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# Multilevel Parameterized Model Order Reduction for Variability Analysis of Circuits

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**Abstract**— A multilevel parameterized model order reduction technique is proposed for variability analysis of large circuits. The original large-scale equations are kept into the multilevel scheme, which allows a full error control about the estimation of stochastic moments. A high computational efficiency is also provided by a set of parameterized reduced order models properly combined with the original large-scale model. Pertinent numerical results are presented to validate the proposed technique.

**Keywords**— Parameterized model order reduction, Multifidelity modeling, Circuits, Variability analysis.

## I. INTRODUCTION

Parameterized model order reduction (PMOR) techniques have been successfully used to obtain efficient models able to accurately describe the behavior of complex systems as a function of frequency/time and design parameters (e.g., layout and material parameters). Over the years, different PMOR techniques have been proposed [1], [2], [3]. PMOR models can be used to significantly speed-up design tasks such as optimization and variability analysis. However, one aspect that deserves high attention is the estimation of the accuracy of PMOR models. This is in general a difficult task and error estimation techniques [4], [5] might not provide accurate information about the actual error.

The lack of accurate information about the error of PMOR models is an evident limitation in design tasks such as variability analysis. An interesting solution to this problem can come from multifidelity modeling methods. The main idea behind multifidelity modeling is to combine models with a different level of accuracy and computational complexity [6], [7]. The original large-scale model and a set of PMOR models can be therefore properly combined.

In this work, a multilevel PMOR technique is proposed for the variability analysis of large circuits. The multifidelity scheme proposed in [7] is used to combine the original large-scale model and a set of PMOR models. This scheme can ensure an error control on the estimation of stochastic moments since the original model is kept into the analysis, while exploiting the efficiency of PMOR models. The number of samples for each model included into the multilevel scheme is chosen in order to maximize efficiency and guarantee a desired accuracy about the stochastic moments estimation. Pertinent numerical results validate the proposed technique.

## II. OUTLINE OF THE PROPOSED METHOD

### A. Formulation of the Circuit Equations

The notion of uncertainty in the circuit response due to uncertainty in circuit design parameters is captured by adapting the modified nodal analysis (MNA) formulation [8] as follows

$$(\mathbf{G}(\boldsymbol{\xi}) + s\mathbf{C}(\boldsymbol{\xi}))\mathbf{X}(s, \boldsymbol{\xi}) = \mathbf{B} \quad (1)$$

$$\mathbf{Y}(s, \boldsymbol{\xi}) = \mathbf{L}\mathbf{X}(s, \boldsymbol{\xi}) \quad (2)$$

where  $\mathbf{G}, \mathbf{C} \in \mathbb{R}^{N \times N}$  are parameter-dependent conductance and susceptance matrices, respectively;  $\mathbf{X}(s, \boldsymbol{\xi}) \in \mathbb{R}^N$  is the vector of MNA variables,  $\mathbf{B} \in \mathbb{R}^N$  is the vector of input signals,  $\mathbf{Y}(s, \boldsymbol{\xi}) \in \mathbb{R}^{N_{out}}$  is the parameter-dependent output response,  $\mathbf{L} \in \mathbb{R}^{N_{out} \times N}$  is the output selection matrices and  $\boldsymbol{\xi} = [\xi_1, \dots, \xi_d]$  is a  $d$ -dimensional vector collecting all the random circuit parameters.

### B. Multi-moment Matching Based PMOR

PMOR techniques aim to find a reduced system whose output matches the few leading moments of the original system output  $\mathbf{Y}(s, \boldsymbol{\xi})$  in (2). In order to find the reduced-order model, the block moments of  $\mathbf{X}(s, \boldsymbol{\xi})$  with respect to frequency (denoted by  $\mathbf{M}_s$ ) and the moments with respect to the random parameters  $\xi_1, \dots, \xi_d$  (denoted by  $\mathbf{M}_{\xi_1}, \dots, \mathbf{M}_{\xi_d}$ ), as well as, the cross-derivative block moments (denoted by  $\mathbf{M}_{s, \xi_1, \dots, \xi_d}$ ) are computed. The moments can be computed using an implicit-moment matching recursive modified Gram-Schmidt-like process [2], [3] to construct a projection matrix  $\mathbf{Q}$  whose columns spans the same subspace of the moments as:

$$\text{colsp}\{\mathbf{Q}\} = \text{colsp}\{\mathbf{M}_s, \mathbf{M}_{\xi_1}, \dots, \mathbf{M}_{\xi_d}, \mathbf{M}_{s, \xi_1, \dots, \xi_d}\} \quad (3)$$

A parameterized reduced order model can be computed using congruent transformation as follows. Define

$$\tilde{\mathbf{X}}(s, \boldsymbol{\xi}) = \mathbf{Q}^T \mathbf{X}(s, \boldsymbol{\xi}) \quad (4)$$

where  $\mathbf{Q} \in \mathbb{R}^{N \times q}$  and  $q \ll N$  denotes the reduced order.

The reduced system can be written as

$$(\tilde{\mathbf{G}}(\boldsymbol{\xi}) + s\tilde{\mathbf{C}}(\boldsymbol{\xi}))\tilde{\mathbf{X}}(s, \boldsymbol{\xi}) = \tilde{\mathbf{B}} \quad (5)$$

$$\tilde{\mathbf{Y}}(s, \boldsymbol{\xi}) = \tilde{\mathbf{L}}\tilde{\mathbf{X}}(s, \boldsymbol{\xi}) \quad (6)$$

where  $\tilde{\mathbf{X}}(s, \boldsymbol{\xi}) = \mathbf{Q}^T \mathbf{X}(s, \boldsymbol{\xi})$ ,  $\tilde{\mathbf{G}} = \mathbf{Q}^T \mathbf{G} \mathbf{Q}$ ,  $\tilde{\mathbf{C}} = \mathbf{Q}^T \mathbf{C} \mathbf{Q}$ ,  $\tilde{\mathbf{B}} = \mathbf{Q}^T \mathbf{B}$  and  $\tilde{\mathbf{L}} = \mathbf{L} \mathbf{Q}$ . Clearly, this reduced model can be used for both frequency- and time-domain analysis.

A set of reduced order models with different order  $q$  can be combined with the original large-scale equations by a multifidelity scheme. Let us consider  $L+1$  levels and therefore  $L+1$  models that provide  $\{\mathbf{Y}_l(s, \boldsymbol{\xi})\}_{l=0}^L$  as corresponding output. We assume the  $L$ -th model is the most accurate and computationally expensive, while the 0-th model is the least accurate and least computationally expensive in the set of  $L+1$  levels. We assume the  $L$ -th model gives what we consider the exact system behaviour  $\mathbf{Y}_L = \mathbf{Y}$ . For our technique, the  $L$ -th model is the original MNA model without any order reduction, all the other models used in the multifidelity scheme are obtained by the PMOR approach described previously using different reduced orders.

### C. Multilevel PMOR for variability analysis

Our contribution targets variability analysis of large circuits. In the variability analysis area, the Monte Carlo (MC) technique has been traditionally used during the design process. MC has a straightforward implementation and provide error measures for the estimation of stochastic quantities, such as mean and variance. However, a large number of simulations might be required, which translates into a high computational cost if the equations to simulate are large-scale. The following equations focus on a scalar function only for ease of notation and they can be used for both frequency- and time-domain quantities of interest. The mean value can be expressed as

$$\mathbb{E}[Y(s, \boldsymbol{\xi})] = \int_{\Gamma} Y(s, \boldsymbol{\xi}) W(\boldsymbol{\xi}) d\boldsymbol{\xi} \quad (7)$$

where  $\mathbb{E}[\cdot]$  represents the expectation operator, the  $Y(s, \boldsymbol{\xi})$  quantity can denote the port voltages  $V_p(s, \boldsymbol{\xi})$  or port currents  $I_p(s, \boldsymbol{\xi})$ .  $W(\boldsymbol{\xi})$  is a joint probability density function.

An estimation of the mean value can be obtained by the MC approach, where a set of random samples based on the joint probability density function of  $\boldsymbol{\xi}$  is used. After the function  $Y(s, \boldsymbol{\xi})$  is sampled in the  $\boldsymbol{\xi}$  space, the collected data can be used to estimate the mean value by sample-based formulas. In this perspective,  $\mathbb{E}[Y(s, \boldsymbol{\xi})]$  can be estimated using  $R$  independent and identically distributed (i.i.d.) samples  $Y_R := (Y(s, \boldsymbol{\xi}_r))_{r=1, \dots, R}$  as

$$\mathbb{E}[Y(s, \boldsymbol{\xi})] \approx m_1(Y_R(s, \boldsymbol{\xi})) = \sum_{r=1}^R \frac{Y(s, \boldsymbol{\xi}_r)}{R} \quad (8)$$

Eq. (8) represents an unbiased estimator of the mean value [9]. The described estimator is based on random sampling. Its mean squared error ( $MSE$ ) can be defined in general as a bias term squared and a variance term (we omit  $(s, \boldsymbol{\xi})$  into  $Y(s, \boldsymbol{\xi})$  and  $Y_R(s, \boldsymbol{\xi})$  for ease of notation in what follows). In the case of an unbiased estimator such as  $m_1$  only the variance term will appear in the  $MSE$

$$MSE(m_1(Y_R)) = \mathbb{E}[(m_1(Y_R) - \mathbb{E}(Y))^2] = \text{Var}(m_1(Y_R)) \quad (9)$$

$$\text{Var}(m_1(Y_R)) = \frac{\text{Var}(Y)}{R} \quad (10)$$

Estimating the needed samples  $R$  to achieve a certain target  $MSE$  is straightforward, since  $R = \frac{\text{Var}(Y)}{MSE(m_1(Y_R))}$ .  $\text{Var}(Y)$  represents the variance of the  $Y$  response and can also be estimated by sample-based formulas, for example using the h-statistics [9]. We note that the MC analysis relies on the most accurate level (model)  $Y_L = Y$ .

The Multilevel Monte Carlo (MLMC) technique generalises the MC technique [7]. The estimation of the mean value of  $Y(s, \boldsymbol{\xi})$  can be obtained as described in [7]

$$\begin{aligned} \mathbb{E}[Y(s, \boldsymbol{\xi})] &\approx m_{1,MLMC}(\{Y_{l,R_l}\}_{l=0}^L) = \\ &m_1(Y_{0,R_0}) + \sum_{l=1}^L m_1(Y_{l,R_l} - Y_{l-1,R_l}) \end{aligned} \quad (11)$$

Considering the assumption  $Y_L = Y$ , the estimator  $m_{1,MLMC}$  is unbiased. Concerning its  $MSE$ , we assume to use i.i.d. samples to sample and evaluate  $Y_0(s, \boldsymbol{\xi})$  and each pair  $[Y_l(s, \boldsymbol{\xi}), Y_{l-1}(s, \boldsymbol{\xi})]$ . The following variance formula for the  $m_{1,MLMC}$  estimator can be obtained [7]

$$\begin{aligned} MSE(m_{1,MLMC}(\{Y_{l,R_l}\}_{l=0}^L)) &= \\ \text{Var}(m_{1,MLMC}(\{Y_{l,R_l}\}_{l=0}^L)) &= \text{Var}(m_1(Y_{0,R_0})) + \\ &+ \sum_{l=1}^L \text{Var}(m_1(Y_{l,R_l} - Y_{l-1,R_l})) = \frac{V_0}{R_0} + \sum_{l=1}^L \frac{V_l}{R_l} \end{aligned} \quad (12)$$

For a chosen  $MSE$  value related to the computation of the mean value of  $Y(s, \boldsymbol{\xi})$ , the number of samples  $\{R_l\}_{l=0}^L$  needed to achieve this error level and minimize the total computational cost of the variability analysis can be found by an analytical formula obtained after a constrained minimisation problem [7]. Considering  $C_0$  as the computational cost to evaluate one sample of  $Y_{0,R_0}$  and  $C_l$  as the computational cost to evaluate one sample of  $(Y_{l,R_l} - Y_{l-1,R_l})$ , the optimal number of samples  $R_l, l = 0, \dots, L$  is equal to [7]

$$R_l = (MSE(m_{1,MLMC}(\{Y_{l,R_l}\}_{l=0}^L)))^{-2} \left( \sum_{l=0}^L \sqrt{V_l C_l} \right) \sqrt{\frac{V_l}{C_l}} \quad (13)$$

Therefore, as in (10), the needed number of samples can be easily computed [7].

The proposed multilevel PMOR approach and MC start from an initial number of samples, then the number of samples needed to satisfy the target  $MSE$  value is evaluated, new simulations are run and this is iterated till the target  $MSE$  value is satisfied.

In this contribution, we focus as theory on the estimation of the mean value. Due to lack of space, the theory extension towards higher order stochastic moments, such as the variance, will be presented in an extended future work. However, we show numerical results for both mean and variance estimation in what follows.

### III. NUMERICAL EXAMPLE

Four coupled microstrip lines, shown in Fig. 1, are used as an example to validate the proposed technique. The MNA description of this circuit has 1204 state variables in (2). The circuit is excited by a trapezoidal voltage source that is connected to the input of the first line with 0.5 ns of rising/falling edge, 3 ns duration, and 1 V amplitude. The uncertainty parameters that were considered for this example are the height  $h$  of the substrate, the width of the lines  $w$ , distance between lines  $d$ , the conductivity  $\sigma$ , and the relative permittivity  $\epsilon_r$ . Their nominal values are 0.5 mm, 2 mm, 2 mm,  $5.6 \cdot 10^7$  S/m, and 4.5, respectively. Each of these parameter is considered to be a uniform random variables varying  $\pm 40\%$  around the nominal value. For each line, there is one resistor of  $50 \Omega$  at the input and a capacitor of 1 pF at the output.

The target  $MSE$  for the proposed technique is equal to  $10^{-5}$ . Three levels were selected: the original model and two PMOR models with reduced order equal to 36 and 12. The standard MC analysis used the original model only. Although a MC-based accuracy validation is not necessary since the proposed technique has a well-defined error control feature, we show in Figs. 2-3 comparison results for the proposed method and MC considering the same target  $MSE$ . A very good agreement can be observed. The number of samples needed by the proposed technique is equal to 175, 1156, 21914 for the original model and the PMOR models of order 36 and 12, respectively. The standard MC analysis needed 12199 samples using the original model. These number of samples are computed using the corresponding formulas described in Section II-C. The proposed method achieved a speed-up equal to 19 with respect to the MC analysis.

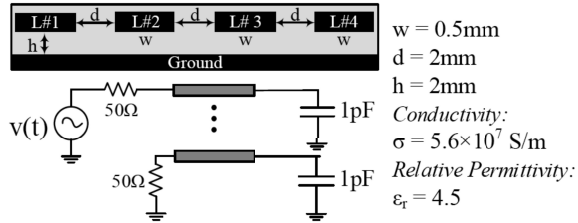


Fig. 1. Circuit example.

### IV. CONCLUSION

A method has been presented to efficiently and accurately perform the variability analysis of circuits. This is based on a multilevel PMOR technique where the original large-scale model and a set of PMOR model are properly combined. This scheme ensures an error control on the estimation of stochastic moments since the original model is kept into the analysis, while exploiting the efficiency of PMOR models. The advantages of the proposed method were demonstrated using a practical interconnect circuit.

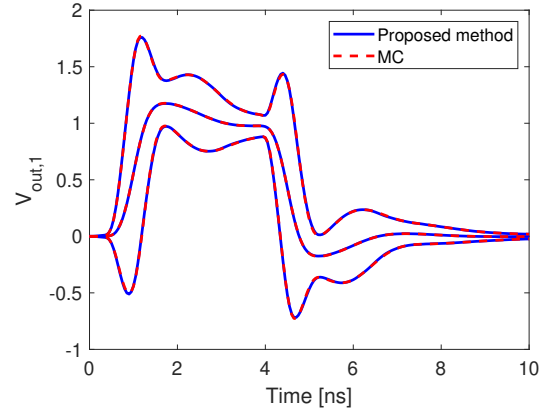


Fig. 2. Mean value and mean value  $\pm 3\sqrt{\text{variance}}$  results of the output voltage of the first line.

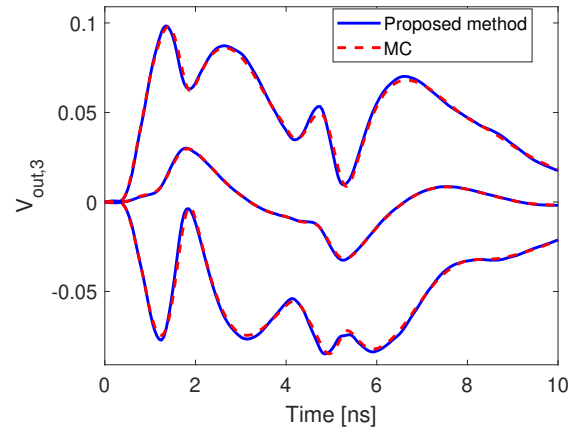


Fig. 3. Mean value and mean value  $\pm 3\sqrt{\text{variance}}$  results of the output voltage of the third line.

### REFERENCES

- [1] L. Daniel, O. C. Siong, L. S. Chay, K. H. Lee, and J. White, "A multiparameter moment-matching model-reduction approach for generating geometrically parameterized interconnect performance models," *IEEE Trans. Comput.-Aided Design Integr. Circuits Syst.*, vol. 23, no. 5, pp. 678–693, May 2004.
- [2] P. Benner and L. Feng, "A robust algorithm for parametric model order reduction based on implicit moment matching," in *Reduced order methods for modeling and computational reduction*. New York: Springer, 2014, vol. 9, ch. 6, pp. 159–185.
- [3] Y. Tao, B. Nouri, M. S. Nakhla, M. A. Farhan, and R. Achar, "Variability analysis via parameterized model order reduction and numerical inversion of laplace transform," *IEEE Transactions on Components, Packaging and Manufacturing Technology*, vol. 7, no. 5, pp. 678–686, 2017.
- [4] L. Feng and P. Benner, "A new error estimator for reduced-order modeling of linear parametric systems," *IEEE Transactions on Microwave Theory and Techniques*, vol. 67, no. 12, pp. 4848–4859, 2019.
- [5] S. Hain, M. Ohlberger, M. Radic, and K. Urban, "A hierarchical a posteriori error estimator for the reduced basis method," *Advances in Computational Mathematics*, vol. 45, no. 5, pp. 2191–2214, 2019.
- [6] S. Koziel and J. W. Bandler, "Reliable microwave modeling by means of variable-fidelity response features," *IEEE Transactions on Microwave Theory and Techniques*, vol. 63, no. 12, pp. 4247–4254, 2015.
- [7] M. B. Giles, "Multilevel monte carlo methods," *Acta Numerica*, vol. 24, p. 259–328, 2015.
- [8] F. N. Najm, *Circuit Simulation*. Hoboken, NJ, USA: Wiley, 2010.
- [9] C. Rose and M. Smith, *mathStatICA: Mathematical Statistics with Mathematica*. Heidelberg: Physica, 2002.