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ANY BAUMSLAG-SOLITAR ACTION ON SURFACES WITH A PSEUDO-ANOSOV ELEMENT HAS A FINITE ORBIT.

NANCY GUELMAN AND ISABELLE LIOUSSE

ABSTRACT. We consider f, h homeomorphisms generating a faithful $BS(1, n)$ -action on a closed surface S , that is, $hfh^{-1} = f^n$, for some $n \geq 2$. According to [GL], after replacing f by a suitable iterate if necessary, we can assume that there exists a minimal set Λ of the action, included in $Fix(f)$.

Here, we suppose that f and h are C^1 in neighbourhood of Λ and any point $x \in \Lambda$ admits an h -unstable manifold $W^u(x)$. Using Bonatti's techniques, we prove that either there exists an integer N such that $W^u(x)$ is included in $Fix(f^N)$ or there is a lower bound for the norm of the differential of h only depending on n and the Riemannian metric on S .

Combining last statement with a result of [AGX], we show that any faithful action of $BS(1, n)$ on S with h a pseudo-Anosov homeomorphism has a finite orbit containing singularities of h ; moreover if f is isotopic to identity it is entirely contained in the singular set of h . As a consequence, there is no faithful C^1 -action of $BS(1, n)$ on the torus with h an Anosov.

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1. INTRODUCTION AND STATEMENTS

In 1962, Gilbert Baumslag and Donald Solitar defined the Baumslag-Solitar groups, which play an important role in combinatorial group theory, geometric group theory and dynamical systems. In this paper we will focus on dynamical aspects. Dynamic of Baumslag-Solitar groups in dimension 1, is well understood:

For actions of Baumslag-Solitar groups on the circle, Burslem and Wilkinson ([BW]) gave a classification, up to conjugacy, of real analytic actions. In [GL2], the authors gave a classification, up to semi-conjugacy of C^1 -actions. Recently Bonatti, Monteverde, Navas and Rivas ([BMNR]) generalized these results proving a classification, up to conjugacy of C^1 -actions.

The question of existence of global fixed points or finite orbits for group actions on surfaces has been extensively studied and one can consider that the abelian case is now well understood: In 1964, in the context of Lie groups, Lima ([Li]) proved that commuting

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vectors fields have a common singularity. In 1989, Bonatti ([Bo]) proved that two sufficiently C^1 -close to identity commuting diffeomorphisms of the 2-sphere have a common fixed point. In 2008, Franks, Handel and Parwani ([FHP]) proved that an abelian group of C^1 -diffeomorphisms isotopic to identity of S has a global fixed point, where S is a closed surface of genus at least 2.

It is natural to ask what are the algebraic conditions for the existence of a global fixed point. In [Pl], Plante extended Lima's result for nilpotent Lie groups and exhibited solvable Lie groups acting without global fixed point on surfaces.

In [HW], Hirsch and Weinstein showed that every analytic action of a connected supersoluble Lie group on a non zero Euler characteristic compact surface has a fixed point.

For fixed points of local actions by Lie groups on surfaces, we refer to Hirsch's survey (see [Hi1]) and [Hi2]).

In [DDF], Druck, Fang and Firmo proved a discrete version of Plante's result on nilpotent groups of diffeomorphisms of the 2-sphere. Very recently, Firmo and Ribón gave topological conditions to insure the existence of finite orbits for nilpotent groups of C^1 -diffeomorphisms of the 2-torus (see [FR]).

In some sense, solvable Baumslag-Solitar groups, $BS(1, n)$, represent the next discrete step: they are non nilpotent discrete subgroups of the supersoluble Lie group \mathcal{A} consisting in homeomorphisms of the real line having the form $x \mapsto ax + b$, $a > 0, b \in \mathbb{R}$ and therefore are metabelian (2-step solvable).

In this direction, A.McCarthy in [McC], proved that the trivial $BS(1, n)$ -action on a compact manifold does not admit C^1 faithful perturbations.

Before stating our results, we give some definitions and properties.

Let $n \in \mathbb{N}$, $n \geq 2$, the solvable Baumslag-Solitar group $BS(1, n)$ is defined by

$$BS(1, n) = \langle a, b \mid aba^{-1} = b^n \rangle .$$

It is well known that $BS(1, n)$ can be represented as the subgroup of \mathcal{A} generated by the two affine maps $f_0(x) = x + 1$ and $h_0(x) = nx$ (where $f_0 \equiv b$ and $h_0 \equiv a$).

In what follows, we will always denote by $\langle f, h \rangle$ an action of $BS(1, n)$ on a surface, meaning that the homeomorphisms f and h satisfy $h \circ f \circ h^{-1} = f^n$.

One can easily check the following properties:

- $h \circ f^k \circ h^{-1} = f^{nk} = (f^k)^n$ for any $k \in \mathbb{Z}$, so $\langle f^k, h \rangle$ is also an action of $BS(1, n)$,
- $h^k \circ f \circ h^{-k} = f^{n^k}$ for any $k \in \mathbb{N}$
- $h^{-k} \circ f^{n^k} \circ h^k = f$ for any $k \in \mathbb{N}$

As consequence we have that:

- $h^{-k}(Fix f) \subset h^{-k}(Fix(f^{n^k})) = Fix(f)$ for any $k \in \mathbb{N}$.

In [GL], the authors of this paper exhibited actions of Baumslag-Solitar groups on \mathbb{T}^2 without finite orbit (this construction extends on any compact surface) and proved the following:

Theorem. [GL] *Let f_0, h be two homeomorphisms, generating a faithful action of $BS(1, n)$ on S . There exists a positive integer N such that $f = f_0^N$ satisfies:*

- f is isotopic to identity,
- f, h generate a faithful action of $BS(1, n)$ on S and
- $Fix f \neq \emptyset$ and contains a minimal set Λ for $\langle f, h \rangle$, that is also a minimal set for h .

The first point is stated in [GL] for $S = \mathbb{T}^2$. For closed orientable surface of genus greater than 1, it is a consequence of the facts that the mapping class group of S does not contain distortion element, according to [FLM] and f is a distortion element. A corollary of this Theorem is that there is no minimal actions of $BS(1, n)$ on closed orientable surfaces.

Remark 1. *According to last Theorem, we may assume without loss of generality that f is isotopic to identity and $Fix(f)$ is not empty, noting that if $\langle f^N, h \rangle$ has a finite orbit F then $\mathcal{O}_f(F)$, the f -orbit of F is a finite orbit of $\langle f, h \rangle$.*

Indeed, $\mathcal{O}_f(F)$ is finite and $h(\mathcal{O}_f(F)) \subset \mathcal{O}_f(F)$ (for $x = f^k(p)$ with $p \in F$, one has $h(x) = h(f^k(p)) = f^{nk}(h(p))$ with $h(p) \in F$).

In [AGX], Alonso, Guelman, Xavier proved that

Theorem. [AGX] *Let $\langle f, h \rangle$ be an action of $BS(1, n)$ on a closed surface S , where f is isotopic to the identity, and h is a (pseudo)-Anosov homeomorphism with stretch factor $\lambda > n$. Then $f = Id$.*

We will now fix some definitions, notations and recall some properties that will be used.

Notations. Let S be a closed connected oriented surface embedded in the 3-dimensionnal Euclidean space \mathbb{R}^3 , endowed with the usual norm denoted by $\| \cdot \|$.

- The distance in S associated to the induced Riemannian metric is denoted by d .
- The injectivity radius of the exponential map associated to d is denoted by ρ .
- The open 2-ball centered at x with radius r with respect to d is denoted by $B_r(x)$ or B_r for the case that we don't need to specify the center.

Remark. In what follows, we always assume that radius of balls are strictly less than ρ .

Let $F : S \rightarrow S$ be a map.

Definition. Let Λ be an F -invariant compact set, we say that F is C^1_Λ if F is C^1 on a neighbourhood W of Λ in S .

Suppose that F is C^1 on an open set W of S .

Notations.

Let $x \in W$, we will denote by:

- $\mathcal{D}F(x) = \mathcal{D}_x F : T_x S \rightarrow \mathbb{R}^3$ the differential at x of F considered as a map $F : S \rightarrow \mathbb{R}^3$,
- $DF(x) = D_x F : T_x S \rightarrow T_{F(x)} S$ the differential at x of F considered as a map $F : S \rightarrow S$ and
- $\| \mathcal{D}_x F \| = \sup \{ \| \mathcal{D}_x F.v \|, v \in T_x, \| v \| = 1 \}$.

We set $\mathcal{S}(F, W) = \sup \{\|\mathcal{D}_x F\|, x \in W\}$.

Definitions.

The C^1 -norm of F on a subset V of W is defined by:

$$\|F\|_V = \sup_{x \in V} (\|F(x)\| + \|\mathcal{D}_x F\|).$$

We say that F is C^1 ϵ -close to identity on V if $\|F - Id\|_V < \epsilon$.

We define $V_\epsilon(F) = \{x \in S : \|(F - Id)(x)\| + \|\mathcal{D}_x(F - Id)\| < \epsilon\}$. Note that this definition implies that F is C^1 at x , for any $x \in V_\epsilon(F)$.

Properties.

There is a constant $C_S \geq 1$ such that for all $x, y \in B_R \subset W$, with $R < \rho$, one has

- (1) $\|x - y\| \leq d(x, y) < C_S \|x - y\|$,
- (2) $d(F(x), F(y)) \leq \mathcal{S}(F, B_R) d(x, y)$.

Theorem 1. *Let f, h be two homeomorphisms, generating a faithful action of $BS(1, n)$ on S , Λ be a minimal set of $\langle f, h \rangle$ included in $Fix f$ and f, h are C^1_Λ .*

1. *Any point x in Λ is an f -elliptic fixed point, in the sense that eigenvalues of $D_x f$ are roots of unit. More precisely, there exists a positive integer N such that the eigenvalue of $D_x f^N$ is 1, for any $x \in \Lambda$.*

2. *Moreover, for all $\epsilon > 0$, there exists $\delta > 0$ and a C^1_Λ -diffeomorphism $f_\epsilon \in \langle f, h \rangle$ such that :*

- f_ϵ, h generate a faithful action of $BS(1, n)$ on S and $\Lambda \subset Fix f_\epsilon \subset Fix(f^N)$, where N is given by previous item.
- $\|f_\epsilon - Id\|_{B_\delta(\Lambda)} \leq \epsilon$, where $B_\delta(\Lambda)$ is the union of balls of center in Λ and radius δ (in other words, $B_\delta(\Lambda) \subset V_\epsilon(f_\epsilon)$).

More precisely, f_ϵ is either f^N or some n^{k_ϵ} -root of f^N .

Thus, following Mc Carthy ([McC]), we can adapt Bonatti's tools ([Bo]) for estimating the norm of the differential of h . More precisely, we prove

Theorem 2. *Let f, h and Λ as in Theorem 1, let W be a neighbourhood of Λ such that f and h are in C^1 on W . Suppose that any point $x \in \Lambda$ has an h -unstable manifold, $W^u(x)$, then either:*

- (1) *there exists $N \in \mathbb{N}$ such that $W^u(x) \subset Fix f^N$, for all $x \in \Lambda$ or*
- (2) $\mathcal{S}(h, W) \geq \frac{n}{C_S}$.

Assumption. *We refer to [FM], for the definition and properties of pseudo-Anosov homeomorphisms and we always assume that a pseudo-Anosov homeomorphism is a C^1 -diffeomorphism except at finitely many points: the singularities of the stable and unstable foliations. We recall that if p is a singularity of a foliation, there exists a smooth chart from a neighbourhood of p to the plane that sends leaves to the level sets of a k prong saddle, with $k \geq 3$.*

Using the result of [AGX], we obtain as a corollary:

Corollary 1. *Any faithful action $\langle f, h \rangle$ of $BS(1, n)$ on S , where f is a C^1 -diffeomorphism isotopic to identity and h is a pseudo-Anosov homeomorphism has a finite orbit. Moreover, this finite orbit is contained in the set of singularities of h and in the set of the fixed points of f .*

Using Remark 1 and algebraic properties of $BS(1, n)$, Corollary 1 can be restated as:

Corollary 2. *Any Baumslag-Solitar action on a surface with a pseudo-Anosov element has a finite orbit.*

Since any pseudo-Anosov homeomorphism of the torus has no singularities, that is, it is an Anosov, then

Corollary 3. *There is no faithful action $\langle f, h \rangle$ of $BS(1, n)$ on the torus, where f is a C^1 -diffeomorphism and h is an Anosov.*

2. PROOF OF THEOREM 1.

Proof of item 1. Let f and Λ as in Theorem 1 and $x_0 \in \Lambda$.

Since Λ is also an h -minimal set, the h -orbit of x_0 is recurrent. Then there exists a subsequence (n_k) ($n_k \rightarrow \infty$) such that $h^{-n_k}(x_0) \rightarrow x_0$.

From $h^{n_k} \circ f \circ h^{-n_k} = f^{n_k}$, we deduce that:

$$Dh^{n_k}(f(h^{-n_k}(x_0))) \circ Df(h^{-n_k}(x_0)) \circ Dh^{-n_k}(x_0) = Df^{n_k}(x_0).$$

Moreover, the points x_0 and $h^{-n_k}(x_0)$ are fixed by f and $(Dh^{-n_k}(x_0))^{-1} = Dh^{n_k}(h^{-n_k}(x_0))$, then:

$$(Dh^{-n_k}(x_0))^{-1} \circ Df(h^{-n_k}(x_0)) \circ Dh^{-n_k}(x_0) = (Df(x_0))^{n_k}.$$

So $Df(h^{-n_k}(x_0))$ and $(Df(x_0))^{n_k}$ have same eigenvalues.

Let us denote by:

ρ^+ [resp. ρ_k^+] the maximum modulus of eigenvalues of $Df(x_0)$ [resp. $Df(h^{-n_k}(x_0))$] and

ρ^- [resp. ρ_k^-] the minimum modulus of eigenvalues of $Df(x_0)$ [resp. $Df(h^{-n_k}(x_0))$].

Hence $\rho_k^+ = (\rho^+)^{n_k}$ and $\rho_k^- = (\rho^-)^{n_k}$.

As f is C^1_Λ and $h^{-n_k}(x_0) \rightarrow x_0$, then $Df(h^{-n_k}(x_0)) \rightarrow Df(x_0)$. Therefore, $\rho_k^+ \rightarrow \rho^+$ and $\rho_k^- \rightarrow \rho^-$.

Consequently, $\rho^+ = \rho^- = 1$ and eigenvalues of $Df(x_0)$ have modulus 1.

Suppose that Λ is infinite. This means that x_0 is not isolated in $Fix f$. We are going to prove that 1 is an eigenvalue of $Df(x_0)$. By contradiction, suppose that 1 is not an eigenvalue of $Df(x_0)$.

Let us introduce local coordinates near x_0 . The map $F = f - Id$ is locally invertible by the Inverse Function Theorem. More precisely, there exist neighborhoods U of x_0 and V of $F(x_0) = (0, 0)$ such that $F : U \rightarrow V$ is a diffeomorphism, hence $F^{-1}(0, 0) = x_0$, which means that x_0 is the unique fixed point of f in U . This contradicts the fact that x_0 is not isolated in $Fix(f)$.

Consequently, if Λ is infinite, 1 is the unique eigenvalue of $Df(x)$ (since f is isotopic to identity, it is also orientation preserving and -1 can not be an eigenvalue of $Df(x)$), for any $x \in \Lambda$.

If Λ is finite, there exists p such that $h^p(x_0) = x_0$. As $Df(x_0)$ and $(Df(h^p(x_0)))^{n^p} = (Df(x_0))^{n^p}$ are conjugate, eigenvalues of $Df(x_0)$ are roots of unity. Thus, there exists a positive integer N such that eigenvalues of $Df^N(x_0)$ are 1. As $Df^N(h^{-j}(x_0))$ and $(Df^N(x_0))^{n^j}$ are conjugate, we get that eigenvalues of $Df^N(h^{-j}(x_0))$ are 1.

Since $\Lambda = \{(h^{-j}(x_0), j \in \mathbb{N})\}$, we get that eigenvalues of $Df^N(x)$ are 1, for any $x \in \Lambda$.

Proof of item 2. We begin by proving

Lemma 2.1. *There exists a positive integer N such that:*

- either $Df^N(x) = Id$ for any $x \in \Lambda$
- or $Df^N(x) \neq Id$ for any $x \in \Lambda$.

Proof of Lemma 2.1. Let $x_0 \in \Lambda$, if $Df^N(x_0) = Id$. As $Df^N(h^{-p}(x_0))$ and $(Df^N(x_0))^{n^p}$ are conjugate, we get that $Df^N(h^{-p}(x_0)) = Id$, for any $p \in \mathbb{N}$. Since $\{h^{-p}(x_0), p \in \mathbb{N}\}$ is dense in Λ , we get that $Df^N(x) = Id$, for any $x \in \Lambda$. \square

We can now prove item 2. Fix $\epsilon > 0$, N given by item 1. We set $\bar{f} = f^N$, the action of $\langle \bar{f} = f^N, h \rangle$ on S is an action of the Baumslag-Solitar group $BS(1, n)$.

Let $x_0 \in \Lambda$.

Case 1 : $D\bar{f}(x_0) = Id$. By Lemma 2.1, for all $x \in \Lambda$, $D\bar{f}(x) = Id$ and we can find an open ball $B_r(x)$ contained in $V_\epsilon(\bar{f})$.

We claim that $\delta = \text{Min}\{r_x, x \in \Lambda\} > 0$, where r_x is the greater number such that $B_r(x) \subset V_\epsilon(\bar{f})$. Indeed, suppose that there exists a sequence x_p such that $r_{x_p} \rightarrow 0$. Without loss of generality, we can suppose that $x_p \rightarrow w$. By Proposition 2.1, $w \in \Lambda$ and $D\bar{f}(w) = Id$. For p sufficiently large $x_p \in B_{\frac{r_w}{2}}(w)$ and then $r_p \geq \frac{r_w}{2}$, that is a contradiction.

Hence the δ -neighborhood of Λ is contained in $V_\epsilon(\bar{f})$ and we conclude by setting $f_\epsilon = \bar{f}$.

Case 2 : $D\bar{f}(x_0) \neq Id$. By Lemma 2.1, $D\bar{f}(x) \neq Id$, for all $x \in \Lambda$.

We can choose in a continuously way an orthonormal basis on each $T_x S$, $x \in \Lambda \cap B$, where B is a small open ball centered at x_0 .

According to item 1 and Lemma 2.1, for all $x \in \Lambda \cap B$ the matrix of $D\bar{f}(x)$ (also denoted by $D\bar{f}(x)$) is conjugate to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ by some matrix A_x that depends continuously on x .

So norms of A_x are uniformly bounded on $\Lambda \cap \bar{B}$.

Let denote $\bar{f}_k = h^{-k} \circ \bar{f} \circ h^k$. As \bar{f} is isotopic to identity and has fixed points, its conjugate \bar{f}_k is isotopic to identity and has fixed points. One can check that:

- (1) \bar{f}_k is a n^k -root of \bar{f} that is $\bar{f}_k^{n^k} = (h^{-k} \circ \bar{f} \circ h^k)^{n^k} = (h^{-k} \circ \bar{f}^{n^k} \circ h^k) = \bar{f}$,
- (2) $\langle \bar{f}_k, h \rangle$ generate a faithful action of $BS(1, n)$ on S , that is $h \circ \bar{f}_k \circ h^{-1} = h^{-k+1} \circ \bar{f} \circ h^{k-1} = h^{-k} \circ (h \circ \bar{f} \circ h^{-1}) \circ h^k = h^{-k} \circ \bar{f}^{n^k} \circ h^k = \bar{f}_k^n$,
- (3) $\text{Fix}(\bar{f}_k) \subset \text{Fix}(\bar{f})$ since $h^{-k}(\text{Fix}(\bar{f})) \subseteq \text{Fix}(\bar{f})$, and

(4) $\Lambda \subset \text{Fix}(\bar{f}_k)$, as $\Lambda \subset \text{Fix}(\bar{f})$ and $h^k(\Lambda) = \Lambda$.

Let $x \in \Lambda \cap B$, since \bar{f} and \bar{f}_k commute, the matrices $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = A_x D\bar{f}(x) A_x^{-1}$ and $A_x D\bar{f}_k(x) A_x^{-1}$ commute. This implies that $A_x D\bar{f}_k(x) A_x^{-1}$ is upper-triangular. But $D\bar{f}(x)$ and $D\bar{f}_k(x)$ are conjugate by $Dh^k(x)$, so $A_x D\bar{f}_k(x) A_x^{-1}$ has a unique eigenvalue 1, then it is of the form $\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$.

Finally $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = A_x D\bar{f}(x) A_x^{-1} = (A_x D\bar{f}_k(x) A_x^{-1})^{n^k} = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}^{n^k} = \begin{pmatrix} 1 & \alpha n^k \\ 0 & 1 \end{pmatrix}$ and then $\alpha = \frac{1}{n^k}$.

Therefore $D_x(\bar{f}_k - Id) = A_x^{-1} \begin{pmatrix} 0 & \frac{1}{n^k} \\ 0 & 0 \end{pmatrix} A_x$.

Since the norms of the A_x are uniformly bounded on $\Lambda \cap \bar{B}$ and we choose orthonormal basis for tangent spaces, we can find an integer k_ϵ such that $\sup_{x \in \Lambda \cap B} \|D_x(\bar{f}_k - Id)\|$ is strictly less than ϵ , for any $k \geq k_\epsilon$.

Hence for any $x_0 \in \Lambda$ we can find an open ball $B_r(x_0)$ contained in $V_\epsilon(\bar{f}_{k_\epsilon})$, therefore, by compactness of Λ there is a positive uniform radius δ such that the δ -neighborhood of Λ is contained in $V_\epsilon(\bar{f}_{k_\epsilon})$.

We conclude by setting $f_\epsilon = \bar{f}_{k_\epsilon}$. This ends the proof of Theorem 1.

3. PROOF OF THEOREM 2.

3.1. Flow-like properties.

In this subsection we will prove a local version of Bonatti flow-like properties for C^1 -closed to identity diffeomorphisms.

Proposition 3.1. *Given $n \in \mathbb{N}$, for all $0 < \eta < 1$, there exists $\epsilon_\eta^n > 0$ such that:*

$$\|f^n(y) - y - n(f(y) - y)\| \leq \eta \|f(y) - y\|,$$

for any $f \in \text{Homeo}(S)$, $\delta \in (0, \frac{\rho}{m})$ and $y \in S$ satisfying $y \in B_\delta(x_0)$ and $B_{m\delta}(x_0) \subset V_{\epsilon_\eta^n}(f)$, where $x_0 \in \text{Fix}(f)$ and $m = (1 + 3C_S(n + 1))$.

Remark 2. *Note that $B_{m\delta}(x_0) \subset V_{\epsilon_\eta^n}(f)$ means that f is C^1 and ϵ_η^n C^1 -close to identity on $B_{m\delta}(x_0)$.*

Proof. Let $0 < \eta < 1$ be given. Proceed using induction on $k = 1, \dots, n$.

The $k = 1$ case is trivial. Choose $\epsilon_\eta^1 = \min \left\{ \frac{\rho}{m}, \frac{\eta}{4n}, \frac{1}{2} \right\}$.

Let $k \in \{1, \dots, n - 1\}$, we suppose that $\forall \eta \in (0, 1)$ there exists ϵ_η^k such that for $f \in \text{Homeo}(S)$, $x_0 \in \text{Fix}(f)$ and $\delta \in (0, \frac{\rho}{m})$ such that $B_{m\delta}(x_0) \subset V_{\epsilon_\eta^k}(f)$, we have

$$\|f^k(y) - y - k(f(y) - y)\| \leq \eta \|f(y) - y\|, \forall y \in B_\delta(x_0).$$

We have to estimate $\|f^{k+1}(y) - y - (k + 1)(f(y) - y)\|$.

$$\|f^{k+1}(y) - y - (k+1)(f(y) - y)\| \leq \|(f - Id)f^k(y) - (f - Id)(y)\| + \|f^k(y) - y - k(f(y) - y)\|.$$

We choose, $\epsilon_\eta^{k+1} \leq \epsilon_\eta^k$, then we get that $\|f^k(y) - y - k(f(y) - y)\| \leq \frac{\eta}{2}\|f(y) - y\|$, for all $y \in B_\delta(x_0)$ and $B_{m\delta}(x_0) \subset V_{\epsilon_\eta^{k+1}}(f)$.

To bound the first term, we first show that $f^k(y) \in B_{m\delta}(x_0)$ provided that $y \in B_\delta(x_0)$ and $B_{m\delta}(x_0) \subset V_{\epsilon_\eta^{k+1}}(f)$. Indeed

$$d(f^k(y), x_0) \leq d(x_0, y) + d(f^k(y), y) \leq d(x_0, y) + C_S \|f^k(y) - y\| \leq d(x_0, y) + C_S(k + \frac{\eta}{2})\|f(y) - y\|$$

by inductive hypothesis. Moreover,

$$\|f(y) - y\| \leq \|f(y) - f(x_0)\| + \|x_0 - y\| \leq \|f\|_{B_\delta(x_0)} d(x_0, y) + d(x_0, y) \leq (\|f\|_{B_\delta(x_0)} + 1)\delta.$$

Since $B_\delta(x_0) \subset V_{\epsilon_\eta^{k+1}}(f)$ and $\epsilon_\eta^{k+1} < 1$, we have $\|f\|_{B_\delta(x_0)} \leq 2$ and therefore we get $\|f(y) - y\| \leq 3\delta$.

Finally $d(f^k(y), x_0) \leq (1 + 3C_S(n + 1))\delta$, that is $f^k(y) \in B_{m\delta}(x_0)$.

The first term can be bounded as follows:

$$\|(f - Id)f^k(y) - (f - Id)(y)\| \leq \|(f - Id)\|_{B_{m\delta}(x_0)} \|f^k(y) - y\| \leq \|(f - Id)\|_{B_{m\delta}(x_0)} (k + \frac{\eta}{2})\|f(y) - y\|$$

by inductive hypothesis. Then

$$\|(f - Id)f^k(y) - (f - Id)(y)\| \leq \epsilon_\eta^k (k + \frac{\eta}{2})\|f(y) - y\|.$$

Moreover $\epsilon_\eta^k (k + \frac{\eta}{2}) = \epsilon_\eta^k k + \epsilon_\eta^k \frac{\eta}{2} \leq \frac{\eta k}{4n} + \frac{\eta}{4}$, by the choice of ϵ_η^1 and the fact that the ϵ_η^k can be chosen to be decreasing in k and in such a way that given $k \leq n$, one has $\epsilon_\eta^k < \epsilon_\eta^k$ for $0 < \eta' < \eta$.

Finally, we can bound the first term by $\frac{\eta}{2}\|f(y) - y\|$ and therefore one has $\|f^{k+1}(y) - y - (k+1)(f(y) - y)\| \leq \eta\|f(y) - y\|$, for all $y \in B_\delta(x_0)$ and $B_{m\delta}(x_0) \subset V_{\epsilon_\eta^{k+1}}(f)$.

Corollary 4. *Given $n \in \mathbb{N}$, $n > 1$, there exists $\epsilon > 0$ such that for any $\delta \in (0, \frac{\rho}{m})$ and $f \in \text{Homeo}(S)$ such that $B_{m\delta}(x_0) \subset V_\epsilon(f)$, any f^n -fixed point $y \in B_\delta(x_0)$ is fixed by f , where $x_0 \in \text{Fix}(f)$ and m is defined as in Proposition 3.1.*

Proof. By Proposition 3.1 for $\eta = 1$, there exists $\epsilon = \epsilon_1^n > 0$ such that:

$$\|f^n(y) - y - n(f(y) - y)\| \leq \|f(y) - y\|$$

for any $f \in \text{Homeo}(S)$, $x_0 \in \text{Fix}(f)$, $y \in B_\delta(x_0)$ and $B_{m\delta}(x_0) \subset V_\epsilon(f)$.

Then $\|f^n(y) - y\| \geq (n - 1)\|f(y) - y\|$, therefore if y is f^n -fixed we have that y is f -fixed.

3.2. Proof of Theorem 2. The key tool of proof of Theorem 2 is the following:

Proposition 3.2. *Let $\delta \in (0, \frac{\rho}{m})$, let h, f, Λ and W as in Theorem 2 verifying that:*

- *for any x, y with $d(x, y) < \delta$, one has $d(f(x), f(y)) < \frac{\rho}{2}$,*
- *$B_\delta(\Lambda)$ and $f(B_\delta(\Lambda))$ are contained in W ,*
- *any point of Λ admits an h -unstable manifold.*

Let $0 < \eta \leq 1$ such that $\|f - Id\|_{B_{m\delta}(\Lambda)} < \epsilon_\eta^n$, where ϵ_η^n and m are given by Proposition 3.1 (i. e. $B_{m\delta}(\Lambda) \subset V_{\epsilon_\eta^n}(f)$).

If $\mathcal{S}(h, W) \leq \frac{n-\eta}{C_s}$, then there exists $r > 0$ such that for any $x \in \Lambda$,

$$W_{loc}^u(x) \cap B_r(x) \subset \text{Fix}(f).$$

Proof. Let $x_0 \in \Lambda$, there exists $p_0 = p_0(x_0, \delta)$ such that for all $y \in W_{loc}^u(x_0) \cap B_\delta(x_0)$, one has $d(h^{-p}(x_0), h^{-p}(y)) < \delta$ for any $p \geq p_0$. Note that $A = \bigcap_{j=0}^{p_0-1} h^j(B_\delta(h^{-j}(x_0)))$ is a non-empty open set that contains x_0 , then it intersects $W_{loc}^u(x)$ in an open arc containing x_0 .

More precisely, by hyperbolicity, there exists a constant $c = c(h)$ such that $B_{\frac{\delta}{c}}(x_0) \cap W_{loc}^u(x_0) \subset h^j(W_{loc}^u(h^{-j}(x_0)) \cap B_\delta(h^{-j}(x_0)))$, $\forall j \in \mathbb{N}$.

Then $B_{\frac{\delta}{c}}(x_0) \cap W_{loc}^u(x_0) \subset A$. From now, we will denote $r = \frac{\delta}{c}$ and $W_{loc}^u(x_0) \cap B_r(x_0)$ by $W_r(x_0)$.

First, we prove that any point $y \in W_r(x_0)$ is f -periodic.

Let $y \in W_r(x_0)$, it is easy to check that for all $p \in \mathbb{N}$, $h^{-p}(y) \in B_\delta(h^{-p}(x_0))$. Indeed, if $p \geq p_0$ we have $d(h^{-p}(x_0), h^{-p}(y)) < \delta$, by definition of p_0 . And if $p < p_0$, by definition of r and A we have that $y \in h^p(B_\delta(h^{-p}(x_0)))$.

If $f(y) = y$, in particular y is f -periodic.

If $f(y) \neq y$, there exists some $l = l(y)$ such that $h^{-l}(y)$ is close enough to $h^{-l}(x_0) \in \text{Fix} f$ to ensure that $\|f(h^{-l}(y)) - h^{-l}(y)\| < \|f(y) - y\|$. Therefore there exists some $p = p(y) \leq l$, such that $\|f(h^{-(p+1)}(y)) - h^{-(p+1)}(y)\| < \|f(h^{-p}(y)) - h^{-p}(y)\|$.

Indeed, by contradiction, let us denote $y_j := h^{-j}(y)$, for $j \in \mathbb{N}$, we would have :

$$\|f(y) - y\| \leq \|f(y_1) - y_1\| \leq \dots \leq \|f(y_l) - y_l\| < \|f(y) - y\|.$$

Notice that as $h^{-(p+1)}(y) \in B_\delta(h^{-(p+1)}(x_0))$, we have that

$$\begin{aligned} d(y_{p+1}, f(y_{p+1})) &\leq d(y_{p+1}, h^{-(p+1)}(x_0)) + d(f(y_{p+1}), h^{-(p+1)}(x_0)) \leq \\ &\leq \delta + d(f(y_{p+1}), f(h^{-(p+1)}(x_0))) \leq \rho. \end{aligned}$$

Since y_{p+1} and $f(y_{p+1})$ are in W , one has

$$d(h(f(y_{p+1})), h(y_{p+1})) \leq \mathcal{S}(h, W) \cdot d(f(y_{p+1}), y_{p+1}).$$

We adapt McCarthy's argument:

$$\|f^n(y_p) - y_p\| = \|hf h^{-1}(y_p) - y_p\| \leq d(hf h^{-1}(y_p), y_p) = d(hf(y_{p+1}), h(y_{p+1})) \leq$$

$$\mathcal{S}(h, W).d(f(y_{p+1}), y_{p+1}) \leq \mathcal{S}(h, W)C_S \|f(y_{p+1}) - y_{p+1}\| < \mathcal{S}(h, W)C_S \|f(y_p) - y_p\|.$$

Since, $y_p \in B_\delta(h^{-p}(x_0))$ and $B_{m\delta}(h^{-p}(x_0)) \subset V_{\epsilon_\eta^n}(f)$, we can apply Proposition 3.1 to obtain

$$\|f^n(y_p) - y_p\| \geq (n - \eta)\|f(y_p) - y_p\| \text{ and consequently}$$

$$(n - \eta)\|f(y_p) - y_p\| < \mathcal{S}(h, W)C_S \|f(y_p) - y_p\|.$$

Therefore, either $(n - \eta) < \mathcal{S}(h, W)C_S$ that contradicts the hypothesis or $\|f(y_p) - y_p\| = 0$ which implies that $y_p = h^{-p}(y)$ is f -fixed, then y is f^{n^p} -fixed. Hence we have proven that any point $y \in W_r(x_0)$ is f -periodic.

Now, we will prove that any point $y \in W_r(x_0)$ is f -fixed.

By contradiction, let $y \in W_r(x_0) \setminus \text{Fix}(f)$ and p_0 be the smallest integer such that y is $f^{n^{p_0}}$ -fixed point. Since $\text{Fix}(f^{n^p}) = h^p(\text{Fix}(f))$, we have that p_0 is the smallest integer such that $h^{-p_0}(y)$ is in $\text{Fix}(f)$.

By definition of p_0 we have that $h^{-(p_0-1)}(y)$ is in $\text{Fix}(f^n) \setminus \text{Fix}(f)$. And by definition of r and A , $h^{-(p_0-1)}(y) \in B_\delta(h^{-(p_0-1)}(x_0))$. This is impossible, according to Corollary 4.

End of the proof of Theorem 2.

Let $\eta > 0$ and ϵ_η^n given by Proposition 3.1. Suppose that $\mathcal{S}(h, W) \leq \frac{n-\eta}{C_S}$.

By Theorem 1, there exists $\delta > 0$ and $f_\eta = f_{\epsilon_\eta^n}$ such that $\|f_\eta - \text{Id}\|_{B_{m\delta}(\Lambda)} < \epsilon_\eta^n$ and f_η, h generate a faithful action of $BS(1, n)$ on S and $\Lambda \subset \text{Fix}(f_\eta)$.

By Proposition 3.2, there exists r such that for all $x \in \Lambda$, $W_r(x) \subset \text{Fix}(f_\eta)$.

Let $x_0 \in \Lambda$, we are going to prove that $W^u(x_0) \subset \text{Fix}(f_\eta)$.

By contradiction, suppose that $W^u(x_0)$ is not contained in $\text{Fix}(f_\eta)$. Therefore, there exists a segment $[a, b]$ in $W^u(x_0)$ and $c \in (a, b)$ such that $[a, c] \subset \text{Fix}(f_\eta)$ and (c, b) is disjoint of $\text{Fix}(f_\eta)$.

Let p be an integer such that $h^{-p}((c, b)) \in B_r(h^{-p}(x_0)) \cap W^u(h^{-p}(x_0)) \subset \text{Fix}(f_\eta)$. Hence $(c, b) \subset \text{Fix}(f_\eta^{n^p})$.

We have proven that there exists a segment $[a, b]$ in $W^u(x_0)$ and $c \in (a, b)$ such that $[a, c] \subset \text{Fix}(f_\eta)$ and $(c, b) \subset \text{Fix}(f_\eta^{n^p}) \setminus \text{Fix}(f_\eta)$.

The set $\mathcal{C} = \bigcup_{k=0}^{n^p-1} f_\eta^k([a, b])$ is a f_η -invariant continuum, one can check that main Theorem of [Wea] applies. Then all but a finite number of points in \mathcal{C} have the same least period. So all but a finite number of points in $[a, b]$ are fixed by f_η , this is a contradiction.

Finally, we have proved that given $\eta > 0$, either $\mathcal{S}(h, W) > \frac{n-\eta}{C_S}$ or $W^u(x_0) \subset \text{Fix}(f_\eta) \subset \text{Fix}(f^N)$.

Therefore, if $W^u(x_0)$ is not included in $\text{Fix}(f^N)$ then for any $\eta > 0$, $\mathcal{S}(h, W) > \frac{n-\eta}{C_S}$.

Let η converges to 0, then if $W^u(x_0)$ is not included in $\text{Fix}(f^N)$ then $\mathcal{S}(h, W) \geq \frac{n}{C_S}$.

4. PROOF OF COROLLARIES 1 AND 2.

In this section, we prove that any Baumslag-Solitar action with a pseudo-Anosov element has a finite orbit.

Proof of Corollary 1.

Let $\langle f, h \rangle$ be a faithful action of $BS(1, n)$ on S . We suppose that f is a C^1 diffeomorphism isotopic to identity and h is pseudo-Anosov homeomorphism. According to Theorem [GL], f admits fixed points.

Let $k \in \mathbb{N}$, f and h^k generate a faithful action of $BS(1, n^k)$ and by Theorem [GL], there is a minimal subset Λ_k of this action that is included in $Fix(f)$. As h^k is a pseudo-Anosov homeomorphism of S , any point admits an h^k -unstable manifold.

Note that for all $k \in \mathbb{N}$, h^k is C^1 on S except at Σ the set of singularities of h .

Suppose that the compact set $Fix(f)$ is disjoint from Σ , then there exists a neighbourhood W of $Fix(f)$ and of Λ_k , for any $k \in \mathbb{N}$ such that f and h^k are C^1 on W .

Moreover $\mathcal{S}(h^k, W) = \sup \{ \|\mathcal{D}_x h^k\|, x \in W \} \leq C_W \lambda^k$ for any $k \in \mathbb{N}$ (see e.g. [FM]).

Hence, we can apply Theorem 2 to f and h^k , and we obtain that either

- (1) there exists $N \in \mathbb{N}$ such that $W^u(x) \subset Fix(f^N)$, for all $x \in \Lambda_k$ or
- (2) $\mathcal{S}(h^k, W) \geq \frac{n^k}{C_S}$.

In the first case, as h^k is pseudo-Anosov homeomorphism, there exists $x \in \Lambda_k$ such that the h^k -unstable manifold of x is dense in S and therefore $Fix(f^N) = S$. Hence $f^N = Id$ and the action is not faithful.

Consequently, it holds that $\mathcal{S}(h^k, W) \geq \frac{n^k}{C_S}$, for all $k \in \mathbb{N}$.

Finally, This implies that $\lambda \geq n$.

According to [AGX], $\lambda \leq n$ and therefore $\lambda = n \in \mathbb{N}$ but this is not possible.

We conclude that $Fix(f) \cap \Sigma \neq \emptyset$, as Σ is a finite h -invariant set, we deduce that the action has a finite orbit contained in $Fix(f) \cap \Sigma$.

Proof of Corollary 2.

Indeed, if an element g of $\langle f, h \rangle = BS(1, n)$ acts on S as a pseudo-Anosov then it is the image of a strict homothety of $BS(1, n)$ (since the homotopy class of the image of a translation has finite order according to [GL]).

Eventually changing g by its inverse and conjugating by a translation element, we can suppose that g is the image of $x \mapsto n^p x$, for some positive integer p , that is $g = h^p$. Therefore $\langle f, g \rangle \simeq BS(1, n^p)$, by Corollary 1, $Fix(f^N)$ (N given by Theorem [GL]) contains a singular periodic point of g and therefore a singular periodic point of h . Hence $\langle f^N, h \rangle$ has a finite orbit and according to the Remark 1, $\langle f, h \rangle$ also has a finite orbit.

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