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Gilbert Reinisch

# Mathematical Definition of the Fine-Structure Constant: a Clue for Fundamental Couplings in Astrophysics

Gilbert Reinisch*

*Université de la Côte d'Azur - Observatoire de la Côte d'Azur*
*06304 Nice Cedex - France*

## Abstract

Astrophysical tests of the stability —or not— of fundamental couplings (e.g. can the numerical value $\sim 1/137$ of the fine-structure constant $\alpha = e^2/\hbar c$ vary with astronomical time?) are a very active area of observational research. Using a specific $\alpha$-free non-relativistic and nonlinear isotropic quantum model compatible with its quantum electrodynamics (QED) counterpart yields the 99% accurate solution $\alpha = 7.364\,10^{-3}$ versus its $7.297...\,10^{-3}$ experimental value. The $\sim 1\%$ error is due to the deliberate use of mean-field Hartree approximation involving lowest-order QED in the calculations. The present theory has been checked by changing the geometry of the model. Moreover, it fits the mathematical solution of the original nonlinear integro-differential Hartree system by use of a rapidly convergent series of nonlinear eigenstates [G. Reinisch, Phys. Lett. A (2024): doi.org/10.1016/j.physleta.2024.129347]. These results strongly suggest the mathematical transcendental nature —e.g. like for $\pi$ or $e$— of $\alpha$'s numerical value $\sim 1/137$ and hence its astrophysical as well as its cosmological stability.

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*Electronic address: Gilbert.Reinisch@oca.eu

Since Dirac [1] [2] and Jordan [3] [4] first suggested it as a possibility, the time variation of the fundamental constants has remained a subject of fascination which motivated numerous theoretical and experimental researches. Landau [5] envisaged the possibility that the fine-structure constant $\alpha \sim 1/137$ could vary with time, due to the renormalization of the electric charge. Data from the natural fission reactors which operated about two billion years ago at Oklo (Gabon) had the potential of providing an extremely tight bound on the variability of $\alpha$ [6]. This bound was revisited and the relative variation of $\alpha$ over this interval of time was found to be $-0.9\,10^{-7} < \Delta\alpha/\alpha < 1.2\,10^{-7}$ while the averaged relative growth of $\alpha$ was estimated between $-6.7\,10^{-17} yr^{-1}$ and $5.0\,10^{-17} yr^{-1}$ [7]. A comprehensive review emphasized the main experimental and observational constraints that have been obtained from atomic clocks, the Oklo phenomenon, Solar system observations, meteorites dating, quasar absorption spectra, stellar physics, pulsar timing, the cosmic microwave background and big bang nucleosynthesis [8]. The deep conceptual importance of carrying out such astrophysical tests of the stability of fundamental couplings has been complemented by some evidence for such a variation [9], coming from high-resolution optical/UV spectroscopic measurements of the fine-structure constant $\alpha$ in absorption systems along the line of sight of bright quasars [10]. However, this opinion is somewhat controversial: the apparent ubiquity, size and general characteristics of the distortions are capable of significantly weakening the evidence for variations in $\alpha$ from quasar absorption lines [11]. More recently, combining four direct measurements related to the value of $\alpha$ 13 billion years ago with existing data, a spatial variation is preferred over a no-variation model at the $3.9\sigma$ level [12]. On the other hand, Feynman [13] was fascinated by the very numerical value $\sim 1/137$ of $\alpha$ which he called *"one of the greatest damn mysteries of physics"*. Therefore, any indication that the numerical value of $\alpha$ could unexpectedly appear in the solution of a non-relativistic (i.e. $\alpha$-free) mathematical description should importantly fuel this controversial issue.

The very first question is: can the fine-structure constant $\alpha = e^2/\hbar c$ appear in the description of a non-relativistic physical process where the velocity of light $c$ is by definition absent? Rather surprisingly, the answer is yes. In [14] [15], an empirical non-relativistic connection between the electronic polarizability of atoms and $\alpha$ is proposed. In [16] [17], the 97.7% visual transparency of graphene –a two-dimensional material with carbon atoms in a honeycomb lattice— is determined solely by $\pi\alpha$. This significant 2.3% absorption of incident white light despite the graphene being only one atom layer thick is a consequence of graphene's unique electronic structure [18].

Actually possible links between non-relativistic physical systems and $\alpha$ are provided by quantum electrodynamics (QED) through its basic hypothesis of virtual photons that mediate in electromagnetic interactions [19] [20]. The celebrated example is the description of classical Coulomb electrostatics (hence non-relativistic) by QED's exchange of a single virtual long-wavelength photon of probability amplitude $\propto \sqrt{\alpha}$ [21] [22]. Specifically: i) The electromagnetic operator $H_{em} = \int \mathbf{j}\mathbf{A} d^3x$ defined by the 4-vector particle current density $\mathbf{j}$ and related potential operator $\mathbf{A}$ is proportional to $e^2/\sqrt{\alpha}$ where $e$ is the electron charge. ii) Therefore the lowest-order probabiliy amplitude to create a virtual photon $|k\rangle$ of given long-wavelength wavenumber $k$ out of the vacuum $|0\rangle$ is:

$$\mathcal{A}_k = \frac{\langle k|H_{em}|0\rangle}{-\hbar\omega_k} \propto \sqrt{\alpha}, \tag{1}$$

since $\omega_k = ck$. iii) Consequently, the total number of virtual photons with all possible wave

numbers that take part in the interaction is:

$$\mathcal{P}_{qed} = \sum_k |\mathcal{A}_k|^2 \propto \alpha. \tag{2}$$

iv) On the other hand, the corresponding energy of this gas of virtual photons is:

$$\mathcal{E}_{qed} = \sum_k |\mathcal{A}_k|^2 \hbar\omega_k \propto e^2. \tag{3}$$

We recover the classical non-relativistic Coulomb energy: it is independent of $\alpha$ (or equivalently of $c$). The use of Feynman-diagram "linked cluster theorem" constitutes a standard procedure to recover (3) as the lowest-order term of the Rayleigh-Schrödinger perturbation series, and hence to evaluate its error. We have indeed [23]:

$$\mathcal{E}_{qed} = \sum_k \Big[ \frac{\langle 0|H_{em}|k\rangle\langle k|H_{em}|0\rangle}{-\hbar\omega_k} - \frac{[\langle 0|H_{em}|k\rangle\langle k|H_{em}|0\rangle]^2}{(-\hbar\omega_k)^3} + 2\frac{[\langle 0|H_{em}|k\rangle\langle k|H_{em}|0\rangle]^3}{(-\hbar\omega_k)^5} + ... \Big]. \tag{4}$$

The negative signs in those denominators in (4) of numerator terms to the odd powers are canceled by the property that the matrix elements of the creation and annihilation operators do have always opposite signs [24]. Thus we obtain from series (4) the next-order correction to (3):

$$\mathcal{E}_{qed} = \sum_k \frac{|\langle k|H_{em}|0\rangle|^2}{\hbar\omega_k} + o\Big[\frac{(e^2/\sqrt{\alpha})^4}{(\hbar c)^3}\Big] = \sum_k |\mathcal{A}_k|^2 \hbar\omega_k + o(e^2\alpha) \propto e^2[1 + o(\alpha)]. \tag{5}$$

Therefore the lowest-order QED description (1-3) yields an error of $\sim 1\%$.

But there is a snag in the above and overcoming this latter is the object of the present work. Contrary to its energy (3), **photon number (2) cannot be properly defined**. Indeed it has a $k^{-1}$ logarithmic singularity at $k \to 0$ in the corresponding integral which unfortunately occurs in the long-wavelength domain of (1) [19]. Therefore *the above lowest-order QED sketch is not self-consistent*.

It is possible to find a specific non-relativistic quantum-electrostatic system whose lowest-order QED description similar to (1-3) yields a well-defined photon number $\mathcal{P}_{qed} \propto \alpha$ free of any long-wavelength singularity. It is provided by quantum-dot Helium [25] [26] [27] and specifically by its couple of lowest-energy electrostatic bound state eigensolutions of the non-relativistic and nonlinear Schrödinger-Poisson (SP) differential system [28] [29]. In a $D = 3$ three-dimensional (3d) radial harmonic potential defined by its angular frequency $\omega$, the discrete real-valued steady-state electron-electron scattering solutions $u_i(X)$ corresponding to the two lowest-energy $S = 0$ eigenstates ("s" states) —ground state $|a)$ and 1rst excited state $|b)$— are defined by the following SP system in appropriate dimensionless units (we use parenthesis instead of brackets in $|a, b)$ to emphasize eigenstate nonlinearity):

$$\left[\frac{d^2}{dX^2} + \frac{D-1}{X}\frac{d}{dX} + C_i(X) - \frac{1}{4}X^2\right] u_i(X) = 0. \tag{6}$$

$$\left[\frac{d^2}{dX^2} + \frac{2}{X}\frac{d}{dX}\right] C_i(X) = u_i^2(X) \quad i = a,\, b, \tag{7}$$

where

$$C_i(X) = \mu_i - e\Phi_i(X), \tag{8}$$

is related to eigenvalue (or chemical potential) $\mu_i$ in units of $\hbar\omega$. It is defined by the nonlinear eigenstate $u_i$, together with its corresponding Coulomb potential $\Phi_i(X)$. The initial conditions are:

$$u_i(0) = u_{i0} \quad ; \quad \Big[\frac{du_i}{dX}\Big]_{X=0} = 0, \tag{9}$$

$$C_i(0) = C_{i0} \quad ; \quad \Big[\frac{dC_i}{dX}\Big]_{X=0} = 0, \tag{10}$$

and the eigenstate's regular boundary condition reads:

$$\lim_{x\to\infty} u_i(X) = 0. \tag{11}$$

For all eigenstates, the normalization condition is:

$$\forall i \quad \int_0^\infty u_i^2(X) X^{D-1} dX = \mathcal{N}. \tag{12}$$

It is indeed defined by the quantum-classical order parameter that only depends on the confining parabolicity $\omega$:

$$\mathcal{N} = \frac{e^2/L}{\hbar\omega} \propto \frac{1}{\sqrt{\omega}}. \tag{13}$$

The characteristic length $L = \sqrt{\hbar/2m\omega}$ is the "harmonic length" of the oscillator. It defines the reduced radial coordinate $X = r/L$. The linear regime $\lim_{\mathcal{N}\to 0} u_i \sim 0$ where the nonlinear effects become negligible in agreement with Eqs (6-7) since they become decoupled corresponds to the strongly-confined case $\omega \to \infty$. Eigenstate non-orthogonality —or overlap— is defined by the inner product in agreement with normalization (12):

$$(a|b) = \frac{1}{\mathcal{N}} \int_0^\infty u_a u_b X^{D-1} dX. \tag{14}$$

In units of $\hbar\omega$, the nonlinear eigenvalues $\mu_i$ can be calculated by use of either the initial conditions $X = 0$ in (8) or the boundary conditions $X \to \infty$ in Poisson equation (7):

$$\mu_i = C_i(0) + e\Phi_i(0) = \lim_{X\to\infty} \Big[C_i(X) + \frac{\mathcal{N}}{X}\Big], \tag{15}$$

where

$$e\Phi_i(0) = \int_0^\infty G(0,X) u_i^2 X^2 dX = \int_0^\infty \frac{1}{X} u_i^2 X^2 dX, = \int_0^\infty u_i^2 X dX, \tag{16}$$

by use of the 3d Green function $G(X', X) = 1/|X' - X|$ of Eq. (7) at $X' = 0$. Equations (15-16) provide an excellent test for the accuracy of the numerical code: we obtained a $10^{-8}$ precision by use of MatLab's ode45 integration code [30].

Now define the electrostatic potential:

$$W(X) = \Phi_b(X) - \Phi_a(X), \tag{17}$$

by use of its two first nonlinear eigenstates $u_{a,b}$ of system (6-12) together with their respective Coulomb potentials $\Phi_{a,b}(X)$ given by Eqs (8) and (15). Apply to it the lowest-order QED

scheme (1-3) with its $\sim 1\%$ error. By use of Poisson equation and with our dimensionless units, potential (17) yields the charge density:

$$\rho(X) = \frac{e}{4\pi\mathcal{N}L^3}[u_b^2(X) - u_a^2(X)]. \tag{18}$$

Use the nonlinear electromagnetic interaction operator defined (in gaussian units) by potential (17) [28]:

$$H_w = \mathcal{N}\int \mathbf{j}\mathbf{A}d^3x = \Big(\frac{2\pi}{L}\Big)^{3/2}\mathcal{N}\sqrt{2\pi\hbar c}\int\Big[\mathbf{a}_{k'}\rho^*_{k'} + \mathbf{a}^+_{k'}\rho_{k'}\Big]\frac{d^3k'}{\sqrt{k'}}. \tag{19}$$

In (19), $\mathbf{j}$ is the 4-vector particle current density and $\mathbf{A}$ the related potential operator; $\mathbf{a}^+_{k'}$ and $\mathbf{a}_{k'}$ are respectively the creation and the annihilation operators of scalar photons with wave vector $k'$ and frequency $\omega_{k'} = ck'$ [24]. We use the stationary configuration where $\mathbf{j}$ reduces to its 4th charge density component (18), while $\rho_{k'}$ is the radial Fourier component of $\rho$. Photon amplitude (1) together with the properties of operators $\mathbf{a}^+$ and $\mathbf{a}$ yield:

$$\mathcal{A}_k = \frac{\langle k|H_w|0\rangle}{-\hbar ck} = -\Big(\frac{2\pi}{L}\Big)^{3/2}\mathcal{N}\sqrt{2\pi\alpha}\frac{n_k}{k^{3/2}}, \tag{20}$$

where $n = \rho/e$ is the particle density defined by (18). Its 3-dimensional radial Fourier component (as a function of the reduced wave number $\kappa = Lk$: hence $\kappa X = kr$) is $n_\kappa = g(\kappa)/(\mathcal{N}\sqrt{8\pi^3})$ with:

$$g(\kappa) = \int_0^\infty \Big[[u_b(X)]^2 - [u_a(X)]^2\Big]\frac{\sin\kappa X}{\kappa X}X^2dX. \tag{21}$$

The lowest-order QED description of the classical non-relativistic electrostatic potential energy $eW$ by use of the charge density (18) yields the photon number:

$$\mathcal{P}_{3d} = \sum_k |\mathcal{A}_k|^2 = \Big(\frac{L}{2\pi}\Big)^3\int_0^\infty |\mathcal{A}_k|^2 4\pi k^2dk = \frac{\alpha}{\pi}\int_0^\infty g^2(\kappa)\frac{d\kappa}{\kappa}, \tag{22}$$

in agreement with (2). But, contrary to (2), *it has no singularity in the long-wavelength limit* $\kappa \to 0$ since $g(0) = 0$ as the result of normalization (12) with D=3. Therefore it can be used to establish a link to some appropriate specific physical property of the non-relativistic —hence $\alpha$-free— nonlinear SP differential system (6-12).

We have the following theorem about nonlinear eigenstate overlap (14) [31] [32]:

$$(a|b) = \frac{eW_{ab}}{\mu_b - \mu_a} = \frac{e\Phi^a_{ab}}{\mu_a - \mu_b} + \frac{e\Phi^b_{ba}}{\mu_b - \mu_a}. \tag{23}$$

The respective Coulomb potentials are labelled here as superscripts when necessary in order to avoid any confusion with the real-valued matrix element $\Phi^i_{ab} = (a|\Phi^i|b) = (b|\Phi^i|a) = \Phi^i_{ba}$. The physical significance of theorem (23) can be illustrated from lowest-order QED (actually first-order time-independent perturbation theory in quantum mechanics [33]). Specifically, matrix element $\Phi^a_{ab}/(\mu_a - \mu_b)$ describes an absorption-like transition; namely, the probability amplitude for the system being in nonlinear ground eigenstate $|a)$ to populate nonlinear excited eigenstate $|b)$ as a result of interaction potential $\Phi_a$ defined by probability density

$|u_a|^2 = u_a^2$ through (7-8) and (15). Similarly, the second term $\Phi_{ba}^b/(\mu_b - \mu_a)$ would define the reverse process induced by interaction potential $\Phi_b$ defined by $u_b^2$; namely, the probability amplitude for the system being in excited eigenstate $|b)$ to populate ground state $|a)$ through an emission-like transition. However, since the system is conservative, there are no external photons to induce these two processes. They are only due to the intrinsic nonlinearity of system (6-12) as shown by theorem (23). Therefore we might equivalently wish to consider them within the QED framework as follows:

1. Nonlinearity yields eigenstate overlap (14);
2. Eigenstate overlap (14) creates virtual QED photons (22);
3. Virtual photons (22) induce eigenstate transitions (23).

These photons interfere [32] in the build-up of the inner product $(a|b)$ as shown by theorem (23). At any extremum of $(a|b)$ —e.g. at the minimum of $(a|b)$, i.e. at the maximum $(a|b)^2_{max}$ of overlap probability $(a|b)^2$ reached at $\mathcal{N} = \mathcal{N}_{max}$—, the two nonlinearly-induced emission/absorption photon amplitudes are in phase and resonantly add up. Such a resonance is defined in QED description (4) by all $\langle 0|H_w|k\rangle\langle k|H_w|0\rangle$ terms in the series that yield the photon gas energy. As a consequence, **we expect photon number (22) to equal** —within the $\sim 1\%$ tolerance defined by (5)— **maximum overlap probability** $(a|b)^2_{max}$ **at** $\mathcal{N} = \mathcal{N}_{max}$. Since the former is proportonal to $\alpha$ while the latter is independent of $\alpha$, we recover the value $(a|b)^2_{max} \sim \alpha$ that has been repeatedly found —though sometimes with questionable arguments— in quantum-dot Helium [28] [32] [34] [35] [36]. Quantitatively, (22) yields at $\mathcal{N} = \mathcal{N}_{max}$ [28]:

$$\alpha = \pi \left[ \frac{(a|b)^2_{max}}{\int_0^\infty g^2(\kappa)\kappa^{-1}d\kappa} \right]_{\mathcal{N}=\mathcal{N}_{max}}. \tag{24}$$

Therefore **the numerical value** $\sim 1/137$ **of** $\alpha$ **is mathematically defined by the sole nonlinear spectral properties of dimensionless eigenvalue differential system (6-12)**: Thus it is a transcendental number of mathematical origin which, like $\pi$ or $e$, should probably be defined by a plethora of other mathematical models and remain invariant over cosmological times.

Two results confirm the quite specific role played in (24) by the maximum overlap probability $(a|b)^2_{max}$ at $\mathcal{N}_{max}$: see fig. 1. Firstly, the direct mathematical solution at $\mathcal{N}_{max}$ of Hartree's original integro-differential system by use of a rapidly convergent series of its nonlinear eigenstates [37]. Secondly, the following $D = 2$ two-dimensional electron gas geometry which strongly modifies both Schrödinger and Poisson equations. Indeed, they respectively become in their dimensionless form ($i = a, b$):

$$\left[ \frac{d^2}{dX^2} + \frac{1}{X}\frac{d}{dX} + C_i - \frac{1}{4}X^2 \right] u_i = 0, \tag{25}$$

$$\left[ \frac{d^2}{dX^2} + \frac{2}{X}\frac{d}{dX} \right] C_i = \frac{u_i^2}{X}. \tag{26}$$

The normalization condition is still given by (12) with $D = 2$. The $X^{-1}$ factor in the source term of Poisson equation (26) is due to Gauss' theorem. When $X \to \infty$, the electrostatic interaction energy $e\Phi_i$ defined by Eqs (8), (15-16) and (26) yields in units of $\hbar\omega$

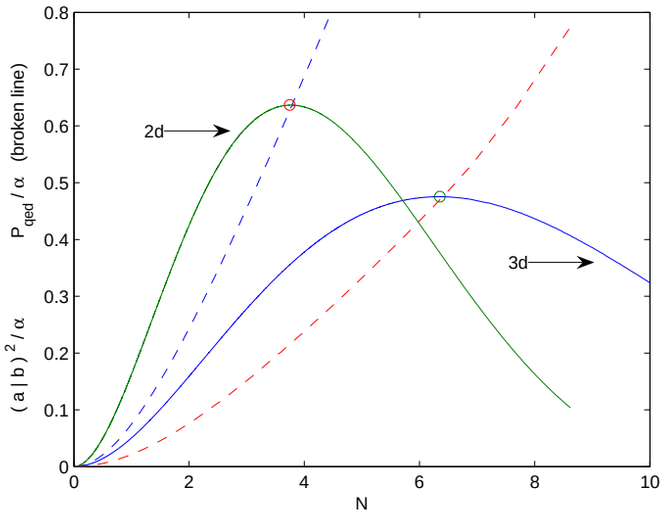


FIG. 1: Photon number $\mathcal{P}_{3d,2d}$ (broken lines) versus square eigenstate overlap $(a|b)^2$ (continuous line) in units of $\alpha$. The circles indicate the maxima of $(a|b)^2$ corresponding to the actual minima of eigenstate overlap $(a|b)$.

$\lim_{X\to\infty} e\Phi_i(X) = X^{-1}\int_0^{X\to\infty} u_i^2(X')X'dX' = \mathcal{N}/X$ through multiplication of (26) by $X^2$ and integration by parts. This result fits normalization (12) with $D = 2$; see also (15). In contrast, droping the $X^{-1}$ factor in the r.h.s. of (26) would yield the 3-dimensional electron density integral $\lim_{X\to\infty} e\Phi_i(X) = X^{-1}\int_0^{\infty} u_i^2(X')X'^2dX'$ instead of the correct 2d one.

In this 2-d electron frame, the charge density (18) now becomes :

$$\rho = \frac{e}{4\pi\mathcal{N}L^2}\Big[u_b^2 - u_a^2\Big], \tag{27}$$

while the corresponding 2d Fourier component of $n = \rho/e$ is $n_\kappa = h(\kappa)/(4\pi\mathcal{N})$ where:

$$h(\kappa) = \int_0^{\infty} [u_b^2(X) - u_a^2(X)]J_0(\kappa X)XdX, \tag{28}$$

and $J_0$ is the Bessel function of the 1rst kind. Therefore the QED probability defined by (20-22) becomes in the present 2-dimensional parabolic electron confinement:

$$\mathcal{P}_{2d} = \frac{\alpha}{2}\int_0^{\infty} h^2(\kappa)\frac{d\kappa}{\kappa}. \tag{29}$$

Since $J_0(0) = 1$ and hence $h(0) = 0$ by use of normalization (12) with D=2, we recover the same regularity of $\mathcal{P}_{2d}$ at $\kappa \sim 0$ as in the 3d electron case (22).

Figure 1 displays the final numerical results. The intersections of photon number $\mathcal{P}_{3d,2d}$ with their respective square eigenstate overlap $(a|b)^2$ occur as expected about the maxima of these latter (circles). In accordance with (22), we have in the 3d case $\mathcal{P}_{3d} = 0.4714\,\alpha$ at maximum eigenstate overlap $(a|b)^2_{max} = 3.47\,10^{-3}$ $(= 0.4755\alpha)$ reached at $\mathcal{N}_{max} = 6.3542$. Hence

the expected $\sim 1\%$ relative error about this intersection, due to the difference between 0.4755 and 0.4714. According to (24), the 3d electron configuration yields $\alpha_{3d} = 3.47\,10^{-3}/0.4714 = 7.361\,10^{-3}$. Similarly, $(a|b)^2_{max} = 4.648\,10^{-3}$ $(= 0.6370\alpha)$ at $\mathcal{N}_{max} = 3.7477$ in the 2d case, while (29) yields $\mathcal{P}_{2d} = 0.6309\,\alpha$. Thus $\alpha_{2d} = 4.648\,10^{-3}/0.6309 = 7.367\,10^{-3}$ in the 2d case. It differs from $\alpha_{3d} = 7.361\,10^{-3}$ by 0.08%. Such an excellent agreement between these two $\alpha$ values validate the present theory and exclude any fortuitous numerical coincidence. Moreover, they both approach the exact value $\alpha = 7.297...\,10^{-3}$ within less than $1\,\%$ error, in perfect agreement with the QED estimation obtained by (5).

Let me conclude:

1. Astrophysical tests of the stability of the fine-structure constant $\alpha \sim 1/137$ are very active in observational research [6]-[12].

2. I obtain $\alpha = 7.364\,10^{-3}$ (vs experimental $\alpha = 7.297...\,10^{-3}$) from the mathematical solution of a dimensionless eigenvalue differential problem derived from the dual description of: i) a stationary nonlinear mean-field interacting quantum system; and ii) its lowest-order QED counterpart.

3. This result strongly suggests the mathematical transcendental nature of $\alpha$ and therefore its astrophysical as well as its cosmological stability.

4. The corner stone of the solution lies in the physical similarity of nonlinear eigenstate overlap with induced eigenstate transitions by QED virtual photons.

5. Despite its 1% relative error due to deliberate simplifications, I believe that the present theory constitutes a pivotal progress towards the solution of what Feynman stressed as "*one of the greatest damn mysteries in physics*" [13].

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