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# INTERVAL EXCHANGE TRANSFORMATIONS GROUPS FREE ACTIONS AND DYNAMICS OF VIRTUALLY ABELIAN GROUPS

NANCY GUELMAN AND ISABELLE LIOUSSE

ABSTRACT. In this paper, we study groups acting freely by IETs. We first note that a finitely generated group admits a free IET action if and only if it is virtually abelian. Then, we classify the free actions of non virtually cyclic groups showing that they are “conjugate” to actions in some specific subgroups  $G_n$ , namely  $G_n \simeq (\mathcal{G}_2)^n \rtimes \mathcal{S}_n$  where  $\mathcal{G}_2$  is the group of circular rotations seen as exchanges of 2 intervals and  $\mathcal{S}_n$  is the group of permutations of  $\{1, \dots, n\}$  acting by permuting the copies of  $\mathcal{G}_2$ .

We also study non free actions of virtually abelian groups and we obtain the same conclusion for any such group that contains a conjugate to a product of restricted rotations with disjoint supports and without periodic points. As a consequence, we get that the group generated by  $f \in G_n$  periodic point free and  $g \notin G_n$  is not virtually nilpotent. Moreover, we exhibit examples of finitely generated non virtually nilpotent subgroups of IETs, some of them are metabelian and others are not virtually solvable.

## 1. INTRODUCTION.

**Definition 1.1.** An **interval exchange transformation (IET)** is a bijective map  $f : [0, 1) \rightarrow [0, 1)$  defined by a finite partition of the unit interval into half-open subintervals and a reordering of these intervals by translations. If a partition has cardinality  $r$ , we say that  $f$  is an **r-IET**. We denote by  $\mathcal{G}$  the group consisting of all IETs.

Circular rotations identify with 2-IETs and an IET  $g$  whose support is a subinterval  $I = [a, b)$  is a **restricted rotation** if the orientation preserving affine map from  $I$  to  $[0, 1)$  conjugates  $g|_I$  to a 2-IET. It is denoted by  $\mathbf{R}_{\alpha, I}$  where  $\alpha \in \frac{[0, b-a]}{b-a \sim 0}$  is represented by  $g(a) - a$ .

An **affine interval exchange transformation (AIET)** is a bijective map  $f : [0, 1) \rightarrow [0, 1)$  defined by a finite partition of the unit interval into half-open subintervals such that the restriction to each of these intervals is an orientation preserving affine map.

An AIET that is a homeomorphism of  $[0, 1)$  is called **PL-homeomorphism**.

A group is said to **act freely** if the only element acting with fixed points is the trivial element, in particular a free action is faithful.

By the classification of finitely generated abelian groups, it is plain that any such group  $G$  admits a free IET action by rotations. But, Hölder’s Theorem (see e.g. Theorem 2.2.32 of [Nav11]) stating that groups that act freely by circle homeomorphisms are abelian is no longer true for free IET actions.

Indeed, it is easy to see that any finite group  $F$  has a free IET action by taking a partition in  $\#F$  subintervals of equal length labelled with the elements of  $F$  and the

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action by left multiplication in the group  $F$ . However, some infinite non abelian groups can also admit free IET actions: we claim that the following IETs  $a$  and  $b$  generate a free and minimal action of the Klein bottle group  $BS(1, -1) := \langle a, b \mid bab^{-1} = a^{-1} \rangle$  provided that  $1, \alpha, \beta_1$  and  $\beta_2$  are rationally linearly independent numbers.

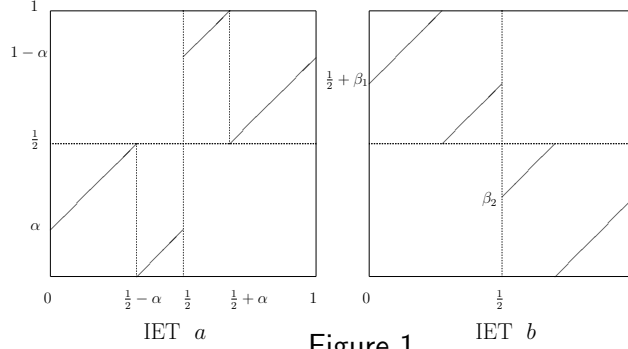


Figure 1

Note that this example is a subgroup of some very specific groups of  $\mathcal{G}$ , namely the groups  $\mathbf{G}_n$  described in the following

**Definition 1.2.** Let  $n$  be a positive integer and  $\mathbb{S}_n = [0, \frac{1}{n}] / 0 = \frac{1}{n}$  be the circle of length  $\frac{1}{n}$ . We define  $\mathbf{G}_n$  as the set of IETs on  $[0, 1)$  that preserve the partition  $[0, 1) = [0, \frac{1}{n}) \cup [\frac{1}{n}, \frac{2}{n}) \dots \cup [\frac{n-1}{n}, 1)$  and whose restrictions to the intervals  $I_i = [\frac{i-1}{n}, \frac{i}{n})$ ,  $i \in \{1, \dots, n\}$ , are IETs with only one interior discontinuity.

For  $g \in \mathbf{G}_n$ , we define  $\sigma_g$  as the element of  $\mathcal{S}_n$  given by  $\sigma_g(i) = j$  if  $g(I_i) = I_j$ .

It follows that, for  $x \in I_i$ , one has  $g(x) = R_{\alpha_i, I_i}(x) + \frac{\sigma_g(i) - i}{n}$ .

We define the **rotation vector** of  $g$  by  $\alpha_g = (\alpha_1, \dots, \alpha_n) = (\alpha_1(g), \dots, \alpha_n(g)) \in \mathbb{S}_n^n$  and we denote  $\mathbf{g} = (\alpha_g, \sigma_g)$ .

The group  $\mathbf{G}_1$  is the group of circular rotations  $\mathbf{R}_\alpha$  ( $\alpha \in \mathbb{S}^1$ ), regarded as 2-IETs.

**Examples 1.** The IET  $a$  and  $b$ , represented above by their graphs, are expressed as:

$$a, b \in \mathbf{G}_2, \quad \left\{ \begin{array}{l} \alpha_a = (\alpha, -\alpha) \\ \sigma_a = \text{Id} \end{array} \right\} \quad \text{and} \quad \left\{ \begin{array}{l} \alpha_b = (\beta_1, \beta_2) \\ \sigma_b \text{ is the transposition } (1, 2) \end{array} \right\}.$$

**Remark 1.3.** A straightforward consequence of this definition is that  $\mathbf{G}_n$  is a group and composing two elements  $f$  and  $g$  of  $\mathbf{G}_n$ , we get  $\left\{ \begin{array}{l} \alpha_{f \circ g} = \sigma_g(\alpha_f) + \alpha_g \\ \sigma_{f \circ g} = \sigma_f \circ \sigma_g \end{array} \right\}$ , where  $\sigma_g$  acts on the vector  $\alpha_f$  by permuting its coordinates. Therefore, the map  $(\alpha, \sigma) : \mathbf{G}_n \rightarrow (\mathbb{S}_n)^n \rtimes \mathcal{S}_n$  is an isomorphism. In particular, the group  $\mathbf{G}_n$  is virtually abelian.

Note that  $\mathbf{G}_n$  can be seen as the isometry group of the disjoint union of  $n$  circles of the same length  $1/n$  and adopting this point of view, it follows from Dahmani, Fujiwara and Guirardel (Proposition 1.2 of [DFG20]) that any finitely generated virtually abelian group is isomorphic to a subgroup of some  $\mathbf{G}_n$ .

In Section 2, we will specify the related construction of [DFG20] in order to provide free actions, this leads to

**Proposition 1.4.** (The Dahmani-Fujiwara-Guirardel Proposition 1.2 revisited) *A finitely generated group admits a free IET action if and only if it is virtually abelian.*

The main result of this paper establishes that this revisited construction is “basically” the only one that provides free IET actions of non virtually cyclic groups. It is related to the dynamical Hölder’s Theorem: “free actions on the circle are topologically semi-conjugate to actions of rotations groups” (see e.g. comments following Theorem 2.2.32 in [Nav11]), which suggests that it makes sense to give a dynamical classification of free actions by elements of  $\mathcal{G}$ . We will do this classification up to conjugacy by maps that are not required to be in  $\mathcal{G}$ , however, as noted by the referees, our conclusion can be reformulated in terms of  $\mathcal{G}$ -conjugation (see Theorem 1’ and its preliminary comments).

Actually our conjugating maps are given by

**Definition 1.5.** An element  $f \in \mathcal{G}$  is **PL  $\circ$   $\mathcal{G}$ -conjugate to  $F \in \mathbf{G}_n$** , if  $f$  is conjugate to  $F$  through a map  $P \circ E$ , where  $E \in \mathcal{G}$  and  $P$  is a PL-homeomorphism such that  $P^{-1}$  is affine on  $[\frac{k-1}{n}, \frac{k}{n})$  for any  $k \in \{1, \dots, n\}$ .

**Theorem 1.** *The image in  $\mathcal{G}$  of any free action of a finitely generated non virtually cyclic group is PL  $\circ$   $\mathcal{G}$ -conjugate to a subgroup of some  $\mathbf{G}_n$ .*

Roughly speaking, being PL  $\circ$   $\mathcal{G}$ -conjugate to a subgroup of some  $\mathbf{G}_n$  can be reformulated as being  $\mathcal{G}$ -conjugate to a subgroup of products of restricted rotations on finitely many disjoint sub-intervals modulo a permutation of them. More precisely,

**Theorem 1’.** *Any free IET-action of a finitely generated non virtually cyclic group is  $\mathcal{G}$ -conjugate to a subgroup of the semi-direct product  $G(P) := R(P) \rtimes \mathcal{S}(P) < \mathcal{G}$  for some partition  $P$  of  $[0, 1)$  in half-open intervals, where  $R(P)$  denotes the group of all products of restricted rotations supported in intervals of  $P$  and  $\mathcal{S}(P)$  the subgroup of the symmetric group on  $P$  consisting of elements that only exchange intervals of the same size.*

The first step in the proof of Theorem 1 is

**Proposition 1.6.** *Any free action of  $\mathbb{Z}^2$  in  $\mathcal{G}$  consists of periodic point free maps having an iterate that is conjugate in  $\mathcal{G}$  to a product of restricted rotations with disjoint supports. In particular its image contains a conjugate to a product of restricted rotations with disjoint supports and without periodic points.*

It is not plain that if all elements of a finitely generated subgroup of  $\mathcal{G}$  have an iterate that is conjugate in  $\mathcal{G}$  to a product of restricted rotations with disjoint supports, then they are simultaneously conjugate to their models. However, dealing with virtually abelian groups, the situation turns out to be much more rigid and this conclusion can be proved only requiring the existence of one conjugate in  $\mathcal{G}$  to a product of restricted rotations with disjoint supports and without periodic points.

The proof of Theorem 1 does not strictly require actions to be free, but rather the presence of specific elements that occur when considering sufficiently large free actions. Moreover, the finite generation condition is involved only when applying Proposition 1.4.

More precisely, the second step is

**Theorem 2.** *Let  $G$  be a virtually abelian subgroup of  $\mathcal{G}$ . If  $G$  contains an element that is conjugate in  $\mathcal{G}$  to a product of restricted rotations with disjoint supports and without periodic points, then  $G$  is PL  $\circ$   $\mathcal{G}$ -conjugate to a subgroup of some  $\mathbf{G}_n$ .*

Some extra dynamical assumptions provide stronger rigidity results:

**Definition 1.7.**

- An IET  $f$  is **totally minimal** if for all  $p \in \mathbb{Z}^*$ , the map  $f^p$  is minimal. More generally:
- An action by elements of  $\mathcal{G}$  of a group  $G$  is **totally minimal** if any finite index subgroup of  $G$  acts minimally on  $[0, 1)$ .

**Remark.** According to [Kea75], an IET that satisfies the Keane I.D.O.C condition is totally minimal.

**Corollary 1.** Let  $G$  be virtually abelian group containing a copy of  $\mathbb{Z}^2$ . Then any faithful totally minimal action of  $G$  in  $\mathcal{G}$  is conjugate in  $\mathcal{G}$  to an action by rotations.

By Schreier Lemma and the classification of finitely generated abelian groups, any finitely generated virtually abelian group is either virtually cyclic or it contains a copy of  $\mathbb{Z}^2$ . Combining this with the theorem of [DFG20] stating that virtually polycyclic IET subgroups are virtually abelian, we get

**Corollary 2.** Let  $G$  be a finitely generated subgroup of  $\mathcal{G}$  that contains a totally minimal IET non conjugate to a rotation then  $G$  is either virtually cyclic or not virtually polycyclic.

Using Theorem 2 and its ingredients of proof, we can exhibit explicit examples of non virtually polycyclic subgroups of  $\mathcal{G}$ . Other examples were known as the lamplighter groups considered in [DFG20], however the situations, we present here, are easy to achieve.

**Corollary 3.** The group  $G$  generated by the following  $f$  and  $g$  is not virtually polycyclic.

- (1)  $f$  is an irrational rotation and  $g$  is not a rotation.
- (2)  $f \in \mathbf{G}_n$  is periodic point free and  $g \notin \mathbf{G}_n$ .

As a consequence of Theorem 2 of [DFG20], for finitely generated IET subgroups, the properties of being virtually abelian, virtually nilpotent or virtually polycyclic are equivalent. This justifies why the groups involved in Corollaries 2 and 3 are described in the abstract as non virtually nilpotent.

Item (1) is the particular case  $n = 1$  of Item (2) and it is then a corollary of Gromov's polynomial growth theorem [Gro81] and Proposition 5.9 of [JMBMdLS18] that claims that a finitely generated subgroup of IET that contains an irrational rotation is either a subgroup of  $\mathbf{G}_1$  or has exponential growth.

In addition, in the last section, we provide examples of finitely generated non virtually polycyclic subgroups of  $\mathcal{G}$  that have extra properties as being metabelian or non virtually solvable. In particular, these subgroups are generated by rotations and torsion elements as the groups considered by Boshernitzan in [Bos16].

This criterium applies to the group  $\langle r, s \rangle$  constructed in Theorem 8.1 of [DFG13] by taking  $s \in \mathbf{G}_3$  (obviously  $r \notin \mathbf{G}_3$ ) and also to the lamplighter groups  $A \wr \mathbb{Z}$  of Proposition 4.1 of [DFG20], since the infinite order generator belongs to  $\mathbf{G}_n$  and the support of the finite order elements can be chosen in such a way that these maps do not belong to  $\mathbf{G}_n$ .

Another motivation for studying the dynamics of the virtually abelian subgroups of  $\mathcal{G}$  comes from the Klein bottle group  $BS(1, -1)$  and its connections with reversible maps: O'Farrell and Short ([OS15]) have pointed out the importance of reversibility in dynamical systems and group theory and they have raised the following question: given a group  $G$ , are all reversible elements of  $G$  reversible by an involution? In [GL19], we study reversible IETs and free actions of  $BS(1, -1)$  play a key role for O'Farrell and Short question. In

particular, Theorem 1 applies to such actions (since the subgroup  $\langle a, b^2 \rangle$  is isomorphic to  $\mathbb{Z}^2$  and of index 2) leading to

**Corollary 4.** *Any free action of  $BS(1, -1)$  by elements of  $\mathcal{G}$  is  $PL \circ \mathcal{G}$ -conjugate to a free action of  $BS(1, -1)$  by elements of some  $\mathbf{G}_n$ .*

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## 2. ALGEBRAIC DESCRIPTION OF FINITELY GENERATED GROUPS ACTING FREELY BY ELEMENTS OF $\mathcal{G}$ : PROOF OF PROPOSITION 1.4

*Proof.* Since translations commute, the orbit of any point under a finitely generated subgroup  $H$  of  $\mathcal{G}$  has polynomial growth. Therefore, if  $H$  acts freely then it has polynomial growth so it is virtually nilpotent by Gromov [Gro81]. According to Novak [Nov09], IET groups do not contain distorted copy of  $\mathbb{Z}$  so, by standard arguments,  $H$  is virtually abelian (this can also be deduced from Theorem 2 of [DFG20] since finitely generated nilpotent groups are polycyclic).

Let  $G$  be a finitely generated virtually abelian group. By the classification of finitely generated abelian groups, there exists  $H \triangleleft G$  isomorphic to some  $\mathbb{Z}^m$  such that  $F = G/H$  is finite. In addition,  $H$  can be represented as a group of circle rotations  $A$  seen as 2-IETs of  $[0, 1)$ . Namely,  $A = \langle R_{\alpha_1}, \dots, R_{\alpha_m} \rangle$  and  $1, \alpha_1, \dots, \alpha_m$  are  $\mathbb{Q}$  linearly independent.

According to Krasner-Kaloujnine (see [KK51])  $G$  embeds in  $H^F \rtimes F$ , where  $F$  acts on  $H^F$  by permuting coordinates. Thus, it suffices to show that  $A^F \rtimes F$  acts freely by elements of some  $\mathbf{G}_n$ .

To any  $(a_s)_{s \in F} \in A^F$  and  $\sigma \in F$ , we associate a map

$$E((a_s), \sigma) : \begin{cases} [0, 1) \times F \rightarrow [0, 1) \times F \\ (x, s) \mapsto (a_s(x), \sigma s) \end{cases}$$

Noticing that  $E((a_s), \sigma) \circ E((a'_s), \sigma') = E((a_{\sigma' s} \circ a'_s), \sigma \sigma')$ , we get that  $E$  induces an injective morphism from  $A^F \rtimes F$  to the group  $\mathbf{MG}_2([0, 1) \times F) := \{E((a_s), \sigma), (a_s) \in (\mathcal{G}_2)^F, \sigma \in F\}$ .

In addition, its image acts freely on  $[0, 1) \times F$  because  $1, \alpha_1, \dots, \alpha_m$  are  $\mathbb{Q}$  linearly independent. Finally,  $\mathbf{MG}_2([0, 1) \times F)$  embeds in  $G_{\#F}$  by subdividing  $[0, 1)$  in  $\#F$  intervals of same length.  $\square$

## 3. DYNAMICAL PROPERTIES OF IETs AND CENTRALIZERS

### 3.1. Dynamical properties of IETs.

#### Definition 3.1.

- The **break point set** of an IET  $f$  is obtained by adding the initial point 0 to the discontinuity set of  $f$ , it is denoted by  $\mathbf{BP}(f)$ . The set consisting of the  $f$ -orbits of points in  $\mathbf{BP}(f)$  is denoted by  $\mathbf{BP}_\infty(f)$ . The **translation set** of  $g$  is  $\{g(x) - x, x \in [0, 1)\}$ .

• The **break point set** of a finitely generated subgroup  $G$  of  $\mathcal{G}$  is  $\mathbf{BP}(G) := \bigcup_{g \in G} \mathbf{BP}(g)$  and its **translation set** is  $\{g(x) - x, g \in G, x \in [0, 1)\}$ .

We let the reader check the following

**Properties 3.2.** *Let  $f, g$  in  $\mathcal{G}$  and  $n \in \mathbb{N}$ .*

- (a)  $\mathbf{BP}(f \circ g) \subseteq \mathbf{BP}(g) \cup g^{-1}(\mathbf{BP}(f))$ ,
- (b)  $\mathbf{BP}(f^{-1}) = f(\mathbf{BP}(f))$  and
- (c)  $\mathbf{BP}(f^n) \subseteq \mathbf{BP}(f) \cup f^{-1}(\mathbf{BP}(f)) \cup \dots \cup f^{-n+1}(\mathbf{BP}(f))$ .

**Remark 3.3.** By these properties,  $\#\mathbf{BP}(f^n)$  is bounded above by  $\#\mathbf{BP}(f) \times |n|$ . However, it is possible for  $\#\mathbf{BP}(f^n)$  to be bounded independently of  $n$ , for IETs of finite order but also for minimal IETs as irrational rotations and then also for all elements of  $\mathbf{G}_n$ , in this case we say that  $f$  is of **bounded break point type**.

**Definition 3.4.** Let  $f \in \mathcal{G}$ .

- Let  $p \in \mathbb{N}^*$ , we denote by  $\text{Per}_p(f)$  the set of points  $x$  such that,  $\mathcal{O}_f(x)$ , the  $f$ -orbit of  $x$  has cardinality  $p$ . The fix point set of  $f$  is  $\text{Fix}(f) = \text{Per}_1(f)$ .
- Let  $V$  be a subset of  $[0, 1)$ , we say that  $V$  is **of type  $\mathcal{M}$**  if it is a non empty finite union of intervals each of the form  $[b, c)$ , with  $b, c$  in  $\mathbf{BP}_\infty(f)$ . A type  $\mathcal{M}$  and  $f$ -invariant set that is minimal for the inclusion among non empty, type  $\mathcal{M}$  and  $f$ -invariant subsets of  $[0, 1)$  is called an  **$f$ -component**.

The well-known decomposition into minimal and periodic components of an IET was first given for measured surface flows by Mayer in 1943 ([May43]) and restated for IETs by Keane ([Kea75]) and Arnoux ([Arn81]). Here, we detail Arnoux formulation (see [Arn81] Proposition p. 20):

**Theorem.** *The interval  $[0, 1)$  can be decomposed as  $[0, 1) = P_1 \cup \dots \cup P_l \cup M_1 \dots \cup M_m$ , where*

- $P_i$  is an  $f$ -periodic component:  $P_i$  is the  $f$ -orbit of an interval  $[b, c)$  with  $b, c$  in  $\mathbf{BP}_\infty(f)$  and all iterates  $f^k$  of  $f$  are continuous on  $[b, c)$ . In particular points in  $P_i$  are periodic of the same period.
- $M_j$  is an  $f$ -minimal component: for any  $x \in M_j$ , the orbit  $\mathcal{O}_f(x)$  is dense in  $M_j$ .

For convenience, the  $f$ -component containing  $x \in [0, 1)$  will be denoted  $\mathbf{M}_f(x)$ .

According to Remark 3.3, it makes sense to give the following

**Definition 3.5.** We define the growth rate of the number of discontinuities for the iterates of an IET  $f$  on the  $f$ -orbit through a given point  $x$  by

$$\mathcal{N}_x(f) = \lim_{n \rightarrow +\infty} \frac{\#\{\mathbf{BP}(f^n) \cap \mathcal{O}_f(x)\}}{n}.$$

**Properties 3.6.** *Let  $f, g$  in  $\mathcal{G}$  and  $x \in [0, 1)$ .*

- (1) If  $\mathbf{BP}(f) \cap \mathcal{O}_f(x) = \emptyset$  then  $\mathcal{N}_x(f) = 0$ .
- (2)  $\mathcal{N}_{g(x)}(g \circ f \circ g^{-1}) = \mathcal{N}_x(f)$ .

*Proof.*

Item (1) follows from Item (c) of Properties 3.2.

Item (2). Let  $N \in \mathbb{N}$ , we write  $\mathbf{A} \stackrel{N}{\subset} \mathbf{B}$  if  $A \subset B \cup F$  where  $\#F \leq N$ .

Let  $C = 2\#\mathbf{BP}(g)$  and  $f_g = g \circ f \circ g^{-1}$ , we have  $\mathcal{O}_{f_g}(g(x)) = g(\mathcal{O}_f(x))$ .

It follows from Item (a) of Properties 3.2 that:

- $\text{BP}(f_g^n) \subseteq \text{BP}(g^{-1}) \cup g(\text{BP}(f^n)) \cup g \circ f^{-n}(\text{BP}(g)) \underset{C}{\subseteq} g(\text{BP}(f^n))$  and
- $\text{BP}(f^n) = \text{BP}(g^{-1} \circ f_g^n \circ g) \subseteq \text{BP}(g) \cup g^{-1}(\text{BP}(f_g^n)) \cup g^{-1} \circ f_g^{-n}(\text{BP}(g^{-1})) \underset{C}{\subseteq} g^{-1}(\text{BP}(f_g^n))$ .

Therefore  $\text{BP}(f_g^n) \cap \mathcal{O}_{f_g}(g(x)) \underset{C}{\subseteq} g(\text{BP}(f^n) \cap g^{-1}(\mathcal{O}_{f_g}(g(x)))) = g(\text{BP}(f^n) \cap \mathcal{O}_f(x))$   
and  $\text{BP}(f^n) \cap \mathcal{O}_f(x) \underset{C}{\subseteq} g^{-1}(\text{BP}(f_g^n) \cap g(\mathcal{O}_f(x))) = g^{-1}(\text{BP}(f_g^n) \cap \mathcal{O}_{f_g}(g(x)))$ .

Finally, we get

$$\#(\text{BP}(f^n) \cap \mathcal{O}_f(x)) - C \leq \#(\text{BP}(f_g^n) \cap \mathcal{O}_{f_g}(g(x))) \leq \#(\text{BP}(f^n) \cap \mathcal{O}_f(x)) + C.$$

We conclude by dividing by  $n$  and taking the limit.  $\square$

In addition,  $\mathcal{N}_x(f)$  provides a characterization of the bounded break point type. More precisely, from [Nov09], we deduce

**Theorem 3.7.** *Adapted version of Theorems 1.1 and 1.2 of [Nov09].*

Let  $f \in \mathcal{G}$ , the following assertions are equivalent

- (1) For any  $x \in [0, 1)$ , we have  $\mathcal{N}_x(f) = 0$ .
- (2) The map  $f$  is of bounded break point type.
- (3) There exists a positive integer  $p$  such that  $f^p$  is conjugate in  $\mathcal{G}$  to a product of restricted rotations of pairwise disjoint supports.
- (4) There exist two positive integers  $p$  and  $n$ , an IET  $E$  and a PL-homeomorphism  $P : [0, 1) \rightarrow [0, 1)$  such that  $\Phi = (P \circ E) \circ f^p \circ (P \circ E)^{-1} \in \mathbf{G}_n$ ,  $\sigma_\Phi = \text{Id}$  and the map  $P^{-1}$  is affine on each  $I_i = [\frac{i-1}{n}, \frac{i}{n})$ .

*Proof.*

Since  $\text{BP}(f^n) = \bigcup_{a \in \text{BP}(f)} \text{BP}(f^n) \cap \mathcal{O}_f(a)$ , the implication (1)  $\implies$  (2) is a direct consequence of Theorem 1.1 of [Nov09] stating that either  $\#\text{BP}(f^n)$  is bounded or  $\#\text{BP}(f^n)$  has linear growth.

The implication (2)  $\implies$  (3) is exactly Theorem 1.2 of [Nov09].

We prove (3)  $\implies$  (4). Let  $E$  be an IET that conjugates  $f^p$  to  $F$ , a product of restricted rotations of pairwise disjoint supports. We decompose  $I$  as a disjoint union of consecutive half open intervals  $J_i$ ,  $i = 1, \dots, n$ , where  $J_i$  is either the support of a restricted rotation in  $F$  or a connected component of  $\text{Fix}(F)$ . We define  $P$  as the PL-homeomorphism such that  $P(J_i) = I_i$  and  $P|_{J_i}$  is affine. It is easily seen that  $\Phi = P \circ F \circ P^{-1} \in \mathbf{G}_n$  and  $\sigma_\Phi = \text{Id}$ .

It remains to prove (4)  $\implies$  (1). As  $\Phi \in \mathbf{G}_n$ , it holds that  $\mathcal{N}_x(\Phi) = 0$  for any  $x \in [0, 1)$ , then  $\mathcal{N}_x(P^{-1} \circ \Phi \circ P) = 0$  since  $P$  is a homeomorphism. Therefore Item (2) of Properties 3.6 implies that  $\mathcal{N}_x(f^p) = 0$  for any  $x \in [0, 1)$ . It follows that  $\mathcal{N}_x(f) = 0$  for any  $x \in [0, 1)$ .

Indeed, let  $x \in [0, 1)$  and  $n \in \mathbb{N}^*$ , we have  $\mathcal{O}_f(x) = \bigcup_{j=0}^{p-1} \mathcal{O}_{f^p}(f^j(x))$  and then

$$\frac{\#(\text{BP}(f^{pn}) \cap \mathcal{O}_f(x))}{pn} \leq \frac{1}{p} \sum_{j=0}^{p-1} \frac{\#(\text{BP}(f^{pn}) \cap \mathcal{O}_{f^p}(f^j(x)))}{n}$$



Taking, the limit when  $n$  goes to infinity, we get that  $\mathcal{N}_x(f) \leq \frac{1}{p} \sum_{j=0}^{p-1} \mathcal{N}_{f^j(x)}(f^p)$ .

Finally, by Item (2) of Properties 3.6, it holds that  $\mathcal{N}_x(f^p) = \mathcal{N}_{f^j(x)}(f^p)$  for any integer  $j$  and then  $\mathcal{N}_x(f) \leq \mathcal{N}_x(f^p) = 0$ .  $\square$

### 3.2. Centralizers.

#### 3.2.1. Generalities.

Let  $G$  be a group and  $f \in G$ . It is well-known that the **centralizer** of  $f$  in  $G$ ,  $\mathbf{C}(f) := \{h \in G : h \circ f = f \circ h\}$ , is a subgroup of  $G$ .

**Lemma 3.8.** *Let  $f, h$  be bijections of a set  $X$  and  $h \in \mathbf{C}(f)$ , then:*

- For any  $x \in X$ ,  $h(\mathcal{O}_f(x)) = \mathcal{O}_f(h(x))$ .
- For any  $p \in \mathbb{N}^*$ , the set of  $f$ -periodic points with exact period  $p$ ,  $\text{Per}_p(f)$ , is  $h$ -invariant.

*Proof.*

• The first item is a direct consequence of the fact that for any  $x \in X$ , one has  $h(f^n(x)) = f^n(h(x))$ .

• Let  $p \in \mathbb{N}^*$ ,  $x \in \text{Per}_p(f)$  means that  $\#\mathcal{O}_f(x) = p$ . By the first item, we get that  $\#\mathcal{O}_f(h(x)) = p$  and therefore  $h(x) \in \text{Per}_p(f)$ .  $\square$

**Lemma 3.9.** *Let  $f, h \in \mathcal{G}$ ,  $h \in \mathbf{C}(f)$  and  $x$  be a non  $f$ -periodic point.*

*Then  $h(x)$  is not  $f$ -periodic and  $h(M_f(x)) = M_f(h(x))$ . In addition, there exists a positive integer  $r$  such that  $h^r(M_f(x)) = M_f(x)$ .*

*Proof.* As  $x$  is not  $f$ -periodic, the orbit  $\mathcal{O}_f(x)$  is dense in  $M_f(x)$ . By Lemma 3.8, the point  $h(x)$  is not  $f$ -periodic and  $h(\mathcal{O}_f(x)) = \mathcal{O}_f(h(x))$ , taking closure we get  $h(M_f(x)) = M_f(h(x))$  even if  $h$  is not continuous.

Indeed, the dynamics of the group  $G$  generated by  $f$  and  $h$  can be lifted, by a Denjoy blow-up, to that of a homeomorphisms group of  $X = [0, 1) \sqcup \text{BP}_\infty^-(G)$ , where  $\text{BP}_\infty^-(G)$  is an abstract copy of  $\text{BP}_\infty(G)$ . To any  $x \in \text{BP}_\infty(G)$  correspond two points of  $X$ :  $x^+ \in [0, 1)$  and  $x^- \in \text{BP}_\infty^-(G)$ . The topology on  $X$  is given by the total order defined on  $X$  by taking the usual order on  $\text{BP}_\infty^-(G)$  and  $[0, 1)$  and setting that for  $x^- \in \text{BP}_\infty^-(G)$  and  $y \in [0, 1)$ :  $x^- < y \iff x^+ \leq y$ . For more details see e.g [Arn81] or [Cor20] §3.2.

In addition, the finiteness of the number of  $f$ -minimal components implies that there exist two distinct integers  $s$  and  $t$  such that  $h^s(M_f(x)) = h^t(M_f(x))$  therefore  $h^{t-s}(M_f(x)) = M_f(x)$ .  $\square$

#### 3.2.2. Centralizer for IETs of bounded break point type.

We begin by recalling

**Properties 3.10.** ([Nov09], Lemma 5.1 and its proof.)

- (1) Two irrational rotations  $R_\alpha$  and  $R_\beta$ , with  $\alpha \neq \beta \pmod{1}$ , are not conjugate in  $\mathcal{G}$ .
- (2) The centralizer in  $\mathcal{G}$  of an irrational rotation is the rotation group  $\mathcal{G}_2$ .

This extends to

**Lemma 3.11.** *Let  $F \in \mathbf{G}_n$  without periodic points then  $\mathbf{C}(F) \subset \mathbf{G}_n$ .*

*Proof.* Let  $F \in \mathbf{G}_n$  without periodic points, noting that  $C(F) \subset C(F^n)$  holds for all  $n \in \mathbb{Z}$  and after replacing  $F$  by some iterate  $F^p$ , it may be assumed that  $\sigma_F = \text{Id}$  and then the intervals  $I_i = [\frac{i-1}{n}, \frac{i}{n})$  are the minimal components of  $F$ , since  $F$  has no periodic points.

Let  $H \in C(F)$ . By Lemma 3.9, there exists a permutation  $\gamma \in \mathcal{S}_n$  such that  $H(I_i) = I_{\gamma(i)}$  and  $H$  conjugates the irrational rotation  $F|_{I_i}$  to the irrational rotation  $F|_{I_{\gamma(i)}}$ . Thus, Item (1) of Properties 3.10 implies that  $\alpha_{\gamma(i)}(F) = \alpha_i(F)$ .

Next, we consider the map  $K$  of  $\mathbf{G}_n$  defined by  $\alpha(K) = 0$  and  $\sigma_K = \gamma$ . It is easily seen that  $K \in C(F)$ , therefore  $K^{-1} \circ H$  is an IET that commutes with  $F$  and fixes each  $I_i$ . By Item (2) of Properties 3.10, one has  $(K^{-1} \circ H)|_{I_i} = R_{\beta_i}$  for any  $i$  meaning that  $K^{-1} \circ H \in \mathbf{G}_n$ , so does  $H$ .  $\square$

More generally, we have

**Proposition 3.12.** *Let  $f$  be a periodic-point free IET that is conjugate to  $F \in \mathbf{G}_n$  by a PL-homeomorphism  $P$  such that  $P|_{I_i}^{-1}$  is an orientation preserving affine map. Then the map  $P$  conjugates  $C(f)$  to a subgroup of  $\mathbf{G}_n$ .*

*Proof.* As in the proof of Lemma 3.11, the map  $f$  has no periodic points then after replacing  $f$  by some iterate, we can suppose that the intervals  $J_i := P^{-1}(I_i)$  are the minimal components of  $f$ . Let  $g \in C(f)$ , according to Lemma 3.9, there exists a permutation  $\gamma \in \mathcal{S}_n$  such that  $g(J_i) = J_{\gamma(i)}$ . In particular  $J_i$  and  $J_{\gamma(i)}$  have same lengths and then the restrictions of  $P$  to  $J_i$  and to  $J_{\gamma(i)}$  are affine with same slopes.

This implies that the a priori AIET  $G = P \circ g \circ P^{-1}$ , which sends  $P(J_i) = I_i$  to  $P(J_{\gamma(i)}) = I_{\gamma(i)}$  for any  $i$ , is actually an IET. Finally  $G$  is an IET that commutes with  $F$  (i.e.  $G \in C(F)$ ), therefore, by Lemma 3.11,  $G$  belongs to  $\mathbf{G}_n$ .  $\square$

#### 4. FREE ACTIONS OF $\mathbb{Z}^2$ BY IET

The aim of this section is to prove Proposition 1.6. By Theorem 3.7, it can be reformulated as: “Any free action of  $\mathbb{Z}^2$  in  $\mathcal{G}$  consists of IETs of bounded break point type.”

This proof, inspired by the description given by Minakawa for the free actions of  $\mathbb{Z}^2$  by circle PL-homeomorphisms (see [Min97]), uses the properties of the rates  $\mathcal{N}_x(f)$ .

As the action  $\langle f, h \rangle$  of  $\mathbb{Z}^2$  is free, for every  $x \in I$  and  $k, q$  distinct integers, one has:

$$\mathcal{O}_f(h^k(x)) \cap \mathcal{O}_f(h^q(x)) = \emptyset.$$

Therefore, as  $\#BP(f)$  is finite, there exists a positive integer  $N_0$  such that:

$$\forall n \geq N_0, \quad \mathcal{O}_f(h^n(x)) \cap BP(f) = \emptyset.$$

Using Item (1) of Properties 3.6, one get  $\mathcal{N}_{h^n(x)}(f) = 0$  for  $n \geq N_0$  and, by Item (2) of Properties 3.6,

$$\mathcal{N}_{h^n(x)}(h^n \circ f \circ h^{-n}) = \mathcal{N}_x(f).$$

By commutativity, it holds that  $\mathcal{N}_{h^n(x)}(h^n \circ f \circ h^{-n}) = \mathcal{N}_{h^n(x)}(f)$ .

Combining these properties, we get  $\mathcal{N}_x(f) = 0$  for any  $x \in I$ . Finally, by Theorem 3.7,  $f$  has bounded break point type.

## 5. PROOF OF THEOREM 2.

After a possible conjugation by an IET, it suffices to prove

**Proposition 5.1.** *Let  $G$  be a non trivial virtually abelian subgroup of  $\mathcal{G}$ . Suppose that there exists  $f_1 \in G$  which is a product of restricted rotations with disjoint supports and without periodic points then  $G$  is conjugate to a subgroup of some  $\mathbf{G}_n$  by the canonical PL-homeomorphism that conjugates  $f_1$  to an element of  $\mathbf{G}_n$ .*

## 5.1. Proof of Proposition 5.1.

Let  $G < \mathcal{G}$  and  $H \triangleleft G$  be an abelian subgroup of  $G$  with finite index,  $p$ .

**Claim 5.2.** *To any  $f \in G$  we can associate a positive integer  $p_f$  in order that the maps  $f^{p_f}$  are pairwise commuting.*

Indeed, let  $f \in G$ , since the set  $G/H$  is finite, there exist integers  $0 \leq s < t$  such that  $f^s$  and  $f^t$  belong to a same class modulo  $H$ , that is  $f^{t-s} \in H$ .

So taking  $p_f = t - s$  leads to the required conclusion.

Consider  $P$  the PL-homeomorphism associated to  $f_1$  as involved for proving (3)  $\implies$  (4) of Theorem 3.7. In particular, the  $f_1$ -components are the intervals  $J_i = P^{-1}(I_i)$  for  $i = 1, \dots, n$ ; they are minimal and  $P|_{J_i}$  has constant slope for any  $i \in \{1, \dots, n\}$ .

Let  $f \in G$  and  $\phi = f \circ f_1 \circ f^{-1}$ . By Claim 5.2, there exist  $p_{f_1}, p_\phi \in \mathbb{N}^*$  such that  $f_1^{p_{f_1}}$  and  $\phi^{p_\phi}$  commute. Thus, Proposition 3.12 applies to  $\phi^{p_\phi} \in \mathcal{C}(f_1^{p_{f_1}})$  and implies that  $\phi^{p_\phi}$  is conjugate by  $P$  to an element of  $\mathbf{G}_n$ , namely  $P \circ \phi^{p_\phi} \circ P^{-1} \in \mathbf{G}_n$ .

Moreover, as  $\phi^{p_\phi}$  is periodic points free, eventually replacing  $p_\phi$  by a multiple, we may assume that the components of  $\phi^{p_\phi}$  are the intervals  $J_i$  and they are minimal.

Therefore, as  $f$  conjugates  $\phi^{p_\phi}$  and  $f_1^{p_{f_1}}$ , the map  $f$  permutes the  $J_i$ . This implies that the a priori AIET  $P \circ f \circ P^{-1}$  is an IET. Indeed, for any  $i = 1, \dots, n$ , it holds that

- the map  $P^{-1}$  sends  $I_i$  to  $J_i$  with constant slope  $\frac{|J_i|}{|I_i|}$ ,
- the IET  $f$  sends  $J_i$  to some  $J_k$  of same length, with constant slope 1 and
- the map  $P$  sends  $J_k$  to  $I_k$  with constant slope  $\frac{|I_k|}{|J_k|} = \frac{|I_i|}{|J_i|}$ .

Now, it remains to prove that  $P \circ f \circ P^{-1} \in \mathbf{G}_n$ .

**Lemma 5.3.** *Let  $n$  be a positive integer and  $T \in \mathbf{G}_n$  be a periodic point free IET. If  $S \in \mathcal{G}$  is such that  $T' = STS^{-1} \in \mathbf{G}_n$  then  $S \in \mathbf{G}_n$ .*

*Proof.*

• If  $n = 1$  then  $T$  and  $T'$  are irrational rotations and  $T'S = ST$ . Identifying  $[0, 1)$  with  $\mathbb{S}^1$ , the maps  $T$  and  $T'$  become continuous and computing break point sets, we get

$$\text{BP}(S) = \text{BP}(T'S) = \text{BP}(ST) = T^{-1}(\text{BP}(S)).$$

Thus  $\text{BP}(S)$  is a finite  $T$ -invariant set so the minimality of  $T$  implies that it is empty and then  $S$  is a rotation.

• If  $n \geq 2$  then eventually changing  $T$  for some iterate, we can suppose that  $\sigma_T$  and  $\sigma_{STS^{-1}}$  are trivial.

- If  $S$  preserves each  $I_j$ , we apply the case  $n = 1$  to the restrictions to  $I_j$  of  $T$  and  $S$ . We conclude that  $S|_{I_j}$  is a rotation and then  $S \in \mathbf{G}_n$ .

- If  $S$  does not preserve each  $I_j$ . As  $S$  conjugates two periodic point free maps in  $\mathbf{G}_n$  there exists a permutation  $\gamma$  such that  $S(I_j) = I_{\gamma(j)}$ . We consider the element  $K \in \mathbf{G}_n$  defined by  $\sigma_K = \gamma$  and  $\alpha_K = 0$ . We have  $K^{-1}S \circ T \circ S^{-1}K \in \mathbf{G}_n$  and  $K^{-1}S$  preserves each  $I_j$ . According to the previous case,  $K^{-1}S \in \mathbf{G}_n$  and therefore  $S \in \mathbf{G}_n$ .  $\square$

We conclude by applying Lemma 5.3 to  $T = P \circ f_1^{p_\phi} \circ P^{-1} \in \mathbf{G}_n$  and  $S = P \circ f \circ P^{-1}$ .

## 6. PROOFS OF THEOREM 1 AND COROLLARIES 1 AND 2.

**Proof of Theorem 1.** Let  $G$  be a finitely generated group acting freely by IET and  $\Gamma < \mathcal{G}$  be the image of a copy of  $\mathbb{Z}^2$  contained in  $G$ . According to Proposition 1.4,  $G$  is virtually abelian. It is plain that  $\Gamma$  also acts freely, then Proposition 1.6 implies that  $\Gamma$  contains IETs that are conjugate in  $\mathcal{G}$  to products of restricted rotations with disjoint supports and without periodic points. Therefore,  $G$  satisfies the hypothesis of Theorem 2 and then its conclusion.

**Proof of Corollary 1.** Let  $G$  be a virtually abelian subgroup of  $\mathcal{G}$  containing a copy  $\langle a, b \rangle$  of  $\mathbb{Z}^2$  and whose action is totally minimal.

Let  $K$  be a finite index abelian normal subgroup of  $G$ . Claim 5.2 implies that there exist integers  $p$  and  $q$  such that  $a^p$  and  $b^q$  generate a copy  $Z$  of  $\mathbb{Z}^2$  in  $K$ . We claim that that  $Z$  acts freely.

Indeed, we argue by contradiction supposing that there exist  $h \in Z \setminus \{\text{Id}\}$  and  $x \in I$  such that  $x \in \text{Fix}(h)$ . As  $K$  is abelian, Lemma 3.8 implies that  $\text{Fix}(h)$  is a non empty  $K$ -invariant set. By totally minimality of  $G$ , all  $K$ -orbits are dense, hence  $\overline{\text{Fix}(h)} = [0, 1)$  and then  $h = \text{Id}$ , a contradiction.

Therefore, by Proposition 1.6,  $G$  contains maps which are conjugate in  $\mathcal{G}$  to products of restricted rotations with disjoint supports and without periodic points. Let  $f_1$  be such an element of  $G$ , Theorem 2 applies to  $G$  and we conclude that there exists a map  $H = P \circ E$  that conjugates  $G$  to a subgroup of some  $\mathbf{G}_n$ , with  $P$  a PL-homeomorphisms,  $E \in \mathcal{G}$  and  $H$  of constant slope on each minimal component of  $f_1$ .

Moreover, as the conjugation by  $H$  preserves the totally minimality (according to classical arguments for topological conjugacy and extra arguments given in the proof of Lemma 3.9 and its footnote), this subgroup is also totally minimal.

We claim that no subgroup of  $\mathbf{G}_n$ ,  $n \geq 2$ , is totally minimal.

Indeed, if  $G < \mathbf{G}_n$  then  $G_0 = \{g \in G \mid \sigma_g = \text{Id}\}$  is a finite index subgroup of  $G$  that preserves any  $I_i$ ,  $i = 1, \dots, n$ . Thus, for  $G_0$  to be minimal it is necessary that  $n = 1$ .

Finally  $n = 1$ , so the map  $f_1$  is minimal and then  $H$  has constant slope 1. In conclusion,  $G$  is conjugate in  $\mathcal{G}$  to a subgroup of  $G_1$ .

**Proof of Corollary 2.** Let  $G$  be a finitely generated subgroup of  $\mathcal{G}$  and  $f \in G$  be a totally minimal IET that is not conjugate to a rotation.

We first note that the action of  $G$  is also totally minimal, since any finite index subgroup of  $G$  contains some non trivial power of  $f$ .

We next argue by contradiction, supposing that  $G$  is virtually polycyclic and not virtually cyclic. By [DFG20],  $G$  is virtually abelian and not virtually cyclic then it contains a

copy of  $\mathbb{Z}^2$ . Therefore, applying Corollary 1, we get that  $f$  is conjugate in  $\mathcal{G}$  to a rotation, a contradiction.

Corollary 2 can also be proven more directly: arguing by contradiction and using [DFG20], we may assume that up to a finite index subgroup,  $G$  is abelian and it contains a totally minimal IET  $f$  non conjugate to a rotation. By [Nov09], the centraliser of  $f$  is cyclic and then  $G$  is cyclic, a contradiction.

## 7. NON VIRTUALLY ABELIAN SUBGROUPS OF $\mathcal{G}$ .

### 7.1. Proof of Corollary 3.

Item (1) is the special case  $n = 1$  in Item (2), so we only prove Item (2): We argue by contradiction supposing that  $G$  is virtually abelian. Hence, according to Claim 5.2, there exist integers  $p$  and  $q$  such that  $f^p \in \mathbf{G}_n$  and  $gf^qg^{-1}$  commute. Therefore  $gf^qg^{-1} \in \mathbf{G}_n$  by Lemma 3.11 and then Lemma 5.3 leads to  $g \in \mathbf{G}_n$ , a contradiction.

### 7.2. Related examples involving arithmetic conditions.

**Notations.** Let  $s \in \mathbb{N}^*$  and  $\alpha_1, \dots, \alpha_s$  be  $s$   $\mathbb{Q}$ -independent irrational numbers. We denote by  $\langle 1, \alpha_1, \dots, \alpha_s \rangle_{\mathbb{Q}}$  the  $\mathbb{Q}$ -vector space generated by 1 and the  $\alpha_i$ 's. We set

- $\mathcal{G}_{\mathbb{Q}} = \{g \in \mathcal{G} \mid g(x) - x \in \mathbb{Q}, \forall x \in I\}$  and
- $\mathcal{G}_{\alpha_1, \dots, \alpha_s} = \{g \in \mathcal{G} \mid g(x) - x \in \langle 1, \alpha_1, \dots, \alpha_s \rangle_{\mathbb{Q}}, \forall x \in I\}$ .

**Corollary 5.** Let  $p, r \in \mathbb{N}^*$ ,  $r < p$  and  $\alpha_1, \dots, \alpha_p$  be  $p$   $\mathbb{Q}$ -independent irrational numbers. If  $f \in \mathcal{G}_{\alpha_1, \dots, \alpha_r}$  is totally minimal not conjugate to a rotation and  $g \in \mathcal{G}_{\alpha_{r+1}, \dots, \alpha_p}$  then the group generated by  $f$  and  $g$  is not virtually polycyclic.

*Proof.* According to Corollary 2,  $G$  is either virtually cyclic or not virtually polycyclic. Suppose, contrary to our claim that  $G$  is virtually cyclic. Hence, there exists  $h \in G$  such that  $Z = \langle h \rangle$  has finite index in  $G$  and then  $f^m = h^n$  and  $g^q = h^t$  for some positive integers  $m, n, q$  and  $t$ . Therefore  $f^{mt} = h^{nt} = g^{qn} \in \mathcal{G}_{\alpha_1, \dots, \alpha_r} \cap \mathcal{G}_{\alpha_{r+1}, \dots, \alpha_p} = \mathcal{G}_{\mathbb{Q}}$  that only contains finite order maps<sup>1</sup>, in particular  $f$  is not totally minimal, a contradiction. Finally,  $G$  is not virtually polycyclic.  $\square$

In the following two subsections, we exhibit examples of finitely generated non virtually polycyclic subgroups of  $\mathcal{G}$  that have extra properties. In the first one, we provide metabelian examples and in the second one, non virtually solvable groups.

### 7.3. Metabelian examples.

Let  $\alpha \in [0, \frac{1}{3}) \setminus \mathbb{Q}$  and  $G$  be the subgroup of  $\mathcal{G}$  generated by  $R_{\alpha}$  and the map  $g \in G_3$  described by Figure 2.

**Proposition 7.1.** *The group  $G$  is metabelian and non virtually polycyclic. More precisely its commutators subgroup  $[G, G]$  is a non finitely generated abelian group whose non trivial elements have order 3.*

**Remark.** *This metabelian group shares many properties with lamplighter groups  $A \wr \mathbb{Z}$ , but its abelianization  $\frac{G}{[G, G]} = \mathbb{Z} \times \frac{\mathbb{Z}}{2\mathbb{Z}}$  whereas the one of a lamplighter group is  $\mathbb{Z}$ .*

<sup>1</sup>Let  $f \in \mathcal{G}_{\mathbb{Q}}$  and  $q$  be the least common multiple of the denominators of the translations of  $f$ , then  $\mathcal{O}_f(x) \subset \{x + \frac{p}{q}, p \in \mathbb{Z}\} \cap [0, 1)$  is finite.

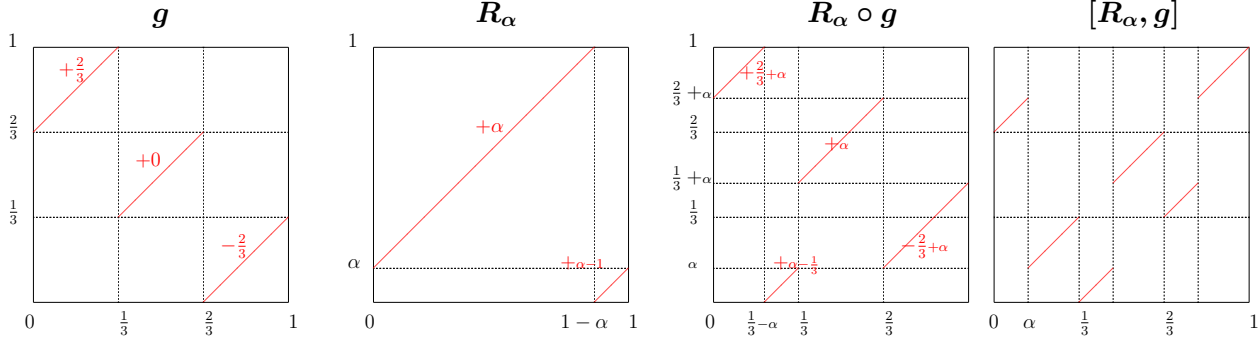


Figure 2

Note that, since  $g$  has order 2, any  $f \in G$  can be written as

$$f = R_\alpha^{p_s} g R_\alpha^{p_{s-1}} g \cdots R_\alpha^{p_1} g R_\alpha^{p_0}, \quad \text{where } p_j \in \mathbb{Z}.$$

**Lemma 7.2.** *For any  $f \in G$ , there exists a unique  $\ell(f) \in \mathbb{Z}$  such that for all  $x \in [0, 1)$ ,*

$$f(x) = x + \ell(f) \alpha + \frac{p_f(x)}{3}, \quad \text{where } p_f(x) \in \mathbb{Z}.$$

*Proof.* We argue by induction on the length  $L_S(f)$  of  $f \in G$  as a word in  $S = \{g, R_\alpha\}$ .

Indeed, if  $L_S(f) = 1$  then either  $f = g$  or  $f = R_{\pm\alpha}$  and the property holds.

Let  $n \in \mathbb{N}^*$ , suppose that  $G$ -elements of length less or equal than  $n$  have the required property. Consider  $f \in G$  with  $L_S(f) = n + 1$ , then either  $f = R_{\pm\alpha} f_0$  or  $f = g f_0$ , where  $L_S(f_0) = n$ . So, for any  $x \in [0, 1)$ , we have either

$$f(x) = f_0(x) \pm \alpha + \epsilon \quad \text{or} \quad f(x) = f_0(x) + \epsilon \frac{2}{3}, \quad \text{where } \epsilon \in \{-1, 0, 1\}.$$

Therefore, either

$$f(x) = x + (\ell(f_0) \pm 1) \alpha + \frac{p_{f_0}(x) + 3\epsilon}{3} \quad \text{or} \quad f(x) = x + \ell(f_0) \alpha + \frac{p_{f_0}(x) + 2\epsilon}{3}.$$

In both cases,  $f$  has the required form and  $\ell(f)$  is unique since  $\alpha$  is irrational.  $\square$

**Lemma 7.3.**

- (1) *The map  $\ell : G \rightarrow \mathbb{Z}$  is a morphism.*
- (2) *Any  $f \in G$  can be written as  $f = P_f R_\alpha^{\ell(f)} = R_\alpha^{\ell(f)} Q_f$  with  $P_f, Q_f \in \ker \ell$ .*
- (3) *The  $\ker \ell$ -orbit of any  $x \in [0, 1)$  consists in at most 3 points.*
- (4) *The group  $\ker \ell$  contains  $[G, G]$  and  $\ker \ell = \mathbf{G}_{\text{per}} = \{f \in G : \exists m \in \mathbb{N}^*, f^m = \text{Id}\}$ .*
- (5) *For any  $\beta \in [0, 1)$ , the map  $[R_\beta, g]$  has order 3.*<sup>2</sup>

*Proof.*

- (1) We obviously have  $\ell(\text{Id}) = 0$  and  $\ell(R_\alpha) = 1$ . Given  $f, h \in G$  and  $x \in [0, 1)$ , the computation of  $hf(x)$  leads to

$$hf(x) = x + (\ell(f) + \ell(h)) \alpha + \frac{p_f(x) + p_h(f(x))}{3}$$

Therefore

$$\ell(hf) = \ell(f) + \ell(h) \quad \text{and} \quad p_{hf}(x) = p_f(x) + p_h(f(x))$$

<sup>2</sup>This will be used for maps in  $G$ , that is for  $\beta = n\alpha \pmod 1$ .

In particular,  $\ell$  is a morphism from  $G$  to  $(\mathbb{Z}, +)$ .

(2) Let  $f \in G$ , one has  $\ell(fR_\alpha^{-\ell(f)}) = \ell(f) + \ell(R_\alpha^{-\ell(f)}) = \ell(f) - \ell(f) = 0$ . Then  $P_f = fR_\alpha^{-\ell(f)} \in \ker \ell$  and  $f = P_f R_\alpha^{\ell(f)}$ .

Similarly, we get  $Q_f = R_\alpha^{-\ell(f)} f \in \ker \ell$  and  $f = R_\alpha^{\ell(f)} Q_f$ .

(3) Let  $x \in [0, 1)$ , the  $\ker \ell$ -orbit of  $x$  is the set  $\{f(x) \mid f \in \ker \ell\}$  and it is included in  $S = \{x + \frac{p}{3}, p \in \mathbb{Z}\} \cap [0, 1)$  which has cardinality 3 since  $S \subset [0, 1)$  and the distance between two distinct elements of  $S$  is  $\frac{1}{3}$  or  $\frac{2}{3}$ .

(4) Since  $\ell$  is a morphism to an abelian group, its kernel contains  $[G, G]$ . In addition, let  $f \in G_{\text{per}}$  of order  $m \in \mathbb{N}^*$  and  $x \in [0, 1)$ . One has

$$f^m(x) = x + m\ell(f)\alpha + \frac{pf^m(x)}{3} = x$$

By irrationality of  $\alpha$ , we get  $\ell(f) = 0$  that is  $f \in \ker \ell$ .

(5) Let  $\beta \in [0, \frac{1}{3})$ , we compute the map  $[R_\beta, g] = R_\beta g R_{-\beta} g$ .

$$\begin{aligned} [\mathbf{0}, \beta) &\xrightarrow[+\frac{2}{3}]{g} [\frac{2}{3}, \frac{2}{3} + \beta) \xrightarrow[-\beta]{R_{-\beta}} [\frac{2}{3} - \beta, \frac{2}{3}) \xrightarrow[+0]{g} [\frac{2}{3} - \beta, \frac{2}{3}) \xrightarrow[\beta]{R_\beta} [\frac{2}{3}, \frac{2}{3} + \beta) \\ [\beta, \frac{1}{3}) &\xrightarrow[+\frac{2}{3}]{g} [\frac{2}{3} + \beta, 1) \xrightarrow[-\beta]{R_{-\beta}} [\frac{2}{3}, 1 - \beta) \xrightarrow[-\frac{2}{3}]{g} [0, \frac{1}{3} - \beta) \xrightarrow[\beta]{R_\beta} [\beta, \frac{1}{3}) \\ [\frac{1}{3}, \frac{1}{3} + \beta) &\xrightarrow[0]{g} [\frac{1}{3}, \frac{1}{3} + \beta) \xrightarrow[-\beta]{R_{-\beta}} [\frac{1}{3} - \beta, \frac{1}{3}) \xrightarrow[+\frac{2}{3}]{g} [1 - \beta, 1) \xrightarrow[\beta-1]{R_\beta} [\mathbf{0}, \beta) \\ [\frac{1}{3} + \beta, \frac{2}{3}) &\xrightarrow[0]{g} [\frac{1}{3} + \beta, \frac{2}{3}) \xrightarrow[-\beta]{R_{-\beta}} [\frac{1}{3}, \frac{2}{3} - \beta) \xrightarrow[0]{g} [\frac{1}{3}, \frac{2}{3} - \beta) \xrightarrow[\beta]{R_\beta} [\frac{1}{3} + \beta, \frac{2}{3}) \\ [\frac{2}{3}, \frac{2}{3} + \beta) &\xrightarrow[-\frac{2}{3}]{g} [0, \beta) \xrightarrow[1-\beta]{R_{-\beta}} [1 - \beta, 1) \xrightarrow[-\frac{2}{3}]{g} [\frac{1}{3} - \beta, \frac{1}{3}) \xrightarrow[\beta]{R_\beta} [\frac{1}{3}, \frac{1}{3} + \beta) \\ [\frac{2}{3} + \beta, 1) &\xrightarrow[-\frac{2}{3}]{g} [\beta, \frac{1}{3}) \xrightarrow[-\beta]{R_{-\beta}} [0, \frac{1}{3} - \beta) \xrightarrow[+\frac{2}{3}]{g} [\frac{2}{3}, 1 - \beta) \xrightarrow[\beta]{R_\beta} [\frac{2}{3} + \beta, 1) \end{aligned}$$

In conclusion,  $[R_\beta, g]$  has support  $[0, \beta) \sqcup [\frac{1}{3}, \frac{1}{3} + \beta) \sqcup [\frac{2}{3}, \frac{2}{3} + \beta)$  and its restriction is periodic of exact period 3. Therefore,  $[R_\beta, g]$  has order 3.

The case  $\beta \in (\frac{2}{3}, 1)$  can be deduced from the previous case by noting that  $\beta' = 1 - \beta \in (0, \frac{1}{3})$  and  $R_\beta = R_{\beta'}^{-1}$ .

Finally, for the case  $\beta \in [\frac{1}{3}, \frac{2}{3}]$ , let us write  $\beta = \frac{1}{3} + \beta_0$  with  $\beta_0 \in [0, \frac{1}{3}]$ . A direct computation similar to that in the case  $\beta \in [0, \frac{1}{3})$  shows that  $[R_\beta, g]$  is continuous on the intervals  $J_1 = [0, \beta_0)$ ,  $J'_1 = [\beta_0, \frac{1}{3})$ ,  $J_2 = [\frac{1}{3}, \frac{1}{3} + \beta_0)$ ,  $J'_2 = [\frac{1}{3} + \beta_0, \frac{2}{3})$ ,  $J_3 = [\frac{2}{3}, \frac{2}{3} + \beta_0)$  and  $J'_3 = [\frac{2}{3} + \beta_0, 1)$ . In addition,  $[R_\beta, g](J_1) = J_2$ ,  $[R_\beta, g](J_2) = J_3$ ,  $[R_\beta, g](J_3) = J_1$  and  $[R_\beta, g](J'_1) = J'_3$ ,  $[R_\beta, g](J'_3) = J'_2$ ,  $[R_\beta, g](J'_2) = J'_1$ . This proves that  $[R_\beta, g]$  has order 3.  $\square$

#### Lemma 7.4.

- (1) The group  $\ker \ell$  is generated by  $\{R_\alpha^n g R_\alpha^{-n}, n \in \mathbb{Z}\}$ .
- (2) The group  $[G, G]$  is generated by  $\{[R_\alpha^n, g], n \in \mathbb{Z}\}$ .

(3) The group  $[G, G]$  coincides with  $[\ker \ell, \ker \ell]$ .

*Proof.*

(1) It is plain that the maps  $R_\alpha^n g R_\alpha^{-n}$ ,  $n \in \mathbb{Z}$  belong to  $\ker \ell$ . We previously noted that any  $f \in G$  can be decomposed as

$$f = R_\alpha^{p_s} g R_\alpha^{p_s-1} g \cdots R_\alpha^{p_1} g R_\alpha^{p_0}, \text{ where } p_j \in \mathbb{Z},$$

and then we can write:

$$f = R_\alpha^{p_s} g \mathbf{R}_\alpha^{-p_s} \mathbf{R}_\alpha^{p_s+p_{s-1}} g R_\alpha^{p_s-2} \cdots R_\alpha^{p_1} g R_\alpha^{p_0}$$

Iterating this process, we get:

$$f = R_\alpha^{p_s} g R_\alpha^{-p_s} R_\alpha^{p_s+p_{s-1}} g R_\alpha^{-(p_s+p_{s-1})} \cdots R_\alpha^{p_s+p_{s-1}+\cdots+p_1} g R_\alpha^{-(p_s+p_{s-1}+\cdots+p_1)} R_\alpha^{\sum_{j=0}^s p_j}$$

Moreover,  $f \in \ker \ell$  if and only if  $\sum_{j=0}^s p_j = 0$ , thus any  $f \in \ker \ell$  is the product of elements of the form  $R_\alpha^p g R_\alpha^{-p}$ .

(2) We first note that for all  $p, q \in \mathbb{Z}$ , we have

$$(*) \quad R_\alpha^p g R_\alpha^{-p} R_\alpha^q g R_\alpha^{-q} = R_\alpha^p g R_\alpha^{-p} \mathbf{g} \mathbf{g} R_\alpha^q g R_\alpha^{-q} = [R_\alpha^p, g] [R_\alpha^q, g]^{-1}.$$

Next, we claim that any element of  $[G, G]$  can be written as a product of an even number of  $R_\alpha^n g R_\alpha^{-n}$  so as a product of  $[R_\alpha^p, g]$  and  $[R_\alpha^q, g]^{-1}$  by (\*).

Indeed, it suffices to prove this property for a commutator  $f = [a, b]$  with  $a, b \in G$ . By Lemma 7.3 (2),  $f$  can be written as

$$f = P_a R_\alpha^{\ell(ab)} Q_b Q_a^{-1} R_\alpha^{-\ell(ab)} P_b^{-1} = P_a R_\alpha^{\ell(ab)} Q_b \mathbf{R}_\alpha^{-\ell(ab)} \mathbf{R}_\alpha^{\ell(ab)} Q_a^{-1} R_\alpha^{-\ell(ab)} P_b^{-1}$$

According to Item (1),  $P_a, P_b, Q_a$  and  $Q_b$  can be written as a product of  $R_\alpha^n g R_\alpha^{-n}$ . Moreover, for any  $c \in G$ ,  $Q_c = R_\alpha^{-\ell(c)} P_c R_\alpha^{\ell(c)}$  so the maps  $P_c, P_c^{-1}, Q_c, Q_c^{-1}$  and their conjugates by  $R_\alpha^m$  ( $m \in \mathbb{Z}$ ) can be decomposed as products of the same number of  $R_\alpha^n g R_\alpha^{-n}$ . Finally,  $f$  can be written as a product of an even number of  $R_\alpha^n g R_\alpha^{-n}$ .

(3) By the previous point, showing that  $[G, G] \subset [\ker \ell, \ker \ell]$  reduces to prove that  $[R_\alpha^n, g] \in [\ker \ell, \ker \ell]$  for any  $n \in \mathbb{Z}$  and we claim that  $[\mathbf{R}_\alpha^n, \mathbf{g}] = [\mathbf{g}, \mathbf{R}_\alpha^n \mathbf{g} \mathbf{R}_\alpha^{-n}]$ .

Indeed, noting that if  $a^2 = b^2 = \text{Id}$  then  $[a, b] = (ab)^2$  and that  $g$  and  $R_\alpha^n g R_\alpha^{-n}$  have order 2, we get  $[g, R_\alpha^n g R_\alpha^{-n}] = (g R_\alpha^n g R_\alpha^{-n})^2 = [g, R_\alpha^n]^2 = [R_\alpha^n, g]^{-2}$ .

In addition, by Lemma 7.3 (5), the map  $[R_\alpha^n, g] = [R_{n\alpha}, g]$  has order 3 then  $[g, R_\alpha^n g R_\alpha^{-n}] = [R_\alpha^n, g]$ .  $\square$

### Lemma 7.5.

(1) Let  $f \in \ker \ell$  and  $J \subset [0, \frac{1}{3})$  such that  $J, J + \frac{1}{3}$  and  $J + \frac{2}{3}$  are continuity intervals of  $f$  then there exists  $\sigma = \sigma_{f,J} \in \mathfrak{S}_3$ , the group of permutations of the set  $\{0, 1, 2\}$  such that  $f(J + \frac{i}{3}) = J + \frac{\sigma(i)}{3}$ .

(2) Let  $f, h \in \ker \ell$  and  $J \subset [0, \frac{1}{3})$  such that  $J, J + \frac{1}{3}$  and  $J + \frac{2}{3}$  are continuity intervals of  $f$  and  $h$  then  $J, J + \frac{1}{3}$  and  $J + \frac{2}{3}$  are continuity intervals of  $fh$  and

$$\sigma_{fh,J} = \sigma_{f,J} \circ \sigma_{h,J}$$



*Proof.*

- (1) Let  $f \in \ker \ell$ , since  $J \subset [0, \frac{1}{3})$  is a continuity interval of  $f$ , the map  $p_f$  is constant on  $J$  equal to some  $p \in \mathbb{Z}$  and  $f(J) = \{x + \frac{p}{3}, x \in J\} = J + \frac{p}{3}$  is contained in  $[\frac{p}{3}, \frac{p+1}{3}) \cap [0, 1)$ . Therefore  $p \in \{0, 1, 2\}$  and  $f(J)$  is either  $J$ ,  $J + \frac{1}{3}$  or  $J + \frac{2}{3}$ .

Analogously, for  $i = 1, 2$ , we get that  $f(J + \frac{i}{3})$  is either  $J$ ,  $J + \frac{1}{3}$  or  $J + \frac{2}{3}$ . This defines the required  $\sigma \in \mathfrak{S}_3$ .

- (2) Let  $f, h \in \ker \ell$  and  $J \subset [0, \frac{1}{3})$  such that  $f$  and  $h$  are continuous on  $J$ ,  $J + \frac{1}{3}$ ,  $J + \frac{2}{3}$ . For any  $i \in \{0, 1, 2\}$ , we have that  $h$  is continuous on  $J + \frac{i}{3}$  and

$$fh(J + \frac{i}{3}) = f\left(J + \frac{\sigma_{h,J}(i)}{3}\right) = J + \frac{\sigma_{f,J}(\sigma_{h,J}(i))}{3}$$

since  $f$  is continuous on the intervals  $J + \frac{i}{3}$ . □

### Proof of Proposition 7.1.

By Item (1) of Corollary 3, the group  $G$  is not virtually polycyclic.

We claim that all maps in  $[G, G] \setminus \{\text{Id}\}$  commute and have order 3.

Indeed, let  $C = [f, h] \in [G, G] \setminus \{\text{Id}\}$  be a non trivial commutator. By Lemma 7.4 (3), we can suppose that  $f, h \in \ker \ell$ . Let us decompose  $[0, 1)$  as a union of triples of intervals  $J$ ,  $J + \frac{1}{3}$ ,  $J + \frac{2}{3}$  with  $J \subset [0, \frac{1}{3})$  such that  $f, h, f^{-1}, h^{-1}$  are continuous on  $J$ ,  $J + \frac{1}{3}$ ,  $J + \frac{2}{3}$ .

Let  $J$  be such an interval, it follows from Item (2) of Lemma 7.5 that

$$\sigma_{[f,h],J} = [\sigma_{f,J}, \sigma_{h,J}] \in [\mathfrak{S}_3, \mathfrak{S}_3] = \mathfrak{A}_3$$

Noting that the non trivial elements of  $\mathfrak{A}_3$  are 3-cycles, we get that  $\sigma_{[f,h],J}$  is either trivial or has order 3 and the same holds for  $C|_J$  by its continuity. As  $C \neq \text{Id}$ , there is some  $J$  such that  $C|_J \neq \text{Id}$  and then  $C$  has order 3.

As  $\mathfrak{A}_3$  is abelian, two commutators in  $[G, G]$  have commuting associate permutations on their common continuity intervals, so they commute.

Finally, we get the claim since  $[G, G]$  is generated by the commutators and the product of two elements of order 3 that commute is either of order 3 or trivial.

In addition,  $[G, G]$  is not finitely generated. This is due to the fact that any finitely generated abelian torsion group has to be finite but the  $[G, G]$ -elements  $[R_\alpha^n, g]$ ,  $n \in \mathbb{Z}$  are pairwise distinct, since the proof of Item (5) of Lemma 7.3 indicates that they have pairwise distinct break point sets.

### 7.4. Non virtually solvable examples.

Let  $n \in \mathbb{N}^*$ ,  $n \geq 5$ , we make the alternating group  $\mathcal{A}_n$  act on  $[0, 1)$  as a subgroup of  $\mathbf{G}_n$  so that the **IET associated to**  $a \in \mathcal{A}_n$ ,  $t_a$ , sends the interval  $J_p = [\frac{p-1}{n}, \frac{p}{n})$  to  $J_{a(p)} = [\frac{a(p)-1}{n}, \frac{a(p)}{n})$  and has a trivial rotation vector. In particular, for any  $x \in [0, 1)$  there exists an integer  $p_a(x)$  such that  $t_a(x) = x + \frac{p_a(x)}{n}$ . Let  $G$  be the subgroup of  $\mathcal{G}$  generated by  $\mathcal{A}_n$  and  $R_\alpha$ ,  $\alpha \in [0, \frac{1}{n}) \setminus \mathbb{Q}$ .

Note that any  $f \in G$  can be written as

$$f = R_\alpha^{p_s} a_s R_\alpha^{p_{s-1}} a_{s-1} \cdots R_\alpha^{p_1} a_1 R_\alpha^{p_0}, \text{ where } a_j \in \mathcal{A}_n \text{ and } p_j \in \mathbb{Z}.$$

**Proposition 7.6.** *The group  $G$  is elementary amenable, non virtually solvable and its commutator subgroup  $[G, G]$  is a locally finite torsion group<sup>3</sup>, moreover  $[G, G]$  is perfect but it is not simple.*

This proposition will be proved after Lemma 7.13.

**Lemma 7.7.** *Let  $f \in G$ . Then there exists a unique  $\ell(f) \in \mathbb{Z}$  such that for all  $x \in [0, 1)$  one has*

$$f(x) = x + \ell(f)\alpha + \frac{p_f(x)}{n}, \text{ where } p_f(x) \in \mathbb{Z}.$$

*Proof.* As in Lemma 7.2, we argue by induction on the length  $L_S(f)$  of  $f \in G$  as a word in  $S = \{\mathcal{A}_n, R_\alpha\}$ . The change being in the case  $f = gf_0$  of the induction step. Here, we have  $f = af_0$  with  $a \in \mathcal{A}_n$  and the involved translation  $\epsilon_{\frac{2}{3}}$  is replaced by  $\frac{pa(x)}{n}$ .  $\square$

**Lemma 7.8.**

- (1) *The map  $\ell : G \rightarrow \mathbb{Z}$  is a morphism.*
- (2) *Any  $f \in G$  can be written as  $f = P_f R_\alpha^{\ell(f)} = R_\alpha^{\ell(f)} Q_f$  where  $P_f, Q_f \in \ker \ell$ .*

The proof is analogous to that of Lemma 7.4.

**Lemma 7.9.**

- (1) *The group  $\ker \ell$  is generated by  $\{R_\alpha^k a R_\alpha^{-k}, k \in \mathbb{Z}, a \in \mathcal{A}_n\}$ .*
- (2) *The group  $\ker \ell$  is perfect and  $[G, G] = \ker \ell$ .*
- (3) *The  $\ker \ell$ -orbit of  $x \in [0, \frac{p}{n})$  is  $\{x + \frac{p}{n}, p = 0, \dots, n-1\}$  and  $\ker \ell = G_{per}$ .*

*Proof.*

- (1) We previously noted that any  $f \in G$  can be written as

$$f = R_\alpha^{p_s} a_s R_\alpha^{p_{s-1}} a_{s-1} \cdots R_\alpha^{p_1} a_1 R_\alpha^{p_0}, \text{ where } p_j \in \mathbb{Z},$$

and as in the proof of Lemma 7.4 we get:

$$f = R_\alpha^{p_s} a_s R_\alpha^{-p_s} R_\alpha^{p_s+p_{s-1}} a_{s-1} R_\alpha^{-(p_s+p_{s-1})} \cdots R_\alpha^{p_s+p_{s-1}+\cdots+p_1} a_1 R_\alpha^{-(p_s+p_{s-1}+\cdots+p_1)} R_\alpha^{\sum_{j=0}^s p_j}$$

Moreover, as  $\ell$  is a morphism  $f \in \ker \ell$  if and only if  $\sum_{j=0}^s p_j = 0$ , thus any  $f \in \ker \ell$  is the product of elements of the form  $R_\alpha^k a R_\alpha^{-k}$ .

- (2) Let  $f \in \ker \ell$ , by the previous point,  $f$  can be written as

$$f = \prod_j R_\alpha^{n_j} a_j R_\alpha^{-n_j} \text{ where } n_j \in \mathbb{Z} \text{ and } a_j \in \mathcal{A}_n$$

As  $\mathcal{A}_n$  is perfect and  $\mathcal{A}_n < \ker \ell$ , any  $a_j$  belongs to  $[\ker \ell, \ker \ell]$ . Finally,  $[\ker \ell, \ker \ell]$  being normal in  $G$ , any  $R_\alpha^{n_j} a_j R_\alpha^{-n_j}$  belongs to  $[\ker \ell, \ker \ell]$  and then  $f$  belongs to  $[\ker \ell, \ker \ell]$ , which establishes that  $\ker \ell$  is perfect.

In addition, as  $\ell$  is a morphism to an abelian group,  $[G, G] < \ker \ell$  and the other inclusion holds since  $\ker \ell$  is perfect.

<sup>3</sup>i.e its finitely generated subgroups are finite

- (3) Let  $x \in [0, \frac{p}{n}]$  and  $f \in \ker \ell$ . Lemma 7.7 gives  $f(x) = x + \frac{p_f(x)}{n}$  for some  $p_f(x) \in \mathbb{Z}$  and  $f(x) \in [0, 1)$  forces  $p_f(x) \in \{0, 1, \dots, n-1\}$ . Moreover, by choosing  $a \in \mathcal{A}_n$  with  $a(1) = p$ , we obtain  $t_a(x) = x + \frac{p-1}{n}$ . Therefore the  $\mathcal{A}_n$ -orbit of  $x$  is  $\{x + \frac{p}{n}, p = 0, \dots, n-1\}$ . Finally,  $\ker \ell$  is contained in  $\mathcal{G}_{\mathbb{Q}}$  so in  $G_{\text{per}}$  and the other inclusion is a consequence of  $\alpha \notin \mathbb{Q}$ .  $\square$

**Definition 7.10.** Let  $f \in \ker \ell$ ,  $x \in [0, \frac{1}{n})$  and  $\beta \in (0, \frac{1}{n})$  such that  $f$  is continuous on the intervals  $J_p(x, \beta) = [x + \frac{p-1}{n}, x + \frac{p-1}{n} + \beta)$  for all  $p = 1, \dots, n$ .

As the translations of  $f$  belong to  $\{\frac{p}{n}, p \in \mathbb{Z}\}$ , the map  $f$  permutes the  $J_p(x, \beta)$  and then there exists  $\omega \in \mathcal{S}_n$  such that

$$f(J_p(x, \beta)) = J_{\omega(p)}(x, \beta) = [x + \frac{\omega(p) - 1}{n}, x + \frac{\omega(p) - 1}{n} + \beta)$$

It is easy to see that  $\omega$  does not depend on the choice of suitable  $\beta$  and it is denoted by  $\omega(f, x)$  and called **the local permutation of  $f$  at  $x$** .

**Proposition 7.11.** Let  $x \in [0, \frac{1}{n})$ .

The map  $\omega_x : \begin{cases} \ker \ell & \rightarrow \mathcal{S}_n \\ f & \mapsto \omega(f, x) \end{cases}$  is a morphism and its image is  $\mathcal{A}_n$ .

*Proof.* Let  $x \in [0, \frac{1}{n})$ , it is obvious that  $\omega(\text{Id}, x) = \text{Id}$ . Let  $f, h \in \ker \ell$  and  $\beta \in (0, \frac{1}{n})$  small enough so that  $f$  and  $h$  are continuous on the intervals  $J_p(x, \beta)$ . Hence, for any  $p \in \{1, \dots, n\}$ , we have

$$h \circ f(J_p(x, \beta)) = h(J_{\omega(f, x)(p)}(x, \beta)) = J_{\omega(h, x)\omega(f, x)(p)}(x, \beta),$$

therefore  $\omega(h \circ f, x) = \omega(h, x) \omega(f, x)$ .

Since  $\ker \ell$  is perfect the image of  $\omega_x$  is contained in  $[\mathcal{S}_n, \mathcal{S}_n] = \mathcal{A}_n$ . In addition, it is easy to check that the IET  $t$  associated to  $\tau \in \mathcal{A}_n$  satisfies  $\omega(t, x) = \tau$  and then we get  $\omega_x(\ker \ell) = \mathcal{A}_n$ .  $\square$

**Proposition 7.12.** For any  $\beta \in (0, \frac{1}{n})$ , the set  $S_\beta$  of maps in  $\ker \ell$  having support in

$$\bigcup_{p=1}^n [\frac{p-1}{n}, \frac{p-1}{n} + \beta)$$

is a normal subgroup of  $\ker \ell$  and its image by  $\omega_0$  is  $\mathcal{A}_n$ .

*Proof.* The normality of  $S_\beta$  is direct consequence of the facts that the translation set of  $\ker \ell$  is  $\{\frac{p}{n}, p \in \mathbb{Z} \cap [-(n-1), n-1]\}$  and the support of  $h \circ f \circ h^{-1}$  is the image by  $h$  of the support of  $f$ . Consequently, the image by  $\omega_0$  of  $S_\beta$  is a normal subgroup of  $\mathcal{A}_n$  which is simple, therefore  $\omega_0(S_\beta)$  is either trivial or equal to  $\mathcal{A}_n$ . Hence, the proof of Proposition 7.12 reduces to prove that  $\omega_0(S_\beta)$  is not trivial. This is provided, taking  $\beta \in (0, \frac{1}{n}) \cap \{\frac{p}{n} + k\alpha, p, k \in \mathbb{Z}\}$  so that  $R_\beta \in G$ , by the following

**Lemma 7.13.** Let  $\beta \in (0, \frac{1}{n})$ ,  $\tau \in \mathcal{A}_n$  and  $t \in \ker \ell$  be its associated IET.

Then the map  $C_{\beta, t} = [R_\beta, t] = R_\beta t R_{-\beta} t^{-1}$  has support in  $\bigcup_{p=1}^n [\frac{p-1}{n}, \frac{p-1}{n} + \beta)$  and its local permutation at 0 is  $\omega(C_{\beta, t}, 0) = [\sigma, \tau]$ , where  $\sigma$  is the  $n$ -cycle  $(1, 2, \dots, n) \in \mathcal{S}_n$ .

*Proof.* We explicitly compute the map  $C_{\beta,t} = [R_\beta, t] = R_\beta t R_{-\beta} t^{-1}$ . We first note that the break point set of  $C_{\beta,t}$  is contained in  $\{\frac{p-1}{n}, \frac{p-1}{n} + \beta; p \in \{1, \dots, n\}\}$  and we have

$$\begin{aligned} \left[\frac{p-1}{n}, \frac{p-1}{n} + \beta\right) &\xrightarrow{t^{-1}} \left[\frac{\tau^{-1}(p)-1}{n}, \frac{\tau^{-1}(p)-1}{n} + \beta\right) \xrightarrow{R_{-\beta}} \left[\frac{\tau^{-1}(p)-1}{n} - \beta, \frac{\tau^{-1}(p)-1}{n}\right) \\ &\xrightarrow{t} \left[\frac{\tau(\tau^{-1}(p)-1)}{n} - \beta, \frac{\tau(\tau^{-1}(p)-1)}{n}\right) \xrightarrow{R_\beta} \left[\frac{\tau(\tau^{-1}(p)-1)}{n}, \frac{\tau(\tau^{-1}(p)-1)}{n} + \beta\right) \end{aligned}$$

and

$$\begin{aligned} \left[\frac{p-1}{n} + \beta, \frac{p}{n}\right) &\xrightarrow{t^{-1}} \left[\frac{\tau^{-1}(p)-1}{n} + \beta, \frac{\tau^{-1}(p)}{n}\right) \xrightarrow{R_{-\beta}} \left[\frac{\tau^{-1}(p)-1}{n}, \frac{\tau^{-1}(p)}{n} - \beta\right) \\ &\xrightarrow{t} \left[\frac{p-1}{n}, \frac{p}{n} - \beta\right) \xrightarrow{R_\beta} \left[\frac{p-1}{n} + \beta, \frac{p}{n}\right) \end{aligned}$$

This proves that  $C_{\beta,t}$  has support in  $\bigcup_{p=1}^n \left[\frac{p-1}{n}, \frac{p-1}{n} + \beta\right)$ . In addition, its local permutation at 0 is given by  $\omega(p) - 1 = \tau(\tau^{-1}(p) - 1)$  that is

$$\omega(p) = \tau(\tau^{-1}(p) - 1) + 1 = \sigma\tau\sigma^{-1}\tau^{-1} = [\sigma, \tau]. \quad \square$$

Note that, if  $\tau \notin \mathbf{C}(\sigma)$  then  $R_\beta$  and  $t$  are not commuting and  $C_{\beta,t}$  is not trivial.

**Proof of Proposition 7.6.** We first prove the properties related to  $[G, G]$ . Lemma 7.9 implies that  $[G, G] = \ker \ell$  is a perfect torsion group and, by Proposition 7.12, it is not simple because of its non trivial normal subgroups  $S_\beta$ .

We claim that the finitely generated subgroups of  $[G, G] = \ker \ell$  are finite.

Indeed, let  $H = \langle m_i \rangle < \ker \ell$  be a finitely generated subgroup of  $\ker \ell$ . By Properties 3.2, the set  $\mathbf{BP}(H)$  is contained in the  $H$ -orbit of the finite set  $\mathbf{UBP}(m_i)$  so it is finite since all  $\ker \ell$ -orbits are finite.

The group  $G$  is elementary amenable since  $G/[G, G] \simeq \mathbb{Z}$  is abelian and  $[G, G] = \ker \ell$  is the direct union of its finitely generated subgroups that are finite.

By Item (1) of Corollary 3,  $G$  is not virtually polycyclic. Moreover, we are going to prove that no finite index subgroup of  $\ker \ell$  is solvable. Indeed, let  $R$  be a finite index subgroup of  $\ker \ell$ . Since a subgroup of finite index always contains a normal subgroup also of finite index, we may suppose that  $R$  is normal in  $G$ . Therefore for any  $x \in [0, \frac{1}{n})$ , the group  $\omega_x(R)$  is either trivial or equal to  $\mathcal{A}_n$ . As  $\ker \ell$  is infinite,  $R$  is not trivial and so there exists  $x \in [0, \frac{1}{n})$  such that  $\omega_x(R)$  is not trivial and then equal to  $\mathcal{A}_n$ .

Finally, since  $\mathcal{A}_n$  is simple,  $\omega_x$  maps any iterated commutators subgroup of  $R$  surjectively to  $\mathcal{A}_n$ . In particular,  $R$  is not solvable and therefore  $\ker \ell$  and then  $G$  are not virtually solvable.

**Remark 7.14.** Due to the SAF-invariant (see [Bos16]), the groups constructed in Sections 7.3 and 7.4, with rationally independent  $\alpha$  are not conjugated in  $\mathcal{G}$ .

Moreover, similar arguments show that Proposition 7.6 holds for the groups generated by  $R_{\alpha_1}, \dots, R_{\alpha_m}$  and  $\mathcal{A}_n$  for  $n \geq 5$ . In particular, these groups are not linear since by Schur Theorem ([Sch11]) any linear torsion group is virtually abelian.

## 8. CONFLICT OF INTEREST

The authors have no conflict of interest to declare that are relevant to this article.

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