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Distortion in groups of Affine Interval Exchange Transformations.

Nancy Guelman¹ and Isabelle Liousse²

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Abstract

In this paper, we study distortion in the group \mathcal{A} of Affine Interval Exchange Transformations (AIET). We prove that any distorted element f of \mathcal{A} has an iterate f^k that is conjugate by an element of \mathcal{A} to a product of infinite order restricted rotations, with pairwise disjoint supports. As consequences, we prove that no Baumslag-Solitar group, $BS(m, n)$ with $|m| \neq |n|$, acts faithfully by elements of \mathcal{A} ; every finitely generated nilpotent group of \mathcal{A} is virtually abelian and there is no distortion element in $\mathcal{A}_{\mathbb{Q}}$, the subgroup of \mathcal{A} consisting of rational AIETs.

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1 Introduction.

In the recent years, notions of distortion have attracted the interest of many people working on geometric group theory as well as rigidity theory (see [11] for a survey).

On one hand, some results established the existence of distorted elements in transformations groups. For instance, D. Calegari and M. Freedman, in [8], showed that all homeomorphisms of spheres are distorted. Moreover, in the case of the unit circle, they proved that every irrational Euclidean rotation is distorted inside the group of $C^{2-\varepsilon}$ -diffeomorphisms for any $\varepsilon > 0$. Requiring smoothness, Avila proved in [1] that irrational rotations are distorted in $Diff^\infty(\mathbb{S}^1)$. In higher dimensions, Milton (see [20], Theorem 1) showed that irrational translations of the d -dimensional torus are distorted in $Diff^\infty(\mathbb{T}^d)$.

On the other hand, a significant consequence of the non existence of distortion is the proof of Zimmer's conjecture in dimension 2: "any action of $SL(3, \mathbb{Z})$ by area preserving diffeomorphisms on a surface has finite image". For instance,

Polterovich ([26]) and Franks-Handel([12]) proved that $Diff_{\mu}^1(\Sigma^2)$ does not contain distortion, where μ is a full support measure on a compact surface Σ^2 .

Novak (in [25]) proved that there is no distortion element in the group of Intervals Exchange Transformations: bijections of the unit interval that are piecewise increasing and isometric.

In this work, we deal with a closely related problem, namely the existence of distortion elements inside the group of AIETs: Affine Intervals Exchange Transformations, denoted by \mathcal{A} . Roughly speaking an AIET is a bijection of the unit interval that is increasing and affine on a finite number of intervals. If the end-points of these intervals and their images are rational points, then the AIET is called a rational AIET. The set of rational AIETs is a subgroup of \mathcal{A} and it is denoted by $\mathcal{A}_{\mathbb{Q}}$.

Finitely generated groups of AIETs have provided several algebraically interesting groups, as for instance classical Thompson's groups F , T and V [9] as well as some of their generalized versions [16], [2], the fundamental groups of orientable surfaces [14], the modular group [24], wreath products of the form $\mathbb{Z} \wr \mathbb{Z} \wr \dots \wr \mathbb{Z}$ ([3], [6], [22], [23]) etc.

Our main result proves that most elements of \mathcal{A} are in fact undistorted.

Theorem 1. *For every distorted element f of \mathcal{A} , there exists a positive integer k such that f^k is conjugate by an element of \mathcal{A} to a product of infinite order restricted rotations, with pairwise disjoint supports.*

This description of distorted elements of \mathcal{A} enables us to prove the following corollaries.

Corollary 1. *The Baumslag-Solitar groups $BS(n, m) = \langle a, b \mid ba^n b^{-1} = a^m \rangle$ with m, n integers and $|m| \neq |n|$ do not act faithfully via elements of \mathcal{A} .*

Corollary 2. *Every torsion free nilpotent subgroup of \mathcal{A} is abelian and every finitely generated nilpotent subgroup of \mathcal{A} is virtually abelian.*

Consequently, the Heisenberg group and thereby $SL(3, \mathbb{Z})$ do not act faithfully via elements of \mathcal{A} .

The last two corollaries were proved by Higman for Thompson's group V (see [16] and [27], chapter 2).

Corollary 3. *There is no distortion element in $\mathcal{A}_{\mathbb{Q}}$.*

The preceding corollary extends to all groups of rational AIETs, results of Bleak-Bowman-Gordon-Graham-Hughes-Matucci-Sapir (see [4] or [7]) and Hmili-Liousse (see [17]) on the non existence of distortion in the Higman-Thompson groups V_n . Its main consequence is that any group G containing distortion elements does not admit faithful actions as rational affine interval exchange transformations. Moreover, if G is almost-simple, such actions have finite image.

This paper is organized as follows:

- In Section 2, definitions and basic facts are given.

In Sections 3 to 8, we establish the propositions that are used in the proof of Theorem 1, given in section 8.

- In Section 3, we prove that any element of \mathcal{A} having semi-hyperbolic periodic orbits is undistorted.
- In Section 4, it is shown that, given f in \mathcal{A} , the sequence whose general term is the number of break points of the iterate f^n of f (for simplicity, this sequence will be called the “number of break points of f^n ” and it will be denoted by $\#BP(f^n)$) is either bounded or grows linearly. As a consequence, for any distortion element the number of break points of f^n is bounded.

In the following sections, we study f in \mathcal{A} without semi-hyperbolic periodic orbit and with bounded number of break points of f^n .

- In section 5 (Theorem 2: ”Extended ”Alternate Version of Li’s Theorem”), we establish that such an f has an iterate that is conjugate by an element of \mathcal{E} to a product of restricted PL-homeomorphisms f_i with bounded numbers of break points of f_i^n .
- For such a PL-homeomorphism f , in Section 6, we apply results of Minakawa [21] to prove that f is PL-conjugate to a PL-homeomorphism B having at most two distinct slopes.
- In Section 7, under the additional assumption that f is distorted, we derive that B is a rotation, by showing that its slopes are 1.
- In Section 8, Theorem 1 is proved.
- Section 9 is devoted to prove Corollaries 1, 2 and 3 of Theorem 1.

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2 Preliminaries.

2.1 Affine Interval Exchange Transformations of $I = [0, 1)$.

Definition 2.1.

A bijection f of $[0, 1)$ is an **Affine Interval Exchange Transformation** (or **AIET**) of $[0, 1)$ if there exists a finite subdivision $0 = a_0 < a_1 < \dots < a_p = 1$ of $[0, 1)$ such that for all $i = 0, \dots, p - 1$, one has $f(x) = \lambda_i x + \beta_i$ for $x \in [a_i, a_{i+1})$, where $\lambda_i \in \mathbb{R}_+^*$ and $\beta_i \in \mathbb{R}$.

¹LIOUSSE, I. & NAVAS, A. *Distortion elements in $PL_+(S^1)$* (2008).

A **break point** is either the initial point 0 or a discontinuity of f or a discontinuity of Df , the derivative of f .

The set of break points of f is denoted $\mathbf{BP}(f)$; it can be decomposed as the union of $\mathbf{BP}_0(f)$, the set consisting of 0 and all discontinuities of f and $\mathbf{BP}_1(f)$, the set of 0 and all discontinuities of Df .

We define $\Delta_f(x) = f_+(x) - f_-(x)$, if $x \in (0, 1)$ and $\Delta_f(0) = f_+(0) - f_-(1)$, where $f_+(a) = \lim_{x \rightarrow a_+} f(x) = f(a)$ and $f_-(a) = \lim_{x \rightarrow a_-} f(x)$.

The λ_i 's are the **slopes** of f .

The **jump** of f at x is defined by $\sigma_f(x) = \frac{D_+f(x)}{D_-f(x)}$ if $x \in (0, 1)$ and $\sigma_f(0) = \frac{D_+f(0)}{D_-f(1)}$ where $Df_+(a) = \lim_{x \rightarrow a_+} \frac{f(x)-f_+(a)}{x-a}$ and $Df_-(a) = \lim_{x \rightarrow a_-} \frac{f(x)-f_-(a)}{x-a}$.

The sets of slopes and jumps of f are denoted respectively by $\Lambda(f)$ and $\sigma(f)$.

An AIET f of $[0, 1)$ is an **IET** if $\Lambda(f) = \{1\}$.

Definition 2.2.

We denote by \mathcal{A} the group consisting of all AIETs of $[0, 1)$.

We denote by \mathcal{E} the group consisting of all IETs of $[0, 1)$.

Remark 1.

A homeomorphism f of the circle $\mathbb{S}^1 = [0, 1]/(0 = 1)$ can be seen as the bijection of $[0, 1)$ defined by $x \mapsto f(x) \pmod{1}$.

Definition 2.3.

When this bijection is an AIET, f is called a **PL-homeomorphism** of $[0, 1)$ even if it may not be continuous at eventually one point of $(0, 1)$.

In what follows, PL-homeomorphisms of \mathbb{S}^1 will be seen as AIETs of $[0, 1)$. For example, the circle rotation by α is viewed as the element R_α of \mathcal{A} with break points 0 and $1 - \alpha$ and given by $f(x) = x + \alpha$ if $x \in [0, 1 - \alpha)$ and $f(x) = x + \alpha - 1$ if $x \in [1 - \alpha, 1)$. We say that R_α is a **rotation** of $[0, 1)$.

Definition 2.4.

An IET f is called a **restricted rotation** if there exists some interval $I = [a, b) \subset [0, 1)$ such that f is supported by I and $f(x) = x + \delta$ if $x \in [a, b - \delta)$ and $f(x) = x + \delta - b + a$ if $x \in [b - \delta, b)$, where $\delta \in \mathbb{R}$, $0 < \delta < b - a$.

An AIET f is called a **restricted PL-homeomorphism** if there exists some interval $I = [a, b) \subset [0, 1)$ such that f is supported by I and $\#(BP_0(f) \cap (a, b)) \leq 1$.

Here are some elementary properties of break points sets.

Property 1. Let f and g be two elements of \mathcal{A} . Then

- $BP(f^{-1}) = f(BP(f))$,
- $BP(f \circ g) \subset BP(g) \cup g^{-1}(BP(f))$,
- $BP(f^n) \subset BP(f) \cup f^{-1}(BP(f)) \cup \dots \cup f^{-(n-1)}(BP(f))$, for all integer $n \geq 0$.

These still hold for $BP_0(f)$.

Unfortunately, such formulas do not hold for $BP_1(f)$; this is due to the following

Property 2. Let f, g in \mathcal{A} and $x \in [0, 1)$, one has

$$\sigma_{f \circ g}(x) = \frac{Df_+(g_+(x))Dg_+(x)}{Df_-(g_-(x))Dg_-(x)} = \frac{Df_+(g_+(x))}{Df_-(g_-(x))} \times \sigma_g(x).$$

If $x \notin BP_0(g)$ then $\sigma_{f \circ g}(x) = \sigma_f(g(x)) \times \sigma_g(x)$.

For $x \in BP_0(g)$, $\sigma_{f \circ g}(x) \neq \sigma_f(g(x)) \times \sigma_g(x)$, in general.

Property 3. If $f \in \mathcal{A}$ and g is a PL-homeomorphism then for all x in $[0, 1)$,

$\sigma_{f \circ g}(x) = \sigma_f(g(x)) \times \sigma_g(x)$ and thereby $BP_1(f \circ g) \subset g^{-1}(BP_1(f)) \cup BP_1(g)$.

Indeed, if g is a PL-homeomorphism and $g_+(x) \neq g_-(x)$ then $g_+(x) = 0$ and $g_-(x) = 1$ therefore $\frac{Df_+(g_+(x))}{Df_-(g_-(x))} = \sigma_f(0)$.

2.2 Interesting subgroups of \mathcal{A} .

Numerous generalizations of Thompson's groups have been defined and studied. We recall, for example, the works of Bieri-Strebel [2] and Higman [16].

Definition 2.5. Let $\Lambda \subset \mathbb{R}^{+*}$ be a multiplicative subgroup and $A \subset \mathbb{R}$ be an additive subgroup invariant by multiplication by elements of Λ and such that $1 \in A$.

We define $V_{\Lambda, A}$ as the subgroup of \mathcal{A} consisting of elements with slopes in Λ , break points and their images in A , and \mathcal{E}_A as the subgroup of \mathcal{E} consisting of elements with break points in A , in fact $\mathcal{E}_A = \mathcal{E} \cap V_{\Lambda, A}$.

The subgroup $A_{\mathbb{Q}}$ of rational AIETs is $V_{\mathbb{Q}_{>0}, \mathbb{Q}}$.

Definition 2.6. Let n be a positive integer, the group $V_{\langle n \rangle, \mathbb{Z}[1/n]}$ is denoted by V_n and called a **Higman-Thompson's group**.

Note that classical Thompson's group V arises as V_2 .

Among many things, Higman (see [16], [27]) proved that V_n is finitely presented and satisfies the conclusions of Corollaries 1 and 2.

Remark 2. As established in [10], Thompson's groups F , T and V are not subgroups of \mathcal{E} .

Remark 3. Similarly, one can define AIETs of any interval $[a, b) \subset \mathbb{R}$. In anticipation of a more general use of results from [18], [25], [21] stated for $[0, 1)$, we note that there exists a unique direct affine map sending $[a, b)$ onto $[0, 1)$. Thus any AIET of $[a, b)$ is affinely conjugate to an AIET of $[0, 1)$ with the same slopes set. However, the arithmetic properties of break points might not be preserved.

2.3 Distortion.

Definition 2.7.

Let Γ be a finitely generated group and $S = \{s_1, \dots, s_r\}$ be a finite generating set of Γ .

The smallest integer l such that $g = s_{i_1}^{\epsilon_1} \dots s_{i_l}^{\epsilon_l}$, with $\epsilon_j \in \{-1, 1\}$ is called the **length** of g relatively to S and it is denoted by $l_S(g)$.

We set $l_S(e) = 0$. The function $l_S : \Gamma \rightarrow \mathbb{N}$ is symmetric ($l_S(g^{-1}) = l_S(g)$) and satisfies $l_S(gh) \leq l_S(g) + l_S(h)$. In particular, for all g in Γ , the sequence $l_S(g^n)$ is sub-additive, thus the sequence $\frac{l_S(g^n)}{n}$ converges. This leads to

Definition 2.8.

We say that g is **distorted** (or **of distortion** or a **distortion element**) in $\Gamma = \langle S \rangle$ if g has infinite order and $\lim_{n \rightarrow +\infty} \frac{l_S(g^n)}{n} = 0$.

Remark 4.

The property of being a distortion element is independent of the choice of generating set S .

Definition 2.9.

More generally, if G is not finitely generated, an element g of G is **distorted** in G if g is a distortion element in some finitely generated subgroup of G .

Properties 2.1.

- Let $g \in G$, the following properties are equivalent
 1. g is distorted in G ,
 2. $\exists N \in \mathbb{Z}^* : g^N$ is distorted in G ,
 3. $\forall N \in \mathbb{Z}^* : g^N$ is distorted in G .
- If $\Phi : \Gamma \rightarrow G$ is a morphism and $g \in \Gamma$ is distorted in Γ then its image $\Phi(g) \in G$ is either of finite order or distorted in G .

3 Semi-hyperbolicity prevents distortion.

Definition 3.1. Let $f \in \mathcal{A}$, we say that $p \in [0, 1]$ is a **semi-hyperbolic periodic point** of period l , if either:

- p is not a break point of f^l , $f^l(p) = p$ and $Df^l \neq 1$ (**hyperbolic**) or
- p is a break point of f^l , $f_+^l(p) = p$ and $Df_+^l(p) \neq 1$ or $f_-^l(p) = p$ and $Df_-^l(p) \neq 1$ (**virtual**).

Proposition 3.1. *If $f \in \mathcal{A}$ has a semi-hyperbolic periodic point then f is undistorted in \mathcal{A} .*

Proof. Let p be a semi-hyperbolic periodic point of f .

Without loss of generality, we can suppose that $f_+(p) = p$ and the right derivative of f at p : $Df_+(p) = \lambda \neq 1$. For clarity, Df_+ will be denoted by D_+f .

By absurd, suppose that f is distorted in a subgroup G of \mathcal{A} generated by $S = \{g_1, \dots, g_s\}$. Then f can be written as $f^n = g_{i_n} \dots g_{i_1}$ with $\lim_{n \rightarrow +\infty} \frac{l_n}{n} = 0$.

We have $D_+f^n(p) = D_+g_{i_n}(p_{l_n}) \dots D_+g_{i_1}(p_1)$, where $p_1 = p$ and $p_j = g_{i_{j-1}} \dots g_{i_1}(p)$, for $j = 2, \dots, l_n$.

Then $(\text{Inf } D_+g_i)^{l_n} \leq D_+f^n(p) \leq (\text{Sup } D_+g_i)^{l_n}$ and

$$\frac{l_n \log(\text{Inf } D_+g_i)}{n} \leq \frac{\log(D_+f^n)(p)}{n} \leq \frac{l_n \log(\text{Sup } D_+g_i)}{n},$$

where $\text{Inf } D_+g_i = \inf_{i,x} D_+g_i(x)$ and $\text{Sup } D_+g_i = \sup_{i,x} D_+g_i(x)$.

As f is distorted, one has $\lim_{n \rightarrow +\infty} \frac{\log(D_+f^n)(p)}{n} = 0$.

On the other hand, since p is a fix point of f , one has $D_+f^n(p) = \lambda^n$ and then $\lim_{n \rightarrow +\infty} \frac{\log(D_+f^n)(p)}{n} = \log \lambda \neq 0$, this is a contradiction. \square

4 Alternative for the growth of the number of break points and characterization of distortion.

We recall that $BP(f)$ the set of break points of f is the union of the two following sets $BP_1(f) = \{a \in [0, 1) : \sigma_f(a) \neq 1\} \cup \{0\}$ and $BP_0(f) = \{a \in [0, 1) : \Delta_f(a) \neq 0\} \cup \{0\}$. We denote by $\#BP_*(f)$ the cardinality of $BP_*(f)$.

4.1 Alternative for the growth of the number of break points.

According to Property 1 (Section 2), one has:

$$\#BP(f^n) \leq \#BP(f) \times n \quad \text{and} \quad \#BP_0(f^n) \leq \#BP_0(f) \times n.$$

The main purpose of this section is to prove the following

Proposition 4.1. *If $f \in \mathcal{A}$ then either*

- $\#BP(f^n)$ has linear growth or
- $\#BP(f^n)$ is bounded.

The proof of Proposition 4.1 will be done in two steps: in Step 1, we prove that if $\#BP_0(f^n)$ is not bounded then $\#BP_0(f^n)$ and therefore $\#BP(f^n)$ have linear growth; in Step 2 assuming that $\#BP_0(f^n)$ is bounded, we establish that if $\#BP_1(f^n)$ is not bounded then $\#BP_1(f^n)$ and therefore $\#BP(f^n)$ have linear

growth. In conclusion, either $\#BP_0(f^n)$ and $\#BP_1(f^n)$ are bounded or $\#BP(f^n)$ has linear growth.

We start by recalling and giving some basic properties and related definitions. Note that for all integer n , the set $BP(f^n)$ is included in the union over all $a \in BP(f)$ of the sets $BP(f^n) \cap \mathcal{O}_f(a)$ (by Property 1), where $\mathcal{O}_f(a) = \{f^n(a), a \in \mathbb{Z}\}$ is the f -orbit of a .

Property 4. *If $a \in BP(f)$ is f -periodic then for all integer n , the set $BP(f^n) \cap \mathcal{O}_f(a)$ is finite and has cardinality less or equal than the period of a .*

Property 5. *Let $a \in BP(f)$.*

1. *If a is not f -periodic then there exists a segment S_a of the orbit of a of the form $S_a = \{b = f^{-p}(a), f^{-p+1}(a), \dots, a, \dots, f^l(a) = c\}$, where b and c belong to $BP(f)$ and for all $k \in \mathbb{N}^*$, $f^{-k}(b) \notin BP(f)$ and $f^k(c) \notin BP(f)$.*

*Such a break point b is called an **initial break point** of f . Therefore:*

2. *If a is an initial break of f point then $S_a = \{a, f(a), \dots, f^{N_a}(a)\}$ and*
 - $\mathcal{O}_f(a) \cap BP(f^n) \subset \{f^{-(n-1)}(a), \dots, f^{N_a}(a)\}$, for all integer $n \geq 0$,
 - $\mathcal{O}_f(a) \cap BP(f^{-n}) \subset \{f(a), \dots, f^{N_a+n}(a)\}$, for all integer $n \geq 0$.

In particular:

- (a) $f^{-k}(a) \notin BP(f^m)$, for all integers $k \geq m \geq 0$,
- (b) $f^k(a) \notin BP(f^m)$, for all integers $m \geq 0$, $k > N_a$ and
- (c) $f^{-k}(a) \notin BP(f^{-p}) = f^p(BP(f^p))$, for all integers $p \geq 0$, $k \geq 0$.

Indeed, as $\#BP(f)$ is finite, if (1) does not hold then there exist some $d \in BP(f)$, m_1 and m_2 distinct integers such that $d = f^{m_1}(a) = f^{m_2}(a)$. This contradicts the non periodicity of a . We derive the second item from Property 1.

Step 1: Alternative for $\#BP_0(f^n)$.

This part is devoted to the proof of

Lemma 4.1. *Let $f \in \mathcal{A}$.*

- *If exists $a \in BP_0(f)$ a non periodic initial break point with $\Delta_{f^{N_a+1}}(a) \neq 0$ then*

$$\#BP_0(f) \times n \geq \#BP_0(f^n) \geq n - N_a.$$

That is $\#BP_0(f^n)$ has linear growth.

- *If for all $a \in BP_0(f)$ non periodic initial break points, $\Delta_{f^{N_a+1}}(a) = 0$, then*

$$BP_0(f^n) \leq \sum_{a \in A} 2N_a + \sum_{a \in B} \text{period}(a),$$

where $A = \{a \in BP_0(f) \text{ non periodic initial}\}$ and $B = \{a \in BP_0(f) \text{ periodic}\}$.

That is $\#BP_0(f^n)$ is bounded.

Let $a \in BP_0(f)$ be a non periodic initial break point and $S_a = \{a, f(a), \dots, f^{N_a}(a)\}$; for simplicity of notation, we set $N = N_a$.

Sublemma 4.1.

- If $\Delta_{f^{N+1}}(a) = 0$ then for all integer $l \geq 1$, $\Delta_{f^{N+l}}(a) = 0$.
- If $\Delta_{f^{N+1}}(a) \neq 0$ then for all integer $l \geq 1$, $\Delta_{f^{N+l}}(a) \neq 0$.

Proof. Let $l \geq 1$, one has

$$\begin{cases} f_+^{N+l}(a) = f^{N+l}(a) = f^{l-1}(f^{N+1}(a)), \\ f_-^{N+l}(a) = f_-^{l-1}(f_-^{N+1}(a)). \end{cases}$$

If $\Delta_{f^{N+1}}(a) = 0$ then $\Delta_{f^{N+l}}(a) = f^{l-1}(f^{N+1}(a)) - f_-^{l-1}(f^{N+1}(a)) = 0$, since f^{l-1} is continuous at $f^{N+1}(a)$ by Property 5 (2b).

If $\Delta_{f^{N+1}}(a) \neq 0$, as f^{l-1} is continuous at $f^{N+1}(a)$, the point $f^{l-1}(f^{N+1}(a))$ can not be equal to $f_-^{l-1}(c)$ for some $c \neq f^{N+1}(a)$. It follows that $f^{N+l}(a) = f^{l-1}(f^{N+1}(a)) \neq f_-^{l-1}(f_-^{N+1}(a)) = f_-^{N+l}(a)$, since $f^{N+1}(a) \neq f_-^{N+1}(a)$. \square

Sublemma 4.2. Let a be an initial break point and n be a positive integer.

$$\Delta_{f^n}(f^{-k}(a)) = \Delta_{f^{n-k}}(a), \text{ for } 0 \leq k \leq n-1.$$

Proof. By Property 5 (2), $\mathcal{O}_f(a) \cap BP(f^n) \subset \{f^{-(n-1)}(a), \dots, a, \dots, f^N(a)\}$. Let us compute $\Delta_{f^n}(f^{-k}(a))$, for $0 \leq k \leq n-1$, we get:

$$\Delta_{f^n}(f^{-k}(a)) = f_+^n(f^{-k}(a)) - f_-^n(f^{-k}(a)) = f^{n-k}(a) - f_-^{n-k}(f_-^k(f^{-k}(a))).$$

Moreover, for all $k \geq 0$, one has $f_-^k(f^{-k}(a)) = a$, according to Property 5 (2a).

$$\text{Finally, } \Delta_{f^n}(f^{-k}(a)) = f^{n-k}(a) - f_-^{n-k}(a) = \Delta_{f^{n-k}}(a). \quad \square$$

We turn now to the proof of Lemma 4.1, estimating $\#BP_0(f^n)$ for a given positive integer n . Combining the two preceding sublemmas, we deduce the following alternative for any initial break point a :

- If $\Delta_{f^{N+1}}(a) \neq 0$ then $\Delta_{f^n}(f^{-k}(a)) \neq 0$, for all $n-k > N$.
Hence $\#(BP_0(f^n) \cap \mathcal{O}_f(a)) \geq n - N$.
- If $\Delta_{f^{N+1}}(a) = 0$ then $\Delta_{f^n}(f^{-k}(a)) = 0$, for all $n-k > N$.
Hence $\#(BP_0(f^n) \cap \mathcal{O}_f(a)) \leq N + (n - (n - N)) = 2N$.

Lemma 4.1 follows immediately from this and Property 4.

Step 2: Alternative for $\#BP(f^n)$.

By Lemma 4.1, if $\#BP_0(f^n)$ is not bounded then $\#BP(f^n)$ has linear growth therefore in this part, we will focus on the case where $\#BP_0(f^n)$ is bounded and prove the following Lemma 4.2 which will complete the proof of Proposition 4.1.

Lemma 4.2. *Let $f \in \mathcal{A}$ with $\#BP_0(f^n)$ bounded.*

- *If exists $a \in BP_1(f)$ a non periodic initial break point such that $\Pi_a \neq 1$, then*

$$\#BP(f) \times n \geq \#BP_1(f^n) \geq n - N_a.$$

Hence $\#BP_1(f^n)$ and $\#BP(f^n)$ have linear growth.

- *If for all $a \in BP_1(f)$ non periodic initial break point, $\Pi_a = 1$, then*

$$\#BP_1(f^n) \leq \sum_{a \in A} 2N_a + \sum_{a \in B} \text{period}(a),$$

where $A = \{a \in BP_1(f) \text{ non periodic initial}\}$ and $B = \{a \in BP_1(f) \text{ periodic}\}$.

Hence $\#BP_1(f^n)$ and $\#BP(f^n)$ are bounded.

Proof. Let $a \in BP(f)$ be a non periodic initial break point and $S_a = \{a, f(a), \dots, f^N(a)\}$ be the smallest segment containing all the break points of f on $\mathcal{O}_f(a)$.

Let $n \geq N + 1$, recall that $BP(f^n) \cap \mathcal{O}_f(a) \subset \{f^{-(n-1)}(a), \dots, a, \dots, f^N(a)\}$.

Let us compute the jump of f^n at the point $f^{-k}(a)$ for $k \geq 0$ and $n-1-k > N$, that is for $0 \leq k < n-1-N$.

Iterating the composition formula given in Property 2, we get:

$$\sigma_{f^n}(f^{-k}(a)) = \frac{Df_+(f_+^{n-1}(f^{-k}(a))) \dots Df_+(f_+^k(f^{-k}(a))) \dots Df_+(f^{-k}(a))}{Df_-(f_-^{n-1}(f^{-k}(a))) \dots Df_-(f_-^k(f^{-k}(a))) \dots Df_-(f^{-k}(a))}.$$

According to Property 5(2c), $f^{-k}(a) = f_-^{-k}(a)$ and therefore $f_-^l(f^{-k}(a)) = f_-^l(f_-^{-k}(a)) = f_-^{l-k}(a)$, for any integer $l \geq 0$; in addition, if $l \leq k$ then $f_-^l(f^{-k}(a)) = f_-^{l-k}(a)$.

Therefore, noting that $n-1-k > N$, we get $\sigma_{f^n}(f^{-k}(a)) = R_1 \times R_2 \times R_3$, where

$$R_1 = \frac{Df_+(f_+^{n-1-k}(a)) \dots Df_+(f_+^{N+1}(a))}{Df_-(f_-^{n-1-k}(a)) \dots Df_-(f_-^{N+1}(a))},$$

$$R_2 = \frac{Df_+(f_+^N(a)) \dots Df_+(a)}{Df_-(f_-^N(a)) \dots Df_-(a)} \quad \text{and}$$

$$R_3 = \frac{Df_+(f^{-1}(a)) \dots Df_+(f^{-k}(a))}{Df_-(f^{-1}(a)) \dots Df_-(f^{-k}(a))}.$$

As $BP_1(f) \cap \mathcal{O}_f(a) \subset S_a$, the third ratio R_3 is trivial.

Since $\#BP_0(f^n)$ is bounded, by Sublemmas 4.1 and 4.2, for all $m \geq N + 1$ the point $f^m(a) = f_+^m(a) = f_-^m(a)$ and does not belong to $BP_1(f)$ (by Property 5 (2)). Thus, the first ratio R_1 is also trivial. Finally,

$$\sigma_{f^n}(f^{-k}(a)) = R_2 = \frac{Df_+(f_+^N(a)) \dots Df_+(a)}{Df_-(f_-^N(a)) \dots Df_-(a)} =: \Pi_a$$

Note that this formula also holds for $n - 1 - k = N$, since the first ratio does not appear in $\sigma_{f^n}(f^{-k}(a))$.

Therefore, the following alternative holds.

- If $\Pi_a \neq 1$ then for all k integer such that $0 \leq k \leq n - 1 - N$, $f^{-k}(a) \in BP_1(f^n)$ and $\#(BP_1(f^n) \cap \mathcal{O}_f(a)) \geq n - N$.
- If $\Pi_a = 1$ then for all k integer such that $0 \leq k \leq n - 1 - N$, $f^{-k}(a) \notin BP_1(f^n)$ and $\#(BP_1(f^n) \cap \mathcal{O}_f(a)) \leq N + (n - (n - N)) = 2N$.

Lemma 4.2 follows immediately from this and Property 4. \square

4.2 Characterization of distortion.

Proposition 4.2. *If f is distorted in \mathcal{A} then $\#BP(f^n)$ is bounded.*

Proof. By contradiction, suppose that $\#BP(f^n)$ is unbounded and f is distorted in a subgroup G of \mathcal{A} generated by $S = \{g_1, \dots, g_s\}$. This means that f^n can be written as $f^n = g_{i_n} \dots g_{i_1}$ with $\lim_{n \rightarrow +\infty} \frac{l_n}{n} = 0$. Therefore, by Property 1, we have:

$$BP(f^n) \subset BP(g_{i_1}) \cup g_{i_1}^{-1}BP(g_{i_2}) \dots \cup (g_{i_{n-1}} \dots g_{i_1})^{-1}BP(g_{i_n}) \text{ then}$$

$$\#BP(f^n) \leq \#BP(g_{i_1}) + \dots + \#BP(g_{i_n}) \leq l_n \max\{\#BP(g_i), i = 1, \dots, s\}$$

and therefore $\frac{l_n}{n} \geq \frac{\#BP(f^n)}{n} (\max\{\#BP(g_i), i = 1, \dots, s\})^{-1}$.

Thus $\lim_{n \rightarrow +\infty} \frac{l_n}{n} > 0$, since $\#BP(f^n)$ has linear growth according to Proposition 4.1, this is a contradiction. \square

5 Extended “alternative version of Li’s Theorem”.

The aim of this section is to prove an extended version of the “Alternate Version of Li’s Theorem” given in [25].

Theorem 2. *Let f in \mathcal{A} without periodic points and with bounded $\#BP(f^n)$ then there exists an integer q such that f^q is conjugate in \mathcal{E} to a product of restricted PL-homeomorphisms of disjoint supports that are minimal when restricted to their respective supports.*

Definition 5.1. Let $f \in \mathcal{A}$. We say that f satisfies the **pair property** if

1. f does not have periodic points,
2. $BP_0(f) = \{\beta_1, \dots, \beta_s, \omega_1, \dots, \omega_s\}$, where any pair (β_i, ω_i) , for $i = 1, \dots, s$, verifies $f(\beta_i) = \omega_i$ and $\beta_i \notin BP_0(f^2)$ and
3. the f -orbits of the β_i ’s are disjoint.

Convention. Eventually re-indexing the ω_i 's, we may suppose that $0 = \omega_1 < \omega_2 < \dots < \omega_s$.

We begin by giving some basic properties.

Property 6.

1. If f has the pair property then any associated pair (β_i, ω_i) , for $i = 1, \dots, s$, verifies
 - (a) $\beta_i \in BP_0(f) \setminus BP_0(f^2)$,
 - (b) $\omega_i \in BP_0(f) \cap BP_0(f^{-1})$.
2. The pair property is invariant under C^0 -conjugation.
3. If f has the pair property with associated pairs (β_i, ω_i) , for $i = 1, \dots, s$, then for any $n \in \mathbb{N}$, f^n has the pair property with associated pairs $(f^{-n}(\omega_i), \omega_i)$, for $i = 1, \dots, s$.

Using these properties, we get $f_-(\beta_i) \in BP_0(f) \cap BP_0(f^{-1}) \cup \{1\}$ so $f_-(\beta_i)$ is an ω_j and we can give

Definition 5.2. Let π be the permutation of $\{1, \dots, s\}$ defined by either

- $\pi(i) = j$ in the case that $f_-(\beta_i) = \omega_j$ or
- $\pi(i) = 1$ when $f_-(\beta_i) = 1$.

Hence, one has $f(\omega_i) = f_-(\omega_{\pi(i)})$, for $i \neq \pi^{-1}(1)$, otherwise $f(\omega_i) = f_-(1)$, for $i = \pi^{-1}(1)$.

Definition 5.3. (see [18]) Let $f \in \mathcal{A}$ with the pair property.

A pair (β_i, ω_i) is said **removable** if either:

- $\omega_{\pi(i)} < \omega_i$ or
- $\omega_{\pi(i)} > \omega_i$ and there exists $\omega_j \in (\omega_i, \omega_{\pi(i)})$.

Lemma 5.1. Let $f \in \mathcal{A}$ without periodic points and with bounded $\#BP_0(f^n)$ then there exists an iterate of f that satisfies the pair property.

Proof. Let $IBP(f)$ be the set of all initial break points of f . According to Section 4, one has $BP_0(f) \subset \bigcup_{a \in IBP(f)} \{a, \dots, f^{N_a}(a)\}$, $\Delta_{f^{N_a+1}}(a) = 0$ and

$$\Delta_{f^n}(f^{-k}(a)) = 0, \text{ for all } k \geq 0, n - k > N_a. \quad (1)$$

Let N be the maximum of the $\{N_a\}$ over all $a \in IBP(f)$ and $F = f^{N+1}$. It holds that $BP_0(F) \cap \mathcal{O}_f(a) \subset \{f^{-N}(a), \dots, a, \dots, f^{N_a}(a)\}$.

After noting that $\Delta_F(f^{-(N+1)+N_a+l}(a)) = 0$, (for $l = 1, \dots, N - N_a + 1$) by formula (1), $BP_0(F) \cap \mathcal{O}_f(a) \subset \bigcup_{l=1}^{N_a} \{f^{-(N+1)+l}(a), f^l(a)\}$.

We claim that any pair $(\beta, \omega) = (f^{-(N+1)+l}(a), f^l(a))$ for $l = 1, \dots, N_a$ satisfies $F(\beta) = \omega$ and $\beta \notin BP_0(F^2)$.

Indeed, obviously $F(\beta) = \omega$, we now compute $\Delta_{F^2}(\beta)$:

$$\Delta_{F^2}(\beta) = \Delta_{f^{2N+2}}(f^{-(N+1)+l}(a)) = 0 \text{ by formula (1) with } n = 2N+2, k = N+1-l.$$

It follows that either $f^{-(N+1)+l}(a) \in BP_0(F)$ and $f^l(a) \in BP_0(F)$ or $f^{-(N+1)+l}(a) \notin BP_0(F)$ and $f^l(a) \notin BP_0(F)$. So $BP_0(F)$ is a finite union of pairs of the form $(f^{-(N+1)+l}(a), f^l(a))$.

Obviously, F satisfies Conditions (1) and (3) of the pair property. \square

Lemma 5.2. *Let $F \in \mathcal{A}$ with the pair property and (β_i, ω_i) be a removable pair of F , then there exists $E \in \mathcal{E}$ such that $\#BP_0(EFE^{-1}) \leq \#BP_0(F) - 2$ and EFE^{-1} has also the pair property. Moreover, $BP_0(E) \subset BP_0(F) \cap BP_0(F^{-1})$.*

Proof.

We begin by considering the case where $\omega_{\pi(i)} < \omega_i$.

Let E be in \mathcal{E} given by $BP_0(E) = \{0, \omega_{\pi(i)}, \omega_i\}$ and with associated permutation $(1, 2, 3) \mapsto (1, 3, 2)$.

According to Property 6(1b), $BP_0(E) \subset BP_0(F) \cap BP_0(F^{-1})$.

One has that

$$BP_0(EFE^{-1}) \subset EF^{-1}(BP_0(E)) \cup E(BP_0(F)) \cup BP_0(E^{-1}).$$

As $F^{-1}(BP_0(E)) \subset BP_0(F)$ and $BP_0(E^{-1}) = E(BP_0(E)) \subset E(BP_0(F))$, it holds that

$$BP_0(EFE^{-1}) \subset E(BP_0(F)).$$

We prove that $E(\beta_i)$ and $E(\omega_i)$ do not belong to $BP_0(EFE^{-1})$ by computing the right and left values at these points:

- $EFE^{-1}(E(\beta_i)) = EF(\beta_i) = E(\omega_i) = \omega_{\pi(i)}$ and since $\beta_i \notin BP_0(E)$,
- $(EFE^{-1})_-(E(\beta_i)) = E_-F_-(\beta_i) = E_-(\omega_{\pi(i)}) = \omega_{\pi(i)}$.

This proves that $E(\beta_i) \notin BP_0(EFE^{-1})$.

- $EFE^{-1}(E(\omega_i)) = EF(\omega_i)$ and
- $(EFE^{-1})_-(E(\omega_i)) = (EFE^{-1})_-(\omega_{\pi(i)}) = (EF)_-(\omega_{\pi(i)})$. In addition, since $F(\omega_i) = F_-(\omega_{\pi(i)})$ does not belong to $BP_0(E)$, we have $(EF)_-(\omega_{\pi(i)}) = E(F(\omega_i))$.

We thus get $EFE^{-1}(E(\omega_i)) = (EFE^{-1})_-(E(\omega_i))$ which proves that $E(\omega_i) \notin BP_0(EFE^{-1})$.

Finally, $\#BP_0(EFE^{-1}) \leq \#BP_0(F) - 2$.

We claim that EFE^{-1} has the pair property with associated pairs of the form $(E(\beta_j), E(\omega_j))$.

Indeed, as $(F^{-2}(\omega_i), \omega_i)$ is a pair for F^2 , it holds that $BP_0(E) \subset BP_0(F^2) \cap BP_0(F^{-2})$ and the same arguments as in the beginning of this proof show that $BP_0(EF^2E^{-1}) \subset E(BP_0(F^2))$.

Obviously $EFE^{-1}(E(\beta_j)) = E(\omega_j)$. Now we argue by contradiction assuming that $E(\beta_j) \in BP_0(EF^2E^{-1})$ then $\beta_j \in E^{-1}(BP_0(EF^2E^{-1})) \subset BP_0(F^2)$, a contradiction.

Clearly EFE^{-1} also satisfies Conditions (1) and (3) of the pair property.

Now, we consider the case where (β_i, ω_i) is a removable pair with the following properties: $\omega_{\pi(i)} > \omega_i$ and there exists $\omega_j \in (\omega_i, \omega_{\pi(i)})$.

We identify 0 to 1 to get a circle and then we cut this circle at the point ω_j . We are in the previous case. This ends the proof of Lemma 5.2. \square

Proof of Theorem 2.

Let $f \in \mathcal{A}$ without periodic points and with bounded $\#BP_0(f^n)$.

By Lemma 5.1, there exists some integer N such that f^N has the pair property.

Starting from f^N and applying Lemma 5.2 a finite number of times, we get that $G = E_r \dots E_1 f^N E_1^{-1} \dots E_r^{-1}$ has the pair property and no removable pair.

Since the associated pairs (β_i, ω_i) of G are not removable, it holds that $\omega_{\pi(i)} > \omega_i$, for any i and that the intervals $(\omega_i, \omega_{\pi(i)})$ are pairwise disjoint and disjoint from the last interval $(\omega_s, 1)$.

Note that $s = \pi^{-1}(1)$, indeed if not then $\omega_s < \omega_{\pi(s)} < 1$, a contradiction.

We claim that for any $1 \leq i < s$, there exists a unique discontinuity point of G in $(\omega_i, \omega_{\pi(i)})$ and this still holds for $(\omega_s, 1)$.

Indeed as $G_+(\omega_i) = G_-(\omega_{\pi(i)})$, the interval $(\omega_i, \omega_{\pi(i)})$ contains at least one point of $BP_0(G)$ and a similar argument shows that the same holds for $(\omega_s, 1)$.

As the associated pairs are unremovable, this discontinuity point is a β_j . Moreover, since the number of β_i 's is exactly the number of intervals of the form $(\omega_i, \omega_{\pi(i)})$ or $(\omega_s, 1)$, then β_j is unique.

Therefore $G([\omega_i, \omega_{\pi(i)}]) = [\omega_j, \omega_{\pi(j)})$, since $G_-(\beta_j) = \omega_{\pi(j)}$ and $G_+(\beta_j) = \omega_j$, if $j \neq \pi^{-1}(1) = s$. If $j = s$, then $G([\omega_i, \omega_{\pi(i)}]) = [\omega_s, 1)$.

This implies that $R = \bigsqcup_{i=1}^{s-1} [\omega_i, \omega_{\pi(i)}) \sqcup [\omega_s, 1)$ is G -invariant. We claim that it is $[0, 1[$.

Indeed, if not, the complementary of R is a finite union of half open intervals that is G -invariant and G is continuous on each interval (since such intervals do not contain β 's and ω 's the discontinuity points of G). Thus, these intervals are periodic which contradicts that $G(f)$ does not have periodic points.

Moreover, there exists an iterate of G such that $G^l([\omega_i, \omega_{\pi(i)}]) = [\omega_i, \omega_{\pi(i)})$, $G^l([\omega_s, 1) = [\omega_s, \omega_1)$ and the restriction of G^l to any $[\omega_i, \omega_{\pi(i)}], i = 1, \dots, s-1$ and to $[\omega_s, 1)$ has just one interior discontinuity point.

Finally, G^l is a product of restricted PL-homeomorphisms Γ_i with disjoint supports and it is conjugated by $E = E_r \dots E_1$ to f^{lN} .

As f and then G^l has no periodic points, by Denjoy's Theorem for Class P circle homeomorphisms (see [15]), each Γ_i is minimal when restricted to its support.

This ends the proof of Theorem 2. \square

Remark 5. Note that the discontinuities of the E_i 's belong the f -orbit of $BP_0(f)$ and the restricted PL-homeomorphisms Γ_i are supported by intervals whose endpoints also are points of the f -orbit of $BP_0(f)$. In particular, if $f \in \mathcal{A}_{\mathbb{Q}}$ then the Γ_i 's are supported by intervals whose endpoints are rational and $E \in \mathcal{A}_{\mathbb{Q}}$.

6 PL conjugation.

Next proposition is due to Minakawa [21].

Proposition 6.1. Let $f \in \mathcal{A}$ be a PL-homeomorphism having bounded $\#BP(f^n)$. Then there exists a PL-homeomorphism $H \in \mathcal{A}$ such that $B := H \circ f \circ H^{-1}$ is an AIET of 2 intervals.

Remark 6. An AIET of 2 intervals B is a PL-homeomorphisms and it is uniquely determined by its slope pair (λ_1, λ_2) , so it is denoted by $\mathbf{B}_{\lambda_1, \lambda_2}$. It verifies that $BP(B) = \{0, B^{-1}(0)\}$, $DB(x) = \lambda_1$ on $[0, B^{-1}(0))$ and $DB(x) = \lambda_2$ on $[B^{-1}(0), 1)$. These maps B_{λ_1, λ_2} were studied in [5]. There it was proven that either:

- $\lambda_1 = \lambda_2$ and B_{λ_1, λ_2} is a rotation or
- $\lambda_1 \neq \lambda_2$ and B_{λ_1, λ_2} is C^0 -conjugate to a rotation R_ρ by a map of the form $x \mapsto \frac{(\omega^x - 1)}{(\omega - 1)}$ for some positive ω distinct from 1.

Remark 7. We shall give a refinement of Minakawa's proof which will enable us to preserve the arithmetic properties of f , this will be explained in Remark 8. An alternative proof using a "PL pair property" can be found in [19].

Proof. As $\#BP(f^n)$ is bounded, Lemma 4.2 indicates that there exists a subset $\{a_i, i \in \mathcal{I}\}$ of $BP_1(f)$ such that $BP_1(f)$ is contained in $\bigsqcup_{i \in \mathcal{I}} \mathcal{S}_i$ with $\mathcal{S}_i = \{f^k(a_i), k = 0, \dots, N_i\}$ and

$$\Pi_{a_i} = \frac{Df_+(f_+^{N_i}(a_i)) \dots Df_+(a_i)}{Df_-(f_-^{N_i}(a_i)) \dots Df_-(a_i)} = 1, \quad \forall i \in \mathcal{I}.$$

Note that $\frac{Df_+(f_+^k(a_i))}{Df_-(f_-^k(a_i))} = \sigma_f(f^k(a_i))$, since $f_+^k(a_i) = f_-^k(a_i)$ or $f_+^k(a_i) = 0$ and $f_-^k(a_i) = 1$.

Then $\Pi_{a_i} = \prod_{c \in \mathcal{S}_i} \sigma_f(c) = 1$, for all $i \in \mathcal{I}$.

We let $H_f = H$ be a PL-homeomorphism of $[0, 1)$ with the following properties:

- The break points of H are the points $f(a_i), \dots, f^{N_i}(a_i)$, for $i \in \mathcal{I}$.

- The associated jumps are $\sigma_H(f^k(a_i)) = \sigma_{f^{N+1}}(f^k(a_i))$ for $k = 1, \dots, N_i$, where $N = \max\{N_i, i \in \mathcal{I}\}$.

Note that $\sigma_H(a_i) = \sigma_{f^{N+1}}(a_i) = \prod_{n=0}^N \sigma_f(f^n(a_i)) = 1$.

At the end of this proof, we will indicate a general lemma (Lemma 6.1) about the existence of PL-homeomorphisms with prescribed break points and slopes. It implies that a necessary and sufficient condition for the existence of such a homeomorphism H is that the product of the H -jumps is trivial, that is

$$\Pi(f) := \prod_{i \in \mathcal{I}, 0 \leq k \leq N_i} \sigma_{f^{N+1}}(f^k(a_i)) = 1.$$

- If $\Pi(f) = 1$, then we can define a map H as above and normalize it by setting $H(0) = 0$.
- If $\Pi(f) \neq 1$, then we add a break point $c \notin \{a_i, \dots, f^{N_i}(a_i)\}$ and require that $\sigma_H(c) = (\Pi(f))^{-1}$; we normalize H by setting $H(c) = 0$.

Now, since f and H are PL-homeomorphisms, Property 3 implies that

- the set $BP_1(H \circ f \circ H^{-1})$ satisfies
$$BP_1(H \circ f \circ H^{-1}) \subset BP_1(H^{-1}) \cup H(BP_1(f)) \cup H \circ f^{-1}(BP_1(H))$$

$$\subset \{H(a_i), \dots, H(f^{N_i}(a_i)), i \in \mathcal{I}\} \cup \{H(c), H(f^{-1}(c))\},$$
- for $i \in \mathcal{I}$, $0 \leq k \leq N_i$, the jump of $H \circ f \circ H^{-1}$ at $H(f^k(a_i))$ is equal to

$$\begin{aligned} \sigma_{H \circ f \circ H^{-1}}(H(f^k(a_i))) &= \frac{\sigma_H(f^{k+1}(a_i)) \times \sigma_f(f^k(a_i))}{\sigma_H(f^k(a_i))} \\ &= \frac{\sigma_{f^{N+1}}(f^{k+1}(a_i)) \times \sigma_f(f^k(a_i))}{\sigma_{f^{N+1}}(f^k(a_i))} = 1, \end{aligned}$$

- $\sigma_{H \circ f \circ H^{-1}}(H(c)) = \Pi(f)$ and
- $\sigma_{H \circ f \circ H^{-1}}(H(f^{-1}(c))) = \Pi(f)^{-1}$.

In conclusion.

- If $\Pi(f) = 1$, the AIET $B = H \circ f \circ H^{-1}$ has no break of slopes, it is a rotation.
- If $\Pi(f) \neq 1$, the AIET $B = H \circ f \circ H^{-1}$ has exactly two breaks of slopes at $0 = H(c)$ and $B^{-1}(0) = H(f^{-1}(c))$, that is $\Lambda(B) = \{\lambda_1, \lambda_2\}$.

□

Lemma 6.1.

Given $0 = c_0 < c_1 < \dots < c_p < 1$ points in $[0, 1)$ and $\sigma_0, \dots, \sigma_p$ positive real numbers such that $\prod_{i=0}^p \sigma_i = 1$, there exists a PL-homeomorphism H such that :

- $BP_0(H) = \{c_0, c_1, \dots, c_p\}$ and
- $\sigma_H(c_i) = \sigma_i$, for $i = 0, \dots, p$.

Moreover, given $d \in [0, 1)$ the map H is unique provided that $H(d) = 0$.

The proof is left to readers, however we indicate some elements of the construction of H . Setting $\Lambda(H) = \{\lambda_1, \dots, \lambda_{p+1}\}$, one has

$$\lambda_i = \sigma_0 \dots \sigma_{i-1} \lambda_1 \text{ and}$$

$$\lambda_1 = (|I_1| + \sigma_1 |I_2| + \dots + \sigma_1 \dots \sigma_p |I_{p+1}|)^{-1} \text{ by computation of the total length of } H([0, 1)).$$

In particular, if c_i and σ_i are rational numbers then $\lambda_i \in \mathbb{Q}$. Moreover, we can choose H such that $H(c_j) \in \mathbb{Q}$, for some c_j and then $H \in \mathcal{A}_{\mathbb{Q}}$.

Note that if $\prod_{i=0}^p \sigma_i \neq 1$ such a H does not exist since it should satisfy that $\lambda_{p+1} = \sigma_0 \dots \sigma_p \lambda_1$ and $\sigma_{p+1} = \frac{\lambda_{p+1}}{\lambda_1}$.

Remark 8. We have described explicitly the conjugating PL-homeomorphism H , we can deduce that if $f \in \mathcal{A}_{\mathbb{Q}}$ then the break points of H and the jumps of H belong to \mathbb{Q} , provided that the point c is chosen in \mathbb{Q} . Therefore, if $f \in \mathcal{A}_{\mathbb{Q}}$ then the conclusions of Proposition 6.1 hold with H and B belonging to $\mathcal{A}_{\mathbb{Q}}$.

7 The case $\Lambda(B) = \{\lambda_1, \lambda_2\}$.

Let $\alpha_1, \dots, \alpha_s$ generating a rank s free abelian multiplicative subgroup Λ of \mathbb{R}^{+*} .

Therefore, given $\lambda \in \Lambda$, there exists a unique $(n_1, \dots, n_s) \in \mathbb{Z}^s$, such that $\lambda = \alpha_1^{n_1} \dots \alpha_s^{n_s}$ and we define $\mathcal{N}_j(\lambda) = n_j$, for all $j \in \{1, \dots, s\}$.

Proposition 7.1. Let $B = B_{\lambda_1, \lambda_2} \in \mathcal{A}$ with $(\lambda_1, \lambda_2) \in \Lambda^2 \setminus \{(1, 1)\}$. There exist $j \in \{1, \dots, s\}$ and $x \in [0, 1)$ such that $\frac{\mathcal{N}_j(D_+ B^n(x))}{n} \rightarrow \nu \neq 0$.

Proof. Noting that B satisfies that $DB(x) = \lambda_1$ on $[0, a)$ and $DB(x) = \lambda_2$ on $[a, 1)$, with $a = B^{-1}(0)$, one has $D_+ B^n(x) = \lambda_1^{N_1(x, n)} \lambda_2^{N_2(x, n)}$, where

$$N_1(x, n) = \#\{x, f(x), \dots, f^{n-1}(x)\} \cap [0, a) = \sum_{k=0}^{n-1} \mathbb{I}_{[0, a)}(f^k(x)) \text{ and}$$

$$N_2(x, n) = \#\{x, f(x), \dots, f^{n-1}(x)\} \cap [a, 1) = \sum_{k=0}^{n-1} \mathbb{I}_{[a, 1)}(f^k(x)).$$

The map B has a unique invariant probability measure μ , since it is C^0 -conjugate to an irrational rotation R_ρ , by [5] (see Remark 6). More precisely, consider h such that $h \circ B \circ h^{-1} = R_\rho$, one has $\mu(A) = \lambda(h(A))$, for all measurable set A . In particular, $\mu([0, a]) = \mu([0, B^{-1}(0)]) = \lambda([h(0), h \circ B^{-1}(0)]) = \lambda([h(0), R_\rho^{-1}(h(0))]) = (1 - \rho)$ and $\mu([a, 1]) = \rho$.

The Birkhoff Ergodic Theorem implies that for μ -almost every point $x \in [0, 1]$, one has

$$\lim_{n \rightarrow +\infty} \frac{N_1(x, n)}{n} = \mu([0, a]) \text{ and } \lim_{n \rightarrow +\infty} \frac{N_2(x, n)}{n} = \mu([a, 1]).$$

Now, let us write λ_1 and λ_2 in the basis $\alpha_1, \dots, \alpha_s$ of Λ that is $\lambda_1 = \alpha_1^{\beta_1} \dots \alpha_s^{\beta_s}$ and $\lambda_2 = \alpha_1^{\delta_1} \dots \alpha_s^{\delta_s}$. We now compute the coordinates $\mathcal{N}_j(D_+ B^n(x))$ of $D_+ B^n(x)$ in this basis.

As $D_+ B^n(x) = \lambda_1^{N_1(x, n)} \lambda_2^{N_2(x, n)} = \alpha_1^{\beta_1 \cdot N_1(x, n) + \delta_1 \cdot N_2(x, n)} \dots \alpha_s^{\beta_s \cdot N_1(x, n) + \delta_s \cdot N_2(x, n)}$, one has

$$\mathcal{N}_j(D_+ B^n(x)) = \beta_j \cdot N_1(x, n) + \delta_j \cdot N_2(x, n).$$

It follows that

$$\frac{\mathcal{N}_j(D_+ B^n(x))}{n} = \beta_j \cdot \frac{N_1(x, n)}{n} + \delta_j \cdot \frac{N_2(x, n)}{n} \rightarrow \beta_j \cdot (1 - \rho) + \delta_j \rho = \rho(\delta_j - \beta_j) + \beta_j.$$

Finally, suppose that $(\lambda_1, \lambda_2) \neq (1, 1)$ then necessary $\lambda_1 \neq \lambda_2$ and there exists j such that $\delta_j \neq \beta_j$. Therefore, $\nu = \rho(\delta_j - \beta_j) + \beta_j \neq 0$, as $\rho \notin \mathbb{Q}$. \square

8 Proof of Theorem 1.

Let f be distorted in \mathcal{A} , as f has no a semi-hyperbolic periodic point, its periodic points are not isolated. Using in addition that $BP_0(f)$ is finite, we get that the set $Per(f)$ of f -periodic points is the union of a finite collection of half open intervals with endpoints in the orbits of $BP_0(f)$. Thereby, there exists some positive integer p such that $Per(f) = Per(f^p) = Fix(f^p)$. It is easy to check that there exists $S \in \mathcal{E}$ whose discontinuities are endpoints of connected components of $Per(f)$ and such that $Fix(Sf^pS^{-1})$ is an interval $P = [0, a)$ and the restriction of Sf^pS^{-1} to $M = [a, 1)$ has no periodic points.

Applying Theorem 2 to the restriction of Sf^pS^{-1} to M , there exist $q \in \mathbb{N}^*$ and $E \in \mathcal{E}$ such that $ES \circ f^{pq} \circ (ES)^{-1} = \prod_{i=1}^p f_i$, where f_i are restricted PL-homeomorphisms with pairwise disjoint supports $I_i = [a_i, b_i)$, $f_i|_{I_i}$ is minimal and $\#BP(f_i^n)$ is bounded (since f is distorted).

Let $i \in \{1 \dots p\}$, applying Proposition 6.1 to $f_i|_{I_i}$, we get that $f_i|_{I_i}$ is conjugate by a PL-homeomorphism H_i of I_i to B_i with $\Lambda(B_i) = \{\lambda_{i,1}, \lambda_{i,2}\}$ and $BP(B_i) = \{a_i, B_i^{-1}(a_i)\}$. Since $f_i|_{I_i}$ is minimal, B_i also is minimal and according to Remark 6, B_i is C^0 -conjugate to an infinite order rotation R_ρ of I_i .

Let $H \in \mathcal{A}$ defined by $H(x) = H_i(x)$, if $x \in I_i$ and $H(x) = x$, if $x \notin \cup I_i$ and let $B = (HES) \circ f^{pq} \circ (HES)^{-1}$.

It is easy to check that $B|_{I_i} = B_i$ and B is distorted in a subgroup $G = \langle g_1, \dots, g_q \rangle$ of \mathcal{A} , since f is distorted in \mathcal{A} .

Let Λ_G be the free abelian multiplicative subgroup Λ of \mathbb{R}^{+*} generated by $\{ Dg_k(x), x \in [0, 1], k \in \{1 \dots q\} \}$. It has finite rank s , we consider a basis $\alpha_1, \dots, \alpha_s$ of it.

Let $i \in \{1 \dots p\}$, note that $\mathcal{N}_j(D_+B^n(y)) = \mathcal{N}_j(D_+B_i^n(y)), \forall y \in I_i$. We suppose that $(\lambda_{i,1}, \lambda_{i,2}) \neq (1, 1)$.

On one hand, by Proposition 7.1, there exist $j \in \{1, \dots, s\}$ and $x \in I_i$ such that $\frac{\mathcal{N}_j(D_+B^n(x))}{n} \rightarrow \nu \neq 0$.

On the other hand, since B is distorted in G , its iterates B^n can be written $B^n = g_{i_{l_n}} \dots g_{i_1}$ with $\lim_{n \rightarrow +\infty} \frac{l_n}{n} = 0$.

Hence $D_+B^n(x) = D_+g_{i_{l_n}}(x_{l_n}) \dots D_+g_{i_1}(x_1)$, where $x_m = g_{i_{m-1}} \dots g_{i_1}(x)$. Then

$$|\mathcal{N}_j(D_+B^n(x))| \leq \sum_{m=1}^{l_n} |\mathcal{N}_j(D_+g_{i_m}(x_m))| \leq l_n S,$$

where $S = \max\{ |\mathcal{N}_j(D_+g_k(y))|, y \in [0, 1], 1 \leq k \leq q \}$.

Finally, $\frac{l_n}{n} \geq \frac{|\mathcal{N}_j(D_+B^n(x))|}{nS} \rightarrow \frac{|\nu|}{S} > 0$, this is a contradiction.

Consequently, for any $i \in \{1 \dots p\}$, $(\lambda_{i,1}, \lambda_{i,2}) = (1, 1)$ and thereby B_i is an infinite order rotation of I_i . Thus B is a product of infinite order restricted rotations with pairwise disjoint supports.

In conclusion, we have proved that when restricted to M , there exists an iterate of f that is conjugate in \mathcal{A} to a product of infinite order restricted rotations with pairwise disjoint supports. We conclude by noting that $f|_{M^c} = Id|_{M^c}$. \square

9 Proof of Corollaries 1, 2 and 3.

9.1 Proof of Corollary 1.

Let a, b in \mathcal{A} such that $ba^m b^{-1} = a^n$ with m, n integers and $|m| \neq |n|$. We will prove that a has finite order.

By absurd, since a is distorted, eventually passing to a power of a and conjugating a and b by an element of \mathcal{A} we can suppose that a is a product of infinite order restricted rotations R_{α_i} of disjoint supports I_i , according to Theorem 1. We

denote $a = \prod_{i=1}^p (R_{\alpha_i}, I_i)$.

The main tool of the proof is the following

Lemma 9.1. *Let $a, b \in \mathcal{A}$ with $a = \prod_{i=1}^p (R_{\alpha_i}, I_i)$ and $ba^m b^{-1} = a^n$ then there exists an integer s such that b^s maps I_i to itself preserving the Lebesgue measure on I_i , denoted by $Leb|_{I_i}$.*

Let $X = \cup I_i$. In what follows we identify a with its restriction to X .

By unique ergodicity of irrational rotations, one has that the ergodic a^p -invariant probabilities on X are $Leb|_{I_i}$, for all $p \in \mathbb{Z}$.

As $Supp(ba^m b^{-1}) = b(Supp(a^m))$ and $Supp(a^n) = Supp(a^m)$, then $b(X) = X$ and b is identified to its restriction to X .

The image by b of an ergodic a^m -invariant measure is an ergodic a^n -invariant measure. Hence, for some permutation σ , $b_*(Leb|_{I_i}) = Leb|_{I_{\sigma(i)}}$.

Thus there exists an integer s such that $b_*^s(Leb|_{I_i}) = Leb|_{I_i}$. \square

The spectrum of an irrational rotation R_α viewed as an IET of I is

$$Sp(R_\alpha, I, Leb|_I) = \langle e^{2i\pi \frac{\alpha}{l}} \rangle \text{ where } l = |I|.$$

A consequence of the identity $b^s a^{m^s} b^{-s} = a^{n^s}$ and Lemma 9.1 is that $b^s|_{I_i}$ sends the generator of $Sp(a^{m^s}, I_i, Leb|_{I_i})$ into a generator of $Sp(a^{n^s}, I_i, Leb|_{I_i})$. Then

$$e^{2i\pi \frac{\alpha_i}{l_i} m^s} = e^{\pm 2i\pi \frac{\alpha_i}{l_i} n^s}.$$

Finally, $\frac{\alpha_i}{l_i} m^s = \pm \frac{\alpha_i}{l_i} n^s \pmod{\mathbb{Z}}$. This is a contradiction since $\frac{\alpha_i}{l_i} \notin \mathbb{Q}$ and $|m| \neq |n|$.

Therefore a has finite order, hence any action of $BS(m, n)$ with $|m| \neq |n|$ by elements of \mathcal{A} is not faithful.

9.2 Proof of Corollary 2.

Let G be a nilpotent subgroup of \mathcal{A} . Suppose by absurd that G is either non abelian torsion free or finitely generated and not virtually abelian.

Since G is nilpotent there exist $u, v \in G$ such that $c = [u, v]$ commutes with u and v . Furthermore, we can choose c of infinite order because G is either non abelian torsion free or finitely generated and not virtually abelian. This implies

Claim 9.1. *For any integers p and q , it holds that $[u^p, v^q] = c^{pq}$.*

In particular $c^{n^2} = [u^n, v^n]$, so c is distorted.

Hence, by Theorem 1, eventually passing to a power of c and conjugating by an element of \mathcal{A} we can suppose that c is a product of infinite order restricted rotations R_{α_i} of disjoint supports I_i . We denote $c = \prod_{i=1}^p (R_{\alpha_i}, I_i)$.

Applying Lemma 9.1 with $m = n$, $a = c$ and $b = u$ [resp. $b = v$], there exists s_u [resp. s_v] such that $u_*^{s_u}(Leb|_{I_i}) = Leb|_{I_i}$ and $v_*^{s_v}(Leb|_{I_i}) = Leb|_{I_i}$.

Therefore $u^{s_u}|_{I_i}$ and $v^{s_v}|_{I_i}$ are IET which commute with the rotation R_{α_i} . Finally, by Lemma 5.1 of [25], $u^{s_u}|_{I_i}$ and $v^{s_v}|_{I_i}$ are rotations so they commute.

According to Claim 9.1, $[u^{s_u}, v^{s_v}] = c^{s_u s_v}$, so $c^{s_u s_v} = Id$ on I_i for any $i = 1, \dots, p$. It follows that $c^{s_u s_v} = Id$. This contradicts that c has infinite order.

9.3 Proof of Corollary 3.

In this section we prove that there is no distortion element in $\mathcal{A}_{\mathbb{Q}}$.

Let $f \in \mathcal{A}_{\mathbb{Q}}$ distorted in $\mathcal{A}_{\mathbb{Q}}$, by Theorem 1, there exist a positive integer s and $H \in \mathcal{A}$, $E \in \mathcal{E}$ and $S \in \mathcal{E}$ such that $(HES)f^{pq}(HES)^{-1} = \prod_{i=1}^p (R_i, I_i)$ is a product of infinite order restricted rotations of disjoint supports I_i .

We first check that the conjugating maps H , E and S are in $\mathcal{A}_{\mathbb{Q}}$.

By the definition of S , the break points of S are endpoints of connected components of $Per(f)$ so they belong to the f -orbit of $BP_0(f)$ which is contained in \mathbb{Q} . Therefore $S \in \mathcal{A}_{\mathbb{Q}}$.

According to Remark 5, $E \in \mathcal{A}_{\mathbb{Q}}$ and the endpoints of the I_i 's are rational. Hence, by Remark 8, $H \in \mathcal{A}_{\mathbb{Q}}$.

Therefore $(HES)f^{pq}(HES)^{-1} \in \mathcal{A}_{\mathbb{Q}}$ and then $R_i \in \mathcal{A}_{\mathbb{Q}}$. This is a contradiction. \square

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