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# In the tradition of Thurston - Geometry and Topology. Introduction

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► **To cite this version:**

Ken'Ichi Ohshika, Athanase Papadopoulos. In the tradition of Thurston - Geometry and Topology. Introduction. Ken'ichi Ohshika; Athanase Papadopoulos. In the tradition of Thurston. Geometry and Topology, Springer Nature, pp.IX-XVI, 2020, 978-3-030-55927-4. hal-04535467

**HAL Id: hal-04535467**

**<https://cnrs.hal.science/hal-04535467>**

Submitted on 6 Apr 2024

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Introduction to the volume:  
*In the tradition of Thurston – Geometry and Topology*

Ken'ichi Ohshika and Athanase Papadopoulos

**Abstract**

In this chapter, we describe in some detail the various topics covered in the volume *In the tradition of Thurston – Geometry and Topology*, edited by K. Ohshika and A. Papadopoulos, published by Springer (2020).

**Keywords** Thurston metric, Teichmüller space, quasymmetric map, quasiconformal map, knot theory, circle packing, complex projective structures, hyperbolic geometry, extremal length, big mapping class group, stratification of spaces of polynomials, co-Minkowski manifold, Anosov representation, Teichmüller spaces, current, complex-analytic moduli problems.

**AMS codes** 14D23, 20F34, 22E25, 30C62, 30F30, 30F35, 30F40, 30F60, 32G13, 37D20, 51F30, 53A15, 55R10, 58J52, 57M05, 57-06, 57M10, 57M50, 57K32, 81T40.

A casual mathematician reading the title of this book may ask: “What is the tradition of Thurston?” He may also ask: “In what field is there a tradition of Thurston?”

Let us start by commenting on these questions.

The second question is easier to answer than the first. In fact, with his personal style, Thurston marked indelibly geometry, topology, group theory and dynamics. His work also had a strong impact on ergodic theory, complex analysis and discrete geometry. It also influenced combinatorics, algebraic geometry, computer science and mathematics education. It is not that Thurston passed from one topic to another, but he considered all these topics as a single one; he was able to see the unity of mathematics. Above all, Thurston inaugurated a new way of understanding, communicating mental images, sharing ideas and writing mathematics. He had a personal and unconventional opinion on what is mathematics about, and on why we do mathematics. He liked to share his mathematics and also his ideas on all sorts of questions. In response to a person asking on MathOverflow: “What can one (such as myself) contribute to mathematics?”, Thurston responded:

It's not mathematics that you need to contribute to. It's deeper than that: how might you contribute to humanity, and even deeper, to the well-being of the world, by pursuing mathematics? Such a question is not possible to answer in a purely intellectual way, because the effects of our actions go far beyond our understanding. We are deeply social and deeply instinctual animals, so much that our well-being depends on many things we do that are hard to explain in an intellectual way. That is why you do well to follow your heart and your passion. [...] The product of mathematics is clarity and understanding. Not theorems, by themselves. [...] The world does not suffer from an oversupply of clarity and understanding (to put it mildly). How and whether specific mathematics might lead to improving the world (whatever that means) is usually impossible to tease out, but mathematics collectively is extremely important. [...] In short, mathematics only exists in a living community of mathematicians that spreads understanding and breathes life into ideas both old and new. The real satisfaction from mathematics

is in learning from others and sharing with others. All of us have clear understanding of a few things and murky concepts of many more. There is no way to run out of ideas in need of clarification. The question of who is the first person to ever set foot on some square meter of land is really secondary. Revolutionary change does matter, but revolutions are few, and they are not self-sustaining—they depend very heavily on the community of mathematicians.

Giving an answer to the first question, on what is the tradition of Thurston, is not easy. Skimming the various chapters of the present volume will hopefully give a good idea of what the mathematics underlying this tradition is about. Our wish in publishing this book and the sequel is to contribute to the perpetuation of Thurston's ideas.

The present volume is divided into 16 chapters, of which we now give an outline.

Chapter 1, by Ken'ichi Ohshika and Athanase Papadopoulos, titled *A glimpse into Thurston's work*, consists of two parts. The first part is an overview of a portion (but definitely not all) of Thurston's work. The topics considered include foliations, contact structures, symplectic structures, 1-dimensional dynamics, 3-manifolds, geometric structures, geometrization of cone-manifolds, Dehn surgery, Kleinian groups, the Thurston norm, holomorphic dynamics, complex projective geometry, circle packings and discrete conformal geometry, word processing in groups, automata, tilings, computer science, mapping class groups, Teichmüller spaces, and fashion design.

The second part of this chapter is a glimpse into some development and impact of Thurston's work. We start by a short report on the proofs of several conjectures which were either formulated by Thurston or in which Thurston's works played a crucial role. This includes the proofs of the Smith conjecture, of Ahlfors's measure-0 conjecture, of Marden's tameness conjecture, of the ending lamination conjecture, of the density conjecture, of the geometrization conjecture, of Waldhausen's conjecture on the structure of fundamental groups of closed irreducible 3-manifolds, on the virtual-Haken conjecture, of the virtual fibering conjecture and of the Ehrenpreis conjecture. We then report on other topics whose development used Thurston's ideas in an essential way. They include works on the Cannon–Thurston maps, on Anti-de Sitter geometry, on linkages, on higher Teichmüller theory and on the Grothendieck–Thurston theory.

Chapter 2, by Ken'ichi Ohshika, titled *A glimpse into Thurston's work*, is a report on the impact of Thurston's work on the Japanese school of topology and geometry, in particular on foliations and on hyperbolic manifolds.

Chapter 3, by Makoto Sakuma, titled is a survey of the impact of Thurston's work on knot theory. The author chose, as a general theme for this chapter, the rigidity and flexibility in 3-manifolds, a reference to Chapter 5 of Thurston's Princeton lecture notes, titled "Flexibility and rigidity of geometric structures." Hyperbolic structures on knot complements constitute the central object of this theory. This is motivated by the fact that, by Thurston's geometrization theorem for Haken manifolds, almost every knot in the 3-sphere is hyperbolic. In this context, the term "rigidity" in this chapter refers to the Mostow–Prasad theorem saying that a complete finite-volume hyperbolic structure on any manifold of dimension  $\geq 3$  is a topological invariant, whereas the term "flexibility" refers to the deformation theory of incomplete or infinite-volume hyperbolic structures.

In his survey, Sakuma reviews successively the main tools that were known before Thurston started working on knot theory, then Thurston's contribution to the subject, then some of the major developments which arose from Thurston's ideas. Among the classical notions that are mentioned, one can find the Alexander polynomial, tools from finite group theory, ideas arising from works of Reidemeister, Schubert, Conway and others on knots, as well as some important theorems in 3-manifold topology. Sakuma then reviews Thurston's extensive use of hyperbolic geometry and the

associated rigidity theorem, the notions of hyperbolic Dehn filling and orbifold fundamental group, deformation spaces of hyperbolic structures equipped with various topologies, character varieties, the classification of mapping classes, the eight 3-dimensional geometries, Thurston's uniformization theorem for Haken manifolds which led to the proof of the Smith conjecture, group actions on trees, the Thurston norm, Thurston's geometrization theorem for orbifolds, his geometrization conjecture for manifolds which became Perelman's theorem, his virtual fibering conjecture proved by Agol, McMullen's Teichmüller polynomial and its relation with the Alexander polynomial, the use of the homology of finite coverings of knot complements, the use of profinite completions of knot groups as knot invariants, and other notions.

At the same time, Sakuma's article constitutes a survey of the growth and development of knot theory in Japan.

Chapter 4 by Sadayoshi Kojima, titled *Thurston's theory of 3-manifolds*, is a survey of Thurston's work on the geometry and topology of 3-manifolds. After mentioning some landmarks from the pre-Thurston era on this subject, in particular, the works of Waldhausen and Jaco–Shalen–Johansson, the review focuses on Thurston's hyperbolic Dehn surgery theorem, his uniformization theorem for Haken manifolds and his work towards the virtual fiber and the virtual-Haken conjectures. The last section of the chapter is a short report on the solution of several conjectures formulated by Thurston, including the geometrization and the virtual fiber conjectures.

Chapter 5, by Philip Bowers, is titled *Combinatorics encoding geometry: The legacy of Bill Thurston in the story of one theorem*. The author gives a comprehensive historical and geometrical survey of the so-called Koebe–Andreev–Thurston theorem on circle packings and its impact. This theorem is the main result proved in Chapter 13 of Thurston's Princeton lecture notes. Thurston's work on circle packings turned out to be at the foundations of the field which became known as *discrete conformal geometry*, now an extremely active field.

After a detailed presentation of the Koebe–Andreev–Thurston theorem, Bowers develops the theory of infinite circle packings on non-compact surfaces, reviewing several related topics and results, including the Rodin–Sullivan discrete approach to the Riemann mapping theorem based on circle packings (a result they obtained based on an idea of Thurston) and the characterization of hyperbolic polyhedra using circle packings, a theory which also originates in Thurston's work, leading to the works of several authors, including Hodgson–Rivin, Schramm, Bao–Bonahon, J. Bowers, K. Pratt, the author himself, and others.

In Chapter 6, titled *On Thurston's parameterization of  $\mathbb{C}P^1$ -structures*, Shinpei Baba, surveys some ideas of Thurston on complex projective structures on surfaces and their moduli spaces. He reviews in particular Thurston's parametrization of the moduli spaces of marked projective structures by the product of measured lamination space with Teichmüller space. Interestingly, this parametrization of a space of 2-dimensional geometric structures involves a passage to dimension 3, namely, the convex hull construction for subsets of the projective plane seen as the boundary of hyperbolic 3-space, in which the theory of pleated surfaces appears as a central object. The author also surveys Goldman's parametrization of the space of marked projective structures with Fuchsian holonomies as well as other notions inspired by Thurston's ideas on complex projective structures.

The convex hull construction mentioned in Chapter 6 is reviewed again in Chapter 7, titled *A short proof of an assertion of Thurston concerning convex hulls*. In this chapter, Graham Smith provides a short proof of the result of Thurston saying that if  $X$  is a closed subset of the Riemann sphere considered as the boundary of 3-dimensional hyperbolic space, then the intrinsic metric of the boundary of the convex hull of  $X$  is hyperbolic. Thurston stated his result and gave a heuristic proof in Chapter 8 of his Princeton lecture notes. The proof proposed by Smith is clear and short.

Chapter 8 by Cyril Lecuire is titled *The double limit theorem and its legacy*. It is a survey of a series of results on deformation spaces of Kleinian groups inspired by two theorems of Thurston: the double limit theorem and the compactness theorem for hyperbolic structures on acylindrical 3-manifolds. These two theorems are used as important steps in Thurston’s proof of his uniformization theorem for Haken manifolds. While outlining the proofs of these two theorems, the author reviews several key tools introduced by Thurston in his work on deformation spaces of Kleinian groups (laminations, train tracks, pleated surfaces, etc.) and at the same time he surveys Thurston’s theorems known under the names of “Broken windows only theorem” and “Window frame bounded theorem.”

Besides reviewing this work of Thurston, the author reports on some historical background, including the classical works of Ahlfors and Bers on quasi-conformal deformations which led to the so-called Ahlfors–Bers coordinates on quasi-conformal deformation spaces. He then reviews several results that are inspired by Thurston’s ideas, including works of Culler, Morgan and Shalen on deformations of hyperbolic manifolds and compactifications of deformation spaces using actions of  $\Lambda$ -trees, Otal’s alternative proof of Thurston’s double limit theorem, works of Ohshika on deformation spaces of Kleinian groups and their boundaries, and others.

In Chapter 9, by Ken’ichi Ohshika and Teruhiko Soma, titled *Geometry and topology of geometric limits. I*, is an answer to a question asked by Thurston. More precisely, the authors provide a complete classification of isometry classes of hyperbolic 3-manifolds corresponding to geometric limits of Kleinian surface groups which are isomorphic to the fundamental group of a finite-type hyperbolic surface. They introduce some invariants for hyperbolic 3-manifolds which are geometric limits of Kleinian surface groups, and they prove that the homeomorphism type and the end invariants of such manifolds determine their isometry type. This result is an analogue of the ending lamination theorem for the case of finitely generated Kleinian groups. It gives a possible solution to problem No. 8 in the list of 24 problems raised by Thurston in his paper *Three dimensional manifolds, Kleinian groups and hyperbolic geometry* (1982),<sup>1</sup> which he formulated as follows: “Analyze limits of quasi-fuchsian groups with accidental parabolic.” As a matter of fact, the result by Ohshika and Soma gives a complete answer to this question in the form it is interpreted by Otal in his survey of Thurston’s paper which appeared in 2014,<sup>2</sup> in which he provides an update of Thurston’s 24 problems.

Chapter 10, by Hyungryul Baik and Keyeongro Kim, titled *Laminar groups and 3-manifolds*, is concerned with that Thurston developed in his 1998 preprint *Three-manifolds, Foliations and Circles, I*, where he associates to a transversely oriented taut foliation on a closed atoroidal 3-manifold a faithful map from the fundamental group of the manifold to the group of orientation-preserving homeomorphisms of the circle, equipped with an invariant lamination. Here, a lamination on the circle is a collection of pairs of points on this circle with an unlinkedness condition. If one realizes these pairs by straight lines in the disc (or, equivalently, by hyperbolic geodesics, if the circle is seen as the boundary of the Poincaré model of the hyperbolic plane), then one gets a lamination in the usual sense. The authors study more generally the notion of laminar groups, that is, groups acting on the circle with invariant laminations. An elementary example of such a group is a surface group, and some fundamental groups of 3-manifold groups are also laminar. Baik and Kim review Thurston’s work on this topic and its development by several authors. They give a condition which

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<sup>1</sup>W. P. Thurston, Three dimensional manifolds, Kleinian groups and hyperbolic geometry, Bull. Amer. Math. Soc. 6 (1982), 357-381.

<sup>2</sup>J.-P. Otal, William P. Thurston: “Three-dimensional manifolds, Kleinian groups and hyperbolic geometry”, Jahresber. Dtsch. Math.-Ver. 116 (2014), 3-20.

insures that a laminar group is not virtually abelian.

Chapter 11, by Viveka Erlandsson and Caglar Uyanik, is titled *Length functions on currents and applications to dynamics and counting*. It is concerned with geodesic currents on closed orientable surfaces of finite type. In this setting, geodesic currents are measure-theoretic objects which generalize closed curves. The authors survey various functions on these spaces, for which they use the generic term “length functions.” The definitions are motivated by analogously defined functions on spaces of closed curves on surfaces, including the length of a closed geodesic for a given Riemannian (possibly singular) metric, the geometric intersection with a fixed filling closed curve or a collection of curves, the word length in the fundamental group, the stable length, etc. The authors then present an overview of the applications of these length functions to counting problems. In particular, they show how to generalize Mirzakhani’s formulae on the asymptotic growth rate of the number of curves of bounded length to the other notions of length. They also study actions of pseudo-Anosov mapping classes on spaces of geodesic currents, showing that such an action has a uniform North-South dynamics, like the one on Teichmüller space equipped with Thurston’s boundary.

Chapter 12, by Javier Aramayona and Nicholas Vlamis, titled *Big mapping class groups: an overview*, is a survey of certain mapping class groups of surfaces of infinite type (which the authors call “big mapping class groups”). This includes the group of compactly supported mapping classes, the Torelli groups, the Teichmüller modular groups (or quasi-conformal mapping class groups) and the asymptotic mapping class groups. The authors address questions related to curve graphs and other complexes associated to infinite-type surfaces, surveying several known results on these complexes, and they study their simplicial automorphism groups, reviewing recent works by several authors on these questions, and formulating an extensive number of open questions, on the algebraic and geometric properties of big mapping class groups. They also mention relations with Thompson groups, a subject which was dear to Thurston.

The questions, techniques and results surveyed in this chapter are all motivated by analogous techniques and results which are known in the context of mapping class groups of surfaces of finite type. The study is based on tools introduced by Thurston on mapping class groups of surfaces of finite type, adapted to this setting.

Chapter 13, by Hideki Miyachi, titled is an exposition of some complex-analytical aspects of Teichmüller space, including extremal length and the Bers boundary, and the Gardiner–Masur boundary, with applications to end invariants and the ending lamination conjecture. One topic discussed in some detail is the use of extremal length geometry in the elaboration of a Poisson integral formula for holomorphic functions and pluriharmonic functions on the Bers compactification, that the author developed in his recent works. He emphasizes the fact that this Poisson formula strengthens the connection between the topological and the complex-analytic aspects of Teichmüller theory, promoting a framework in which the combination of these methods sheds a new unified light on this space.

In Chapter 14, titled *Signatures of monic polynomials*, Norbert A’Campo constructs a real semi-algebraic cell-decomposition of the space of monic real polynomials of fixed degree. The face operators of this cell-decomposition are explicitly given. The classical discriminant of degree- $d$  polynomials, as well as its complement, appear as a union of cells. Since this complement is a classifying space for the braid group, this work provides a finite complex from which we can compute the group cohomology of the braid group with integral coefficients. The work in this chapter is inspired by Thurston’s ideas.

Chapters 15 and 16 concern the impact in pseudo-Riemannian geometry of techniques and tools

introduced by Thurston.

Chapter 15 by Francesco Bonsante and Andrea Seppi, titled *Anti-de Sitter geometry and Teichmüller theory*, is a survey of  $n$ -dimensional anti-de Sitter geometry, with a special emphasis on dimension 3 and on its relation with Teichmüller theory and hyperbolic geometry. This subject was inaugurated in 1990 by G. Mess, who introduced in the study of anti-de Sitter geometry Thurston's ideas and tools that he used in the study of hyperbolic manifolds and Teichmüller spaces. Bonsante and Seppi start by giving a comprehensive introduction to anti-de Sitter geometry. This includes the Klein and quadric models, a study of isometries, geodesics, the universal cover, the boundary at infinity equipped with its conformal structure and Lorentzian metric, polarity, group actions, and Dirichlet domains. The authors emphasize the properties of the special case of dimension 2+1 with the ruling of the boundary surface, surveying Cauchy surfaces and globally hyperbolic timespaces, convexity and convex hull constructions, holonomy and developing maps, and several other notions of anti-de Sitter geometry. They present a self-contained exposition of the results of Mess on the classification of globally hyperbolic anti-de Sitter 3-manifolds containing a Cauchy compact surface and the construction of the Gauss map associated with spacelike surfaces in anti-de Sitter space. They survey the relation between pleated surfaces and the earthquake map, and they recover Mess's proof of Thurston's earthquake theorem. They review recent results on anti-de Sitter 3-dimensional geometry in relation with Teichmüller theory, following the ideas of Mess. They show that the Gauss map is conformal and harmonic. In particular, its image is a minimal surface. They survey the notion of volume in the setting of maximal globally hyperbolic Cauchy compact manifolds, comparing it to the volume of the associated convex core. They describe parameters for various deformation spaces in anti-de Sitter geometry in terms of holomorphic objects, and they apply them to the study of the universal Teichmüller space. They address the question of investigating anti-de Sitter theory with timelike cone singularities and its relation with the Teichmüller theory of hyperbolic surfaces with cone singularities. Some of the results in the last section answer questions raised in the list *Some open questions in anti-de Sitter geometry* (2012).<sup>3</sup>

Chapter 16, by Thierry Barbot and François Fillastre, titled *Quasi-Fuchsian co-Minkowski manifolds*, is concerned with the geometry of co-Minkowski space, that is, the space of spacelike hyperplanes of Minkowski space. The authors study the geometry of this space equipped with the action of affine deformations of co-compact lattices of hyperbolic isometries, highlighting several facets of an analogy between this geometry and that of hyperbolic space equipped with the action of quasi-Fuchsian groups. The authors survey the known results, and at the same time they present new contributions to the theory.

Let us make a short review of these results.

The authors start by introducing a cylindrical affine model of co-Minkowski space which is analogous to the Klein ball model of hyperbolic space. They survey a convex core construction and the definition of a traceless fundamental form giving rise to a unique mean hypersurface (i.e. a surface with a traceless second fundamental form) contained in the convex core. They show that the mean distance between the lower boundary component of the convex core and this mean hypersurface gives rise to an asymmetric norm defined on the space of affine deformations of a certain lattice. By symmetrization, this asymmetric norm leads to a notion of volume of the convex core and to a "mean distance" between the future complete and the past complete flat globally hyperbolic maximal Cauchy compact spacetimes having the same holonomy. In dimension 2+1, this asymmetric norm turns out to be the earthquake norm defined by Thurston, and it is also

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<sup>3</sup>T. Barbot, F. Bonsante, J. Danciger, W. M. Goldman, F. Guéritaud, F. Kassel, K. Krasnov, J.-M. Schlenker, and A. Zeghib. Some open questions in anti-de Sitter geometry. 2012. arXiv:1205.6103.

the total length of the bending lamination of the lower boundary component of the convex core. This allows the authors to give a new proof and a generalization to an arbitrary dimension of a theorem of Thurston saying that the total length function of measured geodesic laminations leads to an asymmetric norm, after identification of a measured lamination with a tangent vector to Teichmüller space using infinitesimal earthquakes. Furthermore, the volume of the convex core turns out to be the sum of the total length of the bending lamination of its boundary, a result which the authors compare with its analogues in hyperbolic and anti-de Sitter spaces.

Finally, Barbot and Fillastre describe a relation between the geometry of co-Minkowski space and the theory of Anosov representations. In particular, they investigate the Anosov character (in the sense of Labourie) of certain representations of hyperbolic groups of isometries of Minkowski space into the isometry group of this space.

We hope that the essays that constitute the various chapters of this volume, besides giving an idea of what Thurston's tradition is, will contribute in keeping it alive. Let us conclude with some words of Thurston, extracted from a brief essay that he wrote on beauty, mathematics and creativity:<sup>4</sup>

Many people think of mathematics as austere and self-contained. To the contrary, mathematics is a very rich and very human subject, an art that enables us to see and understand deep interconnections in the world. The best mathematics uses the whole mind, embraces human sensibility, and is not at all limited to the small portion of our brains that calculates and manipulates with symbols. Through pursuing beauty we find truth, and where we find truth, we discover incredible beauty.

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<sup>4</sup>The essay was distributed on the occasion of a fashion show that took place at the Salon du Carrousel du Louvre in Paris, in March 2010, in which a collection of the Japanese fashion designer Issey Miyake was presented, a collection whose conception was based on Thurston's vision on the eight 3-dimensional geometries.