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# Stability analysis of a class of $2 \times 2$ triangular hyperbolic systems

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**Abstract:** In this paper, the stability analysis of a class of hyperbolic systems with non-diagonalizable principal part and nonuniform coefficients is addressed. We give a stability condition and express the convergence rate that can be obtained. Two cases will be treated independently: with and without a source term. A numerical scheme is designed to illustrate the performance of the Lyapunov stability analysis.

*Keywords:* Linear PDE, Weakly triangular hyperbolic system, Stability analysis, Strict Lyapunov functional.

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## 1. INTRODUCTION

In recent years, significant progress has been made in understanding the stability and control of strictly hyperbolic systems, thanks to the contributions of Bastin and Coron (2016), among others. In particular, control strategies for hyperbolic systems using backstepping with Volterra or Fredholm transformations have been proposed Krstic (2009); Bribiesca-Argomedo and Krstic (2015); Coron et al. (2013). These mathematical advances can then be applied to a wide range of applications, such as drilling processes Bresch-Pietri and Krstic (2014), navigable rivers (Bastin and Coron, 2016, Chapter 8) or wastewater treatment Valentin et al. (2023).

However, as soon as the system is weakly hyperbolic, for instance having a non-diagonalisable principal part with eigenvalues of multiplicity greater than one, only few stability results and even fewer control results are available (see however Alabau-Boussouira (2002); Métivier and Zumbrun (2005)). In this paper, we deal with the case of hyperbolic systems of dimension  $2 \times 2$  when the principal part has a positive eigenvalue of geometric multiplicity equal to two, at least on a part of the spatial domain.

By leveraging Lyapunov analysis, a strict Lyapunov functional is designed which, under certain conditions, allows us to assess the global exponential stability. This Lyapunov functional relies heavily on input-to-state properties Priour and Mazenc (2012) satisfied by our class of system. In addition, we have endeavored to get as close as possible to the optimum bounds Lamare et al. (2016) on the convergence rate and the overshoot. To achieve this, we played with different weights, in particular using polynomial rather than exponential weights in the Lyapunov functional kernels.

Section 2 introduces the system class, along with the accompanying definitions and the statement of the problem. In Section 3, two stability conditions are presented, differentiating between cases with and without a source term. Section 4 illustrates the theoretical findings through the simulation of two academic examples.

*Notation:* Throughout the paper, the partial derivatives in time and space are denoted  $\partial_t$  and  $\partial_x$ , while the classical derivative are  $\dot{\mathcal{V}} = \frac{d}{dt}\mathcal{V}$  and  $\cdot$ . Moreover,  $C_{pw}^k(0, 1; \mathbb{R})$  denotes the space of functions defined from  $[0, 1]$  to  $\mathbb{R}$  for which the  $k - 1$  first derivatives are continuous and the  $k$ -th derivative is piece-wise continuous;  $L^2((0, 1); \mathbb{R})$  denotes the space of square-integrable functions from  $(0, 1)$  to  $\mathbb{R}$ ;  $H^1((0, 1); \mathbb{R}) = \{f : (0, 1) \rightarrow \mathbb{R} \mid f \in L^2((0, 1); \mathbb{R}), f' \in L^2((0, 1); \mathbb{R})\}$  and  $X = L^2((0, 1); \mathbb{R}) \times H^1((0, 1); \mathbb{R})$  equipped with the norm:

$$\|\cdot\|_X^2 : (f, g) \mapsto \|f\|_{L^2((0, 1); \mathbb{R})}^2 + \|g\|_{L^2((0, 1); \mathbb{R})}^2 + \|g'\|_{L^2((0, 1); \mathbb{R})}^2.$$

Given a function  $f : (0, 1) \mapsto \mathbb{R}$ , we denote by  $\bar{f} = \max \left\{ 0, \sup_{x \in (0, 1)} f(x) \right\}$  and  $\underline{f} = \inf_{x \in (0, 1)} f(x)$ .

## 2. PROBLEM STATEMENT

Consider the following class of triangular  $2 \times 2$  hyperbolic system

$$\begin{cases} \partial_t \phi(t, x) = \Lambda(x) (A(x) \partial_x \phi(t, x) + B(x) \phi(t, x)), & x \in (0, 1), \\ \phi(t, 1) = D \phi(t, 0), \\ \phi(0, x) = \phi^0(x) \end{cases} \quad (1)$$

with  $\phi(t, x) = (\phi_1(t, x), \phi_2(t, x)) \in X$  being the state of the system defined for any  $t \geq 0$  and any  $x \in [0, 1]$ ,  $\phi^0 \in X$  a given initial condition satisfying zero-order and first-order compatibility conditions Bastin and Coron (2016).

We have  $\Lambda(x) = \begin{bmatrix} \lambda_1(x) & 0 \\ 0 & \lambda_2(x) \end{bmatrix}$  is such that  $\lambda_{1,2}(x) > 0$ , moreover we assume that  $\lambda_1 \in C_{pw}^0(0, 1; \mathbb{R})$  and  $\lambda_2 \in C_{pw}^1(0, 1; \mathbb{R})$ ,  $A(x) = \begin{bmatrix} 1 & c(x) \\ 0 & 1 \end{bmatrix}$  where  $c \in C_{pw}^0(0, 1; \mathbb{R})$ , the source term  $B(x) = \begin{bmatrix} B_{11}(x) & B_{12}(x) \\ 0 & B_{22}(x) \end{bmatrix}$  is continuous, we assume  $B_{11}, B_{12} \in C_{pw}^0(0, 1; \mathbb{R})$ ,  $B_{22} \in C_{pw}^1(0, 1; \mathbb{R})$  and  $D = \begin{bmatrix} D_{11} & D_{12} \\ 0 & D_{22} \end{bmatrix}$ .

System (1) is worth of interest in cases where the principal part  $A(x)$  is not diagonalizable for some part of the space  $x \in (0, 1)$ , in other words if there exists  $x \in (0, 1)$  such that  $\lambda_1(x) = \lambda_2(x)$  and if for all  $x \in (0, 1)$ ,  $c(x) \neq 0$ .

*Lemma 1.* For any initial condition  $\phi(0) = \phi^0 \in X$ , the system (1) is well-posed and the (weak) solution  $t \mapsto \phi(t; \phi^0)$  belongs to  $C([0, +\infty); X)$  for all  $t \geq 0$ .

**Proof.** See Garetto et al. (2018). ■

*Definition 1.* The origin of system (1) is said to be globally  $\delta$ -exponentially stable if there exists an overshoot  $\kappa \geq 1$  and a decay rate  $\delta > 0$  such that for any initial condition  $\phi^0 \in X$ :

$$\|\phi(t; \phi^0)\|_X^2 \leq \kappa e^{-\delta t} \|\phi^0\|_X^2, \quad \forall t \geq 0.$$

In this paper, we aim at giving sufficient conditions that ensures exponential stability of system (1) in the sense of Definition 1.

### 3. STABILITY ANALYSIS

#### 3.1 First study case : without the source term

In this section, we express a stability condition on system (1) in the case where the source term,  $B$ , is null. This one imposes a restriction on the diagonal terms of  $D$  with respect to the function  $\lambda_2$ .

*Assumption 1.* The function  $\lambda_2$  and the matrix  $D$  satisfy the following inequality

$$\max(D_{11}^2, D_{22}^2) < 1 - \frac{\bar{\lambda}_2}{\lambda_2}. \quad (2)$$

Based on this assumption, we can now give the following result which establishes exponential stability of the origin and gives an estimate on the exponential convergence rate and the overshoot.

*Theorem 1.* Assume  $B = 0$ . Let Assumption 1 holds and let  $\nu := \frac{\bar{\lambda}_2}{\lambda_2 - \bar{\lambda}_2}$ ,  $\bar{\nu} := \frac{1 - D_{22}^2}{D_{22}^2}$ . Then, for any  $\delta < \min\{\frac{\bar{\nu}\lambda_1}{1+\bar{\nu}}, \frac{\bar{\nu}\lambda_2}{1+\bar{\nu}}, \frac{(\bar{\nu}-\nu)\lambda_2}{(1+\bar{\nu})(1+\nu)}\}$ , the origin of system (1) is  $\delta$ -exponentially stable with an overshoot given by

$$\kappa = \frac{\frac{1}{\lambda_1} + \frac{\alpha_{20}}{\lambda_2} + \frac{\alpha_{21}}{\lambda_2}}{\min(\frac{1}{\lambda_1}, \frac{\alpha_{20}}{\lambda_2}, \frac{\alpha_{21}}{\lambda_2})} (1 + \bar{\nu} - \frac{\varepsilon}{2}), \quad 0 < \varepsilon < \bar{\nu} - \nu,$$

and  $\alpha_{20}, \alpha_{21}$  given by (10).

With Assumption 1, it can be checked that  $\nu$ ,  $\bar{\nu}$  and  $\delta$  which appear in the statement of Theorem 1 are all well defined and positive.

*Remark 1.* Note that this result is inspired by Diagne et al. (2012); Prieur and Mazenc (2012) and extends (Bastin and Coron, 2016, Theorem 6.10) for the case of triangular hyperbolic structure. For the diagonal case, i.e.  $c = 0$  and when  $\lambda_2' = 0$ , we recover the well-known stability condition  $|D_{11}| < 1$  and  $|D_{22}| < 1$  (i.e.  $D$  is a Schur matrix).

*Remark 2.* This result is established due to the cascade structure of system (1). With assumption 1, it can be shown that the origin of the first subsystem with state  $\phi_2$  is globally exponentially stable in  $H^1$  norm for any decay rate smaller than  $\bar{\nu}\lambda_2$  whereas the origin of the second part of state  $\phi_1$  is input-to-state stable with respect to the  $H^1$  norm of  $\phi_2$ .

*Remark 3.* The presence of the non diagonal term  $c$  in the system definition forces us to work with the  $L^2 \times H^1$  norm (see Garetto et al. (2018)). We found that when  $D_{21} \neq 0$ , the term  $D_{21}\partial_x\phi_2(0)\partial_x\phi_1(0)$  in the stability analysis cannot be handled. It remains unclear whether the system (1) is stable at the origin when  $D_{21} \neq 0$ . Our numerical observations suggest that the presence of this term significantly affects the convergence properties of the system.

**Proof.** The method used to prove Theorem 1 relies on the construction of a Lyapunov functional inspired by Kharitonov (2012); Krstic (2009). Consider the following functional:

$$\mathcal{V}(\phi) = \int_0^1 \phi^\top(x) P(x) \phi(x) dx + \int_0^1 \partial_x \phi^\top(x) Q(x) \partial_x \phi(x) dx, \quad (3)$$

with matrices  $P(x)$  and  $Q(x)$  given by

$$P(x) = \begin{bmatrix} \frac{1+\mu x}{\lambda_1(x)} & 0 \\ 0 & \alpha_{20} \frac{1+\mu x}{\lambda_2(x)} \end{bmatrix}, \quad Q(x) = \begin{bmatrix} 0 & 0 \\ 0 & \alpha_{21} \frac{1+\mu x}{\lambda_2(x)} \end{bmatrix},$$

and scalars  $\mu, \alpha_{20}, \alpha_{21} > 0$ .

This functional will be decomposed term by term as follows:  $\mathcal{V}(\phi) = \mathcal{V}_1(\phi) + \alpha_{20}\mathcal{V}_{20}(\phi) + \alpha_{21}\mathcal{V}_{21}(\phi)$  with

$$\begin{aligned} \mathcal{V}_1(\phi) &= \int_0^1 \frac{1+\mu x}{\lambda_1(x)} \phi_1^2(x) dx, \\ \mathcal{V}_{20}(\phi) &= \int_0^1 \frac{1+\mu x}{\lambda_2(x)} \phi_2^2(x) dx, \\ \mathcal{V}_{21}(\phi) &= \int_0^1 \frac{1+\mu x}{\lambda_2(x)} (\partial_x \phi_2)^2(x) dx \end{aligned}$$

First, we assess the following inequalities :

$$\begin{aligned} \min\left(\frac{1}{\lambda_1}, \frac{\alpha_{20}}{\lambda_2}, \frac{\alpha_{21}}{\lambda_2}\right) \|\phi\|_X &\leq \mathcal{V}(\phi), \\ \mathcal{V}(\phi) &\leq \left(\frac{1+\mu}{\lambda_1} + \alpha_{20} \frac{1+\mu}{\lambda_2} + \alpha_{21} \frac{1+\mu}{\lambda_2}\right) \|\phi\|_X, \end{aligned} \quad (4)$$

Along  $C^1$  solutions to system (1) with  $B = 0$ , the time derivative of each term of this functional is calculated below.

First, we obtain

$$\begin{aligned} \dot{\mathcal{V}}_{20}(\phi) &= 2 \int_0^1 (1+\mu x) \phi_2(x) \partial_x \phi_2(x) dx, \\ &= -\mu \int_0^1 \phi_2^2(x) dx + (1+\mu) \phi_2^2(1) - \phi_2^2(0) \end{aligned}$$

As  $\phi_2(1) = D_{22}\phi_2(0)$ , and since by Assumption 1,  $|D_{22}| < 1$ , we have

$$\dot{\mathcal{V}}_{20}(\phi) \leq -\mu\|\phi_2\|_{L^2}^2 + (\mu - \bar{\nu})\phi_2^2(1). \quad (5)$$

Second, in the same way, we obtain

$$\begin{aligned} \dot{\mathcal{V}}_1(\phi) &= 2 \int_0^1 (1 + \mu x) \phi_1(x) \partial_x \phi_1(x) dx \\ &\quad + 2 \int_0^1 c(x) (1 + \mu x) \phi_1(x) \partial_x \phi_2(x) dx, \\ &= -\mu \int_0^1 \phi_1^2(x) dx + (1 + \mu) \phi_1^2(1) - \phi_1^2(0) \\ &\quad + 2 \int_0^1 c(x) (1 + \mu x) \phi_1(x) \partial_x \phi_2(x) dx. \end{aligned}$$

For any  $a, b \in L^2((0, 1); \mathbb{R})$  and  $\gamma > 0$ , Young's inequality ensures

$$2 \int_0^1 a(x) b(x) dx \leq \gamma \int_0^1 a^2(x) dx + \frac{1}{\gamma} \int_0^1 b^2(x) dx.$$

Taking  $a(x) = |\phi_1(x)|$  and  $b(x) = |\partial_x \phi_2(x)|$ , the crossed term can be bounded as follows

$$\begin{aligned} &2 \int_0^1 c(x) (1 + \mu x) \phi_1(x) \partial_x \phi_2(x) dx \\ &\leq (1 + \mu) \bar{c} \left( \gamma \int_0^1 \phi_1^2(x) dx + \frac{1}{\gamma} \int_0^1 (\partial_x \phi_2)^2(x) dx \right) \\ &\leq (1 + \mu) \bar{c} \left( \gamma \|\phi_1\|_{L^2}^2 + \frac{1}{\gamma} \|\partial_x \phi_2\|_{L^2}^2 \right) \end{aligned}$$

with  $\bar{c} = \max_{x \in (0, 1)} |c(x)|$ . Hence, the time derivative of  $\mathcal{V}_1$  is bounded by

$$\begin{aligned} \dot{\mathcal{V}}_1(\phi) &\leq (-\mu + (1 + \mu) \bar{c} \gamma) \|\phi_1\|_{L^2}^2 + \frac{(1 + \mu) \bar{c}}{\gamma} \|\partial_x \phi_2\|_{L^2}^2 \\ &\quad + (1 + \mu) (D_{11} \phi_1(0) + D_{12} \phi_2(0))^2 - \phi_1^2(0) \end{aligned}$$

Moreover, Young's inequality ensures

$$\begin{aligned} &(1 + \mu) (D_{11} \phi_1(0) + D_{12} \phi_2(0))^2 - \phi_1^2(0) \\ &\leq (D_{11}^2 - 1) \phi_1^2(0) + D_{12}^2 \phi_2^2(0) + D_{11} D_{12} (a \phi_1^2(0) + \frac{1}{a} \phi_2^2(0)), \\ &\leq (D_{11}^2 - 1 + a |D_{11} D_{12}|) \phi_1^2(0) + (D_{12}^2 + \frac{|D_{11} D_{12}|}{a}) \phi_2^2(0). \end{aligned}$$

Since by Assumption 1,  $|D_{11}| < 1$ , picking  $\mu$  sufficiently small it yields that

$$a := \frac{1 - (1 + \mu) D_{11}^2}{(1 + \mu) |D_{11} D_{12}|} > 0. \quad (6)$$

Consequently, it implies

$$\dot{\mathcal{V}}_1(\phi) \leq (-\mu + \beta \gamma) \|\phi_1\|_{L^2}^2 + \alpha \phi_2^2(1) + \frac{\beta}{\gamma} \|\partial_x \phi_2\|_{L^2}^2, \quad (7)$$

with

$$\begin{aligned} \alpha &= \frac{(1 + \mu)}{D_{22}^2} \left( D_{12}^2 + \frac{|D_{11} D_{12}|}{a} \right) = \left( \frac{1 + \mu}{1 - (1 + \mu) D_{11}^2} \right) \frac{D_{12}^2}{D_{22}^2}, \\ \beta &= (1 + \mu) \bar{c}. \end{aligned}$$

Third, similarly, the time derivative of the last part of the Lyapunov candidate functional yields

$$\begin{aligned} \dot{\mathcal{V}}_{21}(\phi) &= 2 \int_0^1 (1 + \mu x) \partial_x \phi_2(x) \frac{\partial_{xt} \phi_2(x)}{\lambda_2(x)} dx, \\ &= 2 \int_0^1 (1 + \mu x) \partial_x \phi_2(x) \partial_{xx} \phi_2(x) dx \\ &\quad + 2 \int_0^1 (1 + \mu x) \frac{\lambda_2'(x)}{\lambda_2(x)} (\partial_x \phi_2)^2(x) dx, \\ &= -\mu \int_0^1 (\partial_x \phi_2)^2(x) dx + (1 + \mu) \partial_x \phi_2^2(1) - \partial_x \phi_2^2(0) \\ &\quad + 2 \int_0^1 (1 + \mu x) \frac{\lambda_2'(x)}{\lambda_2(x)} (\partial_x \phi_2)^2(x) dx. \end{aligned}$$

As  $\partial_x \phi_2(1) = D_{22} \partial_x \phi_2(0)$  from the compatibility condition, we have

$$\dot{\mathcal{V}}_{21}(\phi) \leq (-\mu + 2(1 + \mu) \frac{\bar{\lambda}_2'}{\bar{\lambda}_2}) \|\partial_x \phi_2\|_{L^2}^2 + (\mu - \bar{\nu}) \partial_x \phi_2^2(1). \quad (8)$$

Considering  $\nu = (1 - \frac{\bar{\lambda}_2'}{\bar{\lambda}_2})^{-1} \frac{\bar{\lambda}_2'}{\bar{\lambda}_2}$  is equivalent to  $\frac{\bar{\lambda}_2'}{\bar{\lambda}_2} = \frac{\nu}{1 + \nu}$  which means that  $(-\mu + 2(1 + \mu) \frac{\bar{\lambda}_2'}{\bar{\lambda}_2}) = \frac{\nu - \mu}{1 + \nu}$ . Thus, from (5), (7) and (8), we end up with

$$\begin{aligned} \dot{\mathcal{V}}(\phi) &\leq (-\mu + \beta \gamma) \|\phi_1\|_{L^2}^2 - \alpha_{20} \mu \|\phi_2\|_{L^2}^2 \\ &\quad + \left( \frac{\beta}{\gamma} + \alpha_{21} \frac{\nu(2 + \mu) - \mu}{1 + \nu} \right) \|\partial_x \phi_2\|_{L^2}^2 \\ &\quad + (\alpha_{20}(\mu - \bar{\nu}) + \alpha) \phi_2^2(1) \\ &\quad + \alpha_{21}(\mu - \bar{\nu}) \partial_x \phi_2^2(1). \end{aligned} \quad (9)$$

To make the last three terms negative, we select the positive scalars  $\mu, \alpha_{20}, \alpha_{21}, \gamma$  as follows

$$\begin{aligned} \mu &= \bar{\nu} - \frac{\varepsilon}{2}, \quad \gamma = \frac{\mu - \bar{\nu} + \varepsilon}{\beta} = \frac{\varepsilon}{2\beta} \\ \alpha_{20} &= \frac{1}{(\bar{\nu} - \mu)} \alpha = \frac{2\alpha}{\varepsilon}, \\ \alpha_{21} &= \frac{1 + \nu}{\mu - \nu(2 + \mu) - (\bar{\nu} - \nu - \varepsilon)} \frac{\beta}{\gamma} = \frac{4(1 + \nu)\beta^2}{\varepsilon^2}. \end{aligned} \quad (10)$$

and sufficiently small scalar  $\varepsilon$  verifying  $\varepsilon < \bar{\nu} - \nu$ . With Assumption 1,  $\frac{\bar{\lambda}_2'}{\bar{\lambda}_2} < 1 - D_{22}^2$ , hence, we clearly check that  $\nu < \mu < \bar{\nu}$  holds. Then, as:

$$\begin{aligned} \|\phi_1\|_{L^2}^2 &\geq \frac{\lambda_1}{1 + \mu} \mathcal{V}_1(\phi) \geq \frac{\lambda_1}{1 + \bar{\nu}} \mathcal{V}_1(\phi), \\ \|\phi_2\|_{L^2}^2 &\geq \frac{\lambda_2}{1 + \mu} \mathcal{V}_{20}(\phi) \geq \frac{\lambda_2}{1 + \bar{\nu}} \mathcal{V}_{20}(\phi), \\ \|\partial_x \phi_2\|_{L^2}^2 &\geq \frac{\lambda_2}{1 + \mu} \mathcal{V}_{21}(\phi) \geq \frac{\lambda_2}{1 + \bar{\nu}} \mathcal{V}_{21}(\phi), \end{aligned}$$

the time derivative of the Lyapunov candidate functional along  $C^1$  solutions of system (1) satisfy

$$\begin{aligned} \dot{\mathcal{V}}(\phi) &\leq -(\bar{\nu} - \varepsilon) \frac{\lambda_1}{1 + \bar{\nu}} \mathcal{V}_1(\phi) - \alpha_{20}(\bar{\nu} - \frac{\varepsilon}{2}) \frac{\lambda_2}{1 + \bar{\nu}} \mathcal{V}_{20}(\phi) \\ &\quad - \alpha_{21} \frac{\bar{\nu} - \nu - \varepsilon}{1 + \bar{\nu}} \frac{\lambda_2}{1 + \bar{\nu}} \mathcal{V}_{21}(\phi) \end{aligned}$$

meaning for sufficiently small scalar  $\varepsilon$  that

$$\dot{\mathcal{V}}(\phi) \leq -\delta \mathcal{V}(\phi), \quad (11)$$

for any  $\delta < \min\{\frac{\bar{\nu} \lambda_1}{1 + \bar{\nu}}, \frac{\bar{\nu} \lambda_2}{1 + \bar{\nu}}, \frac{(\bar{\nu} - \nu) \lambda_2}{(1 + \bar{\nu})(1 + \nu)}\}$ . With Grönwall inequality the above equation (11), and by standard density arguments yields

$$\mathcal{V}(\phi(t; \phi_0)) \leq e^{-\delta t} \mathcal{V}(\phi_0), \quad \forall \phi_0 \in X.$$

To conclude, using the equivalence of norm (4),

$$\|\phi(t; \phi_0)\|_X^2 \leq \kappa e^{-\delta t} \|\phi_0\|_X^2$$

holds with  $\kappa$  and  $\delta$  given explicitly independtly from the initial condition, which allows us to close the proof. ■

### 3.2 Second study case : with the source term

In this section, we consider the case in which the source term  $B$  is non null. The new assumption that we consider is given as follows.

*Assumption 2.* Assume that the functions  $\lambda_2, B$  and the matrix  $D$  satisfy

$$1 - \max \left\{ 2\overline{B_{11}}, \frac{\overline{\lambda_2'}}{\underline{\lambda_2}} + 2\overline{B_{22}} \right\} > \max(D_{11}^2, D_{22}^2). \quad (12)$$

With this assumption, the following theorem can be obtained.

*Theorem 2.* Let Assumption 2 holds and let

$$\begin{aligned} \nu_1 &:= 2\overline{B_{11}}(1 - 2\overline{B_{11}})^{-1}, \\ \nu_2 &:= \frac{\overline{\lambda_2'} + 2\lambda_2\overline{B_{22}}}{\underline{\lambda_2} - \overline{\lambda_2'} - 2\lambda_2\overline{B_{22}}}, \\ \bar{\nu} &:= \frac{1 - D_{22}^2}{D_{22}^2}. \end{aligned} \quad (13)$$

Then, for any  $\delta < \min\{\delta_1, \delta_2\}$  where  $\delta_1 = \frac{\lambda_1(\bar{\nu} - \nu_1)}{(1 + \nu_1)(1 + \bar{\nu})}$  and  $\delta_2 = \frac{\lambda_2(\bar{\nu} - \nu_2)}{(1 + \nu_2)(1 + \bar{\nu})}$ , system (1) is  $\delta$ -exponentially stable with an overshoot given by

$$\kappa = \frac{\frac{1}{\lambda_1} + \frac{\alpha_{20}}{\lambda_2} + \frac{\alpha_{21}}{\lambda_2}}{\min(\frac{1}{\lambda_1}, \frac{\alpha_{20}}{\lambda_2}, \frac{\alpha_{21}}{\lambda_2})} (1 + \bar{\nu} - \frac{\varepsilon}{2}), \quad 0 < \varepsilon < \bar{\nu} - \max\{\nu_1, \nu_2\},$$

and  $\alpha_{20}, \alpha_{21}$  given by (18).

In this second theorem, the stability is obtained under several conditions. As previously, the ratio between diagonal velocities and its derivatives must be small. In addition, the diagonal coefficients of the source term must be upper-bounded (a fortiori when  $B_{11}$  and  $B_{22}$  are negative Assumption 2 reduces to Assumption 1).

*Remark 4.* This result extends (Bastin and Coron, 2016, Theorem 6.10) from strictly hyperbolic to a class of weakly hyperbolic systems. For the strict case, i.e.  $c = 0$ , we fall back to the well-known stability condition  $B_{11}, B_{22} < 0$  ( $B$  a Hurwitz matrix).

*Remark 5.* Note that the case  $c \neq 0$  and  $\lambda_1 = \lambda_2$  imposes the constraint  $B_{21} = 0$  in our Lyapunov analysis. Indeed, when  $B_{21} \neq 0$ , a term in  $\int_0^1 B_{21}(x) \partial_x \phi_2(x) \partial_x \phi_1(x) dx$  appears and cannot be compensated in the absence of negativity on the  $H^1$  norm of the state  $\phi_1$ . A deeper analysis should be carried out to handle full matrices  $B$ .

**Proof.** Note that with Assumption 2,  $\nu_1, \nu_2$  are positive and all the constant involved in the statement of the theorem are well defined. Consider the functional  $\mathcal{V} = \mathcal{V}_1 + \alpha_{20}\mathcal{V}_{20} + \alpha_{21}\mathcal{V}_{21}$  expressed in (3) with scalars  $\mu, \alpha_{20}, \alpha_{21} > 0$  to be fixed later. The equivalence of norm (4) is still verified. Let us now repeat the time derivation of each of the three terms along the  $C^1$  trajectories of system (1) and let us focus on the additional terms that will be denoted  $\mathcal{W}_1, \mathcal{W}_{20}, \mathcal{W}_{21}$  coming from the source term.

First, we obtain

$$\mathcal{W}_{20}(\phi) = 2 \int_0^1 (1 + \mu x) B_{22}(x) \phi_2^2(x) dx,$$

leading to

$$\mathcal{W}_{20}(\phi) \leq 2(1 + \mu) \overline{B_{22}} \|\phi_2\|_{L^2}^2. \quad (14)$$

Second, we obtain

$$\mathcal{W}_1(\phi) = 2 \int_0^1 (1 + \mu x) (B_{11}(x) \phi_1^2(x) + B_{12}(x) \phi_1(x) \phi_2(x)) dx.$$

Applying Young's inequality, for any  $\gamma_1 > 0$ , it leads to

$$\mathcal{W}_1(\phi) \leq (1 + \mu) \left( (2\overline{B_{11}} + \gamma_1 \overline{|B_{12}|}) \|\phi_1\|_{L^2}^2 + \frac{\overline{|B_{12}|}}{\gamma_1} \|\phi_2\|_{L^2}^2 \right), \quad (15)$$

Third, we obtain

$$\begin{aligned} \mathcal{W}_{21} &= 2 \int_0^1 (1 + \mu x) B_{22}(x) \partial_x \phi_2^2(x) dx, \\ &+ 2 \int_0^1 (1 + \mu x) \frac{(\lambda_2 B_{22})'(x)}{\lambda_2(x)} \partial_x \phi_2(x) \phi_2(x) dx. \end{aligned}$$

Applying Young's inequality, for any  $\gamma_2 > 0$ , it leads to

$$\begin{aligned} \mathcal{W}_{21}(\phi) &\leq (1 + \mu) \left( 2\overline{B_{22}} + \gamma_2 \frac{(\lambda_2 B_{22})'}{\lambda_2} \right) \|\partial_x \phi_2\|_{L^2}^2 \\ &+ (1 + \mu) \frac{1}{\gamma_2} \frac{(\lambda_2 B_{22})'}{\lambda_2} \|\phi_2\|_{L^2}^2 \end{aligned} \quad (16)$$

To sum up inequations (9), (14), (15) and (16), we have:

$$\begin{aligned} \dot{\mathcal{V}}(\phi) &\leq [-\mu + 2(1 + \mu) \overline{B_{11}} + \gamma\beta + \gamma_1\beta_1] \|\phi_1\|_{L^2}^2 \\ &+ \left[ \alpha_{20} \left( -\mu + 2(1 + \mu) \overline{B_{22}} + \frac{\alpha_{21}}{\alpha_{20}} \frac{\beta_2}{\gamma_2} \right) + \frac{\beta_1}{\gamma_1} \right] \|\phi_2\|_{L^2}^2 \\ &+ \left[ \alpha_{21} \left( -\mu + (1 + \mu) \left( \frac{\overline{\lambda_2'}}{\underline{\lambda_2}} + 2\overline{B_{22}} \right) + \gamma_2\beta_2 \right) + \frac{\beta}{\gamma} \right] \|\partial_x \phi_2\|_{L^2}^2 \\ &+ (\alpha_{20}(\mu - \bar{\nu}) + \alpha) \phi_2^2(1) + \alpha_{21}(\mu - \bar{\nu}) \partial_x \phi_2^2(1). \end{aligned}$$

with

$$\beta_1 = (1 + \mu) \overline{|B_{12}|}, \quad \beta_2 = (1 + \mu) \frac{(\lambda_2 B_{22})'}{\lambda_2}.$$

From expression (13) we have

$$2\overline{B_{11}} = \frac{\nu_1}{1 + \nu_1}, \quad \frac{\overline{\lambda_2'}}{\underline{\lambda_2}} + 2\overline{B_{22}} = \frac{\nu_2}{1 + \nu_2}.$$

Thus

$$\begin{aligned} \dot{\mathcal{V}}(\phi) &\leq \left[ \frac{\nu_1 - \mu}{1 + \nu_1} + \gamma\beta + \gamma_1\beta_1 \right] \|\phi_1\|_{L^2}^2 \\ &+ \left[ \alpha_{20} \left( \frac{\nu_2 - \mu}{1 + \nu_2} + \frac{\alpha_{21}}{\alpha_{20}} \frac{\beta_2}{\gamma_2} \right) + \frac{\beta_1}{\gamma_1} \right] \|\phi_2\|_{L^2}^2 \\ &+ \left[ \alpha_{21} \left( \frac{\nu_2 - \mu}{1 + \nu_2} + \gamma_2\beta_2 \right) + \frac{\beta}{\gamma} \right] \|\partial_x \phi_2\|_{L^2}^2 \\ &+ (\alpha_{20}(\mu - \bar{\nu}) + \alpha) \phi_2^2(1) + \alpha_{21}(\mu - \bar{\nu}) \partial_x \phi_2^2(1). \end{aligned}$$

Similarly to (10), we begin with setting  $\mu, \gamma, \gamma_1, \gamma_2$  as follows

$$\begin{aligned} \mu &= \bar{\nu} - \frac{\varepsilon}{2}, \quad \gamma = \frac{\varepsilon}{4(1 + \nu_1)\beta}, \\ \gamma_1 &= \frac{\varepsilon}{4(1 + \nu_1)\beta_1}, \quad \gamma_2 = \frac{\varepsilon}{4(1 + \nu_2)\beta_2}. \end{aligned} \quad (17)$$

to obtain

$$\begin{aligned} \dot{\mathcal{V}}(\phi) &\leq \frac{\nu_1 - \bar{\nu} + \varepsilon}{1 + \nu_1} \|\phi_1\|_{L^2}^2 \\ &+ \left[ \alpha_{20} \frac{\nu_2 - \bar{\nu} + \frac{\varepsilon}{2}}{1 + \nu_2} + \alpha_{21} \frac{4(1 + \nu_2)\beta_2^2}{\varepsilon} + \frac{4(1 + \nu_1)\beta_1^2}{\varepsilon} \right] \|\phi_2\|_{L^2}^2 \\ &+ \left[ \alpha_{21} \frac{\nu_2 - \bar{\nu} + \frac{3\varepsilon}{4}}{1 + \nu_2} + \frac{4(1 + \nu_1)\beta^2}{\varepsilon} \right] \|\partial_x \phi_2\|_{L^2}^2 \\ &+ (-\alpha_{20} \frac{\varepsilon}{2} + \alpha) \phi_2^2(1) - \alpha_{21} \frac{\varepsilon}{2} \partial_x \phi_2^2(1). \end{aligned}$$

Finally, we set  $\alpha_{20}, \alpha_{21}$  as

$$\alpha_{20} = \max \left\{ \frac{2\alpha}{\varepsilon}, \alpha_{21} \frac{16(1+\nu_2)^2 \beta_2^2}{\varepsilon^2}, \frac{16(1+\nu_2)(1+\nu_1)\beta_1^2}{\varepsilon^2} \right\}, \quad (18)$$

$$\alpha_{21} = \frac{16(1+\nu_1)(1+\nu_2)\beta^2}{\varepsilon^2}.$$

to achieve

$$\dot{\mathcal{V}}(\phi) \leq \frac{\nu_1 - \bar{\nu} + \varepsilon}{1 + \nu_1} \|\phi_1\|_{L^2}^2 + \alpha_{20} \frac{\nu_2 - \bar{\nu} + \varepsilon}{1 + \nu_2} \|\phi_2\|_{L^2}^2 + \alpha_{21} \frac{\nu_2 - \bar{\nu} + \varepsilon}{1 + \nu_2} \|\partial_x \phi_2\|_{L^2}^2.$$

Then, selecting  $\varepsilon$  small enough, the time derivative of the Lyapunov candidate functional along the trajectories of system (1) satisfies  $\dot{\mathcal{V}}(\phi) \leq -\delta \mathcal{V}(\phi)$  for all  $\delta < \min(\delta_1, \delta_2)$  and the equivalence of norm (4) ends the proof. ■

#### 4. NUMERICAL RESULTS

In this numerical results section, we make use of several challenging test cases to illustrate the performance of the Lyapunov stability analysis approach proposed in the previous section for system (1).

System (1) represents a coupling of two transport equations with negative velocities. In order to design a numerical discretization of the PDEs system, we consider an *up-wind* type *explicit finite difference* scheme that takes into account the directions of waves propagation (see LeVeque (2002)).

We denote the space-time domain  $\mathcal{D} = [0, 1] \times [0, T]$ ,  $T$  being the final time for system (1) dynamics. A discretization of this continuous domain is made; given  $p \in \mathbb{N}^*$  sufficiently large, we introduce the space mesh-size as  $\Delta x = \frac{1}{p+1}$ . Such procedure generates a sequence of grid points  $(x_i, t^n) = (i\Delta x, n\Delta t)$ ,  $i \in \llbracket 0..p+1 \rrbracket$ ,  $n \in \mathbb{N}^*$ . The time-stepping increment  $\Delta t$  will be specified later on.

Approximations of the states of system (1) will be made on the grid points of the mesh, we introduce the following notations:

$$\begin{cases} \phi_k(x_i, t^n) = \phi_{k,i}^n & k \in \{1, 2\} \\ \lambda_{k,i} = \lambda_k(x_i), & k \in \{1, 2\} \\ c_i = c(x_i) \end{cases}$$

##### 4.1 First study case: without the source terms

System (1) is expressed in discrete form as :

$$\begin{cases} \frac{\phi_{1,i}^{n+1} - \phi_{1,i}^n}{\Delta t} = \lambda_{1,i} \frac{\phi_{1,i+1}^n - \phi_{1,i}^n}{\Delta x} + c_i \lambda_{2,i} \frac{\phi_{2,i+1}^n - \phi_{2,i}^n}{\Delta x} \\ \frac{\phi_{2,i}^{n+1} - \phi_{2,i}^n}{\Delta t} = \lambda_{2,i} \frac{\phi_{2,i+1}^n - \phi_{2,i}^n}{\Delta x} \end{cases} \quad (19)$$

Setting:

$$\gamma_{1,i} = \frac{\lambda_{1,i} \Delta t}{\Delta x}, \quad \gamma_{2,i} = \frac{c_i \lambda_{2,i} \Delta t}{\Delta x}$$

we get the numerical scheme :

$$\phi_{1,i}^{n+1} = (1 - \gamma_{1,i}) \phi_{1,i}^n + \gamma_{1,i} \phi_{1,i+1}^n - \gamma_{2,i} \phi_{2,i}^n + \gamma_{2,i} \phi_{2,i+1}^n \quad (20)$$

$$\phi_{2,i}^{n+1} = (1 - \gamma_{2,i}) \phi_{2,i}^n + \gamma_{2,i} \phi_{2,i+1}^n \quad (21)$$

**Lemma 1.** The numerical scheme (17)-(18) is stable<sup>1</sup> under the following Courant–Friedrichs–Lewy (CFL) condition :

$$\Delta t \leq \min\left(\frac{\Delta x}{\lambda_1}, \frac{\Delta x}{\lambda_2}\right) \quad (22)$$

<sup>1</sup> The method must be stable in some appropriate sense, meaning that the small errors made in each time step do not grow too fast in later time steps

Let us now consider system (1) with:

$$\lambda_1(x) = \begin{cases} 10, & x < \frac{1}{2} \\ -4(x - \frac{1}{2})^2 + 10 & \frac{1}{2} \leq x \leq 1 \end{cases}$$

$$\lambda_2(x) = \begin{cases} 10, & x < \frac{1}{2} \\ 4(x - \frac{1}{2})^2 + 10 & \frac{1}{2} \leq x \leq 1 \end{cases}$$

$$c(x) = x + 4, D = \begin{bmatrix} \frac{1}{2} & 1 \\ 0 & \frac{1}{2} \end{bmatrix}, \phi_1^0(x) = 0, \phi_2^0(x) = \frac{x^2}{2} \left(1 - \frac{x}{3}\right) - \frac{2}{3}.$$

In this example,  $\lambda_1(x) = \lambda_2(x)$  for  $x \in [0, \frac{1}{2}]$ , which makes the system weakly hyperbolic.

Fig.1 gives the time-space evolution of the two state variables  $\phi_1$  and  $\phi_2$ . The time evolution of  $\frac{\|\phi\|_X^2}{\|\phi^0\|_X^2}$  is given in Fig.2.

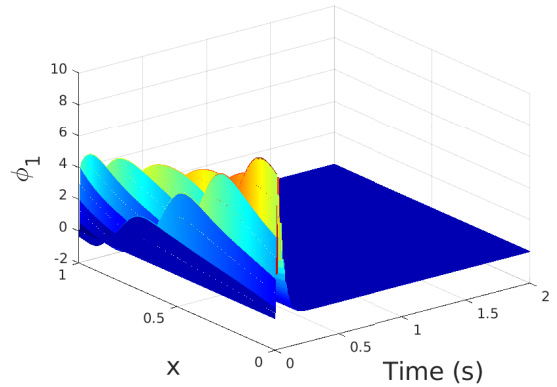


Fig. 1. State  $\phi_1$  over  $(x,t)$  plane, (final time  $T = 2s$ )

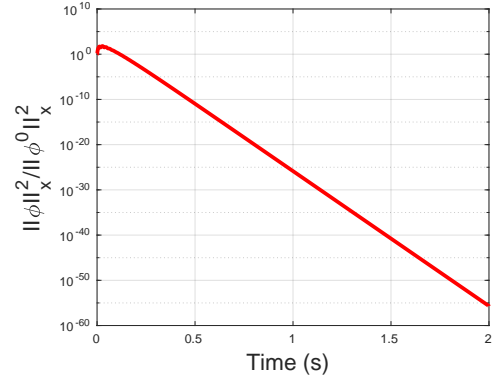


Fig. 2. Time evolution of  $\frac{\|\phi\|_X^2}{\|\phi^0\|_X^2}$  in logarithmic scale

Both states are stabilized to zero as time grows up, the simulations are carried out using the numerical scheme (17)-(18) for 500 grid points in space, with CFL=0.9.

Here, regarding Theorem 1, the assumption

$$\max(D_{11}^2, D_{22}^2) = \frac{1}{4} < 1 - \frac{\lambda_2}{\lambda_1} = \frac{3}{5}$$

is satisfied. Thus, the numerical result corroborates the theoretical one. Theoretically, we expect a decay rate given

by  $\delta_{th} = \min\{\bar{\nu}\bar{\lambda}_1, \bar{\nu}\bar{\lambda}_2, \frac{\bar{\nu}-\nu}{1+\nu}\bar{\lambda}_2\} = \frac{7}{2}$  whereas, numerically, we found  $\delta_{num} = 14.71 > \delta_{th}$ . The difference is due to the conservatism introduced by the proposed Lyapunov analysis.

#### 4.2 Second study case: with the source term

The numerical scheme when adding the source term is written as:

$$\phi_{1,i}^{n+1} = (1 - \gamma_{1,i})\phi_{1,i}^n + \gamma_{1,i}\phi_{1,i+1}^n - \gamma_{2,i}\phi_{2,i}^n + \gamma_{2,i}\phi_{2,i+1}^n + \Delta t(\lambda_{1,i}B_{11,i}\phi_{1,i}^n + \lambda_{1,i}B_{12,i}\phi_{2,i}^n) \quad (20)$$

$$\phi_{2,i}^{n+1} = (1 - \gamma_{2,i})\phi_{2,i}^n + \gamma_{2,i}\phi_{2,i+1}^n + \Delta t\lambda_{2,i}B_{22,i}\phi_{2,i}^n \quad (21)$$

Let us now consider system (1) with  $\lambda_1$ ,  $\lambda_2$ ,  $c$ ,  $D$  and initial conditions that are similar to first example but with a source term  $B = \begin{bmatrix} \frac{x}{6} & c(x) \\ 0 & \frac{x}{6} \end{bmatrix}$

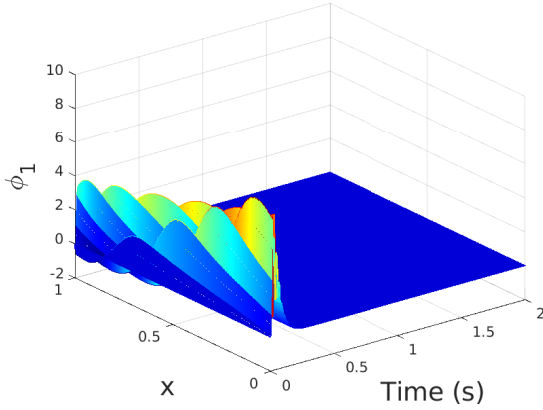


Fig. 3. State  $\phi_1$  over  $(x,t)$  plane, (final time  $T = 2s$ )

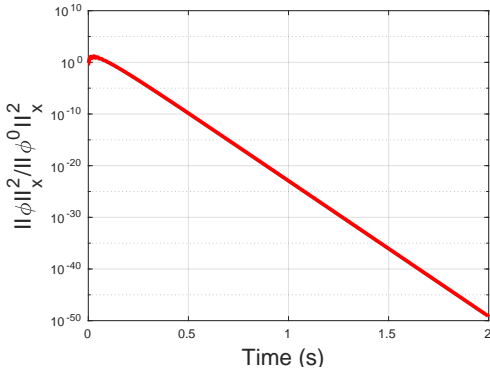


Fig. 4. Time evolution of  $\frac{\|\phi\|_X^2}{\|\phi^0\|_X^2}$  in logarithmic scale

In this situation both states are also stabilized to zero as time grows up. The simulations are carried out using the numerical scheme (20)-(21) for 500 grid points in space, with CFL=0.9.

Here, regarding Theorem 2, the assumption:

$$\max(D_{11}^2, D_{22}^2) = \frac{1}{4} < 1 - \max\left\{2\overline{B_{11}}, \frac{\overline{\lambda_2}}{\overline{\lambda_1}} + 2\overline{B_{22}}\right\} = \frac{4}{15}$$

is satisfied. As stated by Theorem 2, exponential stability is

guaranteed with a decay rate at least equal to  $\delta_{th} = 0.16$ . Hopefully, the numerical result leads to a decay rate of

$$\delta_{num} = 12.89 > \delta_{th}$$

which is better than expected. This confirms the encouraging approach currently proposed by our Lyapunov functionals.

*Remark 6.* Several other tests have been carried out, particularly in the case where matrices  $D$  and  $B$  have a non-null under diagonal element (which we denote respectively  $D_{21}, B_{21}$ , for such situations system (1) is always unstable. We conjecture that closing the loop has a destabilizing effect.

## 5. CONCLUSIONS

To study the stability of weakly hyperbolic systems, where the principal part is non-uniform and non-diagonalizable everywhere, we return to the non-trivial study of triangular structures. Based on a Lyapunov analysis in the  $L^2 \times H^1$  norm, two sufficient conditions for exponential stability emerge: the velocity and source terms have to be small in front of the edge back-reaction. Our refined analysis also allows us to specify upper bounds for decay rate and overshoot values, as illustrated in numerical simulations.

Upcoming works will be dedicated to a non-diagonal Lyapunov functional (full matrix  $P$ ) to reduce the conservatism and tackle the case of systems where  $D_{21}$  and  $B_{21}$  are non null (full matrices  $B, D$ ). Applications to some real physical systems will also be considered in the future.

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