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Introduction to the volume:  
*In the tradition of Thurston II – Geometry and  
Groups*

Ken'ichi Ohshika and Athanase Papadopoulos

**Abstract**

In this chapter, we describe in some detail the various topics covered in the volume *In the tradition of Thurston II – Geometry and Groups*, edited by K. Ohshika and A. Papadopoulos, published by Springer (2022).

**Keywords** complex hyperbolic Kleinian groups, Möbius structures, discrete groups, ends, cone 3-manifolds, critical exponents, cusp opening, Thurston's orbifold geometrisation theorem, Dehn filling, hyperbolic ends,  $\kappa$ -surfaces, Thurston norm, surgery, convex core, quasi-Fuchsian manifold, twisted Alexander polynomial, Seiberg–Witten invariant, Heegaard–Flower homology, representation variety, triangulations of surfaces, cell decompositions of surfaces, hyperbolic Dehn filling, holomorphic dynamics, surgery in representation varieties, Higgs bundles, geodesic triangulations, Koebe–Andreev–Thurston theorem, moduli spaces of singular flat metrics, combination theorems, uniformisation theorem for Haken manifolds, combination theorems, Gromov hyperbolic group, iterations of rational maps, automatic groups, Thurston's characterisation of post-critically finite rational maps, Newton map, right-angled Artin groups, the geometric group theory.

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William Thurston passed away in 2012, ten years ago. Since the beginning of his career as a mathematician, in the early 1970s, Thurston continually introduced new ideas, often reviving classical theories and making them very fresh and inspiring. He reshaped many fields, including group theory, geometry, topology, complex analysis, computer science, combinatorics and others, always trying to go deep in his

understanding of patterns and forms. He was an artist in every sense of the word, creating beauty and sharing it with others. He loved mathematics, and transmitted his way of thinking to several generations of mathematicians who work in a tradition he established, until now and certainly for many years to come.

Our objective in this volume, like in the other volumes of this series to which we have given the generic name “In the tradition of Thurston”, is to make available to a wide community of mathematicians some highlights of the beautiful ideas which Thurston brought at the forefront of mathematical research. The main topics considered here include complex hyperbolic Kleinian groups, Möbius structures, hyperbolic ends and  $\kappa$ -surfaces in hyperbolic space, the Thurston norm, surgeries in representation varieties, triangulations and other cell decompositions of surfaces, combination theorems, iterations of rational maps, automatic groups, and the combinatorics and the geometry of right-angled Artin groups.

To give a more precise idea of the material presented in the volume, let us make an overview of the content of each chapter.

Chapter 2, by Michael Kapovich, is titled *A survey of complex hyperbolic Kleinian groups*. It consists in a survey of the geometry and dynamics of discrete groups of isometries of complex hyperbolic spaces, with an emphasis on the interactions between this theory and the function theory of complex hyperbolic manifolds. Besides a fairly detailed exposition of these theories, the chapter contains an introduction to several basic topics such as discrete subgroups of the group  $PU(n, 1)$  of holomorphic isometries of complex projective space, geometrically finite groups, the theory of ends, cone 3-manifolds, and critical exponents. The chapter also contains a list of open problems on this theme.

In Chap. 3, titled Möbius structures, hyperbolic ends and  $\kappa$ -surfaces in hyperbolic space, Graham Smith studies a generalisation of the correspondence between the boundary of the convex core of a quasi-Fuchsian manifold and the surface at infinity, which was introduced in Thurston’s Princeton 1975–76 lecture notes. The author proves general theorems giving the correspondence between Möbius structures on a surface and hyperbolic ends, the latter being hyperbolic manifolds carrying height functions, that is, locally strictly convex functions whose gradient flow lines are unit speed geodesics and whose sub-level sets are complete. He shows the existence and the uniqueness of the  $\kappa$ -surface corresponding to a given Möbius structure under certain reasonable assumptions, and he constructs operators which allow the passage back and forth between families of hyperbolic ends and Möbius surfaces.

Chapter 4, by Joan Porti, titled *Cone 3-manifolds*, is concerned with the deformation theory of cone 3-manifolds, more precisely, of 3-manifolds equipped with metrics of constant curvature with singularities along embedded graphs. The au-

thor describes the phenomena that occur by deforming the cone angles, including the operations of cusp opening and collapsing, under special restrictions on the cone angles. He reviews the role of cone 3-manifolds with small cone angles appearing in the proof of Thurston’s orbifold geometrisation theorem. A central ingredient in this proof involves a variation of the cone angles and an examination of the phenomena that can occur during this process. The chapter also contains a survey of the theory of geometric convergence of sequences of pointed cone manifolds and a review of the related compactness theorems. At the same time, the author mentions the appearance of cone manifolds in works of several authors other than Thurston, including his own work with Boileau and Leeb, the works of Hodgson and Kerckhoff in their study of Dehn fillings, of Bromberg in his proof of Bers’s density conjecture, of Mednykh and Rasskazov on geometric structures on knot and link complements, and of other authors working in Thurston’s tradition.

In Chap. 5, titled *A survey of the Thurston norm*, Takahiro Kitayama surveys recent progress in 3-dimensional topology which involves the Thurston norm. This is a semi-norm defined on the first cohomology group with real coefficient of a 3-manifold, whose introduction has had a large impact on 3-dimensional topology. In this chapter, the author gives an overview of applications of the Thurston norm, putting a special emphasis on its relation with topological invariants of 3-manifolds such as the (twisted) Alexander polynomial, the Seiberg–Witten invariant, and Heegaard–Flower homology, among others.

In Chap. 6, titled *From hyperbolic Dehn filling to surgeries in representation varieties*, Georgios Kydonakis surveys gluing constructions in representation spaces of surface groups into higher-rank Lie groups, as analogues of the operation of hyperbolic Dehn filling invented by Thurston. The gluing construction in the representation spaces consists in gluing Higgs bundles via the non-abelian Hodge correspondence. Thurston, in his hyperbolic Dehn filling theorem, gave a family of hyperbolic 3-manifolds which were unknown before. In the same way, the gluing construction in the setting of this chapter gives subsets of “model representations” which, presumably, could not be obtained by other methods. The author gives specific examples in the case where the target Lie groups are  $SO(p, p + 1)$  and  $Sp(4, \mathbb{R})$ .

In Chap. 7, titled *Acute geodesic triangulations of manifolds*, Sang-Hyun Kim gives a survey on the problem of existence of acute geodesic triangulations of Riemannian manifolds. In the first part of this chapter, the author surveys known results on acute triangulations of manifolds of dimension greater than 2: Kalai’s theorem on the non-existence of acute geodesic triangulations of manifolds with dimension greater than 5, and some partial results for dimensions 3 and 4. The main part of the chapter deals with the case of surfaces. Starting from Colin de Verdière’s result on the

existence of geodesic triangulations on Riemannian surfaces and Thurston’s famous theorem on the moduli space of flat metrics with cone singularities on the two-sphere, the author presents important topics such as the Koebe–Andreev–Thurston theorem on circle packings and a related result by Hodgson–Rivin on hyperbolic 3-polytopes.

In Chap. 8, titled *Signature calculation of the area Hermitian form on some spaces of polygons*, İsmail Sağlam studies Hermitian forms defined on the moduli spaces of singular flat metrics on the sphere having unit area and with prescribed curvature data at the singular points, arising from the area form. The work is motivated by Thurston’s study of the moduli space of singular flat metrics on the sphere with  $n$  cone-singular points having angles less than  $2\pi$ , where he showed that the area equation induces a Hermitian form of signature  $(1, n-3)$  on the moduli space of such structures. The author in this chapter generalises Thurston’s construction to the case where one singular point has angle greater than  $2\pi$ , and he calculates the signature of the Hermitian form induced by the area form.

In Chap. 9, titled *Equilateral convex triangulations of  $\mathbb{RP}^2$  with three conical points of equal defect*, Mikhail Chernaviskikh, Altan Erdnigor, Nikita Kalinin, and Alexandr Zakharov study convex equilateral triangulations of the real projective plane. The authors determine the top term of the growth function of the number of isometry classes of equilateral triangulations of the real projective plane with at most  $n$  vertices whose vertices have valency 6 except for three with valency 4. The coefficient of the top term is expressed by the Lobachevsky function which appears as the volume of a certain region in  $\mathbb{R}^4$ .

Chapter 10 by Mahan Mj and Sabyasachi Mukherjee, is titled *Combination theorems in groups, geometry and dynamics*. The authors survey a series of results, which they call “combination theorems”, and which are analogues of the Klein–Maskit combination theorems for Kleinian groups. The fields visited include hyperbolic geometry, geometric group theory and holomorphic dynamics. Thurston’s influence on all these fields is highlighted. The survey starts with the tools which Thurston developed in his proof of the uniformisation theorem for Haken manifolds and then pass through combination theorems for Gromov hyperbolic groups due to Bestvina–Feign, Agol–Wise, Haglund–Wise and others. They review the Mahan–Reeves combination theorems in the relative hyperbolicity setting, Thurston’s topological characterisation of rational maps which led to the Douady–Hubbard theory of mating in holomorphic dynamics, and the Mahan–Mukherjee theory of combination of rational maps and Kleinian groups via orbite equivalence.

Chapter 11 by Kevin Pilgrim is titled *On the pullback relation on curves induced by a Thurston map*. We recall here that a Thurston map is an orientation-preserving branched covering of the sphere by itself which is of degree  $\geq 2$  and which is post-

critically finite, that is, the union of the forward orbit of its set of critical points is finite. A key result in the theory of holomorphic dynamics is a theorem of Thurston which provides a topological characterisation of (conjugacy classes of) post-critically finite rational maps of the sphere among Thurston maps. In this chapter, the author presents several results, including works of Floyd, Parry, himself and others on the action of a Thurston map by inverse images on the set of isotopy classes of simple closed curves in the complement of its post-critical set, and he proposes several open questions in this setting.

Chapter 12, by William Floyd, is titled *The pullback map on Teichmüller space induced from a Thurston map*. Thurston's topological characterisation of rational maps is based on a pullback map  $\sigma f$  acting on a certain Teichmüller space associated with a postcritically finite branched covering  $f$  of the Riemann sphere. Thurston proved that such a branched covering is combinatorially equivalent (in some precise sense) to a rational map if and only if  $\sigma f$  has a fixed point. Furthermore, there is a bijection between the fixed points of  $\sigma f$  and the conjugacy classes of rational maps equivalent to  $f$ . In this chapter, the author describes Thurston's characterisation theorem and discusses the recent development in understanding the pullback map  $\sigma f$ , made by Pilgrim, Selinger, Bartholdi, Nekrashevych, Buff, Epstein, Koch and others.

Chapter 13, by Russell Lodge, Yauhen Mikulich and Dierk Schleicher, is titled *A classification of postcritically finite Newton maps*. Here, a Newton map is a rational function  $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  for which there exists some complex polynomial  $p(z)$  such that  $f(z) = z - \frac{p(z)}{p'(z)}$  for all  $z \in \widehat{\mathbb{C}}$ . Newton maps arise in the application of Newton's method to a polynomial, and they form a natural family to be studied from the dynamical perspective. In this chapter, the authors work out, for the family of postcritically finite Newton maps, a complete combinatorial classification in terms of a finite connected graph satisfying certain explicit conditions. The classification relies on Thurston's characterisation and rigidity theorem for postcritically finite branched covers of the sphere. This is a remarkable instance where a whole family of rational functions is subject to a classification.

Chapter 14, by Sarah Rees, is titled *The development of the theory of automatic groups*. After an overview of the basic notions of automatic, bi-automatic and com-bable groups, the author explains why hyperbolic groups and fundamental groups of compact 3-manifolds based on six of Thurston's eight geometries are automatic. She then describes algorithmic aspects of these groups and explains how actions of groups on spaces satisfying various notions of negative curvature can be used to prove automaticity or bi-automaticity. She shows how these results have been used to derive such properties in the setting of mapping class groups, Coxeter groups,

braid groups and other families of Artin groups. All along the chapter, the author surveys the important problems that were tackled by the theory of automatic groups, mentioning several among those which remain open. We recall that the notion of automatic group was introduced by Thurston, after Cannon published his 1984 paper *The combinatorial structure of cocompact discrete hyperbolic groups*, in which the latter showed that Cayley graphs of cocompact discrete groups of isometries of  $n$ -dimensional hyperbolic space have a finite recursive description. Cannon, in the introduction to this paper, expresses his debt to Thurston who promoted a return to geometric considerations for the study of groups, that is, to the classical methods of Dehn and Cayley.

Chapter 15, by Thomas Koberda, is titled *Geometry and combinatorics via right-angled Artin groups*. We recall that for a finite simplicial graph  $\Gamma$ , the associated right-angled Artin group is the group whose generators are the vertices of  $\Gamma$  and whose relations are the commutators corresponding to the edges of  $\Gamma$ . As the author explains, these groups interpolate between free groups and abelian groups. Starting with right-angled Artin groups, he addresses several questions pertaining to the fields of combinatorial group theory, graph theory, complexity theory, mathematical logic, mapping class groups, hyperbolicity and others. Thurston's seal is visible on several of these questions. The author points out relations between objects from these various fields and formulates some open problems. In the last section, he provides a list of questions concerning what remains to be understood on the relationship between combinatorics and algebra, from the perspective of right-angled Artin groups.

The broad range of the contributions in this volume gives only a small idea of the scope of Thurston's impact on mathematics. It is certain that his insight, which is unique in the modern history of mathematics, will continue to influence succeeding generations of mathematicians.

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