$  of  $\gamma(I^-(q) \cap \tilde{S})$  is achronal in  $\widetilde{Ein}_{1,n-1}$ . Indeed,  $D(\gamma(I^-(q) \cap \tilde{S})) = \rho(\gamma)D(I^-(q) \cap \tilde{S})$ . The assertion follows then from Lemma 16.*

**Fact 8.** *The restriction of  $\hat{D}$  to  $W$  is injective. Moreover, the image under  $\hat{D}$  of  $W$  is equal to the Cauchy development of  $D(\gamma(I^-(q) \cap \tilde{S}))$  in  $\widetilde{Ein}_{1,n-1}$ .*



*Proof.* Let  $r, r' \in W$  such that  $\hat{D}(r) = \hat{D}(r')$ . Let  $E_{\hat{\pi}(r)}$  and  $E_{\hat{\pi}(r')}$  be the fibers going through  $r$  and  $r'$ . By definition, their images under  $\hat{D}$  are the fibers of the trivial bundle  $\widetilde{Ein}_{1,n-1} \rightarrow \mathbb{S}^{n-1}$  going through  $\hat{D}(r)$  and  $\hat{D}(r')$ . Since  $\hat{D}(r) = \hat{D}(r')$ , these fibers coincide. We call this fiber  $\Delta$ . Since  $E_{\hat{\pi}(r)}$  and  $E_{\hat{\pi}(r')}$  are inextendible timelike curves, they intersect  $\gamma(I^-(q) \cap \tilde{S})$  in two points  $r_0$  and  $r'_0$ . Hence,  $D(r_0), D(r'_0) \in \Delta \cap D(\gamma(I^-(q) \cap \tilde{S}))$ . But,  $D(\gamma(I^-(q) \cap \tilde{S}))$  is achronal in  $\widetilde{Ein}_{1,n-1}$  (see Remark 7). Thus, the timelike line  $\Delta$  intersects  $D(\gamma(I^-(q) \cap \tilde{S}))$  at most once. Hence,  $D(r_0) = D(r'_0)$ . The restriction of  $D$  to  $\gamma(I^-(q) \cap \tilde{S})$  being injective, we deduce that  $r_0 = r'_0$ . Thus,  $E_{\hat{\pi}(r)} = E_{\hat{\pi}(r')}$ . Since the restriction of  $\hat{D}$  to any causal curve is injective (see Lemma 7), we get  $r = r'$ .

We deduce easily from the injectivity of  $\hat{D}$  on  $W$  that  $\hat{D}(W)$  is contained in the Cauchy development of  $D(\gamma(I^-(q) \cap \tilde{S}))$  in  $\widetilde{Ein}_{1,n-1}$ . This inclusion is clearly a Cauchy-embedding. Since  $W$  is  $\mathcal{C}_0$ -maximal (see Proposition 9), we deduce equality.  $\square$

*Proof of Proposition 11.* Set  $\Sigma := D(I^-(q) \cap \tilde{S})$ . Since  $O(p) \subset U$ , it follows from Fact 7 that  $D(O(p))$  is equal to the shadow  $O(\hat{D}(p), \Sigma)$  of  $\hat{D}(p)$  on  $\Sigma$ . Hence

$$D(\gamma O(p)) = \rho(\gamma)D(O(p)) = \rho(\gamma)O(\hat{D}(p), \Sigma) = O(\rho(\gamma)\hat{D}(p), \rho(\gamma)\Sigma).$$

Hence,  $O(\rho(\gamma)\hat{D}(p), \rho(\gamma)\Sigma) \subset \hat{D}(W)$ . Since  $\hat{D}(W)$  is the Cauchy development of  $\rho(\gamma)\Sigma$  (see Fact 8), it follows that  $\rho(\gamma)\hat{D}(p) \in \hat{D}(W)$ . Thus, there exists a unique point  $p' \in U$  such that  $\hat{D}(p') = \rho(\gamma)\hat{D}(p)$ . It is clear that  $O(p') = \gamma O(p)$ . If there is another point  $p'' \in \mathcal{C}(\tilde{S})$  such that  $O(p'') = \gamma O(p)$ , by Proposition 2, we get  $p' = p''$ . Then, we set  $\gamma.p := p'$ .  $\square$

**Dynamical properties of the action of  $\Gamma$  on  $\mathcal{C}(\tilde{S})$ .** In this paragraph, we prove that the action of  $\Gamma$  on  $\mathcal{C}(\tilde{S})$  is free and properly discontinuous.

**Lemma 17.** *Let  $p$  be a point in the complement of  $\tilde{S}$  in  $\mathcal{C}(\tilde{S})$ . Then, the shadow of  $p$  on  $\tilde{S}$  is a topological disk of dimension  $n - 1$ .*

*Proof.* We can suppose without loss of generality that  $p \in I^+(\tilde{S})$ . Let  $q \in I^+(p)$  and set  $\Sigma := D(I^-(q) \cap \tilde{S})$ . We prove that  $\hat{D}(I^-(q, \mathcal{C}(\tilde{S})))$  is contained in the affine chart  $\text{Mink}_-(\hat{D}(q))$  (see Section 3.3). Suppose there exists  $r \in I^-(q, \mathcal{C}(\tilde{S}))$  such that  $\hat{D}(r) \notin \text{Mink}_-(\hat{D}(q))$ . Then,  $I(\hat{D}(q), \hat{D}(r))$  contains conjugate points. Since the image under  $\hat{D}$  of  $I^-(q, \mathcal{C}(\tilde{S}))$  is causally convex in  $\widetilde{Ein}_{1,n-1}$  (see [9, Cor. 2.8, p. 151]), we deduce that  $I^-(q, \mathcal{C}(\tilde{S}))$  admits a photon whose image under  $\hat{D}$  contains conjugate points. Therefore, by [10, Theorem 10],  $\mathcal{C}(\tilde{S})$  is conformally equivalent to  $\widetilde{Ein}_{1,n-1}$ . Contradiction. Then,  $\hat{D}(p)$  and  $\Sigma$  are contained in  $\text{Mink}_-(\hat{D}(q))$ . As a result, the shadow  $O(\hat{D}(p), \Sigma)$  is the intersection in Minkowski spacetime of the past causal cone of  $\hat{D}(p)$  with  $\Sigma$ . Thus, the map which associates to every past causal direction at  $\hat{D}(p)$ , the intersection of the straight line tangent to this direction with  $\Sigma$  is a homeomorphism. Hence,  $O(\hat{D}(p), \Sigma)$  is a topological  $(n - 1)$ -disk. Since the restriction of  $\hat{D}$  to  $O(p)$  is a diffeomorphism on  $O(\hat{D}(p), \Sigma)$  (see [9, prop. 2.7, p. 151]), the lemma follows.  $\square$

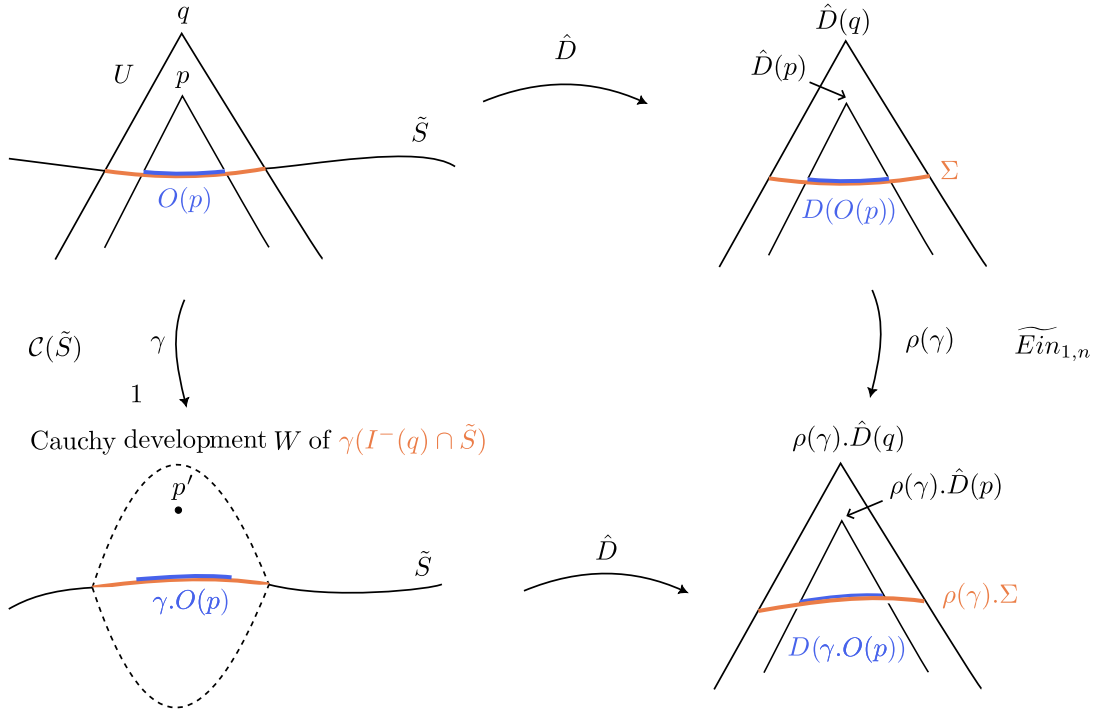


Figure 1: Action of  $\Gamma$  on the Cauchy development of  $\tilde{S}$  in the enveloping space  $E(\tilde{S})$ .

**Proposition 12.** *The action of  $\Gamma$  on  $\mathcal{C}(\tilde{S})$  is free and properly discontinuous.*

*Proof.* Let  $p \in \mathcal{C}(\tilde{S})$  and let  $\gamma \in \Gamma$  such that  $\gamma.p = p$ . Thus,  $\gamma$  preserves  $O(p)$ . Since  $O(p)$  is a topological disk (see Lemma 17), by Brouwer's theorem,  $\gamma$  admits a fixed point in  $O(p)$ . Since the action of  $\Gamma$  on  $\tilde{S}$  is free, we deduce that  $\gamma = id$ . This proves that the action of  $\Gamma$  on  $\mathcal{C}(\tilde{S})$  is free.

Suppose the action of  $\Gamma$  on  $\mathcal{C}(\tilde{S})$  is not properly discontinuous. Then, by [5, Proposition 1], there exist a sequence  $\{p_i\}$  of points of  $\mathcal{C}(\tilde{S})$  converging to some point  $p_\infty \in \mathcal{C}(\tilde{S})$  and a divergent sequence  $\{\gamma_i\}$  of elements of  $\Gamma$  such that  $\{\gamma_i.p_i\}$  converges to some point  $q_\infty \in \mathcal{C}(\tilde{S})$ . Without loss of generality, we can suppose that  $p_\infty, q_\infty \in J^+(\tilde{S})$ .

Let  $p \in I^+(p_\infty)$ . Since  $\lim p_i = p_\infty$ , all the  $p_i$  belong to  $I^-(p)$  except a finite number. Hence,  $O(p_i) \subset O(p)$  for every  $i \geq i_0$  where  $i_0$  is a natural integer. Let  $x_i \in O(p_i)$ . Up to extracting,  $\{x_i\}$  converges to some  $x_\infty \in O(p)$ . Similarly, let  $q \in I^+(q_\infty)$ ; all the  $\gamma_i p_i$  belong to  $O(q)$  except a finite number. Hence,  $\gamma_i O(p_i) = O(\gamma_i p_i) \subset O(q)$ . Therefore, up to extracting,  $\{\gamma_i x_i\}$  converges to some point  $y_\infty \in O(q)$ . This contradicts the properness of the action of  $\Gamma$  on  $\tilde{S}$  (see [5, Proposition 1]).  $\square$

**The  $\mathcal{C}_0$ -maximal extension of  $V$ .**

**Proposition 13.** *The quotient space  $\Gamma \backslash \mathcal{C}(\tilde{S})$  is a  $\mathcal{C}_0$ -maximal extension of  $V$ .*

*Proof.* Let  $i : \tilde{V} \rightarrow E(\tilde{V})$  the embedding of  $\tilde{V}$  in  $E(\tilde{V})$  defined in Section 4. The co-restriction of  $i$  to  $\mathcal{C}(\tilde{S})$  is a Cauchy embedding, denoted by  $\tilde{f}$ . Let  $\pi' : \mathcal{C}(\tilde{S}) \rightarrow \Gamma \backslash \mathcal{C}(\tilde{S})$  be the canonical projection. Since the action of  $\Gamma$  on  $i(\tilde{V})$  coincide with the usual action of  $\Gamma$  on  $\tilde{V}$ , the map  $\tilde{f}$  descends to quotient in a Cauchy embedding  $f : V \rightarrow \Gamma \backslash \mathcal{C}(\tilde{S})$ . It is easy to see that since  $\mathcal{C}(\tilde{S})$  is  $\mathcal{C}_0$ -maximal (see Proposition 9), the quotient  $\Gamma \backslash \mathcal{C}(\tilde{S})$  is also  $\mathcal{C}_0$ -maximal. The proposition follows.  $\square$

**Corollary 5.** *The  $\mathcal{C}_0$ -maximal extension  $\Gamma \backslash \mathcal{C}(\tilde{S})$  is unique up to conformal diffeomorphism.*

*Proof.* Let  $f : V \rightarrow W$  be a Cauchy embedding of  $V$  in a  $\mathcal{C}_0$ -maximal globally hyperbolic conformally flat spacetime  $W$ . It lifts to a Cauchy embedding  $\tilde{f} : \tilde{V} \rightarrow \tilde{W}$ . By Proposition 8,  $\tilde{W}$  admits a conformal copy in  $E(\tilde{V})$  contained in  $\mathcal{C}(\tilde{S})$ , where  $\tilde{S}$  is a Cauchy hypersurface of  $\tilde{V}$  (seen in  $E(\tilde{V})$ ). The inclusion of  $\tilde{W}$  in  $\mathcal{C}(\tilde{S})$  is a Cauchy embedding which descends to the quotient in a Cauchy embedding from  $W$  to  $\Gamma \backslash \mathcal{C}(\tilde{S})$ . By Proposition 13, this last one is surjective. The corollary follows.  $\square$

The proof of Proposition 13 is based on the fact that if the universal covering of a globally hyperbolic spacetime is maximal then this spacetime is maximal. *A priori*, the converse assertion is not true in general. However, Proposition 13 allows to prove that it is true in the conformally flat setting.

**Corollary 6.** *Let  $V$  be a  $\mathcal{C}_0$ -maximal spacetime. Then, the universal covering of  $V$  is  $\mathcal{C}_0$ -maximal.*

*Proof.* Let  $\tilde{S}$  be a Cauchy hypersurface of  $\tilde{V}$ . By Proposition 13, there is a Cauchy embedding  $f$  from  $V$  to  $\Gamma \backslash \mathcal{C}(\tilde{S})$ . This last one lifts to a Cauchy embedding  $\tilde{f} : \tilde{V} \rightarrow \mathcal{C}(\tilde{S})$ . Since  $V$  is maximal,  $f$  is surjective. Therefore,  $\tilde{f}$  is surjective. Since  $\mathcal{C}(\tilde{S})$  is  $\mathcal{C}_0$ -maximal (see Proposition 9),  $\tilde{V}$  is  $\mathcal{C}_0$ -maximal.  $\square$

## 6 $\mathcal{C}_0$ -maximal extensions respect inclusion

In this section, we show that the *functor maximal extension* respect inclusion in the setting of conformally flat spacetimes. More precisely, we establish the following result:

**Theorem 5.** *Let  $V$  be a conformally flat globally hyperbolic spacetime and let  $U$  be a causally convex open subset of  $V$ . Then, there is a conformal embedding from the  $\mathcal{C}_0$ -maximal extension  $\hat{U}$  of  $U$  into the  $\mathcal{C}_0$ -maximal extension  $\hat{V}$  of  $V$ . Moreover, the image of this embedding is causally convex in  $\hat{V}$ .*

We first prove this result in the case where  $V$  is simply-connected in Section 6.1 before dealing with the general case in Section 6.2.

## 6.1 The simply-connected case

We prove Theorem 5 in the case where  $V$  is simply-connected:

**Proposition 14.** *Let  $V$  be a **simply-connected** conformally flat globally hyperbolic spacetime and let  $U$  be a causally convex open subset of  $V$ . Then, there is a conformal embedding from the  $\mathcal{C}_0$ -maximal extension  $\hat{U}$  of  $U$  into the  $\mathcal{C}_0$ -maximal extension  $\hat{V}$  of  $V$ . Moreover, the image is causally convex in  $\hat{V}$ .*

The key idea is to realize the  $\mathcal{C}_0$ -maximal extensions of  $U$  and  $V$  in the enveloping space  $E(V)$  so we can compare them. The proof of Proposition 14 uses the following lemma.

Let  $D : V \rightarrow \widetilde{Ein}_{1,n-1}$  be a developing map.

**Lemma 18.** *The inclusion map  $i : U \hookrightarrow V$  induces a conformal embedding of  $E(U)$  into  $E(V)$  that sends every fiber of  $E(U)$  on a fiber of  $E(V)$  and such that the restriction to every fiber of  $E(U)$  is surjective.*

*Proof.* Consider the foliation of  $V$  by inextensible timelike curves induced by the pull-back by the developing map  $D$  of the vector field  $\partial_t$  on  $Ein_{1,n} \simeq \mathbb{S}^{n-1} \times \mathbb{R}$ . Since  $U$  is causally convex in  $V$ , the intersection of every leaf of  $V$  with  $U$  is an inextensible timelike curve of  $U$ . Therefore, the foliation of  $V$  induces a foliation of  $U$  by inextensible timelike curves. Notice that this foliation coincides with that induced by the pull-back of  $\partial_t$  by the restriction of  $D$  to  $U$ . Let  $\psi : V \rightarrow \mathcal{B}$  and  $\psi_U : U \rightarrow \mathcal{B}_U$  be the canonical projections on the leaf spaces. Then, the map  $\psi \circ i$  descends to the quotient in an embedding  $\bar{i} : \mathcal{B}_U \rightarrow \mathcal{B}$ . Let  $d : \mathcal{B} \rightarrow \mathbb{S}^{n-1}$  (resp.  $d_U : \mathcal{B}_U \rightarrow \mathcal{S}^{n-1}$ ) be the developing map induced by the developing map  $D$  (resp. the restriction of  $D$  to  $U$ ). It is clear that  $d \circ \bar{i} = d_U$ . It follows that the map from  $E(U)$  to  $E(V)$  that sends  $(p, b)$  on  $(p, \bar{i}(b))$  is a conformal embedding that sends every fiber of  $E(U)$  on a fiber of  $E(V)$ . Moreover, the restriction to every fiber of  $E(U)$  is clearly surjective.  $\square$

In other words, Lemma 18 says that  $E(U)$  can be seen as the union of the fibers of  $E(V)$  over some open subset of  $\mathcal{B}$ .

*Proof of Proposition 14.* We identify  $V$  (resp.  $U$ ) with its image in  $E(V)$  (resp.  $E(U)$ ), then we identify  $E(U)$  with its image in  $E(V)$ . Let  $\hat{V}$  (resp.  $\hat{U}$ ) be the  $\mathcal{C}_0$ -maximal extension of  $V$  (resp.  $U$ ). By Proposition 9,  $\hat{V}$  (resp.  $\hat{U}$ ) is the Cauchy development  $\mathcal{C}(S)$  of a Cauchy hypersurface  $S$  of  $V$  (resp.  $\Sigma$  of  $U$ ) in  $E(V)$  (resp.  $E(U)$ ). We prove that the Cauchy development  $\mathcal{C}(\Sigma)$  of  $\Sigma$  in  $E(V)$  is exactly  $\hat{U}$ , then we prove that  $\mathcal{C}(\Sigma) \subset \mathcal{C}(S)$ :

Let  $x \in \hat{U}$  and let  $\hat{\gamma}$  an inextensible causal curve of  $E(V)$  going through  $x$ . The intersection of  $\hat{\gamma}$  with  $E(U)$  is a union of connected components. The component containing  $x$  is an inextensible causal curve of  $E(U)$ , then it intersects  $\Sigma$  in a single point. Hence,  $x \in \mathcal{C}(\Sigma)$ . This proves that  $\hat{U} \subset \mathcal{C}(\Sigma)$ . Actually, this inclusion is a Cauchy embedding. Since  $U$  is  $\mathcal{C}_0$ -maximal, we deduce that  $\hat{U} = \mathcal{C}(\Sigma)$ .

Let  $x \in \mathcal{C}(\Sigma)$  and let  $\hat{\gamma}$  an inextensible causal curve of  $E(V)$  going through  $x$ . By definition, the curve  $\hat{\gamma}$  meets  $\Sigma$ , hence  $V$ . Since  $V$  is causally convex in  $E(V)$ , the intersection of  $\hat{\gamma}$  with  $V$  is an inextensible causal curve of  $V$ , thus it intersects  $S$  in a single point. It follows that  $x \in \mathcal{C}(S)$ . Hence,  $\mathcal{C}(\Sigma) \subset \mathcal{C}(S)$ . The proposition follows.  $\square$

## 6.2 The general case

In the previous section, we proved Theorem 5 in the case where  $V$  is simply-connected. In this section, we prove it for any conformally flat globally hyperbolic spacetime  $V$ . Without loss of generality, we suppose that  $V$  is  $\mathcal{C}_0$ -maximal. Let  $\pi : \tilde{V} \rightarrow V$  be the universal cover of  $V$ . Set  $\Gamma := \pi_1(V)$ . Let  $U'$  be a connected component of  $\pi^{-1}(U)$  and let  $\Gamma'$  be the stabilizer of  $U'$  in  $\Gamma$ .

**Lemma 19.** *The open set  $U'$  is causally convex in  $\tilde{V}$ .*

*Proof.* Let  $p, q \in U'$  and let  $\gamma$  be a causal curve in  $\tilde{V}$  joining  $p$  to  $q$ . The projection  $\pi(\gamma)$  of  $\gamma$  in  $V$  is a causal curve joining the points  $\pi(p)$  and  $\pi(q)$  of  $U$ . Since  $U$  is causally convex in  $V$ , the curve  $\pi(\gamma)$  is contained in  $U$ . Hence,  $\gamma$  is contained in  $U'$ . The lemma follows.  $\square$

Let  $\widehat{U}'$  be the  $\mathcal{C}_0$ -maximal extension of  $U'$ . By Proposition 14,  $\widehat{U}'$  can be conformally identified to a causally convex open subset of  $\tilde{V}$ .

**Lemma 20.** *The stabilizer of  $\widehat{U}'$  in  $\Gamma$  is equal to  $\Gamma'$ .*

*Proof.* Since  $U' \subset \widehat{U}'$ , the stabilizer of  $U'$  is contained in the stabilizer of  $\widehat{U}'$ . Conversely, let  $\gamma$  be an element of  $\Gamma$  stabilizing  $\widehat{U}'$ . Let  $p \in U'$ . Consider a Cauchy hypersurface  $\Sigma$  of  $U'$  going through  $p$ . Let  $\varphi$  be an inextensible causal curve of  $\widehat{U}'$  going through  $p$ . Then,  $\gamma\varphi$  is an inextensible causal curve of  $\widehat{U}'$ . Therefore,  $\gamma\varphi$  intersects  $\Sigma$  in a unique point  $q$ . We prove that  $q = \gamma p$ .

Since  $\widehat{U}'$  is causally convex in  $\tilde{V}$ , the projection  $\pi(\widehat{U}')$  is causally convex in  $V$  and  $\pi(\Sigma)$  is a Cauchy hypersurface of  $\pi(\widehat{U}')$ . Moreover,  $\pi(\varphi)$  is an inextensible causal curve of  $\pi(\widehat{U}')$ , then it meets  $\pi(\Sigma)$  in a single point. Since  $\pi(p), \pi(q) \in \pi(\varphi) \cap \pi(\Sigma)$ , we deduce that  $\pi(p) = \pi(q)$ . Then,  $q = \gamma'p$  with  $\gamma' \in \Gamma$ . Since  $\gamma p, \gamma'p \in \gamma\varphi$ , if  $\gamma \neq \gamma'$ , the causal curve  $\pi(\varphi)$  would be closed. Contradiction. Hence,  $\gamma = \gamma'$ , so  $q = \gamma p$ . This shows that  $\gamma p \in U'$ . Thus,  $\gamma \in \Gamma'$ .  $\square$

The inclusion  $\widehat{U}' \subset \tilde{V}$  descends to the quotient in a conformal embedding from  $\Gamma' \backslash \widehat{U}'$  to  $V$ . Since  $\widehat{U}'$  is causally convex in  $\tilde{V}$ , the image of this embedding is causally convex in  $V$ . It remains to prove the following assertion to conclude.

**Lemma 21.** *The quotient space  $\Gamma' \backslash \widehat{U}'$  is the  $\mathcal{C}_0$ -maximal extension of  $U$ .*

*Proof.* The inclusion  $U' \subset \widehat{U}'$  is a Cauchy embedding which descends to the quotient in a Cauchy embedding from  $U$  to  $\Gamma' \backslash \widehat{U}'$ . We have to prove that  $\Gamma' \backslash \widehat{U}'$  is  $\mathcal{C}_0$ -maximal.

Let  $f$  be a Cauchy embedding from  $\Gamma' \setminus \widehat{U}'$  in a globally hyperbolic conformally flat spacetime  $W$ . Then, the morphism  $f_* : \pi_1(\Gamma' \setminus \widehat{U}') \rightarrow \pi_1(W)$  induced by  $f$  is an isomorphism. Let  $\pi' : \widehat{U}' \rightarrow \Gamma' \setminus \widehat{U}'$  be the canonical projection. It induces an injective morphism  $\pi'_* : \pi_1(\widehat{U}') \rightarrow \pi_1(\Gamma' \setminus \widehat{U}')$ . Hence, the morphism  $(f \circ \pi')_* : \pi_1(\widehat{U}') \rightarrow \pi_1(W)$  is injective. Let  $p : W' \rightarrow W$  the cover such that  $p_*(\pi_1(W')) = (f \circ \pi')_*(\pi_1(\widehat{U}'))$ . Then,  $f$  lifts to a Cauchy embedding  $f' : \widehat{U}' \rightarrow W'$ . Since  $\widehat{U}'$  is  $\mathcal{C}_0$ -maximal,  $f'$  is surjective. Hence,  $f$  is surjective. The lemma follows.  $\square$

## 7 Eikonal functions and $\mathcal{C}_0$ -maximality

### 7.1 Eikonal functions on the sphere

In this section, we characterize causally convex open subsets of  $\widetilde{Ein}_{1,n-1}$  which are  $\mathcal{C}_0$ -maximal in a spatio-temporal decomposition  $\mathbb{S}^{n-1} \times \mathbb{R}$ . We denote by  $d$  the distance on  $\mathbb{S}^{n-1}$  induced by the round metric. By [10, Theorem 10], causally convex open subsets of  $\widetilde{Ein}_{1,n-1}$  with conjugate points are conformally equivalent to  $\widetilde{Ein}_{1,n-1}$ . For this reason, we consider a causally convex open subset  $\Omega$  *without conjugate points*.

Let  $f^+$  and  $f^-$  be the two 1-Lipschitz real-valued functions defined on the projection  $U$  of  $\Omega$  in  $\mathbb{S}^{n-1}$  such that:

$$\Omega = \{(x, t) \in U \times \mathbb{R}, f^-(x) < t < f^+(x)\}.$$

We call  $f$  the common extension of  $f^+$  and  $f^-$  to  $\partial U$ . Let  $g^+, g^-$  be the real-valued functions defined for every  $x \in U$  by:

$$g^+(x) = \inf_{x_0 \in \partial U} \{f(x_0) + d(x, x_0)\} \quad (3)$$

$$g^-(x) = \sup_{x_0 \in \partial U} \{f(x_0) - d(x, x_0)\}. \quad (4)$$

**Proposition 15.** *The causally convex open subset  $\Omega$  of  $\widetilde{Ein}_{1,n-1}$  is  $\mathcal{C}_0$ -maximal if and only if  $f^+$  equals  $g^+$  and  $f^-$  equals  $g^-$ .*

The proof of Proposition 15 uses the following lemma.

**Lemma 22.** *Any 1-Lipschitz real-valued function  $g$  defined on  $U$ , whose extension to  $\partial U$  equals  $f$ , is bounded by  $g^-$  and  $g^+$ .*

*Proof.* Let  $x \in U$  and let  $x_0 \in \partial U$ . We denote by  $\bar{g}$  the extension of  $g$  to  $\partial U$ . Since  $\bar{g}$  is 1-Lipschitz, we have  $\bar{g}(x) - \bar{g}(x_0) \leq d(x, x_0)$ , i.e.  $g(x) - f(x_0) \leq d(x, x_0)$ . Hence,  $g(x) \leq f(x_0) + d(x, x_0)$ . Therefore,  $g(x) \leq f^+(x)$ . From  $-d(x, x_0) \leq \bar{g}(x) - \bar{g}(x_0)$ , we deduce similarly that  $f^-(x) \leq g(x)$ . The lemma follows.  $\square$

*Proof of Proposition 15.* Let us denote  $\Omega'$  the set of points  $(x, t)$  of  $\widetilde{Ein}_{1,n-1}$  such that  $g^-(x) < t < g^+(x)$ . By Lemma 22 and Proposition 5,  $\Omega'$  is a Cauchy-extension of  $\Omega$ .

According to Theorem 5, the  $\mathcal{C}_0$ -maximal extension of  $\Omega$  is conformally equivalent to a causally convex open subset  $\hat{\Omega}$  of  $\widetilde{Ein}_{1,n-1}$  such that  $\Omega \subset \Omega' \subset \hat{\Omega}$  where each inclusion is a Cauchy embedding. Thus, there exist two 1-Lipschitz real-valued functions  $h^-, h^+$  defined on  $U$ , whose extensions to  $\partial U$  equal  $f$ , such that  $\hat{\Omega}$  is the set of points  $(x, t)$  such that  $h^-(x) < t < h^+(x)$ . Hence,  $g^-(x) \leq h^-(x) < h^+(x) \leq g^+(x)$  for every  $x \in U$  (see 22). In other words,  $\hat{\Omega} \subset \Omega'$ . The proposition follows.  $\square$

The domain  $\hat{\Omega}$  in the proof of Proposition 15 is a union of connected component of the dual of the graph of  $f$  (see Proposition 6). Therefore, Proposition 15 can be reformulated as:  *$\mathcal{C}_0$ -maximal causally convex open subsets of  $\widetilde{Ein}_{1,n-1}$  are exactly unions of connected components of duals of closed achronal subsets of  $\widetilde{Ein}_{1,n-1}$ .*

**Definition 14.** *The function  $g^+$  (resp.  $g^-$ ) is called future eikonal (resp. past eikonal).*

**Remark 8.** *This definition is motivated by the classical notion of eikonal function in analysis: a real-valued function  $f$  defined on an open subset  $U$  of  $\mathbb{R}^n$  is called eikonal if  $f$  is differentiable almost everywhere and satisfies the eikonal equation  $\|\nabla f\| = 1$ . It turns out that if  $U$  has a piecewise smooth boundary, then the function  $f(x) := d(x, \partial U)$ , where  $d$  is the usual distance on  $\mathbb{R}^n$ , is eikonal in this sense.*

The following proposition gives a geometrical characterisation of eikonal functions.

**Proposition 16** (Geometrical criterion of eikonality). *A real-valued function  $f^+$  defined on an open subset  $U$  of the sphere  $\mathbb{S}^{n-1}$  is future eikonal if and only if for every  $x \in U$ , there exists a past-directed lightlike geodesic starting from  $(x, f^+(x))$ , entirely contained in the graph of  $f^+$  and with no past endpoint in the graph of  $f^+$ .*

*Proof.* Suppose  $f^+$  is eikonal. We call  $f$  the extension of  $f^+$  to  $\partial U$ . Let  $x \in U$ . There exists  $x_0 \in \partial U$  such that  $f^+(x) = f(x_0) + d(x, x_0)$ . Hence,  $f^+(x) - f(x_0) = d(x, x_0)$ , i.e. the points  $(x, f^+(x))$  and  $(x_0, f(x_0))$  are joined by a past lightlike geodesic  $\varphi$ . This last one is contained in the graph of  $f^+$ . Conversely, let us prove that  $f^+$  equals the function  $g^+$  defined by (3). By Lemma 22, we have  $f^+ \leq g^+$ . Let  $x \in U$ . There exists a past-directed lightlike geodesic  $\varphi$  starting from  $(x, f^+(x))$ , entirely contained in the graph of  $f^+$  and with no past endpoint in the graph of  $f^+$ . In  $\widetilde{Ein}_{1,n-1}$ , the geodesic  $\varphi$  admits a past endpoint  $(x_0, f(x_0))$  with  $x_0 \in \partial U$ . Thus,  $d(x, x_0) = f^+(x) - f(x_0)$ . Hence,  $f^+(x) = f(x_0) + d(x, x_0) \geq g^+(x)$ . Then,  $f^+ = g^+$ .  $\square$

Now, we show that eikonal is a local property.

**Definition 15.** *A real-valued function  $f^+$  defined on an open subset  $U$  of the sphere  $\mathbb{S}^{n-1}$  is locally future eikonal if every point  $x$  of  $U$  admits an arbitrarily small neighborhood  $V_x$  such that the restriction of  $f^+$  to  $V_x$  is future eikonal.*

**Remark 9.** *A locally future eikonal function is locally 1-Lipschitz, thus 1-Lipschitz.*

Let  $f^+$  be a 1-Lipschitz real-valued function defined on an open subset  $U$  of  $\mathbb{S}^{n-1}$ . We call  $f$  its extension to  $\partial U$ .

**Lemma 23.** *If  $f^+$  is future eikonal then  $f^+$  is locally future eikonal.*

*Proof.* Let  $V$  be an open subset of  $U$ . We call  $g$  the extension of  $f^+_{|V}$  to  $\partial V$ . Let  $g^+$  (resp.  $g^-$ ) be the function defined by the expression (3) (resp. (4) after replacing  $U$  by  $V$ ). By Lemma 22, we have  $f^+_{|V} \leq g$ . Let us prove that  $f^+_{|V} \geq g$  so we get the equality. Let  $\Omega$  (resp.  $W$ ) be the  $\mathcal{C}_0$ -maximal causally convex open subset of  $\widetilde{Ein}_{1,n-1}$  bounded by the graphs of  $f^+$  and  $f^-$  (resp.  $g^+$  and  $g^-$ ). The intersection  $W \cap \Omega$  is a causally convex open subset of  $\Omega$ ; its  $\mathcal{C}_0$ -maximal extension is  $W$ . By theorem 5, we get  $W \subset \Omega$ . Hence,  $g^+ \leq f^+_{|V}$ . Thus,  $g^+ = f^+_{|V}$ , then  $f^+_{|V}$  is future eikonal.  $\square$

**Lemma 24.** *If  $f^+$  is locally future eikonal then  $f^+$  is future eikonal.*

*Proof.* We use the criterion given by Proposition 16 to prove that  $f^+$  is eikonal. Let  $x \in U$  and let  $V \subset U$  be a neighborhood of  $x$  such that  $f^+_{|V}$  is future eikonal. Then, there is a past-directed lightlike geodesic  $\varphi$  starting from  $(x, f^+(x))$  with the properties of Proposition 16. In  $\widetilde{Ein}_{1,n-1}$ , the geodesic  $\varphi$  admits a past endpoint  $(x_0, f^+(x_0))$  where  $x_0 \in \partial V$  (we still denote by  $f^+$  its extension to  $\overline{U}$ ). If  $x_0 \in \partial U$ , the lemma is proved. Otherwise, we choose a neighborhood  $V_0 \subset U$  of  $x_0$  such that  $f^+_{|V_0}$  is future eikonal. Again, there is a past-directed lightlike geodesic  $\varphi_0$  starting from  $(x_0, f^+(x_0))$ , entirely contained in the graph of  $f^+_{|V_0}$ , with past endpoint  $(x_1, f^+(x_1))$  where  $x_1 \in \partial V_0$ . The geodesic  $\varphi_0$  extends  $\varphi$ ; indeed, otherwise  $(x_1, f^+(x_1))$  would be in the chronological past of  $(x, f^+(x))$ , i.e. we would have  $d(x, x_1) < f^+(x) - f^+(x_1)$ . This contradicts the fact that  $f^+$  is 1-Lipschitz. Consequently, we extend  $\varphi$  in a lightlike geodesic, entirely contained in the graph of  $f^+$ , with no past endpoint in the graph of  $f^+$ . The lemma follows.  $\square$

We proved the following statement:

**Proposition 17.** *A real-valued function defined on an open subset of the sphere  $\mathbb{S}^{n-1}$  is future eikonal if and only if it is locally future eikonal.*  $\square$

## 7.2 Eikonal functions on a conformally flat Riemannian manifold

Let  $\mathcal{B}$  be a conformally flat Riemannian manifold of dimension  $(n - 1) \geq 2$  and let  $d : \mathcal{B} \rightarrow \mathbb{S}^{n-1}$  be a developing map. The notion of eikonal function on an open subset of the sphere  $\mathbb{S}^{n-1}$  being *local*, it naturally generalizes to functions on an open subset of  $\mathcal{B}$ :

**Definition 16.** *A real-valued function  $f^+$  defined on an open subset  $U$  of  $\mathcal{B}$  is said future eikonal if for every  $x \in U$ , there exists an open neighborhood  $U_x$  of  $x$  contained in  $U$  such that the following hold:*

1. *the restriction of  $d$  to  $U_x$  is injective;*
2. *the map  $f^+ \circ d^{-1}_{|U_x} : d(U_x) \subset \mathbb{S}^{n-1} \rightarrow \mathbb{R}$  is future eikonal.*



Past eikonal functions on  $U$  are defined similarly with the reverse time orientation. The geometrical criterion of eikonality given in Proposition 16 is still valid in this setting.

Eikonality is closely related to  $\mathcal{C}_0$ -maximality. Indeed, let  $E$  be the pull-back of the trivial fiber bundle  $\pi : \widetilde{Ein}_{1,n-1} \rightarrow \mathbb{S}^{n-1}$  by  $d$ :

$$E := \{(p, b) \in \widetilde{Ein}_{1,n-1} \times \mathcal{B} : \pi(p) = d(b)\}.$$

We denote by  $\pi : E \rightarrow \mathcal{B}$  the projection on the second factor. The construction of this fiber bundle is exactly the same than that of the enveloping space in Section 4.2. Then,  $E$  is a conformally flat spacetime of dimension  $n \geq 3$  sharing the same properties than that of the enveloping space. Let  $\hat{D} : E \rightarrow \widetilde{Ein}_{1,n-1}$  be the developing map defined as the projection on the first factor. A causally convex GH open subset  $\Omega$  of  $E$  is given by

$$\Omega = \{(x, t) \in U \times \mathbb{R} : f^-(x) < t < f^+(x)\}$$

where  $f^+$  and  $f^-$  are 1-Lipschitz functions defined on an open subset  $U$  of  $\mathcal{B}$  such that their extensions to  $\partial U$  coincide (see Section 4.3).

**Proposition 18.** *The GH conformally flat spacetime  $\Omega$  is  $\mathcal{C}_0$ -maximal if and only if  $f^+$  is future-eikonal and  $f^-$  past-eikonal.*

*Proof.* Suppose  $\Omega$  is  $\mathcal{C}_0$ -maximal. Let  $x \in U$ . Set  $p_+ = (x, f^+(x))$ . Let  $p \in I^-(p_+) \cap \Omega$ . Since  $\Omega$  is GH, the restriction of  $\hat{D}$  to  $I^+(p, \Omega)$  is injective and its image is causally convex in  $\widetilde{Ein}_{1,n-1}$  (see [9, Prop. 2.7 and Cor. 2.8, p.151]). The  $\mathcal{C}_0$ -maximality of  $\Omega$  implies that  $\hat{D}(I^+(p, \Omega))$  is also  $\mathcal{C}_0$ -maximal (see [9, Prop. 3.6, p.156]). It follows that:

- $\hat{D}(I^+(p, \Omega))$  is the domain bounded by the graph of a future-eikonal function  $g^+$  and the graph of a past-eikonal function  $g^-$  defined on an open subset of  $\mathbb{S}^{n-1}$  (see Proposition 15);
- $d$  is injective on  $U_x := \hat{\pi}(I^+(p, \Omega))$  and  $g^+ = f^+ \circ d|_{U_x}^{-1}$ .

Thus,  $f^+$  is future-eikonal. The proof is similar for  $f^-$  with the reverse time-orientation.

Conversely, suppose that  $f^+$  is future-eikonal and  $f^-$  past-eikonal. Let  $S$  be a Cauchy hypersurface in  $\Omega$ . By Proposition 9, the  $\mathcal{C}_0$ -maximal extension of  $\Omega$  is conformally equivalent to a causally convex open subset  $\hat{\Omega}$  of  $E$ , containing  $\Omega$  and for which  $S$  is a Cauchy hypersurface. Suppose that  $\Omega$  is strictly contained in  $\hat{\Omega}$ . Then, there exists  $x \in U$  such that  $(x, f^+(x)) \in \hat{\Omega}$  or  $(x, f^-(x)) \in \hat{\Omega}$ . Suppose that  $(x, f^+(x)) \in \hat{\Omega}$ . The proof is symmetric if  $(x, f^-(x)) \in \hat{\Omega}$ . Since  $f^+$  is future-eikonal, there exists a past-directed lightlike geodesic  $\varphi$  starting from  $(x, f^+(x))$ , entirely contained in the graph of  $f^+$  and with no endpoint in the graph of  $f^+$ . Then,  $\varphi$  does not intersects  $S$ . Contradiction. Hence,  $\Omega = \hat{\Omega}$ , in other words  $\Omega$  is  $\mathcal{C}_0$ -maximal.  $\square$

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