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# In the tradition of Thurston III, Geometry and Dynamics. Introduction

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► **To cite this version:**

Ken'Ichi Ohshika, Athanase Papadopoulos. In the tradition of Thurston III, Geometry and Dynamics. Introduction. In the tradition of Thurston III, Geometry and Dynamics Geometry and Dynamics, Springer, 2024, 978-3-031-43501-0. 10.1007/978-3-031-43502-7\_1 . hal-04530068

**HAL Id: hal-04530068**

**<https://cnrs.hal.science/hal-04530068>**

Submitted on 2 Apr 2024

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# Chapter 1: Introduction

Ken'ichi Ohshika and Athanase Papadopoulos

## Abstract

In this chapter, we describe in some detail the various topics covered in the volume *In the tradition of Thurston, III*, edited by K. Ohshika and A. Papadopoulos, published by Springer (20224).

**Keywords** Thurston metric, Teichmüller space, quasimetric map, quasiconformal map, Weil–Petersson metric, period embedding, asymptotic counting problems, complex-analytic moduli problems, stacks and moduli problems, aspherical manifold, domination, Gromov order, monotone invariant, Kodaira dimension, Anosov diffeomorphism.

**AMS codes** 14D23, 20F34, 22E25, 30C62, 30F30, 30F35, 30F40, 30F60, 32G13, 37D20, 51F30, 53A15, 55R10, 58J52, 57M05, 57-06, 57M10, 57M50, 57K32, 81T40.

Since his works of the 1970s, William Thurston (1946–2012) have changed the landscape of mathematics, especially the fields of geometry, topology, geometric group theory and dynamics. The aim of the collection *In the tradition of Thurston*, of which the present volume is the third, is to provide the reader with an overview of important aspects of Thurston’s heritage, including ideas he introduced and developments that have taken place in fields closely related to his work.

The topics considered in this volume include Thurston’s asymmetric metric of the Teichmüller spaces of hyperbolic surfaces, with its analogue on the Teichmüller space of Euclidean surfaces, the Weil–Petersson geometry of Teichmüller space, Kleinian groups, holomorphic motions, earthquakes, 3-manifolds, geometric structures, homeomorphism groups of 2-manifolds, dynamics, and orbifolds together with their relations with V-manifolds in algebraic geometry.

Let us review in some detail the contents of each chapter.

The first two chapters that follow this introduction are concerned with Thurston’s asymmetric metric on Teichmüller space. Here, the distance between two hyperbolic metrics on a given surface is defined to be the logarithm of the smallest global Lipschitz constant of a homeomorphism between them which is in the homotopy class of the identity. This metric was introduced by Thurston in his paper *Minimal stretch maps between hyperbolic surfaces*, first circulated in 1985. Thurston’s metric and its generalizations to various settings have been the object of intense and growing research in the last two decades by several authors, and having good surveys on such a topic should be useful to the geometers.

Chapter 2, by Huiping Pan and Weixu Su, is titled *The geometry of the Thurston metric: a survey*. It is a panorama of old and new constructions and results, due to several authors, related to Thurston’s metric. Some of Thurston’s work from his original paper is also reviewed. A central element of this chapter is the construction of extremal Lipschitz maps, called *stretch maps*, between hyperbolic surfaces. These maps, by varying the stretching parameter, give rise to one-parameter families called *stretch lines*. The authors review the behavior at infinity of these lines, making relations between Thurston’s metric and Thurston’s compactification of Teichmüller space. They also discuss the coarse geometry of this metric, describing recent rigidity results concerning its isometry group as well as developments and generalizations of this metric to other settings.

Chapter 3, by Binbin Xu, titled *Thurston’s metric on Teichmüller space of flat tori*, is a survey of results by various authors on Thurston’s metric extended to the case of the Teichmüller space of flat tori, reviewing in particular the relation between best Lipschitz maps and affine maps between flat tori.

Chapter 4 is built on another important theory developed by Thurston in his study of surfaces, namely, the earthquake theorem. Earthquakes are deformations of hyperbolic surfaces which generalize the Fenchel–Nielsen twists along weighted simple closed curves to deformations associated with measured geodesic laminations. Thurston’s *earthquake theorem*, in its original form, states that any two points in the Teichmüller space of a closed hyperbolic surface can be joined by a unique left earthquake. This theorem was used in an essential way by S. Kerckhoff in his 1983 paper *The Nielsen realization problem*. Kerckhoff also provided the first written proof of this theorem. In his paper *Earthquakes in two-dimensional hyperbolic geometry*, published in 1986, Thurston gave another proof of the earthquake theorem in the setting of the universal Teichmüller space (from which one can also recover the case of closed surfaces). In this context, the earthquake theorem says that every orientation-preserving homeomorphism of the unit circle (seen as the boundary of the hyperbolic disc) admits an extension to the hyperbolic disc which is a left (or right) earthquake. In Chapter 4 of the present volume, titled *The Anti-de Sitter proof of Thurston’s earthquake theorem*, Farid Diaf and Andrea Seppi give a new proof, in the setting of anti-de Sitter geometry, of the universal Teichmüller space version of the earthquake theorem. This proof is based on the bi-invariant geometry of the Lie group  $\mathrm{PSL}(2, \mathbb{R})$ , which is also the anti-de Sitter three-space. Diaf and Seppi also recover the original earthquake theorem for closed surfaces that we find in Kerckhoff’s paper.

Chapter 5, by Nicholas Vlamis, is concerned with the geometry of infinite-type surfaces. It is titled *Homeomorphism groups of self-similar 2-manifolds*. Mapping class groups of infinite-type surfaces (sometimes called *big mapping class groups*) have been a subject of intense research in the last few years. In Chapter 5, Vlamis studies homeomorphism groups and mapping class groups of a class of infinite type surfaces which he describes as self-similar 2-manifolds. These are surfaces that exhibit a certain type of homogeneity akin to the 2-sphere and the Cantor set. The theory developed in this chapter is a natural extension of that of homeomorphism groups of the 2-sphere and of the Cantor set. In doing so, Vlamis

generalizes recent results on mapping class groups to the setting of homeomorphism groups. At the same time, he introduces the tools needed for the study of end spaces of 2-manifolds, and he provides the necessary background for the study of the topology of homeomorphism groups and mapping class groups of self-similar 2-manifolds. In particular, he develops the structure theory of stable sets and he obtains strong versions of several recent results on this topic.

Chapter 6, by Eric Schippers and Wolfgang Staubach, is titled *Weil–Petersson Teichmüller theory of surfaces of infinite conformal type*. The Weil–Petersson geometry of Teichmüller space is a rich theory that involves algebraic geometry, complex geometry and analysis, and it has applications in mathematical physics. The topics surveyed by Schippers and Staubach include complex Hilbert manifolds, Kähler geometry, global analysis and generalizations of the period mapping. The authors also discuss the motivations of the theory which originated, at the beginning of the 1980s, in relation with representation theory, and they explain how these developments led to the Kähler geometry of the moduli spaces of surfaces of infinite conformal type, making connections with geometry, analysis, representation theory and physics, in particular fluid mechanics and two-dimensional conformal field theory.

Chapter 7, by Hiroshige Shiga and Toshiyuki Sugawa, is titled *Kleinian groups and geometric function theory*. Here, the authors use the term “geometric function theory” to denote the geometric study of analytic functions and domains in the complex plane and on Riemann surfaces. In particular, they are interested in the complex analytic properties of the region of discontinuity and the limit set of a Kleinian group. They review several kinds of domains in  $\mathbb{R}^n$ , with the Bers embedding of Teichmüller space in the background. This is an embedding which realizes this space as a bounded domain in a complex Banach space. After this review, Shiga and Sugawa discuss several versions of the so-called *modified Bers conjecture* for Fuchsian groups, which concerns the closure of the Bers embedding and which was extended by Sullivan and Thurston to the so-called *density conjecture*. The latter says that every finitely generated Kleinian group is an algebraic limit of geometrically finite Kleinian groups. The conjecture has been proved by Namazi and Souto, and by Ohshika independently. Several hints and developments of these works are given. Results of Sullivan–Thurston, Bers–Royden and Slodkowski on holomorphic motions are also discussed.

Chapter 8, by Ken’ichi Ohshika, is titled *Thurston’s broken windows only theorem revisited*. The title refers to a theorem which appears in Thurston’s preprint *Hyperbolic structures on 3-manifolds III: Deformations of 3-manifolds with incompressible boundary*, which was circulated in 1986 and which constitutes the third of a series of papers in which Thurston gave a proof of his uniformization theorem for Haken manifolds. The *broken windows only theorem* is the main result of that third paper, and, as Ohshika notes, it was intended to be the key part of the proof of the so-called *bounded image theorem*, one of the main steps of the proof of the Haken uniformization theorem. The broken windows only theorem consists of two statements, and Ohshika shows that the second of them is

false. Indeed, he gives a counter-example to this statement. He provides a proof for a weaker version of it, which, however, is not sufficient for the proof of the bounded image theorem. At the same time, Ohshika provides a short summary of the intricate history of the bounded image theorem, for which a weak version, known as the *bounded orbit theorem*, which appears in Thurston’s introductory paper of the series we mentioned, is sufficient for the proof of his uniformization theorem. A proof of the bounded orbit theorem was given by M. Kapovich in his book *Hyperbolic manifolds and discrete groups* published in 2001. Ohshika also speculates on how the second statement in Thurston’s broken windows only theorem was intended to constitute a proof of the bounded image theorem.

Motivated by Thurston’s geometrization picture in dimension three, Christoforos Neofytidis, in Chapter 9, titled *Geometric structures in topology, geometry, global analysis and dynamics*, addresses several questions on manifolds of arbitrary dimension. Some of these questions concern geometric structures in the sense of Thurston, and others involve non-geometric manifolds. The author’s goal is to provide a unified treatment of several problems arising in topology, geometry, global analysis and dynamics. The main problems with which he is concerned are: (i) The existence of an order on the set of homotopy classes of manifolds of a given dimension, which he calls “Gromov order” or “domination relation”: here, a manifold  $M$  *dominates* a manifold  $N$  if there exists a map  $f : M \rightarrow N$  of non-zero degree; (ii) the Gromov–Thurston monotonicity problem for numerical homotopy invariants with respect to the domination relation: here, the author introduces a notion of *geometric Kodaira dimension*; (iii) the existence of Anosov diffeomorphisms, in relation with the Anosov–Smale conjecture stating that all Anosov diffeomorphisms are conjugate to hyperbolic automorphisms of nilmanifolds. Neofytidis shows that Thurston’s geometrization gives a unified approach to these questions.

Chapter 10, by Jayadev Athreya, is titled *Counting problems for invariant point processes*. In this chapter, using linear algebra and the ergodic theory of  $SL(2, \mathbb{R})$ -actions, the author shows how to solve several natural asymptotic counting problems concerning discrete subsets of the plane. The problems arise from the geometry of surfaces. The point of view is axiomatic, and the goal is to give the general important ideas in this theory, with a focus on some key motivating examples. Poisson point processes, which are famous examples of  $SL(2, \mathbb{R})$ -probability invariant measures on the space of discrete subsets of the complex plane, act as motivating examples. Connections with Thurston’s work arise from the latter’s theory of geometric structures on surfaces, where counting problems emanate naturally, in relation with the study of interval exchange maps and quadratic differentials. Athreya also discusses applications of these ideas to the problems of counting holonomies of saddle connections, lattice points, and fine scale distribution in various contexts. Besides being inspired by Thurston’s work, his perspective is also motivated by works of Veech, Eskin–Masur, Marklof, Athreya–Ghosh and others.

Chapter 11, by Juan Martín Pérez and Florent Schaffhauser, is titled *Orbifolds and the modular curve*. The authors start from the observation that the ramification points which appear in the geometric concept of orbifold, developed by Satake and then Thurston, have

an analytic and algebraic counterpart, in a setting where one deals with morphisms that are not *étale*. They recall that in algebraic geometry, there is a theory which is parallel to Thurston's theory of orbifolds and their fundamental groups, and they survey this theory, developing the idea that the two points of view, the algebro-geometric and the Satake–Thurston, can enrich the scope and methods of each other. In particular, they present a construction of the moduli stack of elliptic curves and show that it is in fact an analytic orbifold.

Chapter 12, the last chapter of this volume, by Athanase Papadopoulos, is titled *Footnotes on Thurston's Notes on the Geometry and topology of three-manifolds*. In this chapter, the author provides some historical remarks, addenda and references with comments on some topics tackled by Thurston in his famous notes, *The geometry and topology of three-manifolds*. The various sections in this chapter, considered as footnotes to Thurston's notes, mostly concern the parts of Thurston's notes on hyperbolic geometry, geometric structures, the computation of volumes of hyperbolic polyhedra and the so-called Koebe–Andreiev–Thurston theorem. The author discusses in particular some works of Lobachevsky, Andreiev and Milnor, with an excursus in Dante's cosmology, based on the insight of Pavel Florensky.

The present volume will have a sequel, naturally titled *In the tradition, of Thurston, IV*. We all hope that the collection of articles that constitute this collection will be useful for students and researchers who are interested in Thurston's mathematics.

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