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Abstract: In this paper, the stability analysis of a class of hyperbolic systems with non-diagonalizable principal part and nonuniform coefficients is addressed. We give a stability condition and express the convergence rate that can be obtained. Two cases will be treated independently: with and without a source term. A numerical scheme is designed to illustrate the performance of the Lyapunov stability analysis and two examples are presented.

Keywords: Linear PDE, Triangular Weakly hyperbolic system, Stability analysis, Strict Lyapunov functional.

1. INTRODUCTION

In recent years, significant progress has been made in understanding the stability and control of strictly hyperbolic systems, thanks to the contributions of Bastin and Coron (2016), among others. In particular, control strategies for hyperbolic systems using backstepping with Volterra or Fredholm transformations have been proposed Krstic (2009); Bribiesca-Argomedo and Krstic (2015); Coron et al. (2013). These mathematical advances can then be applied to a wide range of applications, such as drilling processes Bresch-Pietri and Krstic (2014) or navigable rivers (Bastin and Coron, 2016, Chapter 8).

However, as soon as the system of interest is weakly hyperbolic, as in wastewater treatment (Valentin et al. (2023)), for instance having a non-diagonalizable principal part with eigenvalues of multiplicity greater than one, only few stability results and even fewer control results are available (see however Métivier and Zumbrun (2005); Keyfitz and Kranzer (1980)). In this paper, we deal with the case of hyperbolic systems of dimension 2×2 when the principal part has a positive eigenvalue of geometric multiplicity equal to two, at least on a part of the spatial domain.

By leveraging Lyapunov analysis, a strict Lyapunov functional is designed which, under certain conditions, allows us to assess the global exponential stability. This Lyapunov functional relies heavily on input-to-state properties Prieur and Mazenc (2012) satisfied by our class of system. In addition, we have endeavored to get as close as possible to the optimum bounds Lamare et al. (2016) on the convergence rate and the overshoot. To achieve this, we played with different weights, in particular using polynomial rather than exponential weights in the Lyapunov functional kernels.

Section 2 introduces the system class, along with the accompanying definitions and the statement of the problem. In Section 3, two stability conditions are presented, differentiating between cases with and without a source term. Section 4 illustrates the theoretical findings through the simulation of two academic examples.

Notation: Throughout the paper, the partial derivatives in time and space are denoted ∂_t and ∂_x, while the classical derivative are $\dot{V} = \frac{\mathrm{d}}{\mathrm{d}t}V$ and $\lambda' = \frac{\mathrm{d}}{\mathrm{d}x}\lambda$. Moreover, $C^k_{pw}(0,1;\mathbb{R})$ denotes the space of functions defined from $[0,1]$ to $\mathbb{R}$ for which the $k-1$ first derivatives are continuous and the k-th derivative is piece-wise continuous if $k \geq 1$ and the function is piece-wise continuous for $k = 0$; $L^2((0,1);\mathbb{R})$ denotes the space of square-integrable functions from $(0,1)$ to $\mathbb{R}$; $H^1((0,1);\mathbb{R}) = \{f : (0,1) \to \mathbb{R} \mid f \in L^2((0,1);\mathbb{R}),\ f' \in L^2((0,1);\mathbb{R})\}$ and $X = L^2((0,1);\mathbb{R}) \times H^1((0,1);\mathbb{R})$ is equipped with the norm: $\|\cdot\|^2_X : (f,g) \mapsto \|f\|^2_{L^2((0,1),\mathbb{R})} + \|g\|^2_{L^2((0,1);\mathbb{R})} + \|g'\|^2_{L^2((0,1);\mathbb{R})}$. Given a function $f : (0,1) \mapsto \mathbb{R}$, we denote by $\overline{f} = \max\left\{0, \sup_{x\in(0,1)} f(x)\right\}$ and $\underline{f} = \inf_{x\in(0,1)} f(x)$.

2. PROBLEM STATEMENT

Consider the following class of triangular 2x2 hyperbolic system

$$\begin{cases} \partial_t \phi(t,x) = \Lambda(x)\left(A(x)\partial_x \phi(t,x) + B(x)\phi(t,x)\right), x \in (0,1) \\ \phi(t,1) = D\phi(t,0), \\ \phi(0,x) = \phi^0(x) \end{cases} \tag{1}$$

with $\phi(t,\cdot) = (\phi_1(t,\cdot), \phi_2(t,\cdot)) \in \mathcal{D}$ the state of the system defined for any $t \geq 0$ and $\phi^0 \in \mathcal{D}$ any initial condition, where $\mathcal{D} \subset X$ is the domain of the linear operator of system (1), i.e. the subset of X satisfying zero-order

and first-order compatibility conditions, Bastin and Coron (2016). We have $\Lambda(x) = \begin{bmatrix} \lambda_1(x) & 0 \\ 0 & \lambda_2(x) \end{bmatrix}$ such that $\lambda_{1,2}(x) > 0$. Moreover, we assume that $\lambda_1 \in C^0_{pw}(0,1;\mathbb{R})$ and $\lambda_2 \in C^1_{pw}(0,1;\mathbb{R})$, $A(x) = \begin{bmatrix} 1 & c(x) \\ 0 & 1 \end{bmatrix}$ where $c \in C^0_{pw}(0,1;\mathbb{R})$, the source term $B(x) = \begin{bmatrix} B_{11}(x) & B_{12}(x) \\ 0 & B_{22}(x) \end{bmatrix}$ is continuous. We assume $B_{11}, B_{12} \in C^0_{pw}(0,1;\mathbb{R})$, $B_{22} \in C^1_{pw}(0,1;\mathbb{R})$ and $D = \begin{bmatrix} D_{11} & D_{12} \\ 0 & D_{22} \end{bmatrix}$ with $\max(D_{11}^2, D_{22}^2) < 1$ and $D_{12} \in \mathbb{R}$.

System (1) is worth of interest in cases where the principal part $A(x)$ is not diagonalizable for some part of the space $x \in (0,1)$, in other words if there exists $x \in (0,1)$ such that $\lambda_1(x) = \lambda_2(x)$ and if for all $x \in (0,1)$, $c(x) \neq 0$.

Lemma 1. (Well-posedness). For any initial condition ϕ^0 in $\mathcal{D}$, the solution of system (1) of the Cauchy problem $t \mapsto \phi(t;\phi^0)$ is unique and belongs to $C^1([0,+\infty);\mathcal{D})$.

Proof. System (1) is linear and associated to an operator $\mathcal{A}$ in the Hilbert space X. Introducing the usual scalar product $\langle\cdot|\cdot\rangle_X$ in X and applying integration by parts, such operator satisfies for any $\phi \in \mathcal{D}$

$$\left\langle \Lambda^{-1}\mathcal{A}\phi \middle| \phi \right\rangle_X = \langle (B-I)\phi|\phi\rangle_X + \int_0^1 c(x)\phi_1(x)\partial_x\phi_2(x)\mathrm{d}x + \phi^\top(0)(D^\top D - I)\phi(0) + (D_{22}^2 - 1)\partial_x\phi_2^2(0).$$

Based on the fact that D is a Schur matrix and via the Cauchy-Scharz inequality, we obtain $\left\langle \Lambda^{-1}\mathcal{A}\phi \middle| \phi \right\rangle_X \leq \omega \langle\phi|\phi\rangle_X$ for some positive scalar ω and the same occurs for the adjoint operator $\mathcal{A}^*$. According to (Curtain and Zwart, 2020, Corollary 2.3.3), $\mathcal{A}$ generates a strongly continuous semigroup on X. which implies existence of a continuous and unique solution from $[0,\infty)$ to $\mathcal{D}$ (Tucsnak and Weiss, 2009, Proposition 2.3.5).

■

Definition 1. The origin of system (1) is said to be globally δ-exponentially stable if there exists an overshoot $\kappa \geq 1$ and a decay rate $\delta > 0$ such that for any initial condition $\phi^0 \in X$:

$$\|\phi(t;\phi^0)\|_X^2 \leq \kappa\, \mathrm{e}^{-\delta t}\|\phi^0\|_X^2,\ \forall t \geq 0.$$

In this paper, we aim at giving sufficient conditions that ensure exponential stablility of system (1) in the sense of Definition 1.

3. STABILITY ANALYSIS

3.1 First study case : without the source term

In this section, we express a stability condition on system (1) in the case where the source term, B, is null. This one imposes a restriction on the diagonal terms of D with respect to the function λ_2.

Assumption 1. The function λ_2 and the matrix D satisfy the following inequality

$$\max(D_{11}^2, D_{22}^2) < 1 - 2\frac{\overline{\lambda_2'}}{\underline{\lambda_2}}. \tag{2}$$

Based on this assumption, we can now give the following result which establishes exponential stability of the origin and gives an estimate of the exponential convergence rate and the overshoot.

Theorem 1. Assume $B = 0$. Let Assumption 1 holds and let $\nu := \frac{\overline{\lambda_2'}}{\underline{\lambda_2} - \overline{\lambda_2'}}$, $\bar{\nu} := \frac{1-D_{22}^2}{D_{22}^2}$. Then, for any $\delta < \min\{\frac{\bar{\nu}\underline{\lambda_1}}{1+\bar{\nu}}, \frac{\bar{\nu}\underline{\lambda_2}}{1+\bar{\nu}}, \frac{(\bar{\nu}-\nu)\underline{\lambda_2}}{(1+\bar{\nu})(1+\nu)}\}$, the origin of system (1) is δ-exponentially stable with an overshoot given by

$$\kappa = \frac{\frac{1}{\underline{\lambda_1}} + \frac{\alpha_{20}}{\underline{\lambda_2}} + \frac{\alpha_{21}}{\underline{\lambda_2}}}{\min(\frac{1}{\overline{\lambda_1}}, \frac{\alpha_{20}}{\overline{\lambda_2}}, \frac{\alpha_{21}}{\overline{\lambda_2}})}(1 + \bar{\nu} - \tfrac{\varepsilon}{2}), \quad 0 < \varepsilon < \bar{\nu} - \nu,$$

and α_{20}, α_{21} given by (10).

With Assumption 1, it can be checked that ν, $\bar{\nu}$ and δ which appear in the statement of Theorem 1 are all well defined and positive.

Proof. The method used to prove Theorem 1 relies on the construction of a Lyapunov functional inspired by Kharitonov (2012); Krstic (2009). Consider the following functional:

$$\mathcal{V}(\phi) = \int_0^1 \phi^\top(x)P(x)\phi(x)\mathrm{d}x + \int_0^1 \partial_x\phi^\top(x)Q(x)\partial_x\phi(x)\mathrm{d}x, \tag{3}$$

with matrices $P(x)$ and $Q(x)$ given by

$$P(x) = \begin{bmatrix} \frac{1+\mu x}{\lambda_1(x)} & 0 \\ 0 & \alpha_{20}\frac{1+\mu x}{\lambda_2(x)} \end{bmatrix}, \quad Q(x) = \begin{bmatrix} 0 & 0 \\ 0 & \alpha_{21}\frac{1+\mu x}{\lambda_2(x)} \end{bmatrix},$$

and scalars $\mu, \alpha_{20}, \alpha_{21} > 0$.

This functional will be decomposed term by term as follows: $\mathcal{V}(\phi) = \mathcal{V}_1(\phi) + \alpha_{20}\mathcal{V}_{20}(\phi) + \alpha_{21}\mathcal{V}_{21}(\phi)$ with

$$\mathcal{V}_1(\phi) = \int_0^1 \tfrac{1+\mu x}{\lambda_1(x)}\phi_1^2(x)\mathrm{d}x,$$
$$\mathcal{V}_{20}(\phi) = \int_0^1 \tfrac{1+\mu x}{\lambda_2(x)}\phi_2^2(x)\mathrm{d}x,$$
$$\mathcal{V}_{21}(\phi) = \int_0^1 \tfrac{1+\mu x}{\lambda_2(x)}(\partial_x\phi_2)^2(x)\mathrm{d}x$$

First, we assess the following inequalities:

$$\begin{aligned} \min(\frac{1}{\overline{\lambda_1}}, \frac{\alpha_{20}}{\overline{\lambda_2}}, \frac{\alpha_{21}}{\overline{\lambda_2}})\|\phi\|_X \leq \mathcal{V}(\phi), \\ \mathcal{V}(\phi) \leq \left(\frac{1+\mu}{\underline{\lambda_1}} + \alpha_{20}\frac{1+\mu}{\underline{\lambda_2}} + \alpha_{21}\frac{1+\mu}{\underline{\lambda_2}}\right)\|\phi\|_X, \end{aligned} \tag{4}$$

Along C^1 solutions to system (1) with $B = 0$, the time derivative of each term of the function $V(t) = \mathcal{V}(\phi(t))$ is calculated below.

First, we obtain

$$\begin{aligned} \dot{V}_{20} &= 2\int_0^1 (1+\mu x)\phi_2(x)\partial_x\phi_2(x)\mathrm{d}x, \\ &= -\mu\int_0^1 \phi_2^2(x)\mathrm{d}x + (1+\mu)\phi_2^2(1) - \phi_2^2(0), \end{aligned}$$

where the argument t has been removed for simplicity. As $\phi_2(1) = D_{22}\phi_2(0)$, we have

$$\dot{V_{20}}(\phi) = -\mu\|\phi_2\|_{L^2}^2 + (\mu - \bar{\nu})\phi_2^2(1). \tag{5}$$

Second, in the same way, we obtain

$$\begin{aligned}\dot{V}_1 &= 2\int_0^1 (1+\mu x)\phi_1(x)\partial_x\phi_1(x)\mathrm{d}x \\ &\quad +2\int_0^1 c(x)(1+\mu x)\phi_1(x)\partial_x\phi_2(x)\mathrm{d}x, \\ &= -\mu\int_0^1 \phi_1^2(x)\mathrm{d}x + (1+\mu)\phi_1^2(1) - \phi_1^2(0) \\ &\quad +2\int_0^1 c(x)(1+\mu x)\phi_1(x)\partial_x\phi_2(x)\mathrm{d}x.\end{aligned}$$

For any a, b and $\gamma > 0$, Young's inequality (special case) ensures

$$2ab \leq \gamma a^2 + \tfrac{1}{\gamma}b^2$$

Applying the inequality to the crossed term leads

$$\begin{aligned}&2\int_0^1 c(x)(1+\mu x)\phi_1(x)\partial_x\phi_2(x)\mathrm{d}x \\ &\leq (1+\mu)\bar{c}\left(\gamma_1\int_0^1 \phi_1^2(x)\mathrm{d}x + \tfrac{1}{\gamma_1}\int_0^1 (\partial_x\phi_2)^2(x)\mathrm{d}x\right) \\ &\leq (1+\mu)\bar{c}\left(\gamma_1\|\phi_1\|_{L^2}^2 + \tfrac{1}{\gamma_1}\|\partial_x\phi_2\|_{L^2}^2\right)\end{aligned}$$

with $\gamma_1 > 0$, $\bar{c} = \max_{x\in(0,1)} |c(x)|$. Hence, the time derivative of V_1 is bounded by

$$\begin{aligned}\dot{V}_1 \leq\ & (-\mu + (1+\mu)\bar{c}\gamma_1)\|\phi_1\|_{L^2}^2 + \tfrac{(1+\mu)\bar{c}}{\gamma_1}\|\partial_x\phi_2\|_{L^2}^2 \\ & +(1+\mu)(D_{11}\phi_1(0) + D_{12}\phi_2(0))^2 - \phi_1^2(0)\end{aligned}$$

Moreover, for $\gamma_2 > 0$, Young's inequality ensures

$$\begin{aligned}&(1+\mu)(D_{11}\phi_1(0) + D_{12}\phi_2(0))^2 - \phi_1^2(0) \\ &\leq (1+\mu)(\ D_{11}^2\phi_1^2(0) + D_{12}^2\phi_2^2(0) + \\ &|D_{11}D_{12}|(\ \gamma_2\phi_1^2(0) + \tfrac{1}{\gamma_2}\phi_2^2(0)\ ) - \phi_1^2(0), \\ &\leq [(1+\mu)(D_{11}^2 + \gamma_2|D_{11}D_{12}|) - 1]\phi_1^2(0) \\ &\quad + (1+\mu)(D_{12}^2 + \tfrac{|D_{11}D_{12}|}{\gamma_2})\phi_2^2(0).\end{aligned}$$

Since by Assumption 1, $|D_{11}| < 1$, picking μ sufficiently small it yields that

$$\gamma_2 := \frac{1-(1+\mu)D_{11}^2}{(1+\mu)|D_{11}D_{12}|} > 0. \tag{6}$$

Consequently, it implies

$$\dot{V}_1 \leq (-\mu + \beta\gamma_1)\|\phi_1\|_{L^2}^2 + \alpha\phi_2^2(1) + \tfrac{\beta}{\gamma_1}\|\partial_x\phi_2\|_{L^2}^2, \tag{7}$$

with

$$\alpha = \frac{(1+\mu)}{D_{22}^2}\left(D_{12}^2 + \frac{|D_{11}D_{12}|}{\gamma_2}\right) = \left(\frac{1+\mu}{1-(1+\mu)D_{11}^2}\right)\frac{D_{12}^2}{D_{22}^2},$$

$\beta = (1+\mu)\bar{c}$.

Third, similarly, the time derivative of the last part of the Lyapunov candidate functional yields

$$\begin{aligned}\dot{V}_{21} &= 2\int_0^1 (1+\mu x)\partial_x\phi_2(x)\tfrac{\partial_{xt}\phi_2(x)}{\lambda_2(x)}\mathrm{d}x, \\ &= 2\int_0^1 (1+\mu x)\partial_x\phi_2(x)\partial_{xx}\phi_2(x)\mathrm{d}x \\ &\quad +2\int_0^1 (1+\mu x)\tfrac{\lambda_2'(x)}{\lambda_2(x)}(\partial_x\phi_2)^2(x)\mathrm{d}x, \\ &= -\mu\int_0^1 (\partial_x\phi_2)^2(x)\mathrm{d}x + (1+\mu)(\partial_x\phi_2)^2(1) \\ &\quad -(\partial_x\phi_2)^2(0) + 2\int_0^1 (1+\mu x)\tfrac{\lambda_2'(x)}{\lambda_2(x)}(\partial_x\phi_2)^2(x)\mathrm{d}x.\end{aligned}$$

As $\partial_x\phi_2(1) = D_{22}\partial_x\phi_2(0)$ from the compatibility condition, we have

$$\dot{V}_{21} \leq (-\mu + 2(1+\mu)\overline{\tfrac{\lambda_2'}{\lambda_2}})\|\partial_x\phi_2\|_{L^2}^2 + (\mu-\bar{\nu})(\partial_x\phi_2)^2(1) \tag{8}$$

Considering $\nu = (1-2\overline{\frac{\lambda_2'}{\lambda_2}})^{-1}2\overline{\frac{\lambda_2'}{\lambda_2}}$ is equivalent to $2\overline{\frac{\lambda_2'}{\lambda_2}} = \frac{\nu}{1+\nu}$ which means that $(-\mu + 2(1+\mu)\overline{\frac{\lambda_2'}{\lambda_2}}) = \frac{\nu-\mu}{1+\nu}$. Thus, from (5), (7) and (8), we end up with

$$\begin{aligned}\dot{V} \leq\ & (-\mu + \beta\gamma_1)\|\phi_1\|_{L^2}^2 - \alpha_{20}\mu\|\phi_2\|_{L^2}^2 \\ & + \left(\tfrac{\beta}{\gamma_1} + \alpha_{21}\tfrac{\nu-\mu}{1+\nu}\right)\|\partial_x\phi_2\|_{L^2}^2 \\ & + (\alpha_{20}(\mu-\bar{\nu}) + \alpha)\,\phi_2^2(1) \\ & +\alpha_{21}(\mu-\bar{\nu})(\partial_x\phi_2)^2(1).\end{aligned} \tag{9}$$

To make the three first terms negative and the two last terms vanish, we select the positive scalars $\mu, \alpha_{20}, \alpha_{21}, \gamma_1$ as follows

$$\begin{aligned}&\mu = \bar{\nu} - \frac{\varepsilon}{2}, \quad \gamma_1 = \frac{\mu-\bar{\nu}+\varepsilon}{\beta} = \frac{\varepsilon}{2\beta} \\ &\alpha_{20} = \frac{1}{(\bar{\nu}-\mu)}\alpha = \frac{2\alpha}{\varepsilon}, \quad \alpha_{21} = \frac{4(1+\nu)\beta^2}{\varepsilon^2}\end{aligned} \tag{10}$$

and sufficiently small scalar ε verifying $\varepsilon < \bar{\nu} - \nu$. With Assumption 1, $\overline{\frac{\lambda_2'}{\lambda_2}} < 1 - D_{22}^2$, hence, we clearly check that $\nu < \mu < \bar{\nu}$ holds. Then, as:

$$\begin{aligned}\|\phi_1\|_{L^2}^2 &\geq \tfrac{\lambda_1}{1+\mu}\mathcal{V}_1(\phi) \geq \tfrac{\lambda_1}{1+\bar{\nu}}\mathcal{V}_1(\phi), \\ \|\phi_2\|_{L^2}^2 &\geq \tfrac{\lambda_2}{1+\mu}\mathcal{V}_{20}(\phi) \geq \tfrac{\lambda_2}{1+\bar{\nu}}\mathcal{V}_{20}(\phi), \\ \|\partial_x\phi_2\|_{L^2}^2 &\geq \tfrac{\lambda_2}{1+\mu}\mathcal{V}_{21}(\phi) \geq \tfrac{\lambda_2}{1+\bar{\nu}}\mathcal{V}_{21}(\phi),\end{aligned}$$

the time derivative of the Lyapunov candidate functional along C^1 solutions of system (1) satisfy

$$\begin{aligned}\dot{V} \leq\ & -(\overline{\nu}-\varepsilon)\tfrac{\lambda_1}{1+\bar{\nu}}V_1 - \alpha_{20}(\overline{\nu}-\tfrac{\varepsilon}{2})\tfrac{\lambda_2}{1+\bar{\nu}}V_{20} \\ & -\alpha_{21}\tfrac{\bar{\nu}-\nu-\varepsilon}{1+\nu}\tfrac{\lambda_2}{1+\bar{\nu}}V_{21}\end{aligned}$$

meaning for sufficiently small scalar ε that

$$\dot{V} \leq -\delta V, \tag{11}$$

for any $\delta < \min\{\tfrac{\bar{\nu}\lambda_1}{1+\bar{\nu}}, \tfrac{\bar{\nu}\lambda_2}{1+\bar{\nu}}, \tfrac{(\bar{\nu}-\nu)\lambda_2}{(1+\bar{\nu})(1+\nu)}\}$. With Grönwall inequality the above equation (11), and by standard density arguments yields

$$V(t) = \mathcal{V}(\phi(t;\phi_0)) \leq e^{-\delta t}\mathcal{V}(\phi^0)\ ,\ \forall \phi^0 \in X.$$

To conclude, using the equivalence of norm (4),

$$\left\|\phi(t;\phi^0)\right\|_X^2 \leq \kappa e^{-\delta t}\left\|\phi^0\right\|_X$$

holds with κ and δ given explicitely independently from the initial condition, which allows us to close the proof. ■

Remark 1. Note that this result is inspired by Diagne et al. (2012); Prieur and Mazenc (2012) and extends (Bastin and Coron, 2016, Proposition 5.1, item (i) without source terms) for the case of triangular hyperbolic structure. For the diagonal case, i.e. $c = 0$ and when $\lambda_2' = 0$, we recover the well-known stability condition $\max(D_{11}^2, D_{22}^2) < 1$.

Remark 2. This result is established due to the cascade structure of system (1). With assumption 1, it can be shown that the origin of the first subsystem with state ϕ_2 is globally exponentially stable in H^1 norm for any decay rate smaller than $\frac{\bar{\nu}\lambda_2}{(1+\bar{\nu})}$ whereas the origin of the second part of state ϕ_1 is input-to-state stable with respect to the H^1 norm of ϕ_2, Prieur and Mazenc (2012).

Remark 3. The presence of the non diagonal term c in the system definition forces us to work with the $L^2 \times H^1$ norm. We found that when $D_{21} \neq 0$, the term $D_{21}\partial_x\phi_2(0)\partial_x\phi_1(0)$ in the stability analysis cannot be handled. It remains unclear whether the system (1) is stable at the origin when $D_{21} \neq 0$. Our numerical observations suggest that the presence of this term significantly affects the convergence properties of the system but that exponential stability still holds for D_{21} smaller than a small threshold. We could maybe handled this with another candidate Lyapunov functional.

3.2 Second study case : with the source term

In this section, we consider the case in which the source term B is non null. The new assumption that we consider, is given as follows.

Assumption 2. Assume that the functions $\lambda_2(x), \lambda_2'(x)$ and the matrices $B(x), D$ satisfy

$$\max(D_{11}^2, D_{22}^2) < 1 - \max\left\{2\overline{B_{11}}, \frac{2\overline{\lambda_2'}}{\underline{\lambda_2}} + 2\overline{B_{22}}\right\} \tag{12}$$

With this assumption 2, the following theorem 2 can be obtained.

Theorem 2. Let Assumption 2 holds and let

$$\begin{aligned} \nu_1 &:= 2\overline{B_{11}}(1 - 2\overline{B_{11}})^{-1}, \\ \nu_2 &:= \frac{2\overline{\lambda_2'} + 2\underline{\lambda_2}\overline{B_{22}}}{\underline{\lambda_2} - 2\overline{\lambda_2'} - 2\underline{\lambda_2}\overline{B_{22}}}, \\ \bar{\nu} &:= \frac{1 - D_{22}^2}{D_{22}^2}. \end{aligned} \tag{13}$$

Then, for any $\delta < \min\{\delta_1, \delta_2\}$ where $\delta_1 = \frac{\underline{\lambda}_1(\bar{\nu}-\nu_1)}{(1+\nu_1)(1+\bar{\nu})}$ and $\delta_2 = \frac{\underline{\lambda}_2(\bar{\nu}-\nu_2)}{(1+\nu_2)(1+\bar{\nu})}$, system (1) is δ-exponentially stable with an overshoot given by

$$\kappa = \frac{\frac{1}{\underline{\lambda}_1} + \frac{\alpha_{20}}{\underline{\lambda}_2} + \frac{\alpha_{21}}{\underline{\lambda}_2}}{\min(\frac{1}{\overline{\lambda}_1}, \frac{\alpha_{20}}{\overline{\lambda}_2}, \frac{\alpha_{21}}{\overline{\lambda}_2})}(1+\bar{\nu}-\tfrac{\varepsilon}{2}), \quad 0 < \varepsilon < \bar{\nu} - \max\{\nu_1, \nu_2\},$$

and α_{20}, α_{21} given by (18).

In this theorem 2, the stability is obtained under several conditions. As previously, the ratio between between spatial derivative of velocity, and velocity must be small. In addition, the diagonal coefficients of the source term must be upper-bounded (a fortiori when B_{11} and B_{22} are negative, Assumption 2 reduces to Assumption 1).

Proof. Note that with Assumption 2, ν_1, ν_2 are positive and all the constant involved in the statement of the theorem are well defined. Consider the functional $\mathcal{V} = \mathcal{V}_1 + \alpha_{20}\mathcal{V}_{20} + \alpha_{21}\mathcal{V}_{21}$ expressed in (3) with scalars $\mu, \alpha_{20}, \alpha_{21} > 0$ whose expression is given by (10). The equivalence of norm (4) is still verified. Let us now repeat the time derivation of each of the three terms in $V(t) = \mathcal{V}(\phi(t))$ along the C^1 trajectories of system (1) and let us focus on the additional terms that will be denoted W_1, W_{20}, W_{21} coming from the source term.

First, we obtain

$$W_{20} = 2\int_0^1 (1+\mu x)B_{22}(x)\phi_2^2(x)\mathrm{d}x,$$

leading to

$$W_{20} \leq 2(1+\mu)\overline{B_{22}}\|\phi_2\|_{L^2}^2. \tag{14}$$

Second, we obtain

$$W_1 = 2\int_0^1 (1+\mu x)\left(B_{11}(x)\phi_1^2(x) + B_{12}(x)\phi_1(x)\phi_2(x)\right)\mathrm{d}x.$$

Applying Young's inequality, for any $\gamma_3 > 0$, it leads to

$$W_1 \leq (1+\mu)\left((2\overline{B_{11}} + \gamma_3\overline{|B_{12}|})\|\phi_1\|_{L^2}^2 + \frac{\overline{|B_{12}|}}{\gamma_3}\|\phi_2\|_{L^2}^2\right), \tag{15}$$

Third, we obtain

$$\begin{aligned} W_{21} \;=\; & 2\int_0^1 (1+\mu x)B_{22}(x)\partial_x\phi_2^2(x)\mathrm{d}x, \\ & +2\int_0^1 (1+\mu x)\tfrac{(\lambda_2 B_{22})'(x)}{\lambda_2(x)}\partial_x\phi_2(x)\phi_2(x)\mathrm{d}x. \end{aligned}$$

Applying Young's inequality, for any $\gamma_4 > 0$, it leads to

$$\begin{aligned} W_{21} \;\leq\; & (1+\mu)\left(2\overline{B_{22}} + \gamma_4\frac{\overline{(\lambda_2 B_{22})'}}{\underline{\lambda_2}}\right)\|\partial_x\phi_2\|_{L^2}^2 \\ & +(1+\mu)\tfrac{1}{\gamma_4}\frac{\overline{(\lambda_2 B_{22})'}}{\underline{\lambda_2}}\|\phi_2\|_{L^2}^2 \end{aligned} \tag{16}$$

To sum up inequations (9), (14), (15) and (16), we have:

$$\begin{aligned} \dot{V} \leq\; & \left[-\mu + 2(1+\mu)\overline{B_{11}} + \gamma_1\beta + \gamma_3\beta_1\right]\|\phi_1\|_{L^2}^2 \\ & + \left[\alpha_{20}\left(-\mu + 2(1+\mu)\overline{B_{22}} + \tfrac{\alpha_{21}}{\alpha_{20}}\tfrac{\beta_2}{\gamma_4}\right) + \tfrac{\beta_1}{\gamma_3}\right]\|\phi_2\|_{L^2}^2 \\ & + \left[\alpha_{21}\left(-\mu + (1+\mu)(\frac{2\overline{\lambda_2'}}{\underline{\lambda_2}} + 2\overline{B_{22}}) + \gamma_4\beta_2\right) + \tfrac{\beta}{\gamma_1}\right]\|\partial_x\phi_2\|_{L^2}^2 \\ & + \left(\alpha_{20}(\mu - \bar{\nu}) + \alpha\right)\phi_2^2(1) + \alpha_{21}(\mu - \bar{\nu})(\partial_x\phi_2)^2(1). \end{aligned}$$

with

$$\beta_1 = (1+\mu)\overline{|B_{12}|}, \quad \beta_2 = (1+\mu)\tfrac{\overline{(\lambda_2 B_{22})'}}{\underline{\lambda_2}}.$$

From expression (13) we have

$$2\overline{B_{11}} = \tfrac{\nu_1}{1+\nu_1}, \quad \tfrac{2\overline{\lambda_2'}}{\underline{\lambda_2}} + 2\overline{B_{22}} = \tfrac{\nu_2}{1+\nu_2}.$$

Thus

$$\begin{aligned} \dot{V} \leq\; & \left[\tfrac{\nu_1-\mu}{1+\nu_1} + \gamma_1\beta + \gamma_3\beta_1\right]\|\phi_1\|_{L^2}^2 \\ & + \left[\alpha_{20}\left(\tfrac{\nu_2-\mu}{1+\nu_2} + \tfrac{\alpha_{21}}{\alpha_{20}}\tfrac{\beta_2}{\gamma_4}\right) + \tfrac{\beta_1}{\gamma_3}\right]\|\phi_2\|_{L^2}^2 \\ & + \left[\alpha_{21}\left(\tfrac{\nu_2-\mu}{1+\nu_2} + \gamma_4\beta_2\right) + \tfrac{\beta}{\gamma_1}\right]\|\partial_x\phi_2\|_{L^2}^2 \\ & + \left(\alpha_{20}(\mu - \bar{\nu}) + \alpha\right)\phi_2^2(1) + \alpha_{21}(\mu - \bar{\nu})(\partial_x\phi_2)^2(1). \end{aligned}$$

Remark 4. The most restrictive condition is used to fix an upper bound of both terms including $\overline{B_{22}}$. This defines ν_2.

Similarly to (10), we begin with setting $\mu, \gamma_1, \gamma_3, \gamma_4$ as follows

$$\begin{aligned} \mu &= \bar{\nu} - \frac{\varepsilon}{2}, \qquad \gamma_1 = \frac{\varepsilon}{4(1+\nu_1)\beta}, \\ \gamma_3 &= \frac{\varepsilon}{4(1+\nu_1)\beta_1}, \quad \gamma_4 = \frac{\varepsilon}{4(1+\nu_2)\beta_2}. \end{aligned} \tag{17}$$

to obtain

$$\begin{aligned} \dot{V} \leq\; & \tfrac{\nu_1-\bar{\nu}+\varepsilon}{1+\nu_1}\|\phi_1\|_{L^2}^2 \\ & + \left[\alpha_{20}\tfrac{\nu_2-\bar{\nu}+\frac{\varepsilon}{2}}{1+\nu_2} + \alpha_{21}\tfrac{4(1+\nu_2)\beta_2^2}{\varepsilon} + \tfrac{4(1+\nu_1)\beta_1^2}{\varepsilon}\right]\|\phi_2\|_{L^2}^2 \\ & + \left[\alpha_{21}\tfrac{\nu_2-\bar{\nu}+\frac{3\varepsilon}{4}}{1+\nu_2} + \tfrac{4(1+\nu_1)\beta^2}{\varepsilon}\right]\|\partial_x\phi_2\|_{L^2}^2 \\ & + \left(-\alpha_{20}\tfrac{\varepsilon}{2} + \alpha\right)\phi_2^2(1) - \alpha_{21}\tfrac{\varepsilon}{2}(\partial_x\phi_2)^2(1). \end{aligned}$$

Finally, we set α_{20}, α_{21} as

$$\begin{aligned}\alpha_{20} &= \max\left\{\frac{2\alpha}{\varepsilon}, \frac{8(1+\nu_1)(1+\nu_2)[16(1+\nu_2)^2\beta^2\beta_2^2+\varepsilon^2\beta_1^2]}{\varepsilon^4}\right\},\\ \alpha_{21} &= \frac{16(1+\nu_1)(1+\nu_2)\beta^2}{\varepsilon^2}.\end{aligned} \tag{18}$$

to achieve

$$\dot{V} \leq \frac{\nu_1-\bar{\nu}+\varepsilon}{1+\nu_1}\|\phi_1\|_{L^2}^2+\alpha_{20}\frac{\nu_2-\bar{\nu}+\varepsilon}{1+\nu_2}\|\phi_2\|_{L^2}^2+\alpha_{21}\frac{\nu_2-\bar{\nu}+\varepsilon}{1+\nu_2}\|\partial_x\phi_2\|_{L^2}^2.$$

Then, selecting ε small enough, the time derivative of the Lyapunov candidate functional along the trajectories of system (1) satisfies $\dot{V} \leq -\delta V$ for all $\delta < \min(\delta_1, \delta_2)$ and the equivalence of norm (4) ends the proof. ■

Remark 5. Note that the case $c \neq 0$ and $\lambda_1 = \lambda_2$ imposes the constraint $B_{21} = 0$ in our Lyapunov analysis. Indeed, when $B_{21} \neq 0$, a term in $\int_0^1 B_{21}(x)\partial_x\phi_2(x)\partial_x\phi_1(x)\mathrm{d}x$ appears and cannot be compensated in the absence of negativity on the H^1 norm of the state ϕ_1. A deeper analysis should be carried out to handle full matrices $B(x)$.

4. NUMERICAL RESULTS

In this numerical results section, we make use of two challenging test cases to illustrate the performance of the Lyapunov stability analysis approach proposed in the previous section for system (1).
System (1) represents a coupling of two transport equations with negative velocities. In order to design a numerical discretization of the PDEs system, we consider an *upwind* type *explicit finite difference* scheme that takes into account the directions of waves propagation (see LeVeque (2002)).
We denote the space-time domain $\mathscr{D} = [0,1] \times [0,T]$, T being the final time for system (1) dynamics. A discretization of this continuous domain is made; given $p \in \mathbb{N}^*$ sufficiently large, we introduce the space mesh-size as $\Delta x = \frac{1}{p+1}$. Same principle is applied for time discretization. Such procedure generates a sequence of grid points $(x_i, t^n) = (i\Delta x, n\Delta t)$, $i \in [\![0..p+1]\!]$, $n \in \mathbb{N}^*$. The time-stepping increment Δt will be specified later on.

Approximations of the states of system (1) will be made on the grid points of the mesh, we introduce the following notations:

$$\begin{cases}\phi_{k,i}^n = \phi_k(x_i, t^n), & k \in \{1,2\}\\ \lambda_{k,i} = \lambda_k(x_i), & k \in \{1,2\}\\ c_i = c(x_i).\end{cases}$$

4.1 First study case: without the source terms

System (1) is expressed in discrete form as:

$$\begin{cases}\frac{\phi_{1,i}^{n+1}-\phi_{1,i}^n}{\Delta t} = \lambda_{1,i}\frac{\phi_{1,i+1}^n-\phi_{1,i}^n}{\Delta x} + c_i\lambda_{2,i}\frac{\phi_{2,i+1}^n-\phi_{2,i}^n}{\Delta x}\\ \frac{\phi_{2,i}^{n+1}-\phi_{2,i}^n}{\Delta t} = \lambda_{2,i}\frac{\phi_{2,i+1}^n-\phi_{2,i}^n}{\Delta x}\end{cases} \tag{19}$$

Setting:

$$\gamma_{1,i} = \frac{\lambda_{1,i}\Delta t}{\Delta x}, \quad \gamma_{2,i} = \frac{c_i\lambda_{2,i}\Delta t}{\Delta x}$$

we get the numerical scheme :

$$\phi_{1,i}^{n+1} = (1-\gamma_{1,i})\phi_{1,i}^n+\gamma_{1,i}\phi_{1,i+1}^n-\gamma_{2,i}\phi_{2,i}^n+\gamma_{2,i}\phi_{2,i+1}^n \tag{20}$$

$$\phi_{2,i}^{n+1} = (1-\gamma_{2,i})\phi_{2,i}^n+\gamma_{2,i}\phi_{2,i+1}^n \tag{21}$$

Lemma 2. The numerical scheme (20)-(21) is stable [1] under the following Courant–Friedrichs–Lewy (CFL) condition: $\Delta t \leq \Delta x \min(\frac{1}{\lambda_1}, \frac{1}{\lambda_2})$.

Let us now consider system (1) with:

$$\lambda_1(x) = \begin{cases}10, & 0 \leq x < \frac{1}{2}\\ -4(x-\frac{1}{2})^2+10 & \frac{1}{2} \leq x \leq 1\end{cases}$$

$$\lambda_2(x) = \begin{cases}10, & 0 \leq x < \frac{1}{2}\\ 4(x-\frac{1}{2})^2+10 & \frac{1}{2} \leq x \leq 1\end{cases}$$

$c(x) = x+4$, $D = \begin{bmatrix}\frac{1}{2} & 1\\ 0 & \frac{1}{2}\end{bmatrix}$,

$\phi_1^0(x) = -\frac{2}{3}$, $\phi_2^0(x) = \frac{x^2}{2}(1-\frac{x}{3})-\frac{2}{3}$.

In this example, $\lambda_1(x) = \lambda_2(x)$ for $x \in [0.\frac{1}{2}]$, which makes the system weakly hyperbolic.
Fig.1 gives the time-space evolution of the state variable ϕ_1. The time evolution of $\frac{\|\phi\|_X^2}{\|\phi^0\|_X^2}$ is given in Fig.2.

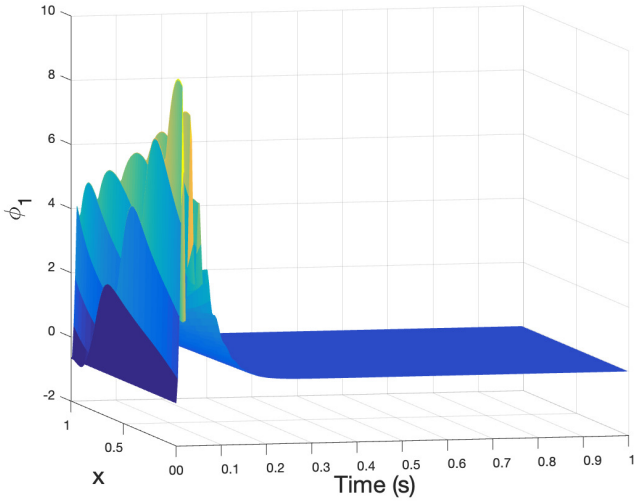

Fig. 1. State ϕ_1 over (t,x) plane, (final time $T = 1$s)

[1] The method must be stable in some appropriate sense, meaning that the small errors made in each time step do not grow too fast in later time steps

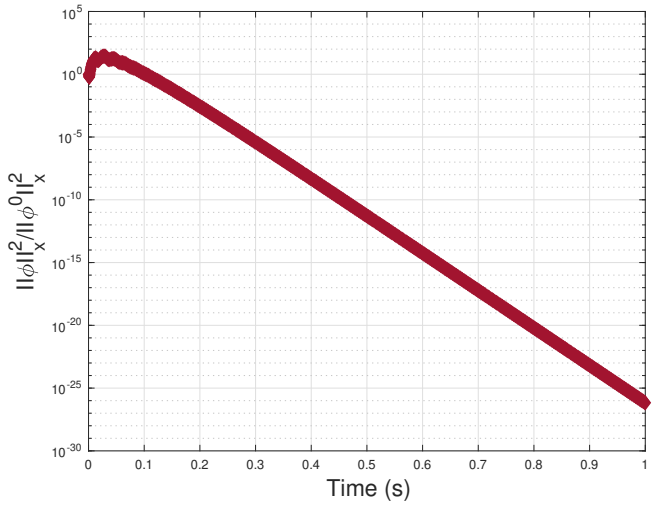

Fig. 2. Time evolution of $\frac{\|\phi\|_x^2}{\|\phi^0\|_x^2}$ in logarithmic scale

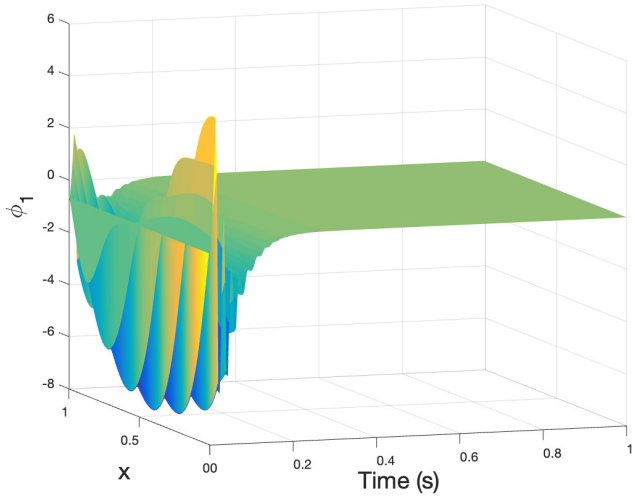

Fig. 3. State ϕ_1 over (t,x) plane, (final time $T = 1$s)

Both states are stabilized to zero as time grows up, the simulations are carried out using the numerical scheme (20)-(21) with 500 grid cells in space under the stability constraint ($\frac{\Delta t}{\Delta x \min(\frac{1}{\bar{\lambda}_1}, \frac{1}{\bar{\lambda}_2})} = 0.9$).

Here, regarding Theorem 1, the assumption

$$\max(D_{11}^2, D_{22}^2) = \tfrac{1}{4} < 1 - \frac{\overline{\lambda_2'}}{\underline{\lambda_2}} = \tfrac{3}{5}$$

is satisfied. Thus, the numerical result corroborates the theoretical one. Theoretically, we expect a decay rate given by $\min\{\frac{\bar{\nu}\underline{\lambda}_1}{1+\bar{\nu}}, \frac{\bar{\nu}\underline{\lambda}_2}{1+\bar{\nu}}, \frac{(\bar{\nu}-\nu)\underline{\lambda}_2}{(1+\bar{\nu})(1+\nu)}\} = \frac{7}{2}$ whereas, numerically, we found $\delta_{num} = 66.46 > \delta_{th}$. The difference is due to the conservatism introduced by the proposed Lyapunov analysis. The numerical overshoot is $\kappa_{num} = 948.86$.

4.2 Second study case: with the source term

The numerical scheme when adding the source term is written as:

$$\phi_{1,i}^{n+1} = (1-\gamma_{1,i})\phi_{1,i}^n + \gamma_{1,i}\phi_{1,i+1}^n - \gamma_{2,i}\phi_{2,i}^n + \gamma_{2,i}\phi_{2,i+1}^n \quad (22)$$

$$+\Delta t(\lambda_{1,i}B_{11,i}\phi_{1,i}^n + \lambda_{1,i}B_{12,i}\phi_{2,i}^n)$$

$$\phi_{2,i}^{n+1} = (1-\gamma_{2,i})\phi_{2,i}^n + \gamma_{2,i}\phi_{2,i+1}^n + \Delta t\lambda_{2,i}B_{22,i}\phi_{2,i}^n \quad (23)$$

Let us now consider system (1) with λ_1, λ_2, c, D and initial conditions that are similar to first example but with a source term $B = \begin{bmatrix} \frac{x}{6} & 100x^3 \\ 0 & \frac{x}{6} \end{bmatrix}$

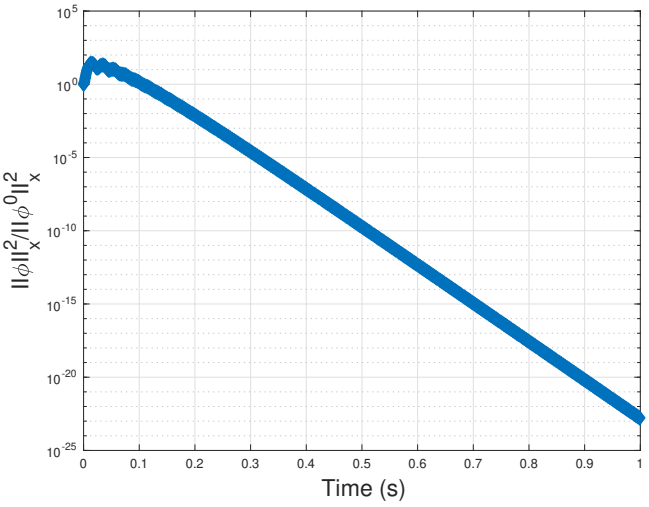

Fig. 4. Time evolution of $\frac{\|\phi\|_x^2}{\|\phi^0\|_x^2}$ in logarithmic scale

In this situation both states are also stabilized to zero as time grows up. The simulations are carried out using the numerical scheme (22)-(23) for 500 grid cells in space, with a similar stability constraint as the first study case.

Here, regarding Theorem 2, the assumption:
$\max(D_{11}^2, D_{22}^2) = \frac{1}{4} < 1 - \max\left\{2\overline{B_{11}}, \frac{\overline{\lambda_2'}}{\underline{\lambda_2}} + 2\overline{B_{22}}\right\} = \frac{4}{15}$ is satisfied. As stated by Theorem 2, exponential stability is guaranteed with a decay rate at least equal to $\delta_{th} = 0.16$. Hopefully, the numerical result leads to a decay rate of

$$\delta_{num} = 58.17 > \delta_{th}$$

which is better than expected and a numerical overshoot $\kappa_{num} = 530.38$ (unlike the first study case, it seems that adding a source term influences the overshoot). This confirms the encouraging approach currently proposed by our Lyapunov functionals.

Remark 6. Several other tests have been carried out, particularly in the case where matrices D and B have a non-

null under diagonal element (which we denote respectively D_{21} and B_{21}, for such situations system (1) is always unstable. We conjecture that closing the loop has a destabilizing effect.

5. CONCLUSIONS

To study the stability of weakly hyperbolic systems, where the principal part is non-uniform and non-diagonalizable everywhere, we return to the non-trivial study of triangular structures. Based on a Lyapunov analysis in the $L^2 \times H^1$ norm, two sufficient conditions for exponential stability emerge: the velocity and source terms have to be small in front of the edge back-reaction. Our refined analysis also allows us to specify upper bounds for decay rate and overshoot values, as illustrated in numerical simulations. Upcoming works will be dedicated to a non-diagonal Lyapunov functional (full matrix P) to reduce the conservatism and tackle the case of systems where D_{21} and B_{21} are non null (full matrices B, D). Applications to some real physical systems will also be considered in the future.

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