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► **To cite this version:**

Simen Å. Ellingsen, Zibo Zheng, Malek Abid, Christian Kharif, Yan Li. Dispersive Wave Focusing on a Shear Current: Part 1-Linear Approximations. *Water Waves*, In press, 10.1007/s42286-024-00085-3 . hal-04535288

HAL Id: hal-04535288

<https://cnrs.hal.science/hal-04535288>

Submitted on 6 Apr 2024

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1 Dispersive wave focusing on a shear current.

2 Part 1: Linear approximations

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13 Abstract

14 We consider the evolution and kinematics during dispersive focusing,
15 for a group of waves propagating atop currents varying with depth.
16 Our analysis assumes long-crested linear waves propagating at arbi-
17 trary angles relative to the current. Although low steepness is assumed,
18 the linear model is often a reasonable approximation for understanding
19 rogue waves. A number of analytical approximate relations are derived
20 assuming different sub-surface current profiles, including linearly varying
21 current, exponentially varying current, and currents of arbitrary depth
22 profile which are weakly sheared following the approximation of Stewart
23 & Joy (*Deep Sea Res. Abs.* **21**, 1974). The orbital velocities are like-
24 wise studied. While shear currents have modest influence on the motion
25 of the envelope of the wave group, they significantly change wave kine-
26 matics. Horizontal orbital velocities are either amplified or suppressed
27 depending on whether the shear is opposing or following, respectively.
28 To illustrate these phenomena we consider a real-world example using
29 velocity profiles and wave spectra measured in the Columbia River
30 estuary. Near the surface at the point where focusing occurs, hori-
31 zontal orbital velocities are respectively increased and decreased by
32 factors of **1.4** and **0.7** for focusing groups propagating on following and

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33 opposing shear (respectively upstream and downstream in the earth-
34 fixed reference system). The implications for the forces a focusing wave
35 group can exert on vessels and installations are profound, emphasizing
36 the importance of considering current profiles in maritime operations.

37 **Keywords:** Wave-shear current interaction, Focused wave group, Wave
38 kinematics

39 **1 Introduction**

40 Rogue waves, characterized as enormous and abrupt waves appearing on the
41 sea's surface, pose a significant threat to maritime activities. These waves
42 which are defined by being far higher than the waves around them, can emerge
43 without warning, occurring both in deep and shallow waters, and they result
44 from various physical mechanisms that concentrate the energy of water waves
45 into a small area. Their occurrence has led to numerous fatalities, injuries, and
46 extensive damages to ships and maritime structures. Among the mechanisms
47 responsible for their formation are dispersive focusing, refraction influenced
48 by variable currents and bottom topography, modulational instability, con-
49 structive wave interference enhanced by second-order interactions, cross-sea
50 interactions, and soliton interactions. For a comprehensive review of these
51 mechanisms, refer to the works by Kharif & Pelinovsky [1], Dysthe et al. [2],
52 and Onorato et al. [3].

53 The main objective of this paper is to analyze the effect of depth-dependent
54 underlying currents on the dispersive focusing of water waves in deep water.
55 Extreme wave events resulting from dispersive focusing or spatiotemporal
56 focusing phenomena can be described as follows: when initially shorter wave
57 packets are positioned in front of longer wave packets with higher group veloc-
58 ities, the longer waves eventually catch up and overtake the shorter waves
59 during the dispersive evolution process. At a fixed location (known as the focus
60 point) and time, the superposition of all these waves leads to the formation
61 of a large-amplitude wave. Subsequently, the longer waves move ahead of the
62 shorter waves, resulting in a decrease in the amplitude of the wave train. In
63 the absence of vorticity, giant waves created by dispersive focusing have been
64 frequently studied experimentally [4–7] and theoretically [8, 9], but studies in
65 the presence of a shear current are very scarce. Kharif et al. [10] investigated
66 the effect of a constant vorticity underlying current on the dispersive focus-
67 ing of a one-dimensional nonlinear wave group propagating in shallow water.
68 Their findings revealed that the presence of constant vorticity increases the
69 maximum amplification factor of the surface elevation as the shear intensity
70 of the current increases. The duration of extreme wave events follows a similar
71 behavior. In narrowband assumption Xin et al. [11] report the different effects
72 of following and opposing shear current on both the extreme and fatigue loads
73 on fixed-bottom offshore slender structures in extreme wave events. The work

74 has shed light on the shear-current modified wave kinematics in the design of
 75 offshore structures, which will be explored further in this work as explained
 76 below.

77 This study is the first part of an investigation of waves focusing dispersively
 78 on vertically sheared currents. The, in some respects, simplest case is treated in
 79 this first part, where the theory is linearised with respect to wave steepness; a
 80 second-order theory is found in part two [12]. Although nonlinear wave effects
 81 are significant for rogue wave situations, a linear approximation has been found
 82 to give reasonable results [13]. Our focus is on investigating the behavior of
 83 focusing wave groups propagating obliquely to the current direction, as well
 84 as wave groups traveling in the same direction as the current, in deep water
 85 (see Figure 1). By analyzing the impact of these depth-dependent underlying
 86 currents on the dispersive focusing of water waves, we aim to enhance our
 87 understanding of the formation and characteristics of rogue waves, contributing
 88 to improved safety measures for maritime activities as highlighted in [11].

89 We consider a range of vertically sheared currents, and of wave shapes
 90 at focus, deriving a series of closed-form approximate results. The currents
 91 we consider include the linear and exponential depth dependence — cases for
 92 which closed-form solutions exist for the linear velocity field — and arbitrary
 93 current profiles which satisfy the weak-shear approximation (fundamentally
 94 that required for the celebrated approximation of Stewart & Joy [14]). Wave
 95 groups focusing to a δ -function singularity, and a narrowband Gaussian packet,
 96 are considered. The approximate formulae derived are, we propose, useful for
 97 their relative simplicity and analytical tractability, for instance for the creation
 98 of focusing waves on sheared currents in numerical and laboratory experiments
 99 (see, e.g., [15]).

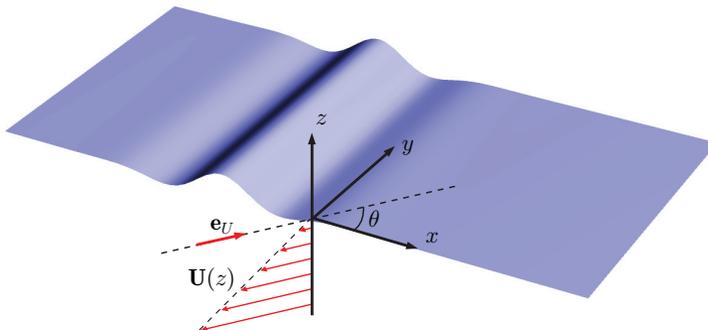


Fig. 1 Geometry: a quasi-2D wave propagating along the x axis. A sub-surface shear current makes an angle θ with the direction of wave propagation. Here $\mathbf{U}(z) = Sze_{\mathbf{U}}$. Currents with opposing and following shear are denoted by $\theta \in [0, \pi/2)$ and $(\pi/2, \pi]$, respectively.

100 A main conclusion of our work, illustrated and quantified through many
 101 examples, is the following: within a linear framework, the presence of shear has
 102 modest effect on the focusing and defocusing of the wave-group envelope, but

103 a large effect on the wave kinematics. The component of the orbital velocities
104 near the surface at the point of focus can be strongly enhanced.

105 2 Theoretical background

106 In this section we review the necessary background theory and phrase it in the
107 formalism we use herein. While not in a strict sense novel, the reexamination
108 of the basics sheds important light on the mechanisms in play which we will
109 refer extensively to in later sections. After defining the problem and geometry,
110 the linear initial-value solution is provided in a suitable form, and standard
111 current profiles and highly useful approximations are briefly recapitulated.

112 2.1 Problem definition

113 We consider a body of water with a free surface which, when undisturbed, is at
114 $z = 0$, and sustaining a shear current which depends arbitrarily on depth. The
115 depth is infinite (some results are generalised to allow finite depth in Appendix
116 A), and we ignore the effects of surface tension. The geometry is sketched in
117 figure 1. The background current has the form

$$\mathbf{U}(z) = \{U_x(z), U_y(z)\} = U(z)\mathbf{e}_U; \quad (1)$$

$$\mathbf{e}_U = \{\cos \theta, \sin \theta\}, \quad (2)$$

118 where \mathbf{e}_U is a unit vector in the xy plane. Without loss of generality we choose
119 the coordinate system which follows the surface of the water so that $U(0) = 0$.
120 As well as simplifying the formalism this choice emphasizes that the effects
121 studies are due to shear rather than surface current. A nonzero surface current
122 is easily worked back into solutions by adding a Doppler shift to frequencies.
123 The angle between wave propagation and current is θ and we shall assume
124 without loss of generality that the wave propagates along the x axis so that
125 $\mathbf{k} \cdot \mathbf{U} = kU_x = kU(z) \cos \theta$. In derivations we shall often retain a general wave
126 vector $\mathbf{k} = \{k_x, 0\}$. We allow k_x to take both signs in derivations, eventually
127 arriving at expressions for waves propagating only in the positive x direction,
128 whereupon we may consider only positive wave numbers. We assume long-
129 crested waves, so the surface elevation is $\zeta(\mathbf{r}, t) = \zeta(x, t)$ where $\mathbf{r} = (x, y)$.
130 We assume here that the current does not change direction with depth, but
131 generalisation to a z -dependent θ is straightforward.

132 As is well known (e.g. [16]), the surface elevation and dispersion relation
133 depend only on \mathbf{U} in the combination $\mathbf{k} \cdot \mathbf{U} = kU_x$, whereas U_y has no influence
134 on ζ . The angle θ thus only plays the role of varying U_x through values between
135 $-U$ and U . We shall see in section 4 that the same is not the case for the
136 velocity field beneath the waves.

137 In this paper we linearise equations and boundary conditions with respect
138 to ζ and its derivatives, as well as orbital velocities — a companion paper
139 considers weakly nonlinear extensions.

2.2 Linear initial-value problem, and solution

We will solve initial value problems in this set-up, a simpler, long-crested version of the theory presented in [17]. The general linear solution can be written in Fourier form with the 3D formulae in Ref. [17] assuming translational symmetry in the y direction, as

$$\zeta(\mathbf{r}, t) = \int_{-\infty}^{\infty} \frac{dk_x}{2\pi} \left[b_+(k_x) e^{-i\omega_+(k_x)t} + b_-(k_x) e^{-i\omega_-(k_x)t} \right] e^{ik_x x} \quad (3)$$

where $\omega_{\pm}(\mathbf{k})$ are the two solutions of the linear dispersion relation for a general wave vector \mathbf{k} , and $b_{\pm}(\mathbf{k}) = b_{\pm}(k_x)$ are spectral weights determined by initial conditions.

In the reference system following the surface current (i.e., $U(0) = 0$), the dispersion relation always has one positive and one negative solution, corresponding to waves propagating in direction \mathbf{k} and $-\mathbf{k}$, respectively, implying that $\omega_+ \geq 0$ and $\omega_- \leq 0$.

Our initial condition is that the shape of the packet is prescribed at focus, $t = 0$, and propagates only in the positive x direction. Let the Fourier transform of ζ at focus be

$$\zeta(x, 0) = \int_{-\infty}^{\infty} \frac{dk_x}{2\pi} \tilde{\zeta}_0(k_x) e^{ik_x x}; \quad \tilde{\zeta}_0(k_x) = \int_{-\infty}^{\infty} dx \zeta(x, 0) e^{-ik_x x}, \quad (4)$$

which with the general linear solution (3) implies

$$b_+(k_x) + b_-(k_x) = \tilde{\zeta}_0(k_x). \quad (5)$$

In order to obtain the appropriate initial shape with only plane waves propagating in the $+x$ direction, we couple the kernel $\exp(ik_x x)$ to $\exp(-i\omega_- t)$ when $k_x < 0$ and to $\exp(-i\omega_+ t)$ when $k_x > 0$:

$$b_{\pm}(k_x) = \tilde{\zeta}_0(k_x) \Theta(\pm k_x), \quad (6)$$

where Θ is the Heaviside unit step function, explicitly

$$\tilde{\zeta}_0(k_x, t) = \tilde{\zeta}_0(k_x) \left[e^{-i\omega_-(k_x)t} \Theta(-k_x) + e^{-i\omega_+(k_x)t} \Theta(k_x) \right]. \quad (7)$$

Substituting $k_x \rightarrow -k_x$ for the b_- term and noticing the well-known symmetry

$$\omega_-(-k_x) = -\omega_+(k_x), \quad (8)$$

the solution may be written

$$\zeta(x, t) = \int_0^{\infty} \frac{dk}{2\pi} \left[\tilde{\zeta}_0(-k) e^{-i\psi} + \tilde{\zeta}_0(k) e^{i\psi} \right], \quad (9)$$

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162 where we now simplify the notation, $k_x \rightarrow k$ and $\omega_+(k_x) \rightarrow \omega(k)$ is the positive-
163 valued frequency. As shorthand, we define the wave phase

$$\psi = \psi(x, t; k) \equiv kx - \omega(k)t, \quad (10)$$

164 frequently written without arguments for succinctness. Since (9) is real-valued,
165 it follows that $\tilde{\zeta}_0(-k) = \tilde{\zeta}_0^*(k)$, so we finally write

$$\zeta(x, t) = 2 \operatorname{Re} \int_0^\infty \frac{dk}{2\pi} \tilde{\zeta}_0(k) e^{i\psi}. \quad (11)$$

166 In the following we shall use this form and therefore assume $k_x = k > 0$, with
167 the exception of derivations where it is sometimes necessary to return to the
168 more fundamental form.

169 In particular, if the shape at $t = 0$ is symmetrical around $x = 0$, $\tilde{\zeta}_0(k)$ is
170 real, hence

$$\zeta(x, t) = 2 \int_0^\infty \frac{dk}{2\pi} \tilde{\zeta}_0(k) \cos \psi(x, t; k). \quad (12)$$

171 **2.3 Vertically sheared currents**

172 We here introduce the classic linear and exponential shear profiles used
173 as canonical examples, and approximate linear theories for arbitrary shear.
174 Known results are briefly reviewed and framed in the formalism we use herein.

175 **2.3.1 Current with constant shear**

176 First we quote well-known results for the simple, linearly depth-dependent
177 current

$$\mathbf{U}(z) = Sz\mathbf{e}_U = Sz\{\cos \theta, \sin \theta\} \quad (13)$$

178 with S the constant shear.

179 Due to the symmetry (8) it is sufficient to consider the positive-valued $\omega(k)$
180 assuming positive k , which we write in the form

$$\omega_\sigma(k) = kc_\sigma(k) = \sqrt{g\chi} - \sigma \quad (14)$$

181 where the shear-modified wave number is

$$\chi(k) = k + \sigma^2/g, \quad (15)$$

182 and

$$\sigma \equiv \frac{1}{2}S \cos \theta. \quad (16)$$

183 The group velocity is

$$c_{g\sigma}(k) = \frac{1}{2} \sqrt{\frac{g}{\chi}}. \quad (17)$$

184 Note for future reference that $c_{g\sigma}(k)$ is symmetrical under $\sigma \rightarrow -\sigma$ while c_σ
185 is not.

186 In the spirit of [17, 18] we may define a Froude-shear number for the linear-
187 shear current based on velocity S/k and length $1/k$,

$$\text{FS}_{\text{lin.}} = \frac{S}{2\sqrt{gk}} \quad (18)$$

188 so that the shear-modified wave number is $\varkappa = k(1 + \text{FS}_{\text{lin.}}^2 \cos^2 \theta)$. This is
189 particularly instructive in narrow-band cases (e.g., long groups with Gaussian
190 envelope) where there is a dominating carrier wave number; see sections 3.4
191 and 4.1 for further details. Here and henceforth a subscript ‘lin.’ indicates the
192 subscribed quantity pertains to the linear current.

193 2.3.2 Current with exponential shear

194 We will frequently make use of the model current with exponential depth
195 profile, which we define

$$\mathbf{U}_{\text{exp}}(z) = U_0(e^{\alpha z} - 1)\mathbf{e}_U = \{U_{x0}, U_{y0}\}(e^{\alpha z} - 1), \quad (19)$$

196 where $\alpha > 0$ is a shear strength and U_{x0} a current strength. This model has
197 been considered for the purposes of wave-current interactions for a very long
198 time, thanks to its similarity to a wind-driven shear-layer (Ekman current)
199 [19].

200 An explicit, exact solution to the linear problem with the exponential cur-
201 rent can be found in terms of hypergeometric functions, which we review in
202 section 4.4. The dispersion relation in this case is, however, implicit but easily
203 calculated numerically.

204 The exponential profile is a particularly useful model in combination with
205 the weak-shear approximation (see section 2.4.1), an approximation which is
206 excellent in the vast majority of oceanographic and coastal flows. Near-surface
207 flows, such as wind-driven Ekman layers or estuarine plumes, are typically
208 reasonably approximated by an exponential, and in this case the linearised
209 weak-shear theory yields a wealth of explicit analytical results, a number of
210 which we derive in this article.

211 2.4 Weak-shear and weak-curvature theory

212 We will summarise the results of theories for dispersion relations and flow fields
213 for an arbitrary current $\mathbf{U}(z)$ satisfying criteria of weak shear and weak cur-
214 vature, respectively. We emphasize that although the former approximation is
215 termed ‘weak shear’ due to the formal requirements for it to be asymptotically
216 accurate, in fact in an oceanic setting the shear can be very strong as in the
217 case of the Columbia River Estuary considered in section 4.5.1, and still give
218 results accurate to within a few percent or less.

219 The weak-shear approximation is in practice that underlying the celebrated
220 approximation of Stewart & Joy [14], typically sufficient in practice while in

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221 cases of extremely strong shear (as effectively felt by a wave of the wave-
 222 length in question), the strong-shear-weak-curvature expressions [20] could be
 223 necessary.

224 As discussed in Ref. [20], a suitable measure of the effective strength of
 225 the current shear is a dimensionless depth-integrated shear, or “directional
 226 shear-Froude number”, $\delta(\mathbf{k})$, defined as

$$\delta(\mathbf{k}) \equiv \frac{1}{c_0(k)} \int_{-\infty}^0 dz U'_x(z) e^{2kz} \equiv \text{FS}_{\text{gen.}} \cos \theta. \quad (20)$$

227 with $c_0(k) = \sqrt{g/k}$ as usual. We use the symbol δ as well as FS to make contact
 228 with previously published theory [17, 18, 20], despite the slight redundancy.
 229 The x component of \mathbf{U} is taken, being the component aligned with the waves,
 230 $\mathbf{k} \cdot \mathbf{U} = kU_x$.

231 That the parameter $\delta(\mathbf{k})$ is a direct generalisation of the shear-Froude num-
 232 ber (18) for linear shear based on the along-wave (here: x) current component,
 233 is easily seen by inserting $U_x(z) = Sz \cos \theta = 2\sigma z$ which gives $\delta(\mathbf{k}) = \delta_{\text{lin.}}(\mathbf{k})$
 234 with

$$\delta_{\text{lin.}}(\mathbf{k}) = 2\sigma \sqrt{\frac{k}{g}} \int_{-\infty}^0 dz e^{2kz} = \frac{\sigma}{\sqrt{gk}} = \text{FS}_{\text{lin.}} \cos \theta. \quad (21)$$

235 For ease of comparison to the linear-shear case above, it is also instructive
 236 for us to define the shear-induced Doppler shift for a wave propagating at an
 237 angle θ :

$$\sigma_\delta(k) = \omega_0(k) \delta(\mathbf{k}) = k \int_{-\infty}^0 dz U'_x(z) e^{2kz} \quad (22)$$

238 a generalisation of σ for the linear current in Eq. (16). We defined $\omega_0 = \sqrt{gk}$.

239 2.4.1 Weak shear

240 A sufficient criterion for the approximate theory of Stewart & Joy [14] and its
 241 generalisations [21, 22] to be accurate is $\delta(\mathbf{k}) \ll 1$ for all k which contribute
 242 significantly (we follow the convention of [23] that \ll and \gg refer to the
 243 absolute values of the quantities compared). The results in references [14, 22]
 244 were derived assuming weak current, $U \ll c$, yet it is shown in ref. [20] that
 245 the true condition of validity is that the shear is weak. (This was suspected by
 246 Kirby & Chen [22] and in fact obliquely discussed already by Skop [21]). After
 247 a partial integration of the original form of the much-used approximation due
 248 to Stewart & Joy [14], it can be written

$$\omega(k) \approx \sqrt{gk} - \sigma_\delta(k) = \omega_0(k) [1 - \delta(\mathbf{k})]. \quad (23)$$

249 We mention in passing that although (20) performs excellently for most typical
 250 ocean and coastal currents concentrated in the near-surface region (e.g. [20,

251 24]), such as the exponential current profile, it does not perform particularly
 252 well for the linear shear case even when shear is moderate [24]; for currents
 253 which are close to linear, the strong-shear approximation in the next section
 254 should be used.

255 Further formulae in the weak-shear approximation will be quoted or derived
 256 later, as they are needed. Explicit expressions for the exponential current in
 257 the context of surface motion are found in section 3.2.1, and weak-shear theory
 258 for the kinematics and orbital velocities may be found in Section 4.2.

259 2.4.2 Strong shear, weak curvature

260 A similar theory allowing $U(z)$ to have arbitrarily strong shear, but weak
 261 curvature, was developed by Ellingsen & Li [20]; the explicit limitation on
 262 curvature may be found therein. Now $\delta(\mathbf{k})$ can be arbitrarily large compared to
 263 unity. The approximate dispersion relation derived in [20] (equation 18) is

$$\omega(k) \approx \omega_0(k)(\sqrt{1 + \delta^2} - \delta) = \sqrt{gk + \sigma_\delta^2} - \sigma_\delta. \quad (24)$$

264 Ellingsen & Li finds no practical situations where (24) performs significantly
 265 worse than (23), and it fares far better when δ is not small compared to unity.
 266 Notice that when the linear current (13) is inserted, one finds $\sigma_\delta(k) = \sigma$
 267 as defined in (16) and the dispersion relation (14) is regained exactly. The
 268 formalism thus bears a close similarity to that with constant shear, in section
 269 2.3.1. Moreover, $\delta \ll 1$ returns the weak-shear dispersion relation (23) to
 270 leading order.

271 The close resemblance in form to the constant shear case makes it natural
 272 to define a generalised function analogous to Eq. (15),

$$\varkappa_\delta(k) = k + \sigma_\delta(k)^2/g. \quad (25)$$

273 whereby (24) can be written $\omega(k) \approx \omega_\delta(k) = \sqrt{g\varkappa_\delta(k)} - \sigma_\delta(k)$.

274 3 Surface motion

275 In this section we derive and analyse a number of potentially useful results for
 276 the moving free surface of a focusing wave groups, including explicit approx-
 277 imate expressions for general and special cases. We assume throughout in
 278 Section 3 that the surface elevation at focus, $\zeta(x, 0)$, is symmetric in x for
 279 simplicity.

280 For purposes of analytical treatment, there are two challenges to contend
 281 with when a vertical shear current is present: the dispersion relation is not
 282 in general given in closed form, and the waves are described by Fourier inte-
 283 grals with no closed-form solutions. We consider in the following a number of
 284 special cases and/or simplifying assumptions which allow useful, closed-form
 285 expressions to be derived.

3.1 General dispersion considerations

We briefly argue why the dispersion relation predicts that vertical shear of $U(z)$ has little effect on the group envelope, but can greatly affect the phase velocity and hence the wave kinematics. We focus now on the simplest case of a linear shear current (13) which is sufficient to illustrate the overall effect of shear.

If now σ is not small compared to \sqrt{k} , the phase velocity in equation (14) depends strongly on θ ; for $\cos \theta > 0$ (opposing shear) the two terms in (14) tend to cancel each other while for $\cos \theta < 0$ (following shear) they add to each other, giving a phase velocity which can be far higher. In contrast, the group velocity is identical under $\theta \rightarrow -\theta$ since $c_{g\sigma}$ depends on $|\cos \theta|$ only.

Bearing in mind that the envelope of a focusing group of waves is governed by the group velocity and its k derivative, the evolution of the group as a whole is largely independent of whether propagation is upstream or downstream. The kinematics of the wave patterns within the focusing group, however, are related to the phase velocity which can be very different depending on the direction θ . To wit, the ratio between phase velocity for opposite directions $\theta = 0$ and $\theta = \pi$ is

$$\frac{c(\theta = \pi)}{c(\theta = 0)} = \frac{\sqrt{1 + \text{FS}_{\text{lin.}}^2} + \text{FS}_{\text{lin.}}}{\sqrt{1 + \text{FS}_{\text{lin.}}^2} - \text{FS}_{\text{lin.}}} = \left(\sqrt{1 + \text{FS}_{\text{lin.}}^2} + \text{FS}_{\text{lin.}} \right)^2 \quad (26)$$

which very significant indeed when $\text{FS}_{\text{lin.}} \sim \mathcal{O}(1)$. In section 4 we study the closely related amplification of horizontal velocities at focus depending on θ .

The situation becomes particularly pointed for strong shear, $\sigma \gg 1$, in which case the phase and group velocities in equations (14) and (17) are

$$c_\sigma(k) = \frac{1}{k}(|\sigma| - \sigma) + \frac{g}{2|\sigma|} + \dots \quad (27a)$$

$$c_{g\sigma}(k) = \frac{g}{2|\sigma|} + \dots \quad (27b)$$

For $\cos \theta > 0$ the group and phase velocities become asymptotically equal and the wave becomes nondispersive, whereas for $\cos \theta < 0$ phase velocity can be many times greater than group velocity.

The effect is illustrated in Fig. 2 where we plot $\eta(x, t)$ at a series of equidistant times as the wave group focuses and defocuses. A short Gaussian packet with carrier wave number k_0 and length L is chosen for improved illustration, as defined and discussed in section 3.4. We plot time in units of $T_{\text{ref}} = \sqrt{L/g}$. The surface elevation $\zeta(x, t)$ was evaluated numerically from Eq. (12). The shear S is constant and made strong for clarity of illustration, $\text{FS}_{\text{lin.}} \cos \theta$ takes the values $-\frac{1}{2}, 0$ and $\frac{1}{2}$ at $k = k_0$. When $\theta = 0$ focusing is characterised by a wave group which slowly varying phase as the group passes through focus. When $\theta = \pi$ on the other hand, crests and troughs move so rapidly that they

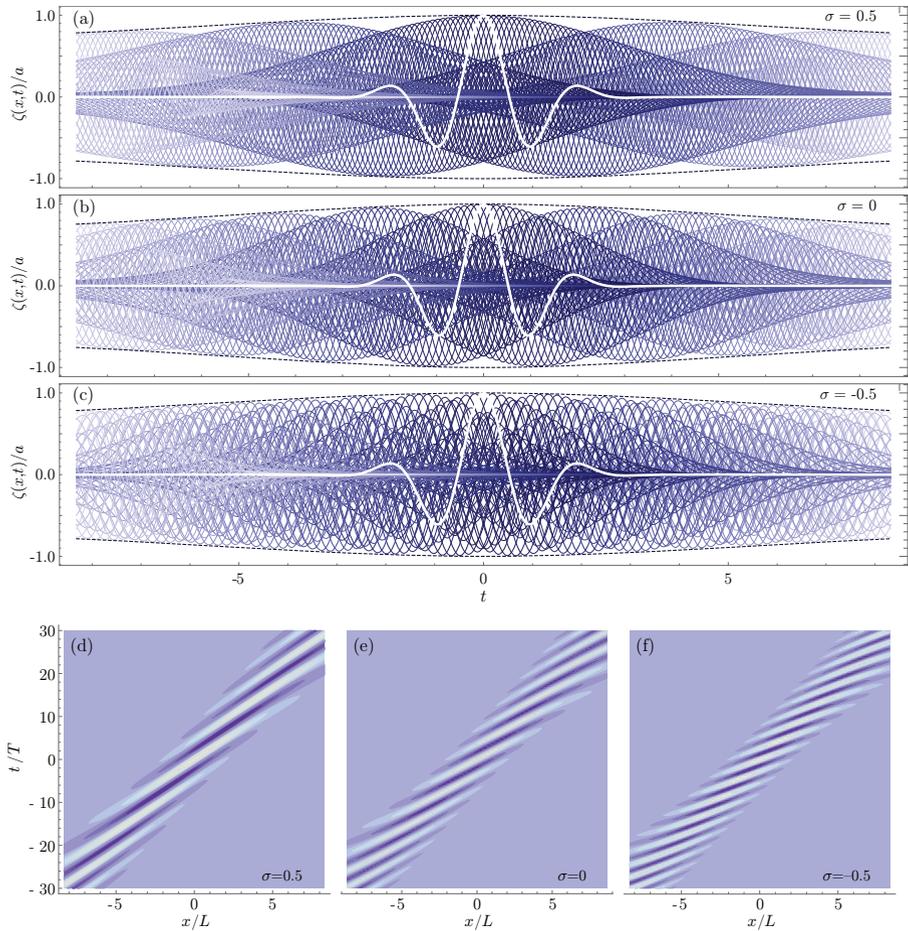


Fig. 2 Illustration of the different kinematic behaviour for waves focusing into a short group, $k_0L = 3$ with a Gaussian envelope of standard deviation L in deep water for opposing, zero and following linear shear. (a,b,c): $\zeta(x, t)/a$ for \tilde{t} from -30 to 30 in steps of 0.25 with graphs growing progressively lighter in colour for increasing $|\tilde{t}|$, and the $t = 0$ (focused) wave group drawn as thicker white lines. The dashed lines are plots of the maximum group height, $\pm L(L^4 + B_0^2 t_g(x)^2)^{-1/4}$, using Eqs. (56), (54) and $t_g(x) = x/c_g(k_0)$. (d,e,f): same, with ζ as shades from darkest to lightest ($\zeta/a = -1$ and $\zeta/a = 1$, respectively), varying in space and time.

320 appear almost chaotic at this time resolution. (An illustration of the shallow
321 water case is given in A.3.)

322 Another case suitable for illustration is the wave which takes the form of
323 a Gaussian soliton at focus,

$$\zeta(x, 0) = ae^{-x^2/2L^2}; \quad \tilde{\zeta}_0(k) = \sqrt{2\pi}aLe^{-\frac{1}{2}k^2L^2}. \quad (28)$$

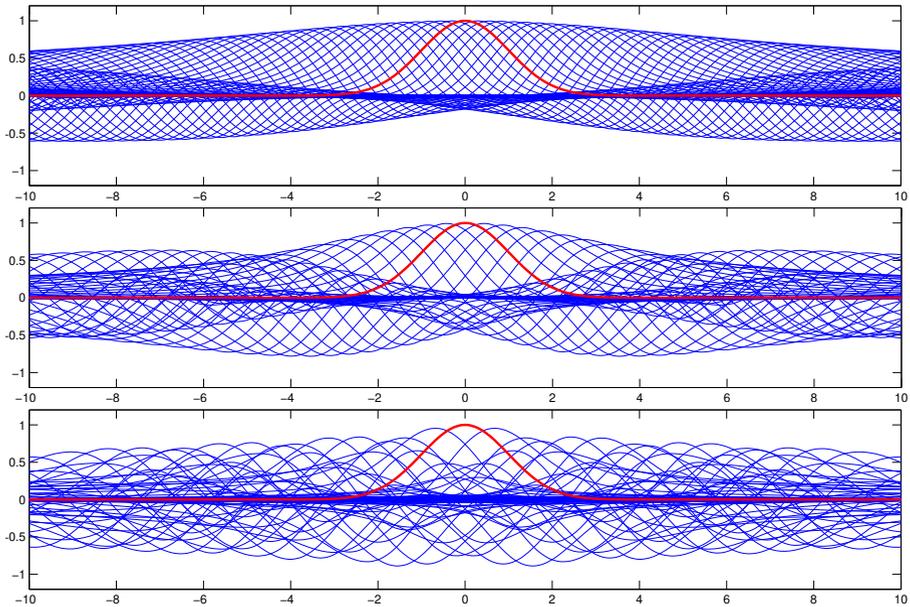


Fig. 3 Illustration of the different kinematic behaviour for waves focusing into a Gaussian soliton of nondimensional width 1 in deep water. The solid graphs show $\zeta(x, t)/a$ for t/T_{ref} from -20 to 20 in steps of 0.5 . Here $T_{\text{ref}}\sigma = 1, 0$ and -1 (top to bottom). The abscissa is x/L , and $\zeta(x, 0)/a$ is shown with thicker, red line.

324 the width of the Gaussian focused shape. The shape is considered by [25],
 325 where an explicit solution is found in the shallow-water case without shear.
 326 The surface elevation according to Eq. (12) is

$$\zeta(x, t) = \sqrt{\frac{2}{\pi}} aL \operatorname{Re} \left[\int_0^{\infty} dk e^{-\frac{1}{2}k^2 L^2 + ikx - i\omega(k)t} \right]. \quad (29)$$

327 The time evolution of a group focusing into a Gaussian soliton with constant
 328 shear in deep water is shown in Fig. 3. The behaviour is once again that the
 329 wave group focusing on following shear ($\cos\theta = -1$) and that on opposing
 330 shear ($\cos\theta = 1$), while sharing the same averaged envelope, behave quite dif-
 331 ferently in a kinematic sense, the former appearing as a single soliton rising
 332 slowly to its maximum and declines again, whereas the latter draws a hec-
 333 tic picture of crests and troughs rapidly replacing each other as the focus is
 334 approached.

325 3.2 Stationary phase approximations

336 Before considering particular cases we derive a general expression for the sta-
 337 tionary phase approximation of the shape of the wave packet sufficiently far
 338 from focus. Assume therefore that $x, t \gg 1$. Formally we write $\psi = t(k\xi - \omega(k))$

339 where

$$\xi \equiv x/t, \quad (30)$$

340 and we assume ξ is moderately large, in the order of $c_g(k)$, then take the
341 asymptotic solution as $|t| \rightarrow \infty$ [23].

342 Equation (12) is rapidly oscillating and dominated by its stationary points
343 when $|t| \rightarrow \infty$. We presume for simplicity that only one such exists, which is the
344 case for for gravity waves except very special and extreme cases; should several
345 stationary points exist, the procedure is simply repeated for each one. Equation
346 (12) has its stationary point at $k = k_{\text{sp}}$ which solves $\psi'(k_{\text{sp}}) = 0$ where a prime
347 here denotes differentiation w.r.t. k . This implies $c_g(k_{\text{sp}}) = \xi$ with $c_g(k_{\text{sp}})$ being
348 the stationary-point group velocity. In general the stationary point must be
349 found numerically, for example with the Direct Integration Method [26] which
350 we will employ later.

351 Now let a subscript ‘sp’ indicate the quantity is evaluated at $k = k_{\text{sp}}$. To
352 connect with formalism in following sections (equation (54) in particular), we
353 define

$$A_{\text{sp}} = \left. \frac{d\omega}{dk} \right|_{k=k_{\text{sp}}} ; \quad B_{\text{sp}}(\xi) = \left. \frac{d^2\omega}{dk^2} \right|_{k=k_{\text{sp}}} . \quad (31)$$

354 Clearly, $A_{\text{sp}} = \omega'_{\text{sp}} = c_g(k_{\text{sp}}) = \xi$, and $\psi''_{\text{sp}} = -\omega''_{\text{sp}} t = -B_{\text{sp}}(\xi)t$ (note that B_{sp}
355 is a function of ξ because k_{sp} is). With the stationary phase approximation
356 (e.g., §6.5 of [23])

$$\begin{aligned} \zeta(x, t) &= \text{Re} \left\{ \frac{1}{\pi} \int_0^\infty dk \tilde{\zeta}_0(k) e^{i\psi(x, t; k)} \right\} \\ &\approx \sqrt{\frac{2}{\pi |B_{\text{sp}}(\xi)t|}} \text{Re} \left\{ \tilde{\zeta}_0(k_{\text{sp}}) \exp \left[i(\psi_{\text{sp}} + \frac{\pi}{4} \text{Sg}(x)) \right] \right\} \Theta(\xi \geq c_{g, \text{min}}), \end{aligned} \quad (32)$$

357 where ‘Sg’ denotes the sign function, Θ is the unit step function, and $c_{g, \text{min}}$ is
358 the smallest value $c_g(k)$ can take. In particular, solutions only exist for $\xi > 0$
359 (bear in mind the assumption $\xi/c_g \sim 1$).

360 In all cases in the following, $\text{Sg}[\psi''_{\text{sp}}] = \text{Sg}(x) = \text{Sg}(t)$ when a stationary
361 point exists, which we therefore assume henceforth.

362 3.2.1 Stationary phase approximation, exponential shear

363 Consider next the case of an exponential current, equation (19). The weak-
364 shear approximation is sure to be accurate in any direction θ if $\delta(\mathbf{k}) \ll 1$
365 where, from equation (20), $\delta(\mathbf{k}) = \delta_\alpha(\mathbf{k})$ with w

$$\delta_\alpha(\mathbf{k}) = \frac{U_{x0}}{c_0(k)} \frac{\alpha}{\alpha + 2k}. \quad (33)$$

366 We might equally refer to δ_α as the Froude-shear number for the exponential
367 case (notation $\text{FS}_{\text{exp}} \cos \theta = \delta_\alpha$, although we will use δ_α in the following).



Fig. 4 Wave surface elevation on an exponential shear current (19) with $U_0/\sqrt{gL} = 0.2$, $\alpha L = 2.5$ and $\theta = 0$. The red dashed line is the stationary phase approximation (32) with the weak exponential shear approximation using equations (35b), (36) and (37). The black solid line is the numerical solution using the Direct Integration Method [26]. The initial wave surface is a Gaussian group $\zeta(x, 0) = a \exp(-\frac{1}{2}x^2/L^2) \cos(k_0x)$ with $k_0L = 1$. The focusing occurs at $t = 0$.

368 For a particular propagation direction θ it is sufficient that $\delta(\mathbf{k}) \ll 1$ for all
 369 significant values of k . The maximum absolute value of δ_α is at $k = \alpha/2$ where

$$\delta_{\alpha, \max} = |U_{x0}| \sqrt{\alpha/8g}. \quad (34)$$

370 Note that the global maximum of δ_α does not depend on the lengthscale L ,
 371 i.e., $\delta_{\max} \ll 1$ guarantees the accuracy of weak-shear theory independently of
 372 the size and shape of the wave group at focus, as should be expected. However,
 373 note that this is a sufficient, not a necessary condition: if αL is much greater
 374 or smaller than unity, δ_α could remain far smaller than its maximal value for
 375 all k which contribute significantly.

376 We note in passing the correspondence with the assumption in Stewart
 377 & Joy's theory of weak current compared to the phase velocity; at $k = \alpha/2$
 378 the condition $\delta_{\max} \ll 1$ can be written $U_{x0}/2c \ll 1$ since the phase velocity

379 is approximately $\sqrt{2g/\alpha}$. This is a (reference system invariant) weak current
 380 assumption: the maximum difference in $U_{x0}(z)$ over the water column is much
 381 smaller than twice the phase velocity.

382 The weak-shear approximation, eq. (23), yields

$$\omega(k) \approx \sqrt{gk} - \frac{U_{x0}\alpha k}{2k + \alpha}; \quad (35a)$$

$$A_{\text{sp}} = \xi = \frac{1}{2} \sqrt{\frac{g}{k_{\text{sp}}}} - \frac{U_{x0}\alpha^2}{(2k_{\text{sp}} + \alpha)^2}; \quad (35b)$$

$$B_{\text{sp}} = -\frac{1}{4k_{\text{sp}}} \sqrt{\frac{g}{k_{\text{sp}}}} + \frac{4U_{x0}\alpha^2}{(2k_{\text{sp}} + \alpha)^3} = -\frac{\xi}{2k_{\text{sp}}} - \frac{U_{x0}\alpha^2(\alpha - 6k_{\text{sp}})}{2k_{\text{sp}}(2k_{\text{sp}} + \alpha)^3}. \quad (35c)$$

383 We wish to solve (35b) with respect to k_{sp} . The assumption behind the weak-
 384 shear approximation is that $\delta \ll 1$ as defined in Eq. (20). In this spirit we
 385 write $\alpha U_{0k} \rightarrow \gamma \alpha U_{0k}$ with γ a ‘‘smallness’’ parameter for bookkeeping we will
 386 eventually take to 1. We expand $k_{\text{sp}} = k_{\text{sp}}^{(0)} + \gamma k_{\text{sp}}^{(1)}$ and solve (35b) in orders
 387 of γ and insert into (35c) to obtain

$$k_{\text{sp}} = \frac{g}{4\xi^2} - \frac{2gU_{x0}\alpha^2\xi}{(g + 2\alpha\xi^2)^2} + \mathcal{O}(\gamma^2); \quad (36)$$

$$B_{\text{sp}}(\xi) = -\frac{2\xi^3}{g} - \frac{8U_{x0}\alpha^2\xi^6(g - 6\alpha\xi^2)}{g(g + 2\alpha\xi^2)^3} + \mathcal{O}(\gamma^2). \quad (37)$$

388 The frequency in the weak-shear stationary phase approximation is found by
 389 inserting (36) into (35a) and retaining terms to $\mathcal{O}(\gamma)$,

$$\omega_{\text{sp}} = \frac{g}{2\xi} - \frac{U_{x0}g\alpha(g + 6\alpha\xi^2)}{2(g + 2\alpha\xi^2)^2} + \mathcal{O}(\gamma^2), \quad (38)$$

390 while the applicability of weak-shear theory is well indicated by the Froude-
 391 shear number at the stationary point,

$$\delta_{\alpha,\text{sp}} = \frac{\alpha\xi U_{x0}}{g + 2\xi^2\alpha} + \mathcal{O}(\gamma^2) \quad (39)$$

392 which has its maximum at $\xi = \sqrt{g/2\alpha}$ where the result (34) is regained.

393 We test the stationary phase surface elevation solution in figure 4. The
 394 red line indicates the arbitrary-accuracy numerical solution using the method
 395 of reference [26], whereas the black is the weak-shear stationary phase solu-
 396 tion, found by inserting (38) and (37) into equation (32). We observe a very
 397 slight phase shift over time because the frequency ω_{sp} is only approximate,
 398 whereas the ‘exact’ and approximate envelope of the group are virtually
 399 indistinguishable.

3.2.2 Stationary phase approximation for linear shear

In the special case of linear shear, an explicit formula is readily derived. Rather than use Eq. (32) we substitute $\varpi = \sqrt{gk + \sigma^2}$ into the dispersion relation (14), $\omega = \omega_\sigma = \varpi - \sigma$, from which it follows that $k(\varpi) = (\varpi^2 - \sigma^2)/g$; $dk = (2\varpi/g)d\varpi$, and according to equation (12) $\zeta(x, t)$ is,

$$\begin{aligned} \zeta(x, t) &= 2 \operatorname{Re} \int_0^\infty \frac{dk}{2\pi} \tilde{\zeta}_0(k) e^{ikx - i\omega_\sigma t} \\ &= \frac{2}{\pi g} \operatorname{Re} \left[e^{-ix\sigma^2/g + i\sigma t} \int_{|\sigma|}^\infty d\varpi \varpi \tilde{\zeta}_0(k(\varpi)) e^{ix\varpi^2/g - i\varpi t} \right]. \end{aligned} \quad (40)$$

Formally we write the exponent as $it\phi(\varpi)$ with $\phi(\varpi) = \varpi^2 x/gt - \varpi$, and consider the asymptote $t \rightarrow \infty$ while assuming $x/t \sim \mathcal{O}(1)$. The stationary phase $\varpi = \varpi_{\text{sp}}$ is

$$\phi'(\varpi_{\text{sp}}) = \left(\frac{2x}{gt} \varpi_{\text{sp}} - 1 \right) \Big|_{\varpi=\varpi_{\text{sp}}} = 0, \quad (41)$$

or, in other words, $\varpi_{\text{sp}} = gt/2x$. (Introduction of the symbol $\xi = x/t$ is not equally handy as in the previous section, and we retain x and t here.) With the stationary phase approximation the integral is (e.g., §6.5 of [23])

$$\int_{|\sigma|}^\infty d\varpi \varpi \tilde{\zeta}_0(k(\varpi)) e^{ix\varpi^2/g - i\varpi t} \approx \sqrt{\frac{\pi g}{|x|}} \frac{t \tilde{\zeta}_0(k_{\text{sp}})}{2x} \exp i \left[-\frac{gt^2}{4x} + \frac{\pi}{4} \operatorname{Sg}(x) \right] \Theta(k_{\text{sp}}) \quad (42)$$

with

$$k_{\text{sp}} = \frac{\varpi_{\text{sp}}^2 - \sigma^2}{g} = \frac{gt^2}{4x^2} - \frac{\sigma^2}{g}. \quad (43)$$

There is no stationary point unless $t/2x > |\sigma|$, hence the unit step function Θ . Thus, taking the real part, the stationary phase approximation to $\zeta(x, t)$ is

$$\zeta(x, t) \approx \sqrt{\frac{g}{\pi|x|}} \frac{t}{x} \operatorname{Re} \left\{ \tilde{\zeta}_0(k_{\text{sp}}) \exp i \left[-\frac{\sigma^2 x}{g} - \frac{gt^2}{4x} + \sigma t + \frac{\pi}{4} \operatorname{Sg}(x) \right] \right\} \Theta(k_{\text{sp}}). \quad (44)$$

The approximation is only nonzero when x and t are either both negative or both positive, as is reasonable since we have assumed propagation towards positive x . In the symmetrical case where $\zeta(x, 0) = \zeta(-x, 0)$, $\tilde{\zeta}_0(k)$ is real and the exponential becomes a cosine.

3.3 Waves focusing to δ -function singularity

Assuming the wave form at focus is a Dirac δ function is the most extreme form of focusing. As is conventional, we overlook the obvious fact that linear theory cannot describe such a wave packet close to its maximum, and regard

the solution some time before and after focusing. The case is particular in the sense that the wave shape at focus has no intrinsic length scale. We write the elevation at focus with the delta function in the limit form [e.g., 27, §7.2]

$$\zeta(x, 0) = a\delta(x/L) = \lim_{\mu \rightarrow 0^+} \frac{a}{\pi} \frac{\mu}{x^2/L^2 + \mu^2} \quad (45)$$

where L is some arbitrary, finite lengthscale for dimensional reasons (in later sections it will play the role of characteristic width of the wave packet at focus). Its obtains physical meaning is only when this singular model flow is compared to whatever real flow it models. The Fourier transform is

$$\tilde{\zeta}_0(k) = aL \lim_{\mu \rightarrow 0^+} e^{-\mu L|k|}. \quad (46)$$

Using (12) we have

$$\zeta(x, t) = 2aL \lim_{\mu \rightarrow 0^+} \int_0^\infty \frac{dk}{2\pi} \cos[kx - \omega(k)t] e^{-\mu Lk}. \quad (47)$$

The role of μ is to render the integral well defined.

For an integral with rapidly oscillating integrand of form

$$\int dq f(q) e^{iX\phi(q)} \quad (48)$$

with $\phi(q) \sim \mathcal{O}(1)$, the stationary phase approximation is accurate for $X \gg 1$ assuming $f(q)$ is significant for $q \sim \mathcal{O}(1)$. Substituting $q = \mu k$ into (47), we observe that $X = x/\mu$, which is very large for any nonzero x . Thus the stationary phase approximation should be sufficiently accurate everywhere, for practical purposes.

From equation (44) the stationary phase approximation for linear shear is

$$\zeta(x, t) = aL \sqrt{\frac{g}{\pi|x|}} \frac{t}{x} \cos \left[\frac{\sigma^2 x}{g} + \frac{gt^2}{4x} - \sigma t - \frac{\pi}{4} \text{Sg}(x) \right] \Theta \left(\frac{gt^2}{2x} - |\sigma|t \right). \quad (49)$$

Corresponding expressions for $\zeta(x, t)$ on other shear currents, including the special case of an exponential currents, are obtained by inserting $\tilde{\zeta}_{\text{sp}} = aL$ into the results in sections 3.2.

The surface elevation for a linear wave focusing towards a δ -function singularity is shown in Fig. 5. Lines show a direct calculation of integral (47) with the integration path rotated slightly into the complex k plane (closed with a non-contributing arc at infinity), ensuring exponential convergence. Let $\tilde{\sigma} = \sigma T_{\text{ref}}$ and $\tilde{t} = t/T_{\text{ref}}$ with reference time $T_{\text{ref}} = \sqrt{L/g}$. Three different shear strengths are shown: $\tilde{\sigma} = -0.5, 0$ and 0.5 with reference time. The circular markers are the values obtained using equation (49); these are indistinguishable from the exact integral in all cases. As discussed in connection



fig/deltapack_sigma_pm0.5.pdf

Fig. 5 Surface elevation ζ/a (x in units of L , t in units of $T_{\text{ref}} = \sqrt{L/g}$) for a wave group focusing to a δ -function singularity on a linear shear current. a) $t = -25 T_{\text{ref}}$, b) $t = -5 T_{\text{ref}}$. Three different shear strengths: $T_{\text{ref}}\sigma = -0.5$ (blue, solid), $T_{\text{ref}}\sigma = 0.5$ (black, dashed), and $T_{\text{ref}}\sigma = 0$ (red, dotted); circular markers show the stationary phase approximation (49).

449 with equation (44), the cases $\tilde{\sigma} = -0.5$ and 0.5 are nearly indistinguishable at
 450 $\tilde{t} = -25$, but differences manifest at the later time $\tilde{t} = -5$.

451 In stark contrast, the behaviour of the wave phase for case $\tilde{\sigma} = 0$ is always
 452 quite distinct from the others, as should be obvious from inspection of the
 453 argument of the cosine in Eq. (49), where the term $\sigma^2 x/g$ is highly significant
 454 when $t/x \sim \sigma/g$. This observation has consequences for creating a focusing
 455 wave in a laboratory with a shear current.

456 An approximate solution in the shallow-water limit, which generalises
 457 results in Refs. [25, 28], is found in appendix A.2.

3.4 Long wave group with Gaussian envelope

We next consider a group with Gaussian envelope of characteristic length L and carrier wave number $k_0 > 0$, i.e.,

$$\zeta(x, 0) = ae^{-x^2/2L^2} \cos(k_0x). \quad (50)$$

We allow the shear current $U(z)$ in equation (1) to be arbitrary and assume $\omega(k)$ and its first and second derivatives are known. Note that k_0L now acts as a bandwidth parameter: The higher k_0L , the narrower the bandwidth: k_0L is approximately the number of wavelengths of the carrier wave within the group. We will assume in derivations that the group is long (i.e., narrowband), $k_0L \gg 1$, yet we will see in the following that narrowband (long-group) approximations are excellent for many practical purposes already at $k_0L = 3 - 5$ which would not in most cases be considered a ‘long’ group.

Taking the Fourier transform of the Gaussian group we obtain with equation (11)

$$\zeta(x, t) = \frac{aL}{\sqrt{2\pi}} \int_0^\infty dk \left[e^{-\frac{1}{2}(k-k_0)^2L^2} + e^{-\frac{1}{2}(k+k_0)^2L^2} \right] \cos[kx - \omega(k)t]. \quad (51)$$

When $k_0L \gg 1$, only the first term in the brackets makes a significant contribution, so, ignoring a term of order $\exp(-\frac{1}{2}k_0^2L^2)$, we may simplify (51) to

$$\zeta(x, t) \approx \frac{aL}{\sqrt{2\pi}} \int_0^\infty dk e^{-\frac{1}{2}(k-k_0)^2L^2} \cos[kx - \omega(k)t]. \quad (52)$$

This simplification becomes suspect for $k_0L \lesssim 3$, depending on the required level of accuracy.

This is the Gaussian group in the sense of [15], prescribing the spatial shape of the wave at focus, slightly different from the definition used in, e.g., [29, 30] where the time series of the wave elevation is specified in the time domain.

The integral (52) gets its significant contributions from near $k = k_0$. The longer the group, i.e., the more periods of the carrier wave it contains, the more focused the integral is around this value. We thus assume the group width L is much larger than a wavelength, i.e., $k_0L \gg 1$. Following [15] we expand the dispersion relation in a Taylor series around $k = k_0$,

$$\omega(k) = \omega(k_0) + A_0(k - k_0) + \frac{1}{2}B_0(k - k_0)^2 + \dots \quad (53)$$

where

$$A_0 = \left. \frac{d\omega}{dk} \right|_{k=k_0}; \quad B_0 = \left. \frac{d^2\omega}{dk^2} \right|_{k=k_0} \quad (54)$$

are found from the dispersion relation, either analytically or numerically using, e.g., the Direct Integration Method [26].

Since k_0L is large the resulting integral is of Laplace type and is approximated as such (see, e.g., § 6.4 of [23]) whereby ζ tends asymptotically to

489

$$\frac{\zeta(x, t)}{a} \approx \frac{L}{\sqrt{2\pi}} \operatorname{Re} \left\{ e^{ik_0x - i\omega(k_0)t} \int_{-\infty}^{\infty} dq e^{-\frac{1}{2}(L^2 + iB_0t)q^2 + iq(x - A_0t)} \right\} \quad (55a)$$

$$= \operatorname{Re} \left\{ \frac{L}{\sqrt{L^2 + iB_0t}} \exp \left[ik_0x - i\omega(k_0)t - \frac{(x - A_0t)^2}{2(L^2 + iB_0t)} \right] \right\} \quad (55b)$$

490 with $q = k - k_0$. Taking the real part readily yields

$$\begin{aligned} \frac{\zeta(x, t)}{a} &\approx \left(\frac{L^4}{L^4 + B_0^2 t^2} \right)^{\frac{1}{4}} \exp \left[-\frac{L^2(x - A_0t)^2}{2(L^4 + B_0^2 t^2)} \right] \\ &\times \cos \left[k_0x - \omega(k_0)t - \frac{1}{2} \arctan \left(\frac{B_0t}{L^2} \right) + \frac{(x - A_0t)^2 B_0t}{2(L^4 + B_0^2 t^2)} \right]. \end{aligned} \quad (56)$$

491 This is the very general result of Ref. [15]. The effect of currents (and other
 492 factors affecting the dispersion, such as finite depth) is only to modify the
 493 expressions for A_0 and B_0 through the more general dispersion relation. For
 494 gravity waves, A_0 is typically positive and B_0 negative. In the cases we con-
 495 sider, the approximation (56) is reasonable already at $k_0L \sim 3$, adequate for
 496 many purposes.

497 3.4.1 Linear shear

498 Turning to our special case of constant shear and deep water, A_0 and B_0 are
 499 easily found from Eq. (14) and may be instructively written in terms of a
 500 shear-modified wave number (see eq. (15))

$$\varkappa_0 = k_0 + \sigma^2/g \quad (57)$$

501 as

$$A_{0,\text{lin.}} = \frac{1}{2}c_0(\varkappa_0); \quad B_{0,\text{lin.}} = -\frac{1}{4}c_0(\varkappa_0)/\varkappa_0 \quad (58)$$

502 with $c_0(k) = \sqrt{g/k}$ as usual. Insertion into (56) gives the approximation of
 503 $\zeta(x, t)$ for a long Gaussian focusing group. Expressions for general water depth
 504 are derived in A.3.

505 Figure 6 compares the approximation (56) to the exact linear solution (52)
 506 for Gaussian groups of two different lengths and strong following and opposing
 507 shear, $\tilde{\sigma} = -0.5$ and $\tilde{\sigma} = 0.5$ for the left and right group on each horizontal line,
 508 respectively. In Fig. 6b a moderately long packet ($k_0L = 10$) is considered, and
 509 the approximation (56) is excellent in all cases, out to having propagated 50
 510 times the initial group width. Surprisingly, Fig. 6a shows how even for a short
 511 package $k_0L = 3$ performs reasonably well especially in the central region of
 512 the group. In accordance with our discussion in Section 3.1, the development
 513 of the envelopes in time is indistinguishable for the two opposite, strong shear
 514 currents, making still images of surface elevations such as these qualitatively
 515 indistinguishable.



fig/GaussGroupComparison_dx_combo_editedbyZ-eps-converted-to.pdf

Fig. 6 Comparison of exact and approximate linear solution for a defocusing Gaussian wave group on linear currents with $\bar{\sigma} = -0.5$ and $\bar{\sigma} = 0.5$ for the packets to the left and right on each horizontal line, respectively, as a function of x measured in number of “widths” L of the Gaussian envelope at focus. The first time (bottommost line of graphs) is at focus, whereupon defocusing is illustrated with time increasing from bottom to top in each panel by intervals $\Delta T = 10L/c_g$ with $c_g(k_0) = A_{0,\text{lin}}$ from Eq. (58) (i.e., the group travels 10 times the ‘envelope width’ between subsequent times). Thin black line: full linear solution (52), thick red graph: approximation (56). a) Short group, $k_0L = 3$, focusing at $x = 0$ and $x = 60L$. b) Long group, $k_0L = 10$ focusing at $x = 0$ and $x = 30L$, respectively.

516 3.4.2 Arbitrary current with weak shear

517 The first two derivatives of $\sigma_\delta(k)$ from equation (22) are

$$\sigma'_\delta(k) = \int_{-\infty}^0 dz (1 + 2kz) U'_x(z) e^{2kz}; \quad (59a)$$

$$\sigma''_\delta(k) = 4 \int_{-\infty}^0 dz z (1 + kz) U'_x(z) e^{2kz}, \quad (59b)$$

518 from which we obtain, by insertion into Eq. (54), the coefficients A_0 and B_0
519 for use in equation (56)

$$A_{0\delta} = \frac{1}{2}c_0(k_0) - \sigma'_{\delta 0}(k_0); \quad B_{0\delta} = -\frac{1}{4}c_0(k_0)/k_0 - \sigma''_{\delta 0}(k_0). \quad (60)$$

520 Comparison with Eq. (58) shows that the first term on the right-hand sides of
521 Eq. (60) are the no-shear expressions, and the remaining terms are corrections
522 due to the weakly sheared current.

523 The quantities σ_δ , σ'_δ and σ''_δ can be written in closed form for a number of
524 different profiles $U_x(z)$ including the linear current (a special case where the
525 weak-shear theory does not perform particularly well [24]) and the exponential
526 which we consider next.

527 Note in passing that with a partial integration of (22) we may write

$$\sigma_\delta(k) = \frac{1}{2}U'_x(0) - \frac{1}{2} \int_{-\infty}^0 dz U''_x(z) e^{2kz}; \quad (61a)$$

$$\sigma'_\delta(k) = - \int_{-\infty}^0 dz z U''_x(z) e^{2kz}; \quad (61b)$$

$$\sigma''_\delta(k) = -2 \int_{-\infty}^0 dz z^2 U''_x(z) e^{2kz}; \quad (61c)$$

528 in other words, σ_δ represents the first-order correction to the wave-averaged
529 shear compared to the surface shear because $U_x(z)$ has nonzero curvature.
530 This is an indication why the weak-curvature theory in section 2.4.2 becomes
531 exact for the linear current which has $U''(z) = 0$, and also why the linear
532 current is a special case where weak-shear theory does not perform very well:
533 the shear correction to A_0 and B_0 for linear shear in equation (58) (which
534 is exact for the linear-current case) is symmetrical under $\sigma \rightarrow -\sigma$, but (60)
535 does not have this symmetry under $\sigma_\delta \rightarrow -\sigma_\delta$. Another way of putting it is
536 that when shear-current corrections in the surface-following system may be
537 treated perturbatively, the first-order correction to the phase velocity is due
538 to mean shear, but for the group velocity it is due to mean curvature. When
539 the curvature vanishes, however, the leading group-velocity correction becomes
540 second order in the average shear. For further discussions, see [20].

541 **3.4.3 Exponential shear. Weak-shear approximation vs** 542 **numerical solution**

543 As a particular example consider the exponential current (19). We will see
544 that a number of useful approximate expressions can be found assuming weak
545 shear and exponential current. Note that even the Columbia River delta shear
546 current, considered to be a very strongly sheared current in this context [31],
547 the weak-shear approximation is sufficient for most practical purposes as we
548 detail in section 3.4.4.

549 We find $A_0 = A_{0\alpha}$ and $B_0 = B_{0\alpha}$ with

$$A_{0\alpha} = \frac{1}{2}c_0(k_0) - \frac{U_{x0}\alpha^2}{(2k_0 + \alpha)^2}; \quad B_{0\alpha} = -\frac{c_0(k_0)}{4k_0} + \frac{4U_{x0}\alpha^2}{(2k_0 + \alpha)^3}. \quad (62)$$

550 In figure 7 the approximate solution (56) with coefficients (62) inserted is
 551 compared with the ‘exact’ numerical solution of the linear-wave initial value
 552 problem. It is striking that although the derivation assumes $k_0L \gg 1$, the
 553 approximation is reasonable already for $k_0L = 3$. Moreover, the shear is here
 554 not extremely weak; from equation (33) we find that for maximally opposing
 555 shear ($\theta = 0$), $\delta(k_0) = 0.10$ and $\delta(k_0) = 0.07$ for $k_0L = 3$ and 10, respectively,
 556 the former of which is slightly higher than that for the Columbia River scenario
 557 we consider in sections 3.4.4 and 4.5.1. This demonstrates the wide applicabil-
 558 ity of the simple closed-form approximation, equations (56) and (62). A slight
 559 phase shift with propagation is observed in both cases in figure 7 due to the
 560 approximate dispersion relation, from equations (23) and (33). In both cases
 561 in figure 7 the envelope is excellently approximated; we quantify this in figure
 562 8 where the decaying height of the defocusing wave group is plotted. Even for
 563 $k_0 = 2$ the agreement is reasonable although this can in no way be called a
 564 ‘narrowband’ wavegroup.

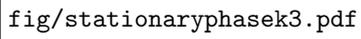
565 3.4.4 Measured current profiles: The Columbia River estuary

566 The flow conditions in the estuary of the Columbia River have been much
 567 studied for a long time (e.g., [32–34]) due to its strong, and strongly sheared,
 568 tidal current, severe wave climate and high shipping traffic. It is also a much
 569 used case for studies of waves interacting with sheared currents in various
 570 contexts (examples include [24, 26, 35–37]).

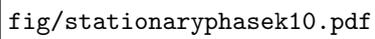
571 In their study of the Columbia River delta, Zippel & Thomson [38] (ZT)
 572 measured simultaneous wave spectra and shear current profiles in the delta of
 573 the Columbia River. An even more strongly sheared current is found among
 574 the measurements of Kilcher & Nash (KN) from the same area [39] (another
 575 set of measurements is described and used in refs [33, 35, 37]). Their respective
 576 current profiles are shown in figure 9. In section 4.5.1 we also make use of the
 577 measured wave spectrum in reference [38], while here we shall use a model
 578 wave group which at focus is slightly more narrowband than the one measured;
 579 this would represent the situation where only the part of the spectrum near
 580 the peak is involved in the focusing, the remainder forming a small-amplitude
 581 random-phase background which we presently ignore.

582 The velocity profiles shown were shifted to the reference frame following the
 583 mean (Eulerian) surface velocity and fitted to an exponential profile $U(z) =$
 584 $U_0[\exp(\alpha z) - 1]$ (see figure 9c) which gives $U_0 = 1.6$ m/s, $\alpha = 0.26$ m⁻¹ for
 585 current ZT and $U_0 = 1.4$ m/s, $\alpha = 0.39$ m⁻¹ for current KN.

586 To study an example of a focusing group we choose reasonable values for
 587 a dispersively focusing wave group in this location — see also section 4.5.1:



fig/stationaryphasek3.pdf



fig/stationaryphasek10.pdf

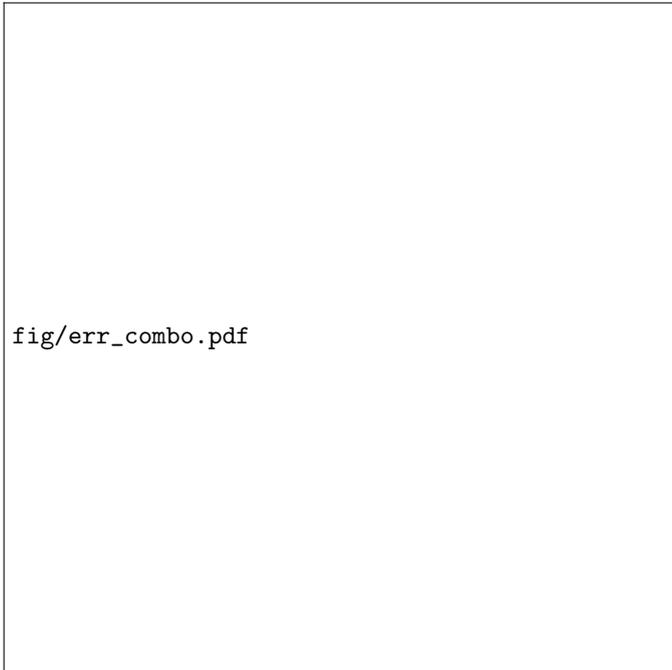


Fig. 8 The decay of envelope amplitude for increasing wave group length k_0L (decreasing spectral bandwidth) on the same exponential shear current as in figure 7. Each point is calculated as the maximum modulus of the Hilbert transform of the analytic surface elevation. Red solid lines: narrow-band weak-shear approximation. Black dash-dot lines: ‘exact’ numerical solution.

588 $k_0 = 0.15$ rad/m, $L = 20$ m, which gives $k_0L = 3.0$. We choose for our example
 589 the average of the two values for U_0 and α , respectively.

590 It is worth pointing out at this stage that these parameters give a shear
 591 Froude number of $\delta_{0\alpha} = 0.11$ (KN) and 0.080 (ZT), respectively, when inserted
 592 into equation (33) ($\delta_{0\alpha} = 0.090$ with the chosen model parameters); in other
 593 words, even though the Columbia River current is frequently used as an exam-
 594 ple of a very strongly sheared current where the effect of shear on the waves
 595 is highly significant, we can safely employ weak-shear theory with errors no
 596 greater than a few percent, less than those from typical measurement uncer-
 597 tainty from field measurements. Moreover, although $k_0L = 3$ is not what one
 598 would refer to as a narrowband wave group, we see from Fig. 7 that narrow-
 599 band weak-shear theory gives a more than adequate approximation of the
 600 surface. Thus we may confidently approximate $\zeta(x, t)$ with the approximate
 601 formula (56) with coefficients (62) inserted.

602 With the mentioned approximation we plot a focusing and defocusing wave
 603 group representative of the Columbia River flow conditions, in figure 10. Albeit
 604 less extreme than for the model linear current, the trend is once again clear: in
 605 the case of opposing shear (the focusing group propagates ‘downstream’ in the
 606 river, in an earth-fixed system) the crests and troughs focus and defocus more



Fig. 9 Shear current profiles of the Columbia River delta current. Measured data by Zippel & Thomson (blue dots) and Kilcher & Nash (red crosses), the lines are exponential functions of form (19) fitted to the data.

607 gently than for the case of following shear ('upstream') where individual crests
 608 and troughs within the group move faster and live shorter. The corresponding
 609 increase in orbital velocities in the latter case is considered and quantified in
 610 section 4.5.1. Once again we notice that the group envelope, represented by
 611 the change in maximum group height with time, varies modestly.

612 3.4.5 Arbitrary current with strong shear

613 Inserting dispersion relation (24) into formula (54) now gives the coefficients
 614 A_0 and B_0 for strong shear in the strong shear, weak curvature approximation
 615 (SSWCA) of Ellingsen & Li [20],

$$A_{0,EL} = \frac{1}{2}c_0(\varkappa_{\delta 0})(1 + 2\sigma_{\delta 0}\sigma'_{\delta 0}/g) - \sigma'_{\delta 0} \quad (63a)$$

$$B_{0,EL} = -\frac{(g + 2\sigma_{\delta 0}\sigma'_{\delta 0})^2}{4\omega_0(\varkappa_{\delta 0})^3} + \frac{\sigma_{\delta 0}^2 + \sigma_{\delta 0}\sigma''_{\delta 0}}{\omega_0(\varkappa_{\delta 0})} - \sigma''_{\delta 0}, \quad (63b)$$

616 where we use the shorthand $\varkappa_{\delta 0} = \varkappa_{\delta}(k_0)$ and $\omega_0(\varkappa_{\delta 0}) = \sqrt{g\varkappa_{\delta 0}} = \varkappa_{\delta 0}c_0(\varkappa_{\delta 0})$;
 617 \varkappa_{δ} was defined in equation (25). If the Froude-shear number δ is small we
 618 obtain (60) to leading order, while assuming linear shear ($\varkappa_{\delta}(k) \rightarrow \varkappa$, and
 619 $\sigma'_{\delta} = \sigma''_{\delta} = 0$) yields expressions (58). The SSWCA should replace the weak-
 620 shear approximation when the shear as seen by the significant waves as



fig/GaussPack_fineres_combo_CR-exp.jpg

Fig. 10 Same illustration as in figure 2, but with an exponential current representative of the Columbia River, shown in figure 9; see main text for further details. The exponential profile (19) is used with $k_0L = 3$, $\alpha L = 6.5$ and $U_0 = 0.107 U_{\text{ref}}$ with $\theta = 0, \pi/2$ and π (top to bottom). The nondimensional time \tilde{t} runs from from -50 to 50 in steps of 0.4 . The dashed lines are plots of the maximum height, $\pm L(L^4 + B_{0\alpha}^2 t_g(x)^2)^{-1/4}$ with $t_g(x) = x/A_{0\alpha}$.

621 extremely strong (by oceanographic standards), i.e., when δ is not small compared to 1, and/or the current shear appears close to constant with depth.
 622 compared to 1, and/or the current shear appears close to constant with depth.
 623 For the exponential current representative of the Columbia River, figure 9,
 624 using expressions (63) instead of (62) gives practically identical results. Deriving
 625 explicit formulae for $A_{0,\text{EL}}$ and $A_{0,\text{EL}}$ with the exponential current (19) is
 626 straightforward, but the resulting expressions are sufficiently bulky that we do
 627 not quote them here. Several realistic situations where the weak-shear theory
 628 is insufficient are mentioned and discussed in reference [20], although these are
 629 not currents which occur in ocean or coastal waters.

4 Wave kinematics and orbital velocities

We proceed now to considering the shear-affected wave orbital motion. We consider three cases, a weakly sheared current in the approximation of [14] (see sections 2.4.1 and 3.4.2), and two cases where an exact solution to the linear problem exists: a current with constant shear and with exponential shear.

Taking a step back to the more general formalism of section 2, we set $\mathbf{k} = \{k_x, 0\}$ (k_x once again takes either sign) and write the orbital velocities of a linear plane wave as

$$\begin{bmatrix} u(x, z, t) \\ v(x, z, t) \\ w(x, z, t) \end{bmatrix} = \text{Re} \int_0^\infty \frac{dk}{2\pi} \begin{bmatrix} \tilde{u}(z, t; k) \\ \tilde{v}(z, t; k) \\ \tilde{w}(z, t; k) \end{bmatrix} e^{ikx} + \text{c.c.} \quad (64)$$

where as before we used that if some function $\varphi(x)$ is real, its Fourier transform satisfies $\tilde{\varphi}(-k_x) = \tilde{\varphi}^*(k_x)$ to only retain positive values of k . Solving the 3-dimensional, linearised Euler equation in Fourier form produces the well-known Rayleigh equation (e.g. [16, 40])

$$\left[\partial_z^2 - k^2 + \frac{\mathbf{k} \cdot \mathbf{U}''(z)}{\omega - \mathbf{k} \cdot \mathbf{U}(z)} \right] \tilde{w}(z, t; \mathbf{k}) = 0 \quad (65)$$

(note that $k = |\mathbf{k}| = |k_x|$ here). Once w is found, the horizontal velocity components $\tilde{\mathbf{u}}_\perp = \{\tilde{u}, \tilde{v}\}$ are obtained using the general relation [26]

$$k^2(\omega - \mathbf{k} \cdot \mathbf{U})\tilde{\mathbf{u}}_\perp = i[\mathbf{k} \cdot \mathbf{U}'\tilde{w} + (\omega - \mathbf{k} \cdot \mathbf{U})\tilde{w}']\mathbf{k} - ik^2\mathbf{U}'\tilde{w}, \quad (66)$$

where the arguments of $\tilde{\mathbf{u}}_\perp(z, t; \mathbf{k})$, $\tilde{w}(z, t; \mathbf{k})$, $\mathbf{U}(z)$ and $\omega(\mathbf{k})$ are understood, and a prime denotes derivative with respect to z . Note in particular that when $\mathbf{k} = \{k, 0\}$, one finds

$$\tilde{u} = i\tilde{w}'/k; \quad (67a)$$

$$\tilde{v} = \frac{-iU'_y(z)}{\omega - kU_x(z)}\tilde{w}. \quad (67b)$$

The eigenvalues of $\omega(k)$ are real provided the denominator in (65) is not zero [41], i.e., no critical layer exists. We shall assume this to be the case, physically implying that no critical layers occur.

Equation (67b) shows how the orbital velocities are modified by the shear current also for $\theta = \pi/2$ (i.e., $\mathbf{k} \cdot \mathbf{U} = kU_x = 0$), even though the surface elevation is equal to that without current in that case (equation (3) shows that ζ is affected by the current only via the dispersion relation $\omega(\mathbf{k})$, in turn obtained as eigenvalues of the Rayleigh equation (65) which depends only on $\mathbf{k} \cdot \mathbf{U}$.)

We now define

$$\tilde{w}(z, t; k) = \tilde{w}(0, t; k)e^{kz}f(z; k) \quad (68)$$

657 with $f(0; k) = 1$. The function f differs from 1 when $U_x(z)$ has curvature (i.e.,
 658 nonzero second derivative) [20] — see e.g., equation (82) below. Thus, from
 659 equation (67a),

$$\tilde{u}(z, t; k) = i\tilde{w}(0, t; k)e^{kz}[f(z; k) + f'(z; k)/k]. \quad (69)$$

660 The kinematic boundary condition gives

$$\tilde{w}(0, t; k) = -i\omega(k)\tilde{\zeta}_0(k)e^{-i\omega(k)t} \quad (70)$$

661 where we used (7), whereby we obtain the general expressions

$$w(x, z, t) = 2 \operatorname{Im} \int_0^\infty \frac{dk}{2\pi} \omega(k) \tilde{\zeta}_0(k) f(z; k) e^{kz} e^{i\psi(x, t; k)}; \quad (71a)$$

$$u(x, z, t) = 2 \operatorname{Re} \int_0^\infty \frac{dk}{2\pi} \omega(k) \tilde{\zeta}_0(k) [f(z; k) + f'(z; k)/k] e^{kz} e^{i\psi(k)t}; \quad (71b)$$

$$v(x, z, t) = -2 \operatorname{Re} \int_0^\infty \frac{dk}{2\pi} \frac{\omega(k) U'_y(z)}{\omega(k) - k U_x(z)} \tilde{\zeta}_0(k) f(z; k) e^{kz} e^{i\psi(k)t}. \quad (71c)$$

662 We define the surface velocity amplification as the ratio of the horizontal
 663 orbital velocity at the (linearised) surface at the point of focus, $u(0, 0, 0)$, with
 664 vs without shear;

$$\operatorname{amp}_0 = \frac{\operatorname{Re} \int_0^\infty dk \omega(k) \tilde{\zeta}_0(k) [1 + f'(0; k)/k]}{\operatorname{Re} \int_0^\infty dk \omega_0(k) \tilde{\zeta}_0(k)} \quad (72)$$

665 with $\omega_0(k) = \sqrt{gk}$ as usual, and noting that $f(z; k) \rightarrow 1$ without shear.

666 In the presence of following shear where $U'_x(z)$ is primarily positive, the
 667 maximum of the horizontal velocity at focus $u(0, z, 0)$ can lie below the surface.
 668 In this case we define a maximum amplification

$$\operatorname{amp}_{\max} = \max_z \left\{ \frac{u(0, z, 0)}{u_0(0, 0, 0)} \right\}, \quad (73)$$

669 where u_0 is the horizontal velocity of the no-current case.

670 4.1 Long Gaussian group (narrow-band)

671 Assume now as in section 3.4 the initial shape $\tilde{\zeta}_0(k)/aL = \sqrt{\pi/2} \exp[-\frac{1}{2}(k -$
 672 $k_0)^2 L^2]$ with $k_0 L \gg 1$. We may restrict ourselves to the upper range of the
 673 water column $|z| \ll k_0 L^2$, which is no significant limitation since velocities,
 674 which decay exponentially as $\exp(k_0 z)$, are negligible when $|z| \sim k_0 L^2 \gg L$.
 675 The Laplace integral approximation becomes identical as in in section 3.4 with

676 expansion around $k = k_0$, giving the orbital velocities as the real part of

$$u \approx \frac{ac(k_0)L}{\sqrt{L^2 + iB_0t}} \exp \left[ik_0x - i\omega(k_0)t - \frac{(x - A_0t)^2}{2(L^2 + iB_0t)} \right] \frac{d}{dz} [f(z; k_0)e^{k_0z}]; \quad (74a)$$

$$v \approx -\frac{ac(k_0)k_0L}{\sqrt{L^2 + iB_0t}} \exp \left[ik_0x - i\omega(k_0)t - \frac{(x - A_0t)^2}{2(L^2 + iB_0t)} \right] \frac{U'_y(z)f(z; k_0)e^{k_0z}}{\omega(k_0) - k_0U_x(z)}; \quad (74b)$$

$$w \approx \frac{-iac(k_0)k_0L}{\sqrt{L^2 + iB_0t}} \exp \left[ik_0x - i\omega(k_0)t - \frac{(x - A_0t)^2}{2(L^2 + iB_0t)} \right] f(z; k_0)e^{k_0z}, \quad (74c)$$

677 with A_0, B_0 as in equation (54). We leave it to the reader write out the real
678 part along the lines of equation (56) if desired. Correction terms of order
679 $(x - A_0t)/k_0L$ enter far from the centre of the group at $x = A_0t$.

680 In the narrowband case it is opportune to also define a surface-shear number

$$\Upsilon_0 = \frac{U'_x(0)}{\omega_0(k_0)} \quad (75)$$

681 as well as a current strength number

$$\mathfrak{U}_0 = \frac{\max[U_x] - \min[U_x]}{c_0(k_0)}. \quad (76)$$

682 Here $\omega_0(k_0) = k_0c_0(k_0) = \sqrt{gk_0}$ as usual, and the functions max and min find
683 extrema with respect to z .

In particular, for linear shear (13)

$$\Upsilon_{0,\text{lin.}} = \frac{2\sigma}{\sqrt{gk_0}} = 2\text{FS}_{\text{lin.}} \cos \theta = 2\delta_{\text{lin.}},$$

684 (σ was defined in equation (16)) whereas \mathfrak{U}_0 is not defined in deep water. For
685 the exponential current profile in the Stewart & Joy weak-shear approximation,
686 see section 4.2.1.

687 4.2 Wave kinematics with arbitrary, weakly sheared 688 current

689 In this section we derive expressions for the orbital velocities under a focusing
690 wave group on an arbitrary, weakly sheared current. Special cases of the final
691 expressions, equations (83), will be simplified further in the following.

692 Consider a focusing wave group on a current $\mathbf{U}(z) = \{U_x, U_y\}(z)$ which is
693 well described by the approximate theory first put forward by Stewart & Joy
694 [14, 42] as described in section 3.4.2. This is typically a very good approxi-
695 mation even in strongly sheared oceanic flows (e.g. [20, 24, 26]). The orbital

696 velocities of a linear plane wave of wave number k of either sign in the xz -
 697 plane for such a situation have been found using assumptions of weak current
 698 [14, 42], although as discussed in section 2.4.1 these approximations are in
 699 fact valid for weakly sheared current, usefully measured via the small-shear
 700 parameter (or Froude number) $\delta(\mathbf{k})$ (see Eq. (20)).

701 The vertical orbital velocity to $\mathcal{O}(\delta)$ is [20, 42]

$$\tilde{w}(z, t; k) = \tilde{w}_0(0, t; k)e^{kz} [1 - \Delta(z; \mathbf{k})]; \quad (77)$$

702 plus terms of $\mathcal{O}(\delta^2)$, \tilde{w}_0 is the vertical velocity without current (which can be
 703 related to $\tilde{\zeta}_0$ via equation (70)), and

$$\Delta(z; \mathbf{k}) \equiv \frac{1}{c_0} \int_{-\infty}^z d\tilde{z} U'_x(\tilde{z}) e^{2k(\tilde{z}-z)}. \quad (78)$$

704 Comparison with (20) reveals that

$$\Delta(0; \mathbf{k}) = \delta(\mathbf{k}), \quad (79)$$

705 hence $\Delta(z; \mathbf{k})$ is a generalisation of the small-shear Froude number $\delta(\mathbf{k})$ but
 706 with contributions only from the wave-aligned current component $U_x(z)$ at
 707 depths greater than $|z|$. (Note: Δ must not be confused with the quantity of
 708 the same name in ref [20]). Clearly $\Delta \sim \mathcal{O}(\delta)$. The dependence of Δ and δ on
 709 \mathbf{k} will often be suppressed. We will need the derivative

$$\Delta'(z) = -2k\Delta(z) + U'_x(z)/c_0. \quad (80)$$

710 The corresponding horizontal velocities, obtained via equation (67), are

$$\tilde{u}(z, t; \mathbf{k}) = i\tilde{w}_0(0, t; k)e^{kz} [1 + \Delta(z; \mathbf{k}) - U'_x(z)/\omega_0], \quad (81a)$$

$$\tilde{v}(z, t; \mathbf{k}) = -i\tilde{w}_0(0, t; k)e^{kz} U'_y(z)/\omega_0 \quad (81b)$$

711 plus terms of order δ^2 .

712 In the formalism of equation (68),

$$f(z; k) = \frac{1 - \Delta(z; \mathbf{k})}{1 - \delta(\mathbf{k})}. \quad (82)$$

713 The function only occurs in equations (71) in the constellation $\omega(k)f(z; k) =$
 714 $\omega_0(k)[1 - \Delta(z; \mathbf{k})]$, using the weak-shear dispersion relation in equation (23).
 715 We shall also require the derivative $\omega f'(z) = 2k\omega_0\Delta(z; \mathbf{k}) - kU'_x(z)$ so that
 716 $\omega f'(0) = 2\omega_0k\delta - kU'_x(0)$.

32 *Wave focusing on a shear current*

717 For a group which at $t = 0$ focuses to a shape $\zeta(x, 0)$ the velocity fields are
718 found by insertion into (71):

$$u(x, z, t) = 2 \operatorname{Re} \int_0^\infty \frac{dk}{2\pi} \omega_0(k) \tilde{\zeta}_0(k) \left[1 + \Delta(z; \mathbf{k}) - \frac{U'_x(z)}{\omega_0} \right] e^{kz+i\psi}; \quad (83a)$$

$$v(x, z, t) = -2U'_y(z) \operatorname{Re} \int_0^\infty \frac{dk}{2\pi} \tilde{\zeta}_0(k) e^{kz+i\psi}; \quad (83b)$$

$$w(x, z, t) = 2 \operatorname{Im} \int_0^\infty \frac{dk}{2\pi} \omega_0(k) \tilde{\zeta}_0(k) [1 - \Delta(z; \mathbf{k})] e^{kz+i\psi}, \quad (83c)$$

719 plus corrections of order δ^2 ; here, $\psi = \psi(x, t; k)$.

720 The surface velocity amplification for an arbitrary $\mathbf{U}(z)$ satisfying $\delta(\mathbf{k}) \ll 1$
721 for all significantly contributing k , from equation (72), is now

$$\operatorname{amp}_{\text{w.s.},0} = 1 + \frac{\int_0^\infty dk \operatorname{Re}\{\tilde{\zeta}_0(k)\} [\omega_0 \delta(\mathbf{k}) - U'_x(0)]}{\int_0^\infty dk \omega_0 \operatorname{Re}\{\tilde{\zeta}_0(k)\}}. \quad (84)$$

722 **4.2.1 Gaussian wave group on exponential weak shear**

723 Consider the same situation as in section 3.4.3: a Gaussian wave packet with
724 carrier wave number k_0 considerably greater than L^{-1} i.e., the group is fairly
725 narrowbanded. The current profile is exponential, (19) in the now familiar
726 weak-shear approximation. A subscript ‘ α ’ will refer to the exponential current
727 as before, and a subscript 0 means evaluation at $k = k_0$. The velocity fields
728 are readily found from (83) by inserting $\{U'_x, U'_y\}(0) = \alpha\{U_{x0}, U_{y0}\}e^{\alpha z}$ and
729 $\Delta(z) = \Delta_\alpha(z)$ where

$$\Delta_\alpha(z) = \delta_\alpha e^{\alpha z} \quad (85)$$

730 with $\delta_\alpha(\mathbf{k})$ from Eq. (33). We define the shorthand

$$\mathbf{a} \equiv \alpha/k_0. \quad (86)$$

731 and note that for the exponential current the definitions (75) and (76) yield

$$\mathfrak{U}_{0\alpha} = \frac{U_{x0}}{c_0(k_0)}; \quad \Upsilon_{0\alpha} = \frac{\alpha U_{x0}}{\omega_0(k_0)} = \mathbf{a} \mathfrak{U}_{0\alpha}; \quad \delta_{\alpha 0} = \frac{\mathbf{a} \mathfrak{U}_{0\alpha}}{\mathbf{a} + 2}. \quad (87)$$

732 We find from equations (74) and (85) — with (82) and noting that $c(k_0) =$
733 $\omega_0(k_0)(1 - \delta_{\alpha 0})/k_0$ in the Stewart & Joy approximation (23) — that the orbital
734 velocities are approximated by

$$u(x, z, t) \approx \operatorname{Re} \left\{ \frac{a\omega_0(k_0)L}{\sqrt{L^2 + iB_{0\alpha}t}} [1 - (1 + \mathbf{a})\delta_{\alpha 0}e^{\alpha z}] e^{\Psi_0(x,z,t)} \right\}; \quad (88a)$$

$$v(x, z, t) \approx -\operatorname{Re} \left\{ \frac{aU'_y(z)L}{\sqrt{L^2 + iB_{0\alpha}t}} e^{\Psi_0(x,z,t)} \right\}; \quad (88b)$$

$$w(x, z, t) \approx \text{Im} \left\{ \frac{a\omega_0(k_0)L}{\sqrt{L^2 + iB_{0\alpha}t}} [1 - \delta_{\alpha 0}e^{\alpha z}] e^{\Psi_0(x, z, t)}, \right\} \quad (88c)$$

735 with the shorthand

$$\Psi_0(x, z, t) = k_0z + ik_0x - i\omega(k_0)t - \frac{(x - A_{0\alpha}t)^2}{2(L^2 + iB_{0\alpha}t)}, \quad (89)$$

736 and $\Upsilon_{0\alpha}$ from equation (87). $A_{0\alpha}$ and $B_{0\alpha}$ were given in equation (62). The
737 approximate expression (88a) is compared to the exact analytical expression
738 presented below in section 4.4.2 in figure 13.

739 The surface amplification is now easily found from equation (88a),

$$\text{amp}_0 = 1 - (1 + \mathbf{a})\delta_{\alpha 0}. \quad (90)$$

740 Equation (88) demonstrates a striking observation mentioned above: when
741 shear is opposing, i.e., $\Upsilon_0 > 0$, the maximum value of u is not necessarily
742 at $z = 0$ but can be positioned below the surface. With the approximate
743 expression (88a), the criterion for the maximum to lie below the surface —
744 that $u'(z) < 0$ at $z = 0$ — is readily found to be

$$(\mathbf{a} + 1)^2\delta_{\alpha 0} > 1 \quad (91)$$

745 or alternatively

$$\mathfrak{U}_{0\alpha} > \frac{\mathbf{a} + 2}{\mathbf{a}(\mathbf{a} + 1)^2} \sim \mathbf{a}^{-2} + \dots \quad (92)$$

746 with the current strength parameter $\mathfrak{U}_{0\alpha}$ from equation (87). The last form
747 is the asymptotic expansion as $\mathbf{a} \gg 1$, which is good to better than 10% for
748 $\mathbf{a} \gtrsim 3.5$. A sufficient criterion for (92) to hold valid asymptotically as $\mathbf{a} \rightarrow \infty$
749 is thus simply that the maximum lies beneath the surface if

$$\mathbf{a}^2\mathfrak{U}_{0\alpha} > 1. \quad (93)$$

750 Note that the large- \mathbf{a} limit is not in contradiction to the weak-shear approxi-
751 mation if $\mathfrak{U}_{0\alpha} \ll 1$ since $\lim_{\mathbf{a} \rightarrow \infty} \delta_{\alpha 0} = \mathfrak{U}_{0\alpha}$.

752 For $U_{x0} > 0$ the maximum value of u is located where $u'(z) = 0$, provided
753 this occurs at a negative z , otherwise it is at $z = 0$. Differentiating (88a) we
754 find the maximum value at focus to be at level $z_{\text{max},\alpha}$ and give amplification
755 $\text{amp}_{\text{max},\alpha}$ as follows,

$$z_{\text{max},\alpha} = \min \left\{ 0, -\frac{1}{\alpha} \ln[(\mathbf{a} + 1)^2\delta_{\alpha 0}] \right\}; \quad (94)$$

$$\text{amp}_{\text{max},\alpha} = \begin{cases} \frac{\mathbf{a}}{1 + \mathbf{a}} [(\mathbf{a} + 1)^2\delta_{\alpha 0}]^{-1/\mathbf{a}}, & \text{if } (\mathbf{a} + 1)^2\delta_{\alpha 0} > 1, \\ 1 - (\mathbf{a} + 1)\delta_{\alpha 0}, & \text{otherwise} \end{cases} \quad (95)$$



Fig. 11 Velocity amplification for an exponential current. Marker shapes indicate values of $\delta_{\alpha 0}$ as quoted in the legend of panel (a), and graphs and markers are colour coded as the legend in panel (b) shows; both legends are common to all panels; we defined $c_{00} = c_0(k_0)$. The solid lines show the maximum amplification, while the dashed lines of the same colour show the surface amplification (visible only when the two are different). The small black dots show the weak-shear narrowband approximation of equation (95). The insets show the shape of the wave group at focus.

756 with $\delta_{\alpha 0}$ from equation (87). Asymptotes for $\mathbf{a} \rightarrow \infty$ are

$$\text{amp}_{\max, \alpha} \approx \begin{cases} 1 - \frac{1 + \ln(\mathfrak{U}_{0\alpha} \mathbf{a}^2)}{\mathbf{a}} + \dots, & \mathfrak{U}_{0\alpha} > 0; \\ -\mathbf{a} \mathfrak{U}_{0\alpha} + 1 + \mathfrak{U}_{0\alpha} + \mathcal{O}(\mathbf{a}^{-1}) & \mathfrak{U}_{0\alpha} < 0, \end{cases} \quad (96)$$

757 while

$$\text{amp}_{\max, \alpha} \approx 1 - \frac{1}{2} \mathbf{a} \mathfrak{U}_{0\alpha} + \dots, \quad \mathbf{a} \rightarrow 0. \quad (97)$$

758 Equation (95) is a main result of this paper. Although derived assuming a
759 narrowband wave package, it approximates the velocity amplification to within
760 a few percent already for relatively broadband (short) wave groups, $k_0 L = 2$.

761 As the inset of figure 11 shows, this Gaussian group is so short as to hardly
762 be referred to as a “group” at all.

763 The close similarity between the four panels of figure 11 shows with clarity
764 that the amplification factor is essentially determined by two nondimensional
765 groups, the relative current strength $\mathfrak{U}_{0\alpha}$ and the relative shear parameter
766 \mathfrak{a} . Moreover, it is striking how closely the simple formulae (95) approximate
767 the numerically calculated amplification for a wide range of parameters of
768 the Gaussian wave group on an exponential profile, even when the underlying
769 assumptions ($k_0L \gg 1$ and $\delta_{\alpha 0} \ll 1$) are clearly violated.

770 4.3 Wave kinematics with arbitrary strongly sheared, 771 weakly curved current

772 When the current $U_{x0}(z)$ is not weakly sheared, i.e., $\delta(\mathbf{k}) \sim \mathcal{O}(1)$, an approx-
773 imate solution for the vertical velocity is found by applying the method of
774 dominant balance to the Rayleigh equation (65) [20]

$$\tilde{w}(z, t; \mathbf{k}) = \tilde{w}_0(0, t; \mathbf{k}) \left[e^{kz} - \frac{1}{k} \int_{-\infty}^z d\tilde{z} \frac{U_x''(\tilde{z})}{c(\mathbf{k}) - U_x(\tilde{z})} e^{k\tilde{z}} \sinh k(z - \tilde{z}) \right] \quad (98)$$

775 with c approximated by equation (24). With (67a) it follows that

$$\tilde{u}(z, t; \mathbf{k}) = i\tilde{w}_0(0, t; \mathbf{k}) \left[e^{kz} - \frac{1}{k} \int_{-\infty}^z d\tilde{z} \frac{U_x''(\tilde{z})}{c(\mathbf{k}) - U_x(\tilde{z})} e^{k\tilde{z}} \cosh k(z - \tilde{z}) \right]. \quad (99)$$

776 When $U_x(z)/c \ll 1$ these expressions can be reduced to equations (77) and
777 (81a), respectively, in the latter case noting that $\cosh \xi = -\sinh \xi + 2 \exp(\xi)$.

778 When an exponential current (19) is inserted, the integral can be evaluated
779 in closed form and expressed in terms of a hypergeometric function. However,
780 the resulting expression is no simpler than the exact solution in this case with
781 no restrictions on shear or curvature, presented in 4.4.2. Due to the relative
782 complexity of the analytical expressions we will not pursue the weak-curvature
783 approximation further for the purposes of kinematics.

784 4.4 Exact linear solutions

785 Exact solutions to the Rayleigh equation (65) exist for nonzero ω only in
786 special situations [16]. We consider two cases: \mathbf{U} being a linear or exponentially
787 decaying function of z .

788 4.4.1 Current with linear shear

789 Consider linear shear, $\mathbf{U}(z) = \{U_x, U_y\} = Sz\{\cos \theta, \sin \theta\}$ and $\omega = \sqrt{\varkappa} - \sigma$,
790 $\varkappa = k + \sigma^2/g$ and $\sigma = \frac{1}{2}S \cos \theta$ as before. The linear-theory orbital velocities
791 for a wave with wave vector $\mathbf{k} = \{k, 0\}$ on such a current are well known. Since
792 $U_x''(z) = 0$, the Rayleigh equation (65) becomes near trivial and following [43]



Fig. 12 Horizontal velocity profiles with linear shear current. The shear profile is expressed as $U_x(z) = 2\sigma z$ with, left to right, $\tilde{\sigma} = \{0.75, 0.5, 0.25, .0.125, 0, -0.125, -0.25, -0.5, -0.75\}$. The wave shape at focus, shown as inset, is $\zeta(x, 0, t)/a = \exp(-\frac{1}{2}x^2/L^2) \cos(k_0x)$ with $k_0L = 3$, i.e., a maximum Froude-shear number $|\text{FS}_{\text{lin.}}| = 0.43$ according to equation (18).

793 (after a rotation of the coordinate system) gives the simple result

$$\tilde{w}(z, t; \mathbf{k}) = \tilde{w}(0, t; \mathbf{k})e^{kz}; \quad \tilde{u}(z; \mathbf{k}) = i\tilde{w}(0, t; \mathbf{k})e^{kz}. \quad (100a)$$

794 This is ostensibly the potential wave solution, which one obtains for a strictly
795 2-dimensional flow with constant vorticity [43], but note that \tilde{v} is not zero;
796 instead

$$\tilde{v}(z, t; \mathbf{k}) = -\frac{iS \sin \theta}{\omega(\mathbf{k}) - kU_x(z)}\tilde{w}(0, t; \mathbf{k})e^{kz}. \quad (101)$$

797 This accords with [43] and describes a shifting and twisting of vortex lines
798 as the plane wave passes. \tilde{v} vanishes when shear is zero or the current is parallel
799 or antiparallel to the wave ($\theta = 0$ or π).

800 Thus for the focusing wave group one obtains (notice that with our
801 convention, $\mathbf{k} = \{k, 0\}$ with $k > 0$)

$$w(x, z, t) = 2 \text{Im} \int_0^\infty \frac{dk}{2\pi} \omega(k) \tilde{\zeta}_0(k) e^{kz} e^{i\psi}; \quad (102a)$$

$$u(x, z, t) = 2 \operatorname{Re} \int_0^\infty \frac{dk}{2\pi} \omega(k) \tilde{\zeta}_0(k) e^{kz} e^{i\psi}; \quad (102b)$$

$$v(x, z, t) = -2 \operatorname{Re} \int_0^\infty \frac{dk}{2\pi} \frac{S\omega(k) \sin \theta}{\omega(k) - kU_{x0}(z)} \tilde{\zeta}_0(k) e^{kz} e^{i\psi}. \quad (102c)$$

802 At first glance it might seem as if u and w are not much affected by the
 803 shear, having as they do the same structure as the expressions sans current.
 804 However, the quantitative effect is highly significant, because as previously
 805 discussed the frequency ω contained in ψ varies very significantly with the
 806 sign of σ when the latter is not very small compared to \sqrt{gk} . Thus, being
 807 proportional to $\omega(k)$, the orbital velocities u, w can be very significantly greater
 808 for $\sigma < 0$ compared to $\sigma > 0$. Secondly an oblique angle between wave and
 809 current makes for significant horizontal motion across the wave plane (also true
 810 for currents of more general depth profile, provided vertical shear is non-zero,
 811 according to equation (67b)).

812 For the linear profile the velocity is always highest at the surface, hence
 813 the velocity amplification is

$$\operatorname{amp}_{\text{lin}} = \frac{\int_0^\infty dk \omega(k) \operatorname{Re}\{\tilde{\zeta}_0(k)\}}{\int_0^\infty dk \omega_0(k) \operatorname{Re}\{\tilde{\zeta}_0(k)\}} \quad (103)$$

814 with $\omega(k)$ from equation (14). The case of a narrowband wave group is con-
 815 sidered in section 4.5, in which case the amplification becomes particularly
 816 simple.

817 The qualitative difference in behaviour during focusing and defocusing was
 818 illustrated in figures 2 and 3. Figure 12 shows the horizontal velocity pro-
 819 files beneath the focus point, $u(0, z, 0)$ for linear shear currents of different
 820 strengths, for a Gaussian wave group with $k_0 L = 3$. The qualitative shape of u
 821 remains the same, except amplified. The surface amplification varies between
 822 approximately 0.65 and 1.52 for the strongest opposing and following shear,
 823 respectively.

824 4.4.2 Current with exponential shear

825 In the case of an exponential current (19), Hughes & Reid [44] showed that the
 826 Rayleigh equation (65) permits the exact solution (see also [45] and Appendix
 827 B of [46])

$$\tilde{w}(z, t; k) = \tilde{W}(t; k) {}_2F_1(\mathcal{A}_-, \mathcal{A}_+; \mathcal{R}; \mathcal{W}(z)) e^{kz} \quad (104)$$

828 with ${}_2F_1$ being the hypergeometric function, $\tilde{W}(t, k)$ follows from free-surface
 829 boundary conditions, and

$$\mathcal{A}_\pm = (k \pm \sqrt{\alpha^2 + k^2})/\alpha; \quad \mathcal{R} = 1 + 2k/\alpha; \quad \mathcal{W}(z) = \frac{kU_{x0}e^{\alpha z}}{\omega + kU_{x0}}. \quad (105)$$



Fig. 13 Horizontal orbital velocity profiles at focus, $u(0, z, 0)$, normalised by the surface value without shear, for a medium-bandwidth focusing wave, $k_0 L = 3$ (a) and $k_0 L = 5$ (b) on an exponential current, equation (19). Insets show the wave group shape at focus. Here $U_{x0} = 0.2 U_{\text{ref}}$ (solid lines) and $-0.2 U_{\text{ref}}$ (dashed lines). Comparison of exact solution [equations (67a) and (106)] (black lines) and the weak-shear narrowband approximation (red lines), equation (88). The values on the top of each curve refer to α/k_0 .

830 The horizontal velocity $\tilde{u}(z, t; k)$ is found from equations (67), $u = i\tilde{w}'/k$,
831 giving

$$\tilde{w}'(z) = k\tilde{w}(z) - \frac{\alpha k U_{x0} \tilde{W}(t; k)}{(\omega + k U_{x0}) \mathcal{R}} e^{(k+\alpha)z} {}_2F_1(\mathcal{A}_- + 1, \mathcal{A}_+ + 1; \mathcal{R} + 1, \mathcal{W}(z)) \quad (106)$$

832 The dispersion relation to find $\omega(k)$ is implicit, given by the combined
833 free-surface boundary condition (see [26])

$$\omega^2 \tilde{w}'(0) - k(gk - \omega \alpha U_{x0}) \tilde{w}(0) = 0. \quad (107)$$

834 whose solution $\omega(k)$ is readily found numerically. Alternatively we may apply
835 the Direct Integration Method [26] directly.

836 The solutions (104) and (106) are exact for linear waves regardless of how
837 strongly sheared the current is. The comparison of the exact solution using the
838 theory in this section and the approximate solution in section 4.2.1 is shown
839 in figure 13. The results demonstrate that the approximate expression gives
840 fairly accurate solution given a relatively weak shear current. Although the
841 exact solution (106) can be used without difficulty, the computation of the
842 hypergeometric function can be time consuming. Therefore, one may consider
843 the approximate expression instead.

844 4.5 Surface velocity amplification for long, Gaussian 845 groups

846 Consider the case of a long, or narrowbanded, Gaussian group with carrier
847 wavenumber k_0 , as considered in section 3.4, i.e., $\tilde{\zeta}_0(k)/a = \sqrt{\pi/2}L \exp[\frac{1}{2}(k -$
848 $k_0)^2 L^2]$, where $k_0 \gg 1$ (as we have seen, $k_0 L \sim 5$ is sufficiently large). At
849 $x = z = t = 0$ the integrals in equation (72) now particularly simple in the
850 Laplace approximation, and we obtain

$$\text{amp}_0 \approx \frac{c(k_0)}{c_0(k_0)} \left[1 + \frac{1}{k_0} f'(0; k_0) \right]. \quad (108)$$

851 In other words a typical value of the amplification is, to leading order, the
852 ratio of phase velocities at the carrier frequency with and without shear, which
853 as discussed in section 3.1 can vary greatly for realistic shear currents, with a
854 correction term playing a role if the profile $U_x(z)$ has significant curvature.

855 In particular, for linear shear $U_x(z) = Sz$, we have $f'(z) = 0$ (e.g. [17]) and
856 with (14),

$$\text{amp}_{\text{lin.shear}} \approx \sqrt{1 + \frac{\sigma^2}{\omega_0^2}} - \frac{\sigma}{\omega_0} = \sqrt{1 + \text{FS}_{\text{lin.}}^2 \cos^2 \theta} - \text{FS}_{\text{lin.}} \cos \theta \quad (109)$$

857 where the shear-Froude number was defined in (18), taken here at $k = k_0$.
858 This accords with equation (26) which was based on the difference in phase
859 velocity only.

860 4.5.1 Velocity and amplification in real conditions: the 861 Columbia River Estuary

862 We consider now the current and wave climate measured in the estuary of the
863 Columbia River, as detailed in section 3.4.4. To make the study as realistic as
864 possible, we devise a focusing wave based on the wave spectrum reported in
865 reference [38] and generate a wave field from this. The power spectrum we use
866 is shown in figure 14 (a).



Fig. 14 (a) Power energy spectrum measured in the Columbia River delta [38], (b) initial-value spectrum $\tilde{\zeta}_0(k)$ based on $S(f)$. Circular markers: measured data. Solid line: fitted function.

867 We devise a smooth initial-value spectrum $\tilde{\zeta}_0(k)$ based on the measured
 868 wave spectrum $S(f)$ with $f = \omega/2\pi$ is the wave frequency in cycles per second.
 869 In this paragraph only, we use dimensional units without an asterisk. Unlike for
 870 generating a random sea state [47], for a focusing wave group we can obtain an
 871 initial wave elevation spectrum $\tilde{\zeta}_0(k)$ which resembles that which is measured,
 872 as follows. The discrete measured values $\{f_i, S_i\}$ are transformed to a set of
 873 discrete value pairs $\{k_i, \tilde{\zeta}_i\}$,

$$\sqrt{2S_i\Delta f} = \tilde{\zeta}_i\Delta k \quad (110a)$$

$$\tilde{\zeta}_i = \frac{\sqrt{2S_i\Delta f}}{\Delta\omega} \left(\frac{d\omega}{\Delta k} \right)_i = \frac{\sqrt{2S_i\Delta f}}{4\pi^2\Delta f} \frac{g}{2f_i} \quad (110b)$$

874 where we used $\Delta\omega/\Delta k \approx d\omega/dk$, Δf is the distance between frequency values
 875 f_i , and we used $\omega = \sqrt{gk}$. We finally fit the set of points $\{k_i, \tilde{\zeta}_i\}$ to the spectrum
 876 of a broadband Gaussian group, (51), $\tilde{\zeta}_0(k) = \sum_{\pm} a \exp[-L^2(k \pm k_0)^2/2]$,
 877 which gives $k_0 = 0.13$ rad/m and $1/L = 0.087$ /m. The measured spectrum $S(\omega)$
 878 is shown in panel (a) of figure 14, and the smooth $\tilde{\zeta}_0(k)$ is plotted along with
 879 the measurements, transformed with equation (110b) in figure 14(b).

880 The governing nondimensional parameters are thus $\mathfrak{U}_{0\alpha} = 0.18$, $\mathfrak{a} = 2.03$
 881 (ZT) and $\mathfrak{U}_{0\alpha} = 0.16$, $\mathfrak{a} = 3.05$ (KN). The shear-Froude number for current ZT
 882 and KN are $\delta_{\alpha 0} = 0.092$ and 0.095 , respectively, so although these currents are
 883 frequently referred to as very strongly sheared in the context of natural flows
 884 [31, 34, 35], the conditions lie safely in the weak-shear regime where Stewart &
 885 Joy's approximation (sections 2.4.1 and 4.2) can be used. Moreover, although
 886 the wave shape at focus is so broadband as to hardly be called a group (resem-
 887 bling the shape in the inset of figure 11a), the narrow-band approximation
 888 (88a) is an excellent approximation to the velocity profiles at focus.

889 The resulting horizontal orbital velocity profile $u(z, 0, 0)$ at focus is shown
 890 in figure 15(a) for the wave group propagating downstream, across and
 891 upstream on the Columbia River current as seen by an earth-fixed observer,
 892 corresponding to, respectively, maximally opposing, zero and maximally fol-
 893 lowing shear, i.e., $\theta = 0$, $\theta = \pi/2$ and $\theta = \pi$. (Bear in mind that velocities in
 894 our formalism are measured in the system following the mean surface velocity;
 895 see figure 9).

896 The surface shape at focus is identical by construction, and the envelope
 897 of the focusing and defocusing groups are virtually indistinguishable, yet the
 898 difference in maximum orbital velocity is dramatic. The wave-induced orbital
 899 velocity beneath the focus point, $u(0, 0, 0)$, is increased and reduced by factors
 900 of approximately 1.4 and 0.7 compared to the no-shear case for groups propa-
 901 gating upstream and downstream on the river, respectively. This corresponds
 902 to the wave-induced dynamic pressure, at $x = t = 0$, $p_{\text{dyn}} = \frac{1}{2}\rho u^2$, being
 903 approximately doubled and halved, respectively, very significantly affecting
 904 the force exerted on vessels and constructions. (For balance is worth bearing
 905 in mind that waves propagating against the current are typically higher than
 906 those propagating downstream in this location [38]).

907 5 Conclusions

908 We have considered the linearised theory of focusing long-crested wave groups
 909 on shear currents of arbitrary vertical depth dependence, allowing an arbi-
 910 trary angle between the current and wave propagation. Although limited in
 911 steepness, a number of insights into the way groups of waves focus and defo-
 912 cus can be gained. We derive a large number of approximate relations which
 913 can explicitly reveal the underlying physics, while at the same time being use-
 914 ful tools in their own right, for instance for the generation of focusing wave
 915 groups in a wave tank along the lines of [15].

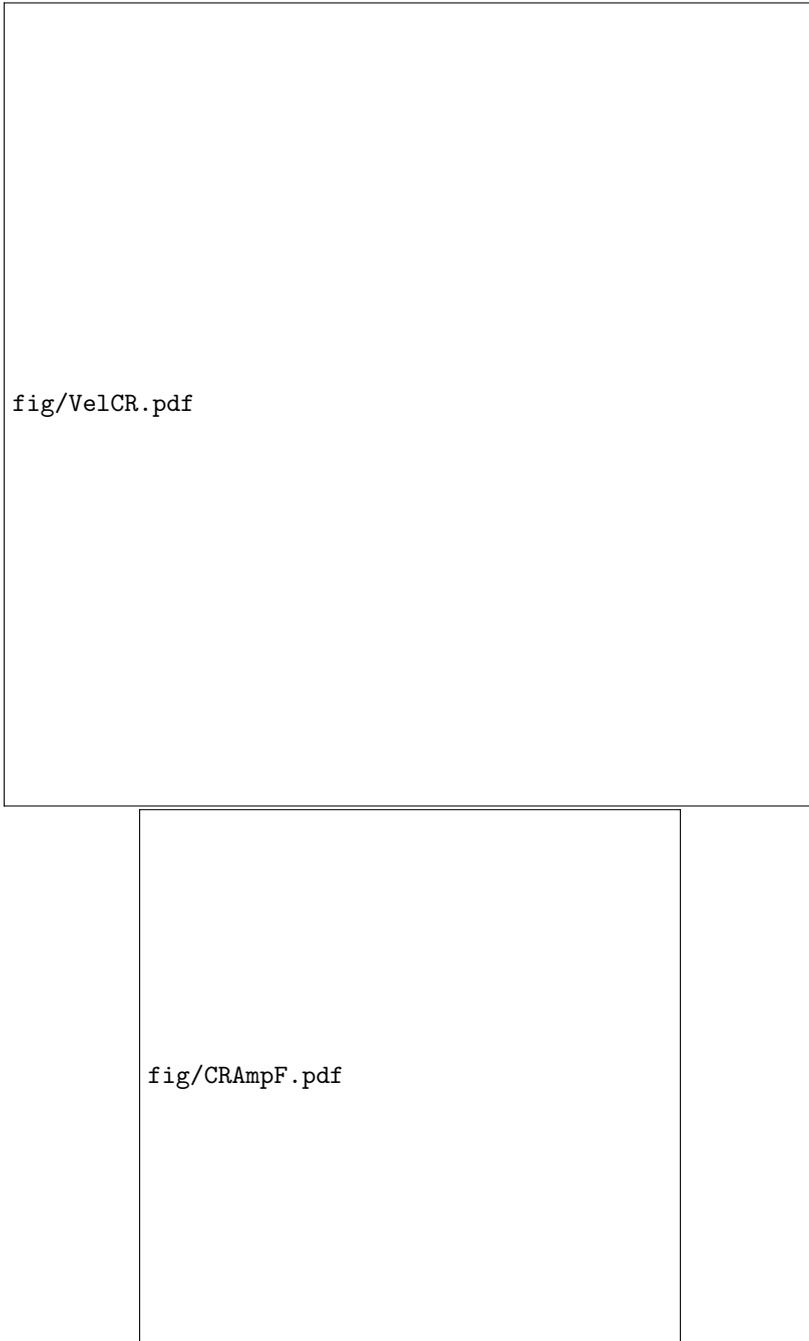


Fig. 15 Top: Horizontal orbital velocity $u(0, z, 0)$ beneath the focus of a medium-bandwidth Gaussian group ($k_0 L = 1.48$) in the presence of measured currents in the Columbia River Estuary, as measured by Zippel & Thomson (ZT, [38]) and Kilcher & Nash (KN, [39]), shifted so that $U(0) = 0$ and approximated by exponentials $U(z) = U_0[\exp(\alpha z) - 1]$. Values for $|U_{x0}|$, α , a and $\delta_{\alpha 0}$ for ZT and KN are found in the main text. Black lines show ‘exact’ numerical solutions, red lines are the weak-shear-narrowband approximation, equation (88a). Solid, thick, and dashed lines refer to $\cos \theta = 0, \pi/2$ and π , respectively. Bottom: velocity amplification as a function of the angle θ between wave propagation direction and current for the same two velocity profiles.

Particular attention is paid to two groups of currents: the current varying linearly with depth, and currents of arbitrary depth dependence whose effect on waves may be approximated using the theory by Stewart & Joy's [14] and others, the criterion for which is that the depth-averaged shear is, in the appropriate sense, sufficiently weak. For the linear current, exact and readily tractable solutions exist, allowing several classical results without shear to be generalised. The "weak-shear" assumption behind the latter class of currents is not a significant limitation in practice, since the vast majority of oceanographic currents and wave spectra satisfy the appropriate criterion of validity. For instance in the Columbia River delta which we consider as an example, the shear is frequently described as being very strongly sheared [31, 34, 35], yet remains safely within the weak-shear assumption's range of validity. The assumption is already in widespread use in ocean modelling (e.g. [35]).

For the much used model of a current profile varying exponentially with depth — modelled as $U(z) = U_0[e^{\alpha z} - 1]$ with $\alpha > 0$ — the weak-shear approximation yields a number of broadly applicable, simple and closed-form approximate relations for the surface elevation of a progressing wave group, and its concomitant orbital velocity field.

Particular attention is paid to Gaussian wave groups which at focus takes the shape of a carrier wave (wavenumber k_0) with a Gaussian envelope of width L : $\zeta_0(x) \propto \exp(-\frac{1}{2}x^2/L^2) \cos k_0x$. Assuming a long group — i.e., narrowband in Fourier space — we may derive a wealth of relations which can describe a wide range of realistic situations. Strikingly, the group does not need to be particularly long (or narrowband): the narrowband results are excellent for most practical purposes already $k_0L = 3$, a group which at the point of focus mainly consists of a single tall crest with deep troughs either side.

A key observation from studying the developing wave group is that while the shear current has modest effect on the evolution of the group envelope, the behaviour inside the group is far more affected. In opposing shear individual crests rise slowly and take longer to traverse the length of the group, while following shear causes the wave behaviour inside the group to appear more volatile, with individual crests and troughs rising and falling more rapidly.

Regarding the orbital wave motion beneath the surface, this difference in behaviour depending on the direction of sub-surface shear becomes even more important. For following shear, horizontal orbital velocities are significantly amplified compared to the case sans shear, and reduced in opposing shear. The amplification can significantly alter the wave forces acting on a body encountering the focusing group.

We derive a simple approximate relation for the velocity amplification beneath the focus point of a Gaussian wave group atop an exponential velocity profile, as a function of two nondimensional groups of parameters: the relative current strength $U_0\sqrt{k_0/g}$, and the vertical vs horizontal rate of variation, α/k_0 .

960 For illustration of these observed phenomena in a practical setting we have
 961 considered waves in the mouth of the Columbia River, where depth-resolved
 962 current measurements as well as measured wave spectra are available [38, 39].
 963 For a focusing wave with the same spectrum as that measured, i.e., hav-
 964 ing the same surface elevation $\zeta(x, 0)$ at the point of focus, the horizontal
 965 orbital velocity beneath the crest is increased by approximately 40% for fol-
 966 lowing shear (i.e., propagating upstream in an earth-fixed frame) compared
 967 to a depth-uniform current, whereas for opposing shear (downstream propa-
 968 gation) the amplitude is reduced by about a factor 0.72. This corresponds to
 969 the wave-induced dynamic pressure being approximately doubled and halved,
 970 respectively, greatly affecting the forces such a focused group will exert on
 971 vessels and structures.

972 **A Linear focusing theory in shallow water**

973 **A.1 Linear shear**

974 The dispersion relation with finite depth h is now (e.g. [43]),

$$\omega(k) = \sqrt{gk \tanh kh + (\sigma \tanh kh)^2} - \sigma \tanh kh. \quad (111)$$

975 Following Refs. [25, 28], the surface elevation integral (11) can be solved
 976 approximately in shallow water by expanding $\omega(k)$ in powers of h and k

$$\begin{aligned} \omega(k) = & kc_h - k\sigma h + \frac{1}{2}k\sigma^2 g^{-\frac{1}{2}} h^{\frac{3}{2}} - g^{-\frac{3}{2}} \left(\frac{1}{6}g^2 k^3 + \frac{1}{8}k\sigma^4 \right) h^{\frac{5}{2}} \\ & + \frac{1}{3}k^3 \sigma h^3 - g^{-\frac{3}{2}} \sigma^2 \left(\frac{1}{4}g^2 k^2 - \frac{1}{16}\sigma^4 \right) h^{\frac{7}{2}} + \mathcal{O}(h^{\frac{9}{2}}) \end{aligned} \quad (112a)$$

$$\equiv w_1 k - \frac{1}{3}w_3 k^3 + \dots \quad (112b)$$

977 with

$$w_1 = c_h - h\sigma + \frac{1}{2}\sigma^2 g^{-\frac{1}{2}} h^{\frac{3}{2}} - \frac{1}{8}g^{-\frac{3}{2}} k\sigma^4 h^{\frac{5}{2}} + \mathcal{O}(h^3); \quad (113a)$$

$$w_3 = \frac{1}{2}c_h h^2 - \sigma h^3 + \mathcal{O}(h^{\frac{7}{2}}), \quad (113b)$$

978 where $c_h = \sqrt{gh}$ is the phase (and group) velocity in the shallow-water limit
 979 in absence of a shear current.

980 **A.2 δ -function singularity in shallow water with linear 981 current profile**

982 Consider the case of a group focusing to the delta function singularity, equation
 983 (47), where the expansion (111) is inserted. We shall need the linear and cubic
 984 in k .

985 Evaluating the integral (47), the surface shape at any time t is then,
 986 approximately,

$$\zeta(x, t) = \frac{aL}{(w_3t)^{1/3}} \text{Ai} \left[\frac{x - w_1t}{(w_3t)^{1/3}} \right]. \quad (114)$$

987 This is a direct generalisation of the result of refs. [25, 28] including a constant
 988 shear.

989 A.3 Gaussian wave group, arbitrary depth

990 For linear shear S and general depth h , the expressions for $A_0 = \omega'(k_0)$ and
 991 $B_0 = \omega''(k_0)$ which may be inserted into (56) are

$$A_0 = -\frac{h\sigma}{C_0^2} + \frac{C_0\mathcal{S}_0 + (2\kappa - k_0)h}{2C_0^2\sqrt{\mathcal{T}_0}} \sqrt{\frac{g}{\kappa}}; \quad (115a)$$

$$B_0 = \frac{2h^2\sigma\mathcal{T}_0}{C_0^2} - \sqrt{g\kappa\mathcal{T}_0} \left[\frac{1}{4\kappa^2} \left(\frac{hk_0}{C_0\mathcal{S}_0} - 1 \right)^2 + \frac{h^2}{C_0^2} \left(2 - \frac{k_0}{\kappa} \right) \right], \quad (115b)$$

992 with shorthand

$$C_0 \equiv \cosh k_0h, \quad \mathcal{S}_0 \equiv \sinh k_0h, \quad \mathcal{T}_0 \equiv \tanh k_0h = \mathcal{S}_0/C_0, \quad (116)$$

993 and shear-modified wave number

$$\kappa = k_0 + \sigma^2\mathcal{T}_0/g \longrightarrow \begin{cases} k_0 + \sigma^2/g & k_0h \rightarrow \infty \\ k_0(1 + \sigma^2h/g) & k_0h \rightarrow 0 \end{cases}. \quad (117)$$

994 (Note $\omega(k_0) = \sqrt{g\kappa\mathcal{T}_0} - \sigma\mathcal{T}_0$ in this formalism).

995 In the shallow water regime, $k_0h \equiv \xi_0 \ll 1$ we obtain

$$A_0 = \sqrt{gh} - v_\sigma \xi_0 + \frac{1}{2}(v_\sigma^2/c_{00})\xi_0^{\frac{3}{2}} - \frac{1}{2}c_{00}(1 + \frac{1}{4}v_\sigma^4/c_{00}^4)\xi_0^{\frac{5}{2}} + v_\sigma \xi_0^3 + \dots \quad (118a)$$

$$k_0B_0 = -c_{00}\xi_0^{\frac{5}{2}} + 2v_\sigma \xi_0^3 + \dots \quad (118b)$$

996 with $c_{00} = c_0(k_0) = \sqrt{g/k_0}$ and $v_\sigma = \sigma/k_0$. For example, from equation
 997 (56) one sees that for a Gaussian package the time it takes for the group
 998 to change significantly is $t \sim L^2/B_0 \approx (k_0L^2/c_{00})\xi_0^{-\frac{5}{2}}$, by which time the
 999 group has traveled of order $A_0t/\lambda_0 \sim (k_0^2L^2/2\pi)\xi_0^{-2}$ wavelengths of the carrier
 1000 wave. However, notice that the leading-order correction to the phase and group
 1001 velocities are order $\text{FS}_{\text{lin}}\xi_0^{\frac{1}{2}}$ and can be significantly affected by the shear even
 1002 when ξ_0 is not extremely small. This is illustrated in Figure 16 in moderately
 1003 shallow water, $\xi_0 = 0.2$: The group at focus (red graph) does not change shape
 1004 perceptibly, but the phase velocity is clearly increased for $\sigma < 0$ and decreased
 1005 for $\sigma > 0$.



Fig. 16 Waves focusing into a group with a Gaussian envelope in moderately shallow water, $k_0 h = 0.2$. The solid graphs show $\eta(x, t)/a$ for $\sqrt{gk_0}t$ from -45 to 45 in steps of 0.5 with the shape at $t = 0$ shown in red. Here $S = \sqrt{gk_0}$, $k_0 L = 3$, and $\theta = 0$ (top), $\pi/2$ (centre) and π (bottom).

1006 **Conflict of interest.** corresponding author avers that there is no conflict
1007 of interest.

1008 **Data availability.** No new data was generated in the research reported, and
1009 all equations necessary to reproduce the results are included.

1010 **Acknowledgments.** SÅE was supported in part by the European Union
1011 (ERC, WaTurSheD, project 101045299) and the Research Council of Norway
1012 (project 325114). ZZ acknowledges the support from China Scholarship Council
1013 through project 201906060137. YL acknowledges the financial support from
1014 the Research Council of Norway through a FRIPRO mobility grant (project
1015 no. 287398) and a POS-ERC grant (project no. 342480). Views and opinions
1016 expressed are those of the authors only and do not necessarily reflect those of

1017 the European Union or the European Research Council. Neither the European
1018 Union nor the granting authority can be held responsible for them.

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