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# Dual Representation of Unbounded Dynamic Concave Utilities<sup>\*</sup>

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## Abstract

In several unbounded linear spaces containing all bounded endowments, we present the dynamic concave utilities (hence the dynamic convex risk measure) as the solutions of backward stochastic differential equations (BSDEs) with unbounded terminal values by applying some updated existence and uniqueness results on unbounded solutions of scalar BSDEs with generators of linear growth, super-linear growth, sub-quadratic growth and quadratic growth. The Legendre-Fenchel transform (dual representation) of convex functions, the de la Vallée-Poussin theorem, Young's inequality and Gronwall's inequality are the main tools to prove these representation results.

*Keywords:* Dynamic concave utilities, Dynamic convex risk measure, Dual representation, Backward stochastic differential equation, Unbounded terminal value.

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## 1. Introduction

Fix a nonnegative real number  $T > 0$  and an integer  $d \geq 1$ . Assume that  $(B_t)_{t \in [0, T]}$  is a standard  $d$ -dimensional Brownian motion defined on some complete probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ ,  $(\mathcal{F}_t)_{t \geq 0}$  is its natural and augmented filtration, and  $\mathcal{F}_T = \mathcal{F}$ . The equality and inequality between random elements are understood to hold  $\mathbb{P}$ -a.s. For each  $t \in [0, T]$ , let  $L^\infty(\mathcal{F}_t)$  denote the set of  $\mathcal{F}_t$ -measurable scalar bounded random variables, and let  $\mathcal{A}(\mathcal{F}_t)$  denote a general linear space of  $\mathcal{F}_t$ -measurable scalar random variables containing  $L^\infty(\mathcal{F}_t)$ . Suppose that  $\mathcal{A}(\mathcal{F}_s) \subset \mathcal{A}(\mathcal{F}_t)$  for each  $0 \leq s \leq t \leq T$ . By a dynamic concave utility function defined on  $\mathcal{A}(\mathcal{F}_T)$ , we mean a family of time-parameterized operators  $\{U_t(\cdot), t \in [0, T]\}$  mapping from  $\mathcal{A}(\mathcal{F}_T)$  to  $\mathcal{A}(\mathcal{F}_t)$  and satisfying the following properties for each  $t \in [0, T]$ :

- (i) **Monotonicity:**  $U_t(\xi) \geq U_t(\eta)$  for each  $\xi, \eta \in \mathcal{A}(\mathcal{F}_T)$  such that  $\xi \geq \eta$ ;
- (ii) **Translation invariance:**  $U_t(\xi + \eta) = U_t(\xi) + \eta$  for each  $\xi \in \mathcal{A}(\mathcal{F}_T)$  and  $\eta \in \mathcal{A}(\mathcal{F}_t)$ ;
- (iii) **Concavity:**  $U_t(\theta\xi + (1 - \theta)\eta) \geq \theta U_t(\xi) + (1 - \theta)U_t(\eta)$  for all  $\xi, \eta \in \mathcal{A}(\mathcal{F}_T)$  and  $\theta \in (0, 1)$ ;
- (iv) **Time consistency:**  $U_s(\xi) = U_s(U_t(\xi))$  for each  $\xi \in \mathcal{A}(\mathcal{F}_T)$  and  $s \in [0, t]$ .

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We denote by  $x \cdot y$  the Euclidean inner product of two vectors  $x, y \in \mathbb{R}^d$ . For a bounded endowment  $\xi \in L^\infty(\mathcal{F}_T)$  and for a lower semi-continuous convex function  $f : \mathbb{R}^d \rightarrow \mathbb{R}_+ \cup \{+\infty\}$  such that  $f(0) = 0$  and  $\liminf_{|q| \rightarrow +\infty} f(q)/|q|^2 > 0$ , we define

$$U_t(\xi) := \text{essinf} \left\{ \mathbb{E}_{\mathbb{Q}^q} \left[ \xi + \int_t^T f(q_s) ds \middle| \mathcal{F}_t \right] \middle| \mathbb{Q}^q \sim \mathbb{P} \right\}, \quad t \in [0, T], \quad (1.1)$$

where  $\mathbb{E}_{\mathbb{Q}^q}[\cdot | \mathcal{F}_t]$  is the expectation operator conditional to the sigma field  $\mathcal{F}_t$  under the probability measure  $\mathbb{Q}^q$  equivalent to  $\mathbb{P}$  and

$$\mathbb{E} \left[ \frac{d\mathbb{Q}^q}{d\mathbb{P}} \middle| \mathcal{F}_t \right] = \exp \left\{ \int_0^t q_s \cdot dB_s - \frac{1}{2} \int_0^t |q_s|^2 ds \right\}, \quad t \in [0, T].$$

It is easy to verify that the time-parameterized operator  $\{U_t(\cdot), t \in [0, T]\}$  defined via (1.1) constitutes a dynamic concave utility function defined on  $L^\infty(\mathcal{F}_T)$ . And, as shown in [15], all dynamic concave utility functions are of a similar form under some mild additional assumptions. Furthermore, by Theorems 2.1-2.2 in [12] we know that there exists an  $(\mathcal{F}_t)$ -progressively measurable square-integrable  $\mathbb{R}^d$ -valued process  $(Z_t)_{t \in [0, T]}$  such that  $(U_t(\xi), Z_t)_{t \in [0, T]}$  is the unique bounded solution of the following scalar backward stochastic differential equation (BSDE in short):

$$Y_t = \xi - \int_t^T g(Z_s) ds + \int_t^T Z_s \cdot dB_s, \quad t \in [0, T], \quad (1.2)$$

where

$$g(z) := \sup_{q \in \mathbb{R}^d} (z \cdot q - f(q)) \geq 0, \quad \forall z \in \mathbb{R}^d$$

is the conjugate concave function of  $f$  with  $g(0) = 0$ , and  $\limsup_{|z| \rightarrow +\infty} g(z)/|z|^2 < +\infty$ .

In the other words, the dynamic concave utility function defined on  $L^\infty(\mathcal{F}_T)$  via (1.1) admits a dual representation as the solution to BSDE (1.2). It is well known from [4] and [5] that if the endowment  $\xi$  has only exponential moments of any order, which maybe unbounded, then BSDE (1.2) admits a unique adapted solution  $(Y_t, Z_t)_{t \in [0, T]}$  such that  $\sup_{t \in [0, T]} |Y_t|$  has exponential moments of any order, and that the associated comparison theorem and stability theorem on the solutions of preceding BSDEs hold. Several questions are naturally asked: for any endowment  $\xi$  admitting only exponential moments of any order, whether (1.1) can be well-defined? In this setting, does the dual representation still hold true? Can these assertions be extended to some larger linear space of endowments under some suitable assumptions on the function  $f$ ? In the present paper, we shall give some affirmative answers to these questions. In particular, we consider the general case of core function  $f$ , which can depend on  $(\omega, t)$ .

Note that up to a sign, a dynamic concave utility is actually equivalent to a dynamic convex risk measure. They are both important conceptions in finance mathematics, which have been extensively investigated for example in [2, 3, 8, 9, 10, 11, 15, 16, 17, 18, 23, 24, 25, 26, 28, 29, 33, 34, 35]. Consequently, all conclusions of the present paper are available for the dynamic convex risk measures.

The rest of this paper is organized as follows. In the next section, we introduce some notations, spaces and several updated existence and uniqueness results of adapted solutions to scalar BSDEs with generators of linear growth, super-linear growth, sub-quadratic growth and quadratic growth, which will be used in this paper. In section 3, we state our dual representation results of the dynamic concave utilities

defined on several unbounded linear spaces of endowments via the solutions of preceding BSDEs with unbounded terminal values, and provide some remarks and examples to illustrate our theoretical results. Finally, in section 4 we give the detailed proof of these results. Before that, two useful propositions are presented and verified, which may be available in other places.

## 2. Preliminaries

First of all, we introduce some notations and spaces used in this paper. Let  $\mathbb{R}_+ := [0, +\infty)$ , and for  $a, b \in \mathbb{R}$ , let  $a \wedge b := \min\{a, b\}$ ,  $a^+ := \max\{a, 0\}$  and  $a^- := -\min\{a, 0\}$ . Let  $\mathbf{1}_A(x)$  represent the indicator function of set  $A$ . For a convex function  $f : \mathbb{R}^d \rightarrow \mathbb{R}$ , denote by  $\partial f(z_0)$  its subdifferential at  $z_0$ , which is the non-empty convex compact set of elements  $u \in \mathbb{R}^d$  such that

$$f(z) - f(z_0) \geq u \cdot (z - z_0), \quad \forall z \in \mathbb{R}^d.$$

For  $p, \mu > 0$  and  $t \in [0, T]$ , we denote by  $L^p(\mathcal{F}_t)$  the set of  $\mathcal{F}_t$ -measurable scalar random variables  $\eta$  such that  $\mathbb{E}[|\eta|^p] < +\infty$ , and define the following three spaces of  $\mathcal{F}_t$ -measurable scalar random variables:

$$L(\ln L)^p(\mathcal{F}_t) := \{\eta \in \mathcal{F}_t \mid \mathbb{E}[|\eta|(\ln(1 + |\eta|))^p] < +\infty\},$$

$$L \exp[\mu(\ln L)^p](\mathcal{F}_t) := \{\eta \in \mathcal{F}_t \mid \mathbb{E}[|\eta| \exp(\mu(\ln(1 + |\eta|))^p)] < +\infty\}$$

and

$$\exp(\mu L^p)(\mathcal{F}_t) := \{\eta \in \mathcal{F}_t \mid \mathbb{E}[\exp(\mu|\eta|^p)] < +\infty\}.$$

It is not hard to verify that these spaces become smaller when the parameter  $\mu$  or  $p$  increases, and for each  $\mu, \bar{\mu}, r > 0$  and  $0 < p < 1 < q$ ,

$$L^\infty(\mathcal{F}_t) \subset \exp(\mu L^r)(\mathcal{F}_t) \subset L \exp[\bar{\mu}(\ln L)^q](\mathcal{F}_t) \subset L \exp[\mu \ln L](\mathcal{F}_t) = L^{1+\mu}(\mathcal{F}_t)$$

and

$$L^q(\mathcal{F}_t) \subset L \exp[\mu(\ln L)^p](\mathcal{F}_t) \subset L(\ln L)^r(\mathcal{F}_t) \subset L^1(\mathcal{F}_t).$$

For each  $p, \mu > 0$  and  $0 < \bar{p} \leq 1 < \tilde{p}$ , it is clear that the following spaces

$$L \exp[\mu(\ln L)^{\bar{p}}](\mathcal{F}_t), \quad \bigcap_{\bar{\mu} > 0} L \exp[\bar{\mu}(\ln L)^{\bar{p}}](\mathcal{F}_t), \quad \bigcup_{\bar{\mu} > 0} L \exp[\bar{\mu}(\ln L)^{\tilde{p}}](\mathcal{F}_t) \quad \text{and} \quad \bigcap_{\bar{\mu} > 0} \exp(\bar{\mu} L^p)(\mathcal{F}_t)$$

are all linear spaces containing  $L^\infty(\mathcal{F}_t)$ . Note that under the conditions without causing confusion, the  $\sigma$ -algebra  $(\mathcal{F}_T)$  is usually omitted in these notations on the spaces of random variables.

Denote by  $\Sigma_T$  the set of all  $(\mathcal{F}_t)$ -stopping times  $\tau$  taking values in  $[0, T]$ . For an  $(\mathcal{F}_t)$ -adapted scalar process  $(X_t)_{t \in [0, T]}$ , we say that it belongs to class (D) if the family of random variables  $\{X_\tau : \tau \in \Sigma_T\}$  is uniformly integrable. Finally, let us recall some updated results on scalar BSDEs. Consider the following scalar BSDE:

$$Y_t = \xi - \int_t^T g(s, Z_s) ds + \int_t^T Z_s \cdot dB_s, \quad t \in [0, T], \quad (2.1)$$

where  $\xi$  is called the terminal value, which is an  $\mathcal{F}_T$ -measurable scalar random variable, the random function

$$g(\omega, t, z) : \Omega \times [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R},$$

which is  $(\mathcal{F}_t)$ -progressively measurable for each  $z \in \mathbb{R}^d$ , is called the generator of (2.1), and the pair of  $(\mathcal{F}_t)$ -progressively measurable processes  $(Y_t, Z_t)_{t \in [0, T]}$  taking values in  $\mathbb{R} \times \mathbb{R}^d$  is called an adapted solution of (2.1) if  $\mathbb{P} - a.s.$ ,  $t \mapsto Y_t$  is continuous,  $t \mapsto |g(t, Z_t)| + |Z_t|^2$  is integrable, and (2.1) holds.

Throughout the whole paper, we will be given two  $(\mathcal{F}_t)$ -progressively measurable nonnegative scalar processes  $(h_t)_{t \in [0, T]}$  and  $(\bar{h}_t)_{t \in [0, T]}$ , and four positive constants  $\gamma, \lambda, c > 0$  and  $\alpha \in (1, 2)$ . Let  $\alpha^* > 2$  represent the conjugate of  $\alpha$ , i.e.,

$$\frac{1}{\alpha} + \frac{1}{\alpha^*} = 1.$$

Furthermore, we introduce the following assumptions on the generator  $g$ .

(H0)  $d\mathbb{P} \times dt - a.e.$ ,  $g(\omega, t, \cdot)$  is convex.

(H1)  $g$  has a quadratic growth in  $z$ , i.e.,  $d\mathbb{P} \times dt - a.e.$ , for each  $z \in \mathbb{R}^d$ , we have

$$|g(\omega, t, z)| \leq \bar{h}_t(\omega) + \frac{\gamma}{2}|z|^2.$$

(H2)  $g$  has a sub-quadratic growth in  $z$ , i.e.,  $d\mathbb{P} \times dt - a.e.$ , for each  $z \in \mathbb{R}^d$ , we have

$$|g(\omega, t, z)| \leq \bar{h}_t(\omega) + \gamma|z|^\alpha.$$

(H3)  $g$  has a super-linear growth in  $z$ , i.e.,  $d\mathbb{P} \times dt - a.e.$ , for each  $z \in \mathbb{R}^d$ , we have

$$|g(\omega, t, z)| \leq \bar{h}_t(\omega) + \gamma|z|(\ln(e + |z|))^\lambda.$$

(H4)  $g$  has a linear growth in  $z$ , i.e.,  $d\mathbb{P} \times dt - a.e.$ , for each  $z \in \mathbb{R}^d$ , we have

$$|g(\omega, t, z)| \leq \bar{h}_t(\omega) + \gamma|z|.$$

**Remark 2.1.** Both assumptions (H0) and (H4) yield that  $d\mathbb{P} \times dt - a.e.$ , for each  $\theta \in (0, 1)$  and  $z_1, z_2 \in \mathbb{R}^d$ , we have

$$\begin{aligned} g(\omega, t, z_1) &= g\left(\omega, t, \theta z_2 + (1 - \theta) \frac{z_1 - \theta z_2}{1 - \theta}\right) \\ &\leq \theta g(\omega, t, z_2) + (1 - \theta) \left(\bar{h}_t(\omega) + \gamma \frac{|z_1 - \theta z_2|}{1 - \theta}\right) \\ &= \theta g(\omega, t, z_2) + (1 - \theta) \bar{h}_t(\omega) + \gamma|z_1 - \theta z_2|, \end{aligned}$$

and then, by letting  $\theta \rightarrow 1$  in the last inequality,

$$|g(\omega, t, z_1) - g(\omega, t, z_2)| \leq \gamma|z_1 - z_2|,$$

i.e.,  $g$  is Lipschitz with respect to  $z$ .

**Proposition 2.2.** The following several assertions hold true.

(i) Assume that the generator  $g$  satisfies assumptions (H0) and (H1), and that

$$|\xi| + \int_0^T \bar{h}_t dt \in \bigcap_{\mu > 0} \exp(\mu L).$$

Then, BSDE (2.1) admits a unique adapted solution  $(Y_t, Z_t)_{t \in [0, T]}$  such that

$$\sup_{t \in [0, T]} |Y_t| \in \bigcap_{\mu > 0} \exp(\mu L).$$

(ii) Assume that the generator  $g$  satisfies assumptions (H0) and (H2), and that

$$|\xi| + \int_0^T \bar{h}_t dt \in \bigcap_{\mu > 0} \exp(\mu L^{\frac{2}{\alpha^*}}).$$

Then, BSDE (2.1) admits a unique adapted solution  $(Y_t, Z_t)_{t \in [0, T]}$  such that

$$\sup_{t \in [0, T]} |Y_t| \in \bigcap_{\mu > 0} \exp(\mu L^{\frac{2}{\alpha^*}}).$$

(iii) Assume that the generator  $g$  satisfies assumptions (H0) and (H3), and that

$$|\xi| + \int_0^T \bar{h}_t dt \in \bigcap_{\mu > 0} L \exp[\mu (\ln L)^{(\frac{1}{2} + \lambda) \vee (2\lambda)}].$$

Then, BSDE (2.1) admits a unique adapted solution  $(Y_t, Z_t)_{t \in [0, T]}$  such that for each  $\mu > 0$ ,

the process  $\left( |Y_t| \exp\left(\mu (\ln(1 + |Y_t|))^{(\frac{1}{2} + \lambda) \vee (2\lambda)}\right) \right)_{t \in [0, T]}$  belongs to class (D).

Moreover, if there exists a constant  $\bar{\mu} > 0$  such that

$$|\xi| + \int_0^T \bar{h}_t dt \in L \exp[\bar{\mu} (\ln L)^{1+2\lambda}],$$

then for any positive constant  $\tilde{\mu} < \bar{\mu}$ , we have

$$\sup_{t \in [0, T]} |Y_t| \in L \exp[\tilde{\mu} (\ln L)^{1+2\lambda}].$$

(iv) Assume that the generator  $g$  satisfies assumptions (H0) and (H4), and that for some constant  $\mu > \mu_0 := \gamma \sqrt{2T}$ , we have

$$|\xi| + \int_0^T \bar{h}_t dt \in L \exp[\mu (\ln L)^{\frac{1}{2}}].$$

Then, BSDE (2.1) admits a unique adapted solution  $(Y_t, Z_t)_{t \in [0, T]}$  such that for some  $\tilde{\mu} > 0$ ,

the process  $\left( |Y_t| \exp\left(\tilde{\mu} \sqrt{\ln(1 + |Y_t|)}\right) \right)_{t \in [0, T]}$  belongs to class (D).

Moreover, if

$$|\xi| + \int_0^T \bar{h}_t dt \in \bigcap_{\bar{\mu} > 0} L \exp[\bar{\mu} (\ln L)^{\frac{1}{2}}],$$

then for any  $\bar{\mu} > 0$ , we have

the process  $\left( |Y_t| \exp\left(\bar{\mu} \sqrt{\ln(1 + |Y_t|)}\right) \right)_{t \in [0, T]}$  belongs to class (D).

*Proof.* Assertion (i) is a direct consequence of Corollary 6 in [5], and assertion (ii) follows immediately from Theorem 3.1 in [20]. Furthermore, according to Theorem 2.4 of [21] together with Doob's maximal inequality for martingales, we can easily derive assertion (iii). Finally, in view of Remark 2.1, assertion (iv) can be obtained by applying Theorem 3.1 of [27] and Theorem 2.5 of [6]. The interested readers are also referred to [19], [31] and [22] for more details. The proof is complete.  $\square$

### 3. Statement of the main result

In the rest of this paper, we always assume that the random core function

$$f(\omega, t, q) : \Omega \times [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$$

is  $(\mathcal{F}_t)$ -progressively measurable for each  $q \in \mathbb{R}^d$ , and that  $f$  satisfies the following assumption:

(A0)  $d\mathbb{P} \times dt - a.e.$ ,  $f(\omega, t, \cdot)$  is lower semi-continuous and convex, and there exists an  $(\mathcal{F}_t)$ -progressively measurable  $\mathbb{R}^d$ -valued process  $(\bar{q}_t)_{t \in [0, T]}$  and a constant  $k \geq 0$  such that  $d\mathbb{P} \times dt - a.e.$ ,

$$|\bar{q}_t(\omega)| \leq k \quad \text{and} \quad f(\omega, t, \bar{q}_t(\omega)) \leq h_t(\omega).$$

Recalling that  $(h_t)_{t \in [0, T]}$  is a given  $(\mathcal{F}_t)$ -progressively measurable nonnegative scalar process.

In particular, if  $d\mathbb{P} \times dt - a.e.$ ,  $f(\omega, t, \cdot)$  is a convex function taking values in  $\mathbb{R}$  and  $f(\omega, t, 0) \equiv 0$ , then assumption (A0) is naturally satisfied.

For each endowment  $\xi \in \mathcal{A}(\mathcal{F}_T)$  and random core function  $f$ , we define the following process space:

$$\begin{aligned} \mathcal{H}(\xi, f) := & \left\{ \begin{array}{l} (q_t)_{t \in [0, T]} \text{ is an } (\mathcal{F}_t)\text{-progressively measurable } \mathbb{R}^d\text{-valued process :} \\ \int_0^T |q_s|^2 ds < +\infty \quad \mathbb{P} - a.s., \quad \mathbb{E}_{\mathbb{Q}^q} \left[ |\xi| + \int_0^T (h_s + |f(s, q_s)|) ds \right] < +\infty, \\ \text{with } L_t^q := \exp \left( \int_0^t q_s \cdot dB_s - \frac{1}{2} \int_0^t |q_s|^2 ds \right), \quad t \in [0, T], \\ \text{being a uniformly integrable martingale, and } \frac{d\mathbb{Q}^q}{d\mathbb{P}} := L_T^q \end{array} \right\}. \end{aligned} \quad (3.1)$$

In the present paper, we aim to study that under what conditions on the endowment  $\xi$  and the random core function  $f$ , the following time-parameterized operator

$$U_t(\xi) := \operatorname{ess\,inf}_{q \in \mathcal{H}(\xi, f)} \mathbb{E}_{\mathbb{Q}^q} \left[ \xi + \int_t^T f(s, q_s) ds \middle| \mathcal{F}_t \right], \quad t \in [0, T], \quad (3.2)$$

is well-defined, and admits a dual representation via the adapted solution of the following BSDE

$$Y_t = \xi - \int_t^T g(s, Z_s) ds + \int_t^T Z_s \cdot dB_s, \quad t \in [0, T], \quad (3.3)$$

where the generator  $g$  of BSDE (3.3) is the Legendre-Fenchel transform of  $f$ , i.e.,

$$g(\omega, t, z) := \sup_{q \in \mathbb{R}^d} (z \cdot q - f(\omega, t, q)), \quad \forall (\omega, t, z) \in \Omega \times [0, T] \times \mathbb{R}^d. \quad (3.4)$$

Hence, the operator  $\{U_t(\cdot), t \in [0, T]\}$  defined via (3.2) constitutes a dynamic concave utility function defined on some linear spaces bigger than  $L^\infty(\mathcal{F}_T)$ . It is clear from (3.4) and (A0) that  $d\mathbb{P} \times dt - a.e.$ ,  $g(\omega, t, \cdot)$  is a convex function defined on  $\mathbb{R}^d$ , and

$$g(\omega, t, z) \geq z \cdot \bar{q}_t(\omega) - f(\omega, t, \bar{q}_t(\omega)) \geq -k|z| - h_t(\omega), \quad \forall z \in \mathbb{R}^d. \quad (3.5)$$

Let us further present the following assumptions on the core function  $f$ , all of which can ensure that the dual function  $g$  of  $f$  satisfies that  $d\mathbb{P} \times dt - a.e.$ ,  $g(\omega, t, z) < +\infty$  for each  $z \in \mathbb{R}^d$ .

(A1)  $d\mathbb{P} \times dt - a.e.$ , for each  $q \in \mathbb{R}^d$ , we have

$$f(\omega, t, q) \geq \frac{1}{2\gamma} |q|^2 - h_t(\omega).$$

(A2)  $d\mathbb{P} \times dt - a.e.$ , for each  $q \in \mathbb{R}^d$ , we have

$$f(\omega, t, q) \geq \gamma^{-\frac{1}{\alpha-1}} |q|^{\alpha^*} - h_t(\omega).$$

(A3)  $d\mathbb{P} \times dt - a.e.$ , for each  $q \in \mathbb{R}^d$ , we have

$$f(\omega, t, q) \geq c \exp\left(2\gamma^{-\frac{1}{\lambda}} |q|^{\frac{1}{\lambda}}\right) - h_t(\omega).$$

(A4)  $d\mathbb{P} \times dt - a.e.$ , for each  $q \in \mathbb{R}^d$ , we have

$$f(\omega, t, q) \geq -h_t(\omega) \quad \text{and} \quad f(\omega, t, q) \equiv +\infty \quad \text{in the case of } |q| > \gamma.$$

The main result of the paper is stated as follows.

**Theorem 3.1.** *We have the following dual representation results.*

(i) *Assume that the core function  $f$  satisfies assumptions (A0) and (A1), and that*

$$|\xi| + \int_0^T h_t dt \in \bigcap_{\mu > 0} \exp(\mu L).$$

*Then, the process  $\{U_t(\xi), t \in [0, T]\}$  defined via (3.2) is well-defined, the generator  $g$  defined in (3.4) satisfies assumptions (H0) and (H1) with  $\bar{h}_t = h_t + k^2/(2\gamma)$ , and there exists an  $(\mathcal{F}_t)$ -progressively measurable  $\mathbb{R}^d$ -valued process  $(Z_t)_{t \in [0, T]}$  such that  $(Y_t := U_t(\xi), Z_t)_{t \in [0, T]}$  is the unique adapted solution of BSDE (3.3) satisfying*

$$\sup_{t \in [0, T]} |Y_t| \in \bigcap_{\mu > 0} \exp(\mu L).$$

*Moreover, the operator  $\{U_t(\cdot), t \in [0, T]\}$  defined via (3.2) constitutes a dynamic concave utility function defined on  $\bigcap_{\mu > 0} \exp(\mu L)$ .*

(ii) *Assume that the core function  $f$  satisfies assumptions (A0) and (A2), and that*

$$|\xi| + \int_0^T h_t dt \in \bigcap_{\mu > 0} \exp(\mu L^{\frac{2}{\alpha^*}}).$$

*Then, the process  $\{U_t(\xi), t \in [0, T]\}$  defined via (3.2) is well-defined, the generator  $g$  defined in (3.4) satisfies assumptions (H0) and (H2) with*

$$\bar{h}_t = h_t + \gamma^{-\frac{1}{\alpha-1}} k^{\alpha^*},$$

*and there exists an  $(\mathcal{F}_t)$ -progressively measurable  $\mathbb{R}^d$ -valued process  $(Z_t)_{t \in [0, T]}$  such that  $(Y_t := U_t(\xi), Z_t)_{t \in [0, T]}$  is the unique adapted solution of BSDE (3.3) satisfying*

$$\sup_{t \in [0, T]} |Y_t| \in \bigcap_{\mu > 0} \exp(\mu L^{\frac{2}{\alpha^*}}).$$

*Moreover, the operator  $\{U_t(\cdot), t \in [0, T]\}$  defined via (3.2) constitutes a dynamic concave utility function defined on  $\bigcap_{\mu > 0} \exp(\mu L^{\frac{2}{\alpha^*}})$ .*



(iii) Assume that the core function  $f$  satisfies assumptions (A0) and (A3), and that

$$|\xi| + \int_0^T h_t dt \in \bigcup_{\mu > 0} L \exp[\mu(\ln L)^{1+2\lambda}].$$

Then, the process  $\{U_t(\xi), t \in [0, T]\}$  defined via (3.2) is well-defined, the generator  $g$  defined in (3.4) satisfies assumptions (H0) and (H3) with

$$\bar{h}_t = h_t + \exp\left(2\gamma^{-\frac{1}{\lambda}}|k|^{\frac{1}{\lambda}}\right) + C_{c,\gamma,\lambda},$$

where  $C_{c,\gamma,\lambda}$  is a positive constant depending only on  $(c, \gamma, \lambda)$ , and there exists an  $(\mathcal{F}_t)$ -progressively measurable  $\mathbb{R}^d$ -valued process  $(Z_t)_{t \in [0, T]}$  such that  $(Y_t := U_t(\xi), Z_t)_{t \in [0, T]}$  is the unique adapted solution of BSDE (3.3) satisfying

$$\sup_{t \in [0, T]} |Y_t| \in \bigcup_{\mu > 0} L \exp[\mu(\ln L)^{1+2\lambda}].$$

Moreover, the operator  $\{U_t(\cdot), t \in [0, T]\}$  defined via (3.2) constitutes a dynamic concave utility function defined on  $\cup_{\mu > 0} L \exp[\mu(\ln L)^{1+2\lambda}]$ .

(iv) Assume that the core function  $f$  satisfies assumptions (A0) with  $k \leq \gamma$  and (A4), and that

$$|\xi| + \int_0^T h_t dt \in \bigcap_{\bar{\mu} > 0} L \exp[\bar{\mu}(\ln L)^{\frac{1}{2}}].$$

Then, the process  $\{U_t(\xi), t \in [0, T]\}$  defined via (3.2) is well-defined, the generator  $g$  defined in (3.4) satisfies assumptions (H0) and (H4) with  $\bar{h}_t = h_t$ , and there exists an  $(\mathcal{F}_t)$ -progressively measurable  $\mathbb{R}^d$ -valued process  $(Z_t)_{t \in [0, T]}$  such that  $(Y_t := U_t(\xi), Z_t)_{t \in [0, T]}$  is the unique adapted solution of BSDE (3.3) such that for each  $\bar{\mu} > 0$ ,

$$\text{the process } \left( |Y_t| \exp\left(\bar{\mu} \sqrt{\ln(1 + |Y_t|)}\right) \right)_{t \in [0, T]} \text{ belongs to class (D).}$$

Moreover, the operator  $\{U_t(\cdot), t \in [0, T]\}$  defined via (3.2) constitutes a dynamic concave utility function defined on  $\cap_{\bar{\mu} > 0} L \exp[\bar{\mu}(\ln L)^{\frac{1}{2}}]$ .

**Example 3.2.** We would like to introduce several practical examples of unbounded endowments in a financial market. Let  $(X_t)_{t \in [0, T]}$  be the unique adapted solution of the following SDE:

$$dX_t = b(t, X_t)dt + \sigma(t, X_t) \cdot dB_t, \quad t \in [0, T]; \quad X_0 = x_0,$$

where  $x_0 \in \mathbb{R}$  is a given constant and both  $b(t, x) : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  and  $\sigma(t, x) : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}^d$  are measurable functions satisfying that for each  $x_1, x_2 \in \mathbb{R}^2$  and  $t \in [0, T]$ , we have

$$|b(t, x_1) - b(t, x_2)| + |\sigma(t, x_1) - \sigma(t, x_2)| \leq c|x_1 - x_2| \quad \text{and} \quad |b(t, 0)| + |\sigma(t, 0)| \leq c.$$

We consider an endowment  $\xi$  which equals to  $X_T$ . By classical theory of SDEs we know that

$$\xi := X_T \in L^2 \subset \cap_{\mu > 0} L \exp[\mu(\ln L)^{\frac{1}{2}}].$$

Moreover, if it is also supposed that  $|\sigma(t, x)| \leq c$  for each  $(t, x) \in [0, T] \times \mathbb{R}$ , then there exists two positive constants  $c_1$  and  $c_2$  depending only on  $(c, T)$  such that

$$\mathbb{E} \left[ \exp \left( c_1 \sup_{t \in [0, T]} |X_t|^2 \right) \right] \leq c_2 \exp(c_2 |x_0|^2),$$

which indicates that for each  $\lambda > 0$ ,

$$\xi := X_T \in \bigcap_{\mu > 0} \exp(\mu L) \subset \bigcap_{\mu > 0} \exp(\mu L^{\frac{2}{\alpha^*}}) \subset \cup_{\mu > 0} L \exp[\mu(\ln L)^{1+2\lambda}] \subset L^2.$$

Finally, if we let  $d = 1$ ,  $b(t, x) := bx$  and  $\sigma(t, x) := \sigma x$  for two positive constants  $b$  and  $\sigma$ , then we have

$$X_t = x_0 \exp \left( bt - \frac{1}{2} \sigma^2 t + \sigma B_t \right), \quad t \in [0, T].$$

Thus, for a typical European call option  $\xi$  defined by  $(X_T - K)^+$  with the previously agreed strike price  $K > 0$ , we can conclude that for each  $\lambda \in (0, 1/2]$ ,

$$\xi := (X_T - K)^+ \in \cup_{\mu > 0} L \exp[\mu(\ln L)^{1+2\lambda}].$$

**Example 3.3.** We present the following several examples to which Theorem 3.1 can apply.

- (i) Let  $(\bar{q}_t)_{t \in [0, T]}$  be an  $(\mathcal{F}_t)$ -progressively measurable  $\mathbb{R}^d$ -valued process such that  $d\mathbb{P} \times dt$ -a.e.,  $|\bar{q}_t| \leq \gamma$  and let the core function  $f$  be defined as follows:  $\forall (\omega, t, q) \in \Omega \times [0, T] \times \mathbb{R}^d$ ,

$$f(\omega, t, q) := \begin{cases} h_t(\omega), & q = \bar{q}_t(\omega); \\ +\infty, & \text{otherwise.} \end{cases}$$

It is clear that  $f$  satisfies assumptions (A0) and (A4), and if

$$\mathbb{E}_{\mathbb{Q}^{\bar{q}}} \left[ \xi + \int_0^T h_s ds \right] < +\infty, \quad (3.6)$$

then

$$U_t(\xi) := \operatorname{ess\,inf}_{q \in \mathcal{H}(\xi, f)} \mathbb{E}_{\mathbb{Q}^q} \left[ \xi + \int_t^T f(s, q_s) ds \middle| \mathcal{F}_t \right] = \mathbb{E}_{\mathbb{Q}^{\bar{q}}} \left[ \xi + \int_t^T h_s ds \middle| \mathcal{F}_t \right], \quad t \in [0, T]. \quad (3.7)$$

On the other hand, it is easy to verify that the convex conjugate function of  $f$  is the following:

$$\forall (\omega, t, z) \in \Omega \times [0, T] \times \mathbb{R}^d, \quad g(\omega, t, z) := \sup_{q \in \mathbb{R}^d} (z \cdot q - f(\omega, t, q)) = \bar{q}_t(\omega) \cdot z - h_t(\omega).$$

According to Girsanov's theorem, we know that when (3.6) is satisfied, the process  $\{U_t(\xi)\}_{t \in [0, T]}$  in (3.7) is just the first process of the unique solution of BSDE (3.3) with this generator  $g$ . We remark that (iv) of Theorem 3.1 indicates that the above assertion holds when (3.6) is replaced with

$$|\xi| + \int_0^T h_t dt \in \bigcap_{\mu > 0} L \exp[\mu(\ln L)^{\frac{1}{2}}].$$

- (ii) Let the core function  $f$  be defined as follows:  $\forall (\omega, t, q) \in \Omega \times [0, T] \times \mathbb{R}^d$ ,

$$f(\omega, t, q) := \begin{cases} 0, & |q| \leq \gamma; \\ +\infty, & |q| > \gamma. \end{cases}$$

It is clear that  $f$  satisfies assumptions (A0) and (A4) with  $h_t \equiv 0$ , and for each  $\xi \in L^2$ , we have

$$U_t(\xi) := \operatorname{ess\,inf}_{q \in \mathcal{H}(\xi, f)} \mathbb{E}_{\mathbb{Q}^q} \left[ \xi + \int_t^T f(s, q_s) ds \middle| \mathcal{F}_t \right] = \operatorname{ess\,inf}_{q \in \mathbb{R}^d: |q| \leq \gamma} \mathbb{E}_{\mathbb{Q}^q} [\xi | \mathcal{F}_t], \quad t \in [0, T]. \quad (3.8)$$

On the other hand, it is easy to verify that the convex conjugate function of  $f$  is the following:

$$\forall (\omega, t, z) \in \Omega \times [0, T] \times \mathbb{R}^d, \quad g(\omega, t, z) := \sup_{q \in \mathbb{R}^d} (z \cdot q - f(\omega, t, q)) = \gamma |z|.$$

It follows from Lemma 3 of [7] that for each  $\xi \in L^2$ , the process  $\{U_t(\xi)\}_{t \in [0, T]}$  in (3.8) is just the first process of the unique solution of BSDE (3.3) with this generator  $g$ . We remark that (iv) of Theorem 3.1 indicates that the above assertion holds when  $\xi \in \cap_{\mu > 0} L \exp[\mu(\ln L)^{\frac{1}{2}}]$ .

(iii) Let the core function  $f$  be defined as follows:  $\forall (\omega, t, q) \in \Omega \times [0, T] \times \mathbb{R}^d$ ,

$$f(\omega, t, q) := \begin{cases} \frac{|q|^2}{2}, & |q| \leq \gamma; \\ +\infty, & |q| > \gamma. \end{cases}$$

It is clear that  $f$  satisfies assumptions (A0) and (A4) with  $h_t \equiv 0$ . It is not hard to verify that the convex conjugate function of  $f$  is the following:  $\forall (\omega, t, z) \in \Omega \times [0, T] \times \mathbb{R}^d$ ,

$$g(\omega, t, z) := \sup_{q \in \mathbb{R}^d} (z \cdot q - f(\omega, t, q)) = \begin{cases} \frac{|z|^2}{2}, & |z| \leq \gamma; \\ \gamma |z| - \frac{\gamma^2}{2}, & |z| > \gamma, \end{cases}$$

and that this generator  $g$  satisfies assumptions (H0) and (H4). Thus, if  $\xi \in \cap_{\mu > 0} L \exp[\mu(\ln L)^{\frac{1}{2}}]$ , the conclusions in (iv) of Theorem 3.1 can be applied.

(iv) Let the core function  $f$  be defined as follows:  $\forall (\omega, t, q) \in \Omega \times [0, T] \times \mathbb{R}^d$ ,

$$f(\omega, t, q) := e^{|q|} + h_t(\omega).$$

It is clear that  $f$  satisfies assumptions (A0) and (A3) with  $(c, \gamma, \lambda) = (1, 2, 1)$ . It is not hard to verify that the convex conjugate function of  $f$  is the following:  $\forall (\omega, t, z) \in \Omega \times [0, T] \times \mathbb{R}^d$ ,

$$g(\omega, t, z) := \sup_{q \in \mathbb{R}^d} (z \cdot q - f(\omega, t, q)) = |z|(\ln |z| - 1) - h_t(\omega),$$

and that  $g$  satisfies assumptions (H0) and (H3). Thus, if  $\xi + \int_0^T h_t dt \in \cup_{\mu > 0} L \exp[\mu(\ln L)^3]$ , then the conclusions in (iii) of Theorem 3.1 can be applied.

(v) Let the core function  $f$  be defined as follows:  $\forall (\omega, t, q) \in \Omega \times [0, T] \times \mathbb{R}^d$ ,

$$f(\omega, t, q) := \frac{1}{4}|q|^4 + h_t(\omega).$$

It is clear that  $f$  satisfies assumptions (A0) and (A2) with  $(\alpha, \alpha^*, \gamma) = (\frac{4}{3}, 4, \sqrt[3]{4})$ . It is not hard to verify that the convex conjugate function of  $f$  is the following:  $\forall (\omega, t, z) \in \Omega \times [0, T] \times \mathbb{R}^d$ ,

$$g(\omega, t, z) := \sup_{q \in \mathbb{R}^d} (z \cdot q - f(\omega, t, q)) = \frac{3}{4}|z|^{\frac{4}{3}} - h_t(\omega),$$

and that this generator  $g$  satisfies assumptions (H0) and (H2). Thus, if  $\xi + \int_0^T h_t dt \in \cap_{\mu > 0} \exp(\mu L^{\frac{1}{2}})$ , then the conclusions in (ii) of Theorem 3.1 can be applied.

(vi) Let the core function  $f$  be defined as follows:  $\forall(\omega, t, q) \in \Omega \times [0, T] \times \mathbb{R}^d$ ,

$$f(\omega, t, q) := \frac{1}{2\gamma} |q|^2.$$

It is clear that  $f$  satisfies assumptions (A0) and (A1) with  $h_t \equiv 0$ . On the other hand, it is easy to verify that the convex conjugate function of  $f$  is the following:  $\forall(\omega, t, z) \in \Omega \times [0, T] \times \mathbb{R}^d$ ,

$$g(\omega, t, z) := \sup_{q \in \mathbb{R}^d} (z \cdot q - f(\omega, t, q)) = \frac{\gamma}{2} |z|^2.$$

It follows from [4] that for each  $\xi \in \exp(\gamma L)$ , BSDE (3.3) with the above generator  $g$  admits a unique adapted solution  $(Y_t, Z_t)_{t \in [0, T]}$  such that  $\{\exp(\gamma |Y_t|)\}_{t \in [0, T]}$  belongs to class (D), and the process  $Y$  can be explicitly expressed as follows:

$$Y_t = \frac{1}{\gamma} \ln(\mathbb{E}[\exp(\gamma \xi) | \mathcal{F}_t]), \quad t \in [0, T].$$

Thus, according to (i) of Theorem 3.1, we can conclude that for each  $\xi \in \cap_{\mu > 0} \exp(\mu L)$ ,

$$U_t(\xi) := \operatorname{ess\,inf}_{q \in \mathcal{H}(\xi, f)} \mathbb{E}_{\mathbb{Q}^q} \left[ \xi + \frac{1}{2\gamma} \int_t^T |q_s|^2 ds \middle| \mathcal{F}_t \right] = \frac{1}{\gamma} \ln \mathbb{E}[\exp(\gamma \xi) | \mathcal{F}_t], \quad t \in [0, T].$$

(vii) Let  $d = 1$  and the core function  $f$  be defined as follows:  $\forall(\omega, t, q) \in \Omega \times [0, T] \times \mathbb{R}$ ,

$$f(\omega, t, q) := \begin{cases} +\infty, & q < 1; \\ q - 1, & 1 \leq q \leq 2; \\ \frac{q^2}{4}, & q > 2. \end{cases}$$

It is clear that  $f$  satisfies assumptions (A0) and (A1) with  $\gamma = 2$  and  $h_t \equiv 0$ . It is not hard to verify that the convex conjugate function of  $f$  is the following:  $\forall(\omega, t, z) \in \Omega \times [0, T] \times \mathbb{R}$ ,

$$g(\omega, t, z) := \sup_{q \in \mathbb{R}^d} (z \cdot q - f(\omega, t, q)) = \begin{cases} z, & z < 1; \\ z^2, & z \geq 1, \end{cases}$$

and that this generator  $g$  satisfies assumptions (H0) and (H1). Thus, if  $\xi \in \cap_{\mu > 0} \exp(\mu L)$ , then the conclusions in (i) of Theorem 3.1 can be applied.

**Remark 3.4.** We make the following remarks on Theorem 3.1.

(i) As stated in the introduction, the conclusions in (i) and (iv) of Theorem 3.1 have been respectively explored in [12] and [15] for the special case that the endowment  $\xi$  is a bounded random variable, i.e.,  $\xi \in L^\infty$ . It is further assumed that  $f$  is independent of  $(\omega, t)$  and takes values in  $\mathbb{R}_+$  with  $f(0) = 0$  in [15]. We would like to emphasize that Theorem 3.1 illustrates that the dual representation, as the unique adapted solution of BSDEs, of several class of dynamical concave utilities holds still for unbounded endowments and general random core functions. In particular, this makes it possible to compute the unbounded dynamical concave utilities via solving the solution of BSDEs by numerical algorithms such as Monte Carlo method.

(ii) In Theorem 3.1, the integrability requirement on  $\xi + \int_0^T h_t dt$  is independent of two constants  $\gamma$  and  $c$ . This brings convenience to the application of Theorem 3.1. In fact, when the core function  $f$  is given, one can pick proper constants  $\gamma$  and  $c$  so that either of (A1)-(A4) holds for  $f$ .

(iii) The space  $L^p$  ( $p > 1$ ) is smaller than the space of  $\cap_{\mu>0} L \exp[\mu(\ln L)^{\frac{1}{2}}]$  and bigger than the space of  $\cup_{\mu>0} L \exp[\mu(\ln L)^{1+2\lambda}]$ . It is interesting to find some appropriate growth conditions on the core function  $f$ , which are weaker than (A4) and stronger than (A3), such that the dual representation in Theorem 3.1 holds for the endowments  $\xi \in L^2$  or  $\xi \in L^p$  ( $p > 1$ ). The existence of these conditions is unsolved up to now.

(iv) The proof of Theorem 3.1 is enlightened by [12], [15] and in particular [13], where all of the Legendre-Fenchel transform of a convex function, the de la vallée-Poussin theorem, Young's inequality and Gronwall's inequality play important roles. See the next section for more details. In addition, from the proof of Theorem 3.1, it can be observed that the boundedness assumptions on the process  $(\bar{q}_t)_{t \in [0, T]}$  in (A0) can be appropriately weakened in stating (i)-(iii) of Theorem 3.1.

#### 4. Proof of the main result

In this section, we will give the proof of Theorem 3.1. Before that, let us establish two useful technical propositions, which are interesting in their own right.

First of all, let us recall the known Young inequality. Let  $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be a strictly increasing continuous function with  $f(0) = 0$ , and  $g$  be the inverse function of  $f$ . Then, we have

$$xy \leq \int_0^x f(s)ds + \int_0^y g(s)ds \leq xf(x) + yg(y), \quad \forall x, y \geq 0. \quad (4.1)$$

Based on this elementary inequality, we easily prove the following proposition.

**Proposition 4.1.** *Let  $\mu, \delta > 0$  be two arbitrarily given positive constants. Then, we have*

$$xy \leq x \exp\left(\frac{x^\delta}{\mu^\delta}\right) + \mu y (\ln(1+y))^{\frac{1}{\delta}}, \quad \forall x, y \geq 0 \quad (4.2)$$

and

$$xy \leq x \exp\left(\frac{(\ln(1+x))^\delta}{\mu^\delta}\right) + y \exp\left(\mu(\ln(1+y))^{\frac{1}{\delta}}\right), \quad \forall x, y \geq 0. \quad (4.3)$$

Furthermore, for each  $q > 1$  and  $\varepsilon > 0$ , there exists a constant  $C_{q, \varepsilon, \mu, \delta} > 0$  depending only on  $(q, \varepsilon, \mu, \delta)$  such that

$$xy \leq \varepsilon \exp\left(\frac{qx^\delta}{\mu^\delta}\right) + \mu y (\ln(1+y))^{\frac{1}{\delta}} + C_{q, \varepsilon, \mu, \delta}, \quad \forall x, y \geq 0 \quad (4.4)$$

and in the case of  $\delta > 1$ ,

$$xy \leq \varepsilon \exp\left(\frac{q(\ln(1+x))^\delta}{\mu^\delta}\right) + y \exp\left(\mu(\ln(1+y))^{\frac{1}{\delta}}\right) + C_{q, \varepsilon, \mu, \delta}, \quad \forall x, y \geq 0. \quad (4.5)$$

In particular, we have

$$xy \leq \mu \exp\left(\frac{x}{\mu}\right) + \mu y \ln(1+y), \quad \forall x, y \geq 0, \quad (4.6)$$

and for each  $q > 1$ , there exists a constant  $\bar{C}_{\mu, q} > 0$  depending only on  $(\mu, q)$  such that

$$y \exp(x) \leq \bar{C}_{\mu, q} \exp\left(\frac{qx^2}{\mu^2}\right) + y \exp\left(\mu \sqrt{\ln(1+y)}\right), \quad \forall x, y \geq 0. \quad (4.7)$$

In addition, for  $\delta > 1$ , we have

$$xy \leq \mu x^\delta + \frac{1}{\mu^{\frac{1}{\delta-1}}} y^{\delta^*}, \quad \forall x, y \geq 0, \quad (4.8)$$

where  $\delta^* := \delta/(\delta - 1)$  is the conjugate of  $\delta$ , i.e.,  $1/\delta + 1/\delta^* = 1$ .

*Proof.* Inequalities (4.2) and (4.3) follow immediately by picking

$$f(x) = \exp\left(\frac{x^\delta}{\mu^\delta}\right) - 1 \quad \text{and} \quad f(x) = \exp\left(\frac{(\ln(1+x))^\delta}{\mu^\delta}\right) - 1,$$

respectively, in (4.1). Observe that for each  $q > 1$  and  $\varepsilon > 0$ , we have

$$\lim_{x \rightarrow +\infty} \frac{\varepsilon \exp\left(\frac{(q-1)x^\delta}{\mu^\delta}\right)}{x} = +\infty, \quad \forall \delta > 0$$

and

$$\lim_{x \rightarrow +\infty} \frac{\varepsilon \exp\left(\frac{(q-1)(\ln(1+x))^\delta}{\mu^\delta}\right)}{x} = +\infty, \quad \forall \delta > 1.$$

Then, there exists a constant  $k_{q,\varepsilon,\mu,\delta} > 0$  depending only on  $(q, \mu, \delta, \varepsilon)$  such that for each  $x \geq k_{q,\varepsilon,\mu,\delta}$ ,

$$x \leq \varepsilon \exp\left(\frac{(q-1)x^\delta}{\mu^\delta}\right), \quad \forall \delta > 0$$

and

$$x \leq \varepsilon \exp\left(\frac{(q-1)(\ln(1+x))^\delta}{\mu^\delta}\right), \quad \forall \delta > 1.$$

Thus, the desired inequalities (4.4) and (4.5) follow immediately from (4.2) and (4.3) together with the last two inequalities. By picking

$$f(x) = \exp\left(\frac{x}{\mu}\right) - 1$$

in (4.1), we have (4.6). Furthermore, note that for each  $q > 1$  and  $x \geq 0$ , we have

$$\begin{aligned} 0 &\leq \ln(1 + \exp(x)) - \ln(\exp(x)) = \ln(1 + \exp(-x)) \leq \ln 2, \\ (\ln(1 + \exp(x)))^2 &\leq (\ln 2 + x)^2 \leq \left(1 + \frac{2}{q-1}\right) (\ln 2)^2 + \frac{q+1}{2} x^2 \end{aligned}$$

and

$$\exp(x) \leq \exp\left(\frac{q-1}{2\mu^2} x^2 + \frac{\mu^2}{2(q-1)}\right).$$

By letting  $\exp(x)$  instead of  $x$  and  $\delta = 2$  in (4.3), inequality (4.7) follows immediately by picking

$$\bar{C}_{\mu,q} := \exp\left(\frac{\mu^2}{2(q-1)} + \frac{(q+1)(\ln 2)^2}{(q-1)\mu^2}\right).$$

Finally, by picking  $f(x) = \mu x^{\delta-1}$  in (4.1), we get (4.8). The proof is then complete.  $\square$

**Remark 4.2.** We make the following remarks regarding several inequalities in Proposition 4.1.

(i) Inequality (4.6) can be compared to the well-known Fenchel inequality:

$$xy \leq \exp(x) + y(\ln y - 1), \quad \forall x \in \mathbb{R}, \quad \forall y > 0. \quad (4.9)$$

It is easy to check that the inequality (4.6) plays the same role as (4.9) in [13] and [14].

(ii) Inequality (4.7) can be compared to the following one

$$y \exp(x) \leq \exp\left(\frac{x^2}{\mu^2}\right) + \exp(\mu^2)y \exp\left(\mu\sqrt{\ln(1+y)}\right), \quad \forall x \in \mathbb{R}, \forall y \geq 0, \forall \mu > 0, \quad (4.10)$$

which comes from Lemma 2.4 in [27]. Note that the claim of  $\mu > \gamma\sqrt{2T}$  for some certain positive constant  $\gamma > 0$  is equivalent to that of  $\mu > q\gamma\sqrt{2T}$  for some suitable constant  $q > 1$ . It is not hard to verify that (4.7) plays the same role as (4.10) in [27], [6], [19], [31] and [21].

(iii) Inequality (4.8) can be compared to the classical Young inequality

$$xy \leq \frac{1}{\delta}x^\delta + \frac{1}{\delta^*}y^{\delta^*}, \quad \forall x, y \geq 0, \forall \delta > 1. \quad (4.11)$$

It seems to be more convenient to use (4.8) than (4.11).

Furthermore, by Gronwall's inequality we can verify the following proposition.

**Proposition 4.3.** *Let  $(q_t)_{t \in [0, T]}$  is an  $(\mathcal{F}_t)$ -progressively measurable  $\mathbb{R}^d$ -valued process such that  $\mathbb{P}$ -a.s.,  $\int_0^T |q_s|^2 ds < +\infty$ . Define its stochastic exponential by*

$$L_t^q := \exp\left(\int_0^t q_s \cdot dB_s - \frac{1}{2} \int_0^t |q_s|^2 ds\right), \quad t \in [0, T].$$

Then, we have

$$\mathbb{E}[L_T^q \ln(1 + L_T^q)] \leq \frac{1}{2} \mathbb{E}\left[\int_0^T L_t^q |q_t|^2 dt\right] + \ln 2 \quad (4.12)$$

and

$$\mathbb{E}\left[L_T^q [\ln(1 + L_T^q)]^{\frac{\alpha^*}{2}}\right] \leq C_{\alpha, T} \mathbb{E}\left[\int_0^T L_t^q |q_t|^{\alpha^*} dt\right] + C_{\alpha, T}, \quad (4.13)$$

where  $C_{\alpha, T}$  is a positive constant depending only on  $(\alpha, T)$ . Furthermore, for each  $\mu, \varepsilon > 0$ , there exists a positive constant  $\tilde{C}_{\mu, \varepsilon, \gamma, \lambda, T} > 0$  depending only on  $(\mu, \varepsilon, \gamma, \lambda, T)$  such that

$$\mathbb{E}\left[L_T^q \exp\left(\mu [\ln(1 + L_T^q)]^{\frac{1}{1+2\lambda}}\right)\right] \leq \varepsilon \mathbb{E}\left[\int_0^T L_s^q \exp\left(2\gamma^{-\frac{1}{\lambda}} |q_s|^{\frac{1}{\lambda}}\right) ds\right] + \tilde{C}_{\mu, \varepsilon, \gamma, \lambda, T}. \quad (4.14)$$

*Proof.* It suffices to consider the case that those terms in the right hand side of (4.12)-(4.14) are finite.

Note that

$$dL_t^q = L_t^q q_t \cdot dB_t, \quad t \in [0, T]. \quad (4.15)$$

We first verify (4.12). Define  $l(x) = x \ln(1 + x)$  for each  $x > 0$ . Then, for each  $x > 0$ , we have

$$l'(x) = \ln(1 + x) + \frac{x}{1 + x} > 0 \quad \text{and} \quad 0 < l''(x) = \frac{1}{1 + x} + \frac{1}{(1 + x)^2} < \frac{1}{x}.$$

It then follows from Itô's formula that, in view of (4.15),

$$dl(L_t^q) = l'(L_t^q) L_t^q q_t \cdot dB_t + \frac{1}{2} l''(L_t^q) (L_t^q)^2 |q_t|^2 dt \leq l'(L_t^q) L_t^q q_t \cdot dB_t + \frac{1}{2} L_t^q |q_t|^2 dt, \quad t \in [0, T]. \quad (4.16)$$

For each  $n \geq 1$ , define the following  $(\mathcal{F}_t)$ -stopping time:

$$\tau_n := \inf\{t \geq 0 : \int_0^t (l'(L_s^q) L_s^q)^2 |q_s|^2 ds \geq n\} \wedge T.$$

Note that  $\tau_n \rightarrow T$  as  $n$  tends to infinity. By (4.16) we deduce that for each  $n \geq 1$ ,

$$\mathbb{E}[l(L_{\tau_n}^q)] - \ln 2 \leq \frac{1}{2} \mathbb{E} \left[ \int_0^{\tau_n} L_t^q |q_t|^2 dt \right].$$

By sending  $n$  to infinity in the last inequality and using Fatou's lemma, we can easily obtain (4.12).

We now prove (4.13). Define  $\bar{l}(x) = (e+x)[\ln(e+x)]^{\frac{\alpha^*}{2}}$  for each  $x \in \mathbb{R}_+$ . Then, for each  $x \in \mathbb{R}_+$ , we have

$$\bar{l}(x) = [\ln(e+x)]^{\frac{\alpha^*}{2}} + \frac{\alpha^*}{2} [\ln(e+x)]^{\frac{\alpha^*}{2}-1} > 0$$

and

$$0 < \bar{l}''(x) = \frac{\alpha^*}{2} [\ln(e+x)]^{\frac{\alpha^*}{2}-1} \frac{1}{e+x} + \frac{\alpha^*(\alpha^*-2)}{4} [\ln(e+x)]^{\frac{\alpha^*}{2}-2} \frac{1}{e+x} < \frac{(\alpha^*)^2}{4} \frac{[\ln(e+x)]^{\frac{\alpha^*}{2}-1}}{x}.$$

Furthermore, for each  $x, y \in \mathbb{R}_+$ , by Young's inequality we get

$$[\ln(e+x)]^{\frac{\alpha^*}{2}-1} y^2 = \left( [\ln(e+x)]^{\frac{\alpha^*}{2}} \right)^{\frac{\alpha^*-2}{\alpha^*}} \left( y^{\alpha^*} \right)^{\frac{2}{\alpha^*}} \leq \frac{\alpha^*-2}{\alpha^*} [\ln(e+x)]^{\frac{\alpha^*}{2}} + \frac{2}{\alpha^*} y^{\alpha^*}.$$

It then follows from Itô's formula that for each  $t \in [0, T]$ ,

$$\begin{aligned} d\bar{l}(L_t^q) &= \bar{l}'(L_t^q) L_t^q q_t \cdot dB_t + \frac{1}{2} \bar{l}''(L_t^q) (L_t^q)^2 |q_t|^2 dt \\ &\leq \bar{l}'(L_t^q) L_t^q q_t \cdot dB_t + L_t^q \left( \frac{1}{8} \alpha^* (\alpha^* - 2) [\ln(e + L_t^q)]^{\frac{\alpha^*}{2}} + \frac{\alpha^*}{4} |q_t|^{\alpha^*} \right) dt \\ &\leq \bar{l}'(L_t^q) L_t^q q_t \cdot dB_t + \left( \frac{1}{8} \alpha^* (\alpha^* - 2) \bar{l}(L_t^q) + \frac{\alpha^*}{4} L_t^q |q_t|^{\alpha^*} \right) dt. \end{aligned} \quad (4.17)$$

For each  $n \geq 1$ , define the following  $(\mathcal{F}_t)$ -stopping time:

$$\tau_n := \inf \{ t \geq 0 : \int_0^t (\bar{l}'(L_s^q) L_s^q)^2 |q_s|^2 ds \geq n \} \wedge T.$$

Note that  $\tau_n \rightarrow T$  as  $n$  tends to infinity. By (4.17), we deduce that for each  $n \geq 1$  and  $t \in [0, T]$ ,

$$\begin{aligned} \mathbb{E}[\bar{l}(L_{t \wedge \tau_n}^q)] - e &\leq \frac{1}{8} \alpha^* (\alpha^* - 2) \mathbb{E} \left[ \int_0^{t \wedge \tau_n} \bar{l}(L_s^q) ds \right] + \frac{\alpha^*}{4} \mathbb{E} \left[ \int_0^{t \wedge \tau_n} L_s^q |q_s|^{\alpha^*} ds \right] \\ &\leq \frac{1}{8} \alpha^* (\alpha^* - 2) \int_0^t \mathbb{E}[\bar{l}(L_{s \wedge \tau_n}^q)] ds + \frac{\alpha^*}{4} \mathbb{E} \left[ \int_0^T L_s^q |q_s|^{\alpha^*} ds \right]. \end{aligned}$$

Applying Gronwall's inequality to the last inequality yields that for each  $n \geq 1$ ,

$$\mathbb{E}[\bar{l}(L_{t \wedge \tau_n}^q)] \leq \left( \frac{\alpha^*}{4} \mathbb{E} \left[ \int_0^T L_s^q |q_s|^{\alpha^*} ds \right] + e \right) \exp \left( \frac{1}{8} \alpha^* (\alpha^* - 2) t \right), \quad t \in [0, T].$$

By letting  $t = T$  and sending  $n$  to infinity in the last inequality, and then by using Fatou's lemma, we can obtain the desired inequality (4.13).

Finally, we verify (4.14). Given  $\mu > 0$  and define  $\tilde{l}(x) = (\tilde{k} + x) \exp \left( \mu [\ln(\tilde{k} + x)]^{\frac{1}{1+2\lambda}} \right)$  for each  $x \in \mathbb{R}_+$ , where  $\tilde{k}$  is a fixed positive constant large enough such that for each  $x \in \mathbb{R}_+$ ,

$$\tilde{l}'(x) = \tilde{l}(x) \frac{\frac{\mu}{1+2\lambda} + [\ln(\tilde{k} + x)]^{\frac{2\lambda}{1+2\lambda}}}{(\tilde{k} + x) [\ln(\tilde{k} + x)]^{\frac{2\lambda}{1+2\lambda}}} > 0$$



and

$$\begin{aligned}
0 < \tilde{l}''(x) &= \tilde{l}(x) \frac{\frac{\mu}{1+2\lambda} \left[ \ln(\tilde{k} + x) + \frac{\mu}{1+2\lambda} [\ln(\tilde{k} + x)]^{\frac{1}{1+2\lambda}} - \frac{2\lambda}{1+2\lambda} \right]}{(\tilde{k} + x)^2 [\ln(\tilde{k} + x)]^{\frac{1+4\lambda}{1+2\lambda}}} \\
&< \frac{2\mu}{1+2\lambda} \frac{\exp\left(\mu [\ln(\tilde{k} + x)]^{\frac{1}{1+2\lambda}}\right)}{x [\ln(\tilde{k} + x)]^{\frac{2\lambda}{1+2\lambda}}}.
\end{aligned} \tag{4.18}$$

Furthermore, it follows from (4.2) with  $(\mu, \delta) = (\gamma^2, \frac{1}{2\lambda})$  that there exists a positive constant  $\bar{k}_{\lambda, \gamma}$  depending only on  $\lambda$  such that for each  $x > 1$  and  $y \geq 0$ ,

$$xy \leq y \exp\left(\gamma^{-\frac{1}{\lambda}} y^{\frac{1}{2\lambda}}\right) + \gamma^2 x [\ln(1+x)]^{2\lambda} \leq 2\gamma^2 x (\ln x)^{2\lambda} + y \exp\left(\gamma^{-\frac{1}{\lambda}} y^{\frac{1}{2\lambda}}\right) + \bar{k}_{\lambda, \gamma},$$

and then for each  $x, y \in \mathbb{R}_+$  we have, in view of  $\tilde{k}$  being large enough,

$$\begin{aligned}
\frac{\exp\left(\mu [\ln(\tilde{k} + x)]^{\frac{1}{1+2\lambda}}\right)}{[\ln(\tilde{k} + x)]^{\frac{2\lambda}{1+2\lambda}}} y^2 &\leq 2\gamma^2 \frac{\exp\left(\mu [\ln(\tilde{k} + x)]^{\frac{1}{1+2\lambda}}\right)}{[\ln(\tilde{k} + x)]^{\frac{2\lambda}{1+2\lambda}}} \left(\mu [\ln(\tilde{k} + x)]^{\frac{1}{1+2\lambda}}\right)^{2\lambda} \\
&\quad + y^2 \exp\left(\gamma^{-\frac{1}{\lambda}} y^{\frac{1}{\lambda}}\right) + \bar{k}_{\lambda, \gamma} \\
&= 2\gamma^2 \mu^{2\lambda} \exp\left(\mu [\ln(\tilde{k} + x)]^{\frac{1}{1+2\lambda}}\right) + y^2 \exp\left(\gamma^{-\frac{1}{\lambda}} y^{\frac{1}{\lambda}}\right) + \bar{k}_{\lambda, \gamma}.
\end{aligned} \tag{4.19}$$

It then follows from Itô's formula together with (4.18) and (4.19) that for each  $t \in [0, T]$ ,

$$\begin{aligned}
d\tilde{l}(L_t^q) &= \tilde{l}'(L_t^q) L_t^q q_t \cdot dB_t + \frac{1}{2} \tilde{l}''(L_t^q) (L_t^q)^2 |q_t|^2 dt \leq \tilde{l}'(L_t^q) L_t^q q_t \cdot dB_t \\
&\quad + \frac{\mu L_t^q}{1+2\lambda} \left( 2\gamma^2 \mu^{2\lambda} \exp\left(\mu [\ln(k + L_t^q)]^{\frac{1}{1+2\lambda}}\right) + |q_t|^2 \exp\left(\gamma^{-\frac{1}{\lambda}} |q_t|^{\frac{1}{\lambda}}\right) + \bar{k}_{\lambda, \gamma} \right) dt \\
&\leq \tilde{l}'(L_t^q) L_t^q q_t \cdot dB_t + \frac{\mu}{1+2\lambda} \left( 2\gamma^2 \mu^{2\lambda} \tilde{l}(L_t^q) + L_t^q |q_t|^2 \exp\left(\gamma^{-\frac{1}{\lambda}} |q_t|^{\frac{1}{\lambda}}\right) + L_t^q \bar{k}_{\lambda, \gamma} \right) dt.
\end{aligned} \tag{4.20}$$

For each  $n \geq 1$ , define the following  $(\mathcal{F}_t)$ -stopping time:

$$\tau_n := \inf\{t \geq 0 : \int_0^t \left(\tilde{l}'(L_s^q) L_s^q\right)^2 |q_s|^2 ds \geq n\} \wedge T.$$

Note that  $\tau_n \rightarrow T$  as  $n$  tends to infinity. By (4.20) we can deduce that for each  $n \geq 1$  and  $t \in [0, T]$ ,

$$\begin{aligned}
\mathbb{E}[\tilde{l}(L_{t \wedge \tau_n}^q)] - \tilde{k} \exp\left(\mu (\ln \tilde{k})^{\frac{1}{1+2\lambda}}\right) \\
\leq \frac{2\gamma^2 \mu^{2\lambda+1}}{1+2\lambda} \mathbb{E} \left[ \int_0^{t \wedge \tau_n} \tilde{l}(L_s^q) ds \right] + \frac{\mu}{1+2\lambda} \mathbb{E} \left[ \int_0^{t \wedge \tau_n} L_s^q |q_s|^2 \exp\left(\gamma^{-\frac{1}{\lambda}} |q_s|^{\frac{1}{\lambda}}\right) ds \right] + \frac{\mu \bar{k}_{\lambda, \gamma} T}{1+2\lambda} \\
\leq \frac{2\gamma^2 \mu^{2\lambda+1}}{1+2\lambda} \int_0^t \mathbb{E} \left[ \tilde{l}(L_{s \wedge \tau_n}^q) \right] ds + \frac{\mu}{1+2\lambda} \mathbb{E} \left[ \int_0^T L_s^q |q_s|^2 \exp\left(\gamma^{-\frac{1}{\lambda}} |q_s|^{\frac{1}{\lambda}}\right) ds \right] + \frac{\mu \bar{k}_{\lambda, \gamma} T}{1+2\lambda}.
\end{aligned}$$

Applying Gronwall's inequality to the last inequality yields that for each  $n \geq 1$ ,

$$\begin{aligned}
\mathbb{E}[\tilde{l}(L_{t \wedge \tau_n}^q)] &\leq \left( \frac{\mu}{1+2\lambda} \mathbb{E} \left[ \int_0^T L_s^q |q_s|^2 \exp\left(\gamma^{-\frac{1}{\lambda}} |q_s|^{\frac{1}{\lambda}}\right) ds \right] + \frac{\mu \bar{k}_{\lambda, \gamma} T}{1+2\lambda} + \tilde{k} \exp\left(\mu (\ln \tilde{k})^{\frac{1}{1+2\lambda}}\right) \right) \\
&\quad \times \exp\left(\frac{2\gamma^2 \mu^{2\lambda+1}}{1+2\lambda} t\right), \quad t \in [0, T].
\end{aligned}$$

Letting  $t = T$  and sending  $n$  to infinity in the last inequality, and then using Fatou's lemma, we obtain that there exists a positive constant  $C_{\mu, \gamma, \lambda, T} > 0$  depending only on  $(\mu, \gamma, \lambda, T)$  such that

$$\mathbb{E} \left[ L_T^q \exp\left(\mu [\ln(1 + L_T^q)]^{\frac{1}{1+2\lambda}}\right) \right] \leq C_{\mu, \gamma, \lambda, T} \mathbb{E} \left[ \int_0^T L_s^q |q_s|^2 \exp\left(\gamma^{-\frac{1}{\lambda}} |q_s|^{\frac{1}{\lambda}}\right) ds \right] + C_{\mu, \gamma, \lambda, T}. \tag{4.21}$$

Observe that for each  $\varepsilon > 0$ ,

$$\lim_{x \rightarrow +\infty} \frac{\varepsilon \exp\left(\gamma^{-\frac{1}{\lambda}} |x|^{\frac{1}{\lambda}}\right)}{C_{\mu, \gamma, \lambda, T} x^2} = +\infty.$$

There exists a positive constant  $\bar{C}_{\varepsilon, \mu, \gamma, \lambda, T} > 0$  depending only on  $(\varepsilon, \mu, \gamma, \lambda, T)$  such that for each  $x \in \mathbb{R}^+$ ,

$$C_{\mu, \gamma, \lambda, T} x^2 \exp\left(\gamma^{-\frac{1}{\lambda}} |x|^{\frac{1}{\lambda}}\right) \leq \varepsilon \exp\left(2\gamma^{-\frac{1}{\lambda}} |x|^{\frac{1}{\lambda}}\right) + \bar{C}_{\varepsilon, \mu, \gamma, \lambda, T}. \quad (4.22)$$

The inequality (4.14) follows immediately by combining (4.21) and (4.22). The proof is completed.  $\square$

**Remark 4.4.** In the above proof, by letting  $q_s \mathbf{1}_{[0, t]}(s) \equiv 0$  and replacing the mathematical expectation with the conditional mathematical expectation with respect to  $\mathcal{F}_t$ , we can also verify that for each  $t \in [0, T]$ ,

$$\mathbb{E} \left[ \frac{L_T^q}{L_t^q} \ln \left( 1 + \frac{L_T^q}{L_t^q} \right) \middle| \mathcal{F}_t \right] \leq \frac{1}{2} \mathbb{E} \left[ \int_t^T \frac{L_s^q}{L_t^q} |q_s|^2 ds \middle| \mathcal{F}_t \right] + \ln 2, \quad (4.23)$$

$$\mathbb{E} \left[ \frac{L_T^q}{L_t^q} \left[ \ln \left( 1 + \frac{L_T^q}{L_t^q} \right) \right]^{\frac{\alpha^*}{2}} \middle| \mathcal{F}_t \right] \leq C_{\alpha, T} \mathbb{E} \left[ \int_t^T \frac{L_s^q}{L_t^q} |q_s|^{\alpha^*} ds \middle| \mathcal{F}_t \right] + C_{\alpha, T} \quad (4.24)$$

and

$$\mathbb{E} \left[ \frac{L_T^q}{L_t^q} \exp \left\{ \mu \left[ \ln \left( 1 + \frac{L_T^q}{L_t^q} \right) \right]^{\frac{1}{1+2\lambda}} \right\} \middle| \mathcal{F}_t \right] \leq \varepsilon \mathbb{E} \left[ \int_t^T \frac{L_s^q}{L_t^q} \exp \left( 2\gamma^{-\frac{1}{\lambda}} |q_s|^{\frac{1}{\lambda}} \right) ds \middle| \mathcal{F}_t \right] + \tilde{C}_{\mu, \varepsilon, \gamma, \lambda, T}. \quad (4.25)$$

We are now in the position to prove Theorem 3.1.

#### 4.1. Proof of Assertion (i) of Theorem 3.1

Assume that the core function  $f$  satisfies (A0) and (A1), and that

$$|\xi| + \int_0^T h_t dt \in \bigcap_{\mu > 0} \exp(\mu L). \quad (4.26)$$

First of all, we show that  $(\bar{q}_t)_{t \in [0, T]} \in \mathcal{H}(\xi, f)$ , and then the space  $\mathcal{H}(\xi, f)$  is nonempty. Indeed, since the process  $(\bar{q}_t)_{t \in [0, T]}$  is bounded, the stochastic exponential

$$L_t^{\bar{q}} := \exp \left( \int_0^t \bar{q}_s \cdot dB_s - \frac{1}{2} \int_0^t |\bar{q}_s|^2 ds \right), \quad t \in [0, T]$$

is a uniformly integrable martingale, which has moments of any order. From (A0) and (A1) we can verify that  $d\mathbb{P} \times dt - a.e.$ ,  $|f(t, \bar{q}_t)| \leq h_t$ , and then by (4.26) and the fact that  $\bigcap_{\mu > 0} \exp(\mu L) \subset L^2$ ,

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}^{\bar{q}}} \left[ |\xi| + \int_0^T (h_s + |f(s, \bar{q}_s)|) ds \right] &\leq 2 \mathbb{E}_{\mathbb{Q}^{\bar{q}}} \left[ |\xi| + \int_0^T h_s ds \right] = 2 \mathbb{E} \left[ \left( |\xi| + \int_0^T h_s ds \right) L_T^{\bar{q}} \right] \\ &\leq 2 \left\{ \mathbb{E} \left[ \left( |\xi| + \int_0^T h_s ds \right)^2 \right] \right\}^{\frac{1}{2}} \left\{ \mathbb{E} \left[ (L_T^{\bar{q}})^2 \right] \right\}^{\frac{1}{2}} < +\infty. \end{aligned} \quad (4.27)$$

Hence,  $(\bar{q}_t)_{t \in [0, T]} \in \mathcal{H}(\xi, f)$ .

Furthermore, for each  $(q_t)_{t \in [0, T]} \in \mathcal{H}(\xi, f)$ , its stochastic exponential

$$L_t^q := \exp \left( \int_0^t q_s \cdot dB_s - \frac{1}{2} \int_0^t |q_s|^2 ds \right), \quad t \in [0, T]$$

is a uniformly integrable martingale. By (A1) and (3.1) we have for each  $t \in [0, T]$ ,

$$\begin{aligned} \frac{1}{2\gamma} \mathbb{E} \left[ \int_t^T \frac{L_s^q}{L_t^q} |q_s|^2 ds \middle| \mathcal{F}_t \right] &= \frac{1}{2\gamma} \mathbb{E} \left[ \left( \int_t^T |q_s|^2 ds \right) \frac{L_T^q}{L_t^q} \middle| \mathcal{F}_t \right] = \frac{1}{2\gamma} \mathbb{E}_{\mathbb{Q}^q} \left[ \int_t^T |q_s|^2 ds \middle| \mathcal{F}_t \right] \\ &\leq \mathbb{E}_{\mathbb{Q}^q} \left[ \int_t^T (h_s + f(s, q_s)) ds \middle| \mathcal{F}_t \right] \\ &\leq \mathbb{E}_{\mathbb{Q}^q} \left[ \int_0^T (h_s + |f(s, q_s)|) ds \middle| \mathcal{F}_t \right] < +\infty. \end{aligned} \quad (4.28)$$

Then, by (A1) and (4.6) with  $\mu = 1/\gamma$  we can deduce that for each  $t \in [0, T]$ ,

$$\begin{aligned} &\mathbb{E}_{\mathbb{Q}^q} \left[ \xi + \int_t^T f(s, q_s) ds \middle| \mathcal{F}_t \right] \\ &\geq -\mathbb{E} \left[ \left( |\xi| + \int_t^T h_s ds \right) \frac{L_T^q}{L_t^q} \middle| \mathcal{F}_t \right] + \frac{1}{2\gamma} \mathbb{E} \left[ \left( \int_t^T |q_s|^2 ds \right) \frac{L_T^q}{L_t^q} \middle| \mathcal{F}_t \right] \\ &\geq -\frac{1}{\gamma} \mathbb{E} \left[ \exp \left\{ \gamma \left( |\xi| + \int_t^T h_s ds \right) \right\} \middle| \mathcal{F}_t \right] - \frac{1}{\gamma} \mathbb{E} \left[ \frac{L_T^q}{L_t^q} \ln \left( 1 + \frac{L_T^q}{L_t^q} \right) \middle| \mathcal{F}_t \right] \\ &\quad + \frac{1}{2\gamma} \mathbb{E} \left[ \int_t^T \frac{L_s^q}{L_t^q} |q_s|^2 ds \middle| \mathcal{F}_t \right]. \end{aligned} \quad (4.29)$$

Combining the last inequality and (4.23), we know that for each  $t \in [0, T]$ ,

$$\mathbb{E}_{\mathbb{Q}^q} \left[ \xi + \int_t^T f(s, q_s) ds \middle| \mathcal{F}_t \right] \geq -\frac{1}{\gamma} \mathbb{E} \left[ \exp \left\{ \gamma \left( |\xi| + \int_t^T h_s ds \right) \right\} \middle| \mathcal{F}_t \right] - \frac{1}{\gamma} \ln 2, \quad (4.30)$$

which means, in view of (4.26), that the process  $U_t(\xi)$  in (3.2) is well defined.

In the sequel, by (A1) we deduce that  $d\mathbb{P} \times dt - a.e.$ , for each  $z \in \mathbb{R}^d$ ,

$$g(\omega, t, z) = \sup_{q \in \mathbb{R}^d} \{ z \cdot q - f(\omega, t, q) \} \leq \sup_{q \in \mathbb{R}^d} \left\{ z \cdot q - \frac{1}{2\gamma} |q|^2 + h_t(\omega) \right\} \leq \frac{\gamma}{2} |z|^2 + h_t(\omega),$$

which together with (3.5) yields that the generator  $g$  defined in (3.4) satisfies assumptions (H0) and (H1) with  $\bar{h}_t = h_t + k^2/(2\gamma)$ . It then follows from (i) of Proposition 2.2 and (4.26) that BSDE (3.3) admits a unique adapted solution  $(Y_t, Z_t)_{t \in [0, T]}$  such that

$$\sup_{t \in [0, T]} |Y_t| \in \bigcap_{\mu > 0} \exp(\mu L). \quad (4.31)$$

To show the dual representation, we need to further verify that  $U_t(\xi) = Y_t$  for each  $t \in [0, T]$ , where  $U_t(\xi)$  is defined in (3.2). We first prove that for each  $(q_t)_{t \in [0, T]} \in \mathcal{H}(\xi, f)$ , it holds that

$$\mathbb{E}_{\mathbb{Q}^q} \left[ \xi + \int_t^T f(s, q_s) ds \middle| \mathcal{F}_t \right] \geq Y_t, \quad t \in [0, T]. \quad (4.32)$$

According to (A0) and (3.4), by the dual representation of a convex function, we know that

$$f(\omega, t, q) = \sup_{z \in \mathbb{R}^d} (z \cdot q - g(\omega, t, z)), \quad \forall (\omega, t, q) \in \Omega \times [0, T] \times \mathbb{R}^d. \quad (4.33)$$

For each  $n \geq 1$  and  $t \in [0, T]$ , define the following stopping time:

$$\tau_n^t := \inf \{ s \geq t : \int_t^s |Z_u|^2 du \geq n \} \wedge T. \quad (4.34)$$

It follows from (3.3) and (4.33) that for each  $n \geq 1$ ,

$$\begin{aligned} Y_t &= Y_{\tau_n^t} - \int_t^{\tau_n^t} g(s, Z_s) ds + \int_t^{\tau_n^t} Z_s \cdot dB_s \\ &= Y_{\tau_n^t} + \int_t^{\tau_n^t} (Z_s \cdot q_s - g(s, Z_s)) ds + \int_t^{\tau_n^t} Z_s \cdot dB_s^q \\ &\leq Y_{\tau_n^t} + \int_t^{\tau_n^t} f(s, q_s) ds + \int_t^{\tau_n^t} Z_s \cdot dB_s^q, \quad t \in [0, T], \end{aligned}$$

where the shifted Brownian motion  $B_t^q := B_t - \int_0^t q_s ds$ ,  $t \in [0, T]$  is a standard  $d$ -dimensional Brownian motion under the new probability measure  $\mathbb{Q}^q$  in view of Girsanov's theorem. Taking the mathematical expectation conditioned on  $\mathcal{F}_t$  with respect to the probability  $\mathbb{Q}^q$  in the last inequality, we obtain that for each  $n \geq 1$ ,

$$Y_t \leq \mathbb{E}_{\mathbb{Q}^q} \left[ Y_{\tau_n^t} + \int_t^{\tau_n^t} f(s, q_s) ds \middle| \mathcal{F}_t \right], \quad t \in [0, T]. \quad (4.35)$$

On the other hand, by (4.12), (4.28) with  $t = 0$  and (3.1), we know that

$$\mathbb{E} [L_T^q \ln(1 + L_T^q)] < +\infty, \quad \mathbb{E}_{\mathbb{Q}^q} \left[ \int_0^T |f(s, q_s)| ds \right] < +\infty,$$

and then, in view of (4.6) with  $\mu = 1$  and (4.31),

$$\mathbb{E}_{\mathbb{Q}^q} \left[ \sup_{t \in [0, T]} |Y_t| \right] = \mathbb{E} \left[ \left( \sup_{t \in [0, T]} |Y_t| \right) L_T^q \right] \leq \mathbb{E} \left[ \exp \left( \sup_{t \in [0, T]} |Y_t| \right) \right] + \mathbb{E} [L_T^q \ln(1 + L_T^q)] < +\infty.$$

Thus, the desired assertion (4.32) follows by sending  $n$  to infinity in (4.35) and applying Lebesgue's dominated convergence theorem.

Next, we set  $\tilde{q}_s \in \partial g(s, Z_s)$  for each  $s \in [0, T]$  and prove that  $(\tilde{q}_t)_{t \in [0, T]} \in \mathcal{H}(\xi, f)$  and

$$Y_t = \mathbb{E}_{\mathbb{Q}^{\tilde{q}}} \left[ \xi + \int_t^T f(s, \tilde{q}_s) ds \middle| \mathcal{F}_t \right], \quad t \in [0, T]. \quad (4.36)$$

Since  $f(s, \tilde{q}_s) = Z_s \cdot \tilde{q}_s - g(s, Z_s)$ , by (A1) we have for each  $s \in [0, T]$ ,

$$g(s, Z_s) \leq Z_s \cdot \tilde{q}_s - \frac{1}{2\gamma} |\tilde{q}_s|^2 + h_s \leq \gamma |Z_s|^2 + \frac{1}{4\gamma} |\tilde{q}_s|^2 - \frac{1}{2\gamma} |\tilde{q}_s|^2 + h_s = \gamma |Z_s|^2 - \frac{1}{4\gamma} |\tilde{q}_s|^2 + h_s,$$

and then

$$\int_0^T |\tilde{q}_s|^2 ds \leq 4\gamma \int_0^T (\gamma |Z_s|^2 + h_s - g(s, Z_s)) ds < +\infty.$$

Moreover, for each  $n \geq 1$ , define the following stopping time:

$$\sigma_n := \inf \{ t \geq 0 : \int_0^t (|Z_s|^2 + |\tilde{q}_s|^2) ds \geq n \} \wedge T. \quad (4.37)$$

and define

$$L_t^{\tilde{q}} := \exp \left( \int_0^t \tilde{q}_s \cdot dB_s - \frac{1}{2} \int_0^t |\tilde{q}_s|^2 ds \right), \quad t \in [0, T] \quad \text{and} \quad \frac{d\mathbb{Q}_n^{\tilde{q}}}{d\mathbb{P}} := L_{\sigma_n}^{\tilde{q}}.$$

Then,  $\mathbb{Q}_n^{\tilde{q}}$  is a probability measure on  $(\Omega, \mathcal{F}_T)$  equivalent to  $\mathbb{P}$  for each  $n \geq 1$ . Letting  $q_s = \tilde{q}_s \mathbf{1}_{[0, \sigma_n]}(s)$  in (4.12) of Proposition 4.3, we obtain that for each  $n \geq 1$ ,

$$\mathbb{E} [L_{\sigma_n}^{\tilde{q}} \ln(1 + L_{\sigma_n}^{\tilde{q}})] \leq \frac{1}{2} \mathbb{E} \left[ \int_0^{\sigma_n} L_t^{\tilde{q}} |\tilde{q}_t|^2 dt \right] + \ln 2. \quad (4.38)$$

Since  $f(s, \tilde{q}_s) = Z_s \cdot \tilde{q}_s - g(s, Z_s)$ , it follows from (3.3) that for each  $n \geq 1$ ,

$$\begin{aligned} Y_0 &= Y_{\sigma_n} - \int_0^{\sigma_n} g(s, Z_s) ds + \int_0^{\sigma_n} Z_s \cdot dB_s \\ &= Y_{\sigma_n} + \int_0^{\sigma_n} (Z_s \cdot \tilde{q}_s - g(s, Z_s)) ds + \int_0^{\sigma_n} Z_s \cdot dB_s^{\tilde{q}} \\ &= Y_{\sigma_n} + \int_0^{\sigma_n} f(s, \tilde{q}_s) ds + \int_0^{\sigma_n} Z_s \cdot dB_s^{\tilde{q}}, \quad t \in [0, T], \end{aligned}$$

where  $B_t^{\tilde{q}} := B_t - \int_0^{t \wedge \sigma_n} \tilde{q}_s ds$ ,  $t \in [0, T]$  is a standard  $d$ -dimensional Brownian motion under the probability measure  $\mathbb{Q}_n^{\tilde{q}}$  via Girsanov's transform. Taking the conditional mathematical expectation with respect to  $\mathcal{F}_t$  under  $\mathbb{Q}_n^{\tilde{q}}$  in the last inequality, we obtain that for each  $n \geq 1$ ,

$$Y_0 = \mathbb{E}_{\mathbb{Q}_n^{\tilde{q}}} \left[ Y_{\sigma_n} + \int_0^{\sigma_n} f(s, \tilde{q}_s) ds \right]. \quad (4.39)$$

On the other hand, by (A1), (4.6) with  $\mu = \frac{1}{2\gamma}$  and (4.38), we deduce that

$$\begin{aligned} &\mathbb{E}_{\mathbb{Q}_n^{\tilde{q}}} \left[ \xi + \int_0^{\sigma_n} f(s, \tilde{q}_s) ds \right] \\ &\geq -\mathbb{E} \left[ \left( |\xi| + \int_0^{\sigma_n} h_s ds \right) L_{\sigma_n}^{\tilde{q}} \right] + \frac{1}{2\gamma} \mathbb{E} \left[ \left( \int_0^{\sigma_n} |\tilde{q}_s|^2 ds \right) L_{\sigma_n}^{\tilde{q}} \right] \\ &\geq -\frac{1}{2\gamma} \mathbb{E} \left[ \exp \left\{ 2\gamma \left( |\xi| + \int_0^{\sigma_n} h_s ds \right) \right\} \right] - \frac{1}{2\gamma} \mathbb{E} \left[ L_{\sigma_n}^{\tilde{q}} \ln(1 + L_{\sigma_n}^{\tilde{q}}) \right] + \frac{1}{2\gamma} \mathbb{E} \left[ \int_0^{\sigma_n} L_s^{\tilde{q}} |\tilde{q}_s|^2 ds \right] \\ &\geq -\frac{1}{2\gamma} \mathbb{E} \left[ \exp \left\{ 2\gamma \left( |\xi| + \int_0^T h_s ds \right) \right\} \right] + \frac{1}{4\gamma} \mathbb{E} \left[ \int_0^{\sigma_n} L_s^{\tilde{q}} |\tilde{q}_s|^2 ds \right] - \frac{\ln 2}{2\gamma}. \end{aligned} \quad (4.40)$$

In view of (4.26), (4.39) and (4.40), there exists a positive constant  $C > 0$  independent of  $n$  such that

$$\sup_{n \geq 1} \mathbb{E} \left[ \int_0^{\sigma_n} L_s^{\tilde{q}} |\tilde{q}_s|^2 ds \right] \leq C$$

and then, in view of (4.38),

$$\sup_{n \geq 1} \mathbb{E} \left[ L_{\sigma_n}^{\tilde{q}} \ln(1 + L_{\sigma_n}^{\tilde{q}}) \right] \leq \frac{C}{2} + \ln 2 < +\infty. \quad (4.41)$$

According to the de La Vallée Poussin lemma and the last inequality, we deduce that the random variable sequence  $(L_{\sigma_n}^{\tilde{q}})_{n=1}^{\infty}$  is uniformly integrable, and then  $\mathbb{E}[L_T^{\tilde{q}}] = 1$ , so  $(L_t^{\tilde{q}})_{t \in [0, T]}$  is a uniformly integrable martingale. Furthermore, by applying Fatou's lemma and (4.41), we obtain

$$\mathbb{E} \left[ L_T^{\tilde{q}} \ln(1 + L_T^{\tilde{q}}) \right] \leq \liminf_{n \rightarrow \infty} \mathbb{E} \left[ L_{\sigma_n}^{\tilde{q}} \ln(1 + L_{\sigma_n}^{\tilde{q}}) \right] < +\infty. \quad (4.42)$$

To show that  $(\tilde{q}_t)_{t \in [0, T]} \in \mathcal{H}(\xi, f)$ , we have to prove that

$$\mathbb{E}_{\mathbb{Q}^{\tilde{q}}} \left[ |\xi| + \int_0^T (h_s + |f(s, \tilde{q}_s)|) ds \right] < +\infty. \quad (4.43)$$

Indeed, by (A1), (4.6) with  $\mu = 1$ , (4.26) and (4.42) we have

$$\begin{aligned} &\mathbb{E}_{\mathbb{Q}^{\tilde{q}}} \left[ |\xi| + \int_0^T (h_s + f^-(s, \tilde{q}_s)) ds \right] \\ &\leq 2\mathbb{E}_{\mathbb{Q}^{\tilde{q}}} \left[ |\xi| + \int_0^T h_s ds \right] \leq 2\mathbb{E} \left[ \exp \left( |\xi| + \int_0^T h_s ds \right) \right] + 2\mathbb{E} \left[ L_T^{\tilde{q}} \ln(1 + L_T^{\tilde{q}}) \right] < +\infty. \end{aligned} \quad (4.44)$$

By a similar computation to (4.39), we deduce that for each  $n \geq 1$ ,

$$Y_t = \mathbb{E}_{\mathbb{Q}^{\bar{q}}} \left[ Y_{\tau_n^t} + \int_t^{\tau_n^t} f(s, \tilde{q}_s) ds \middle| \mathcal{F}_t \right], \quad t \in [0, T]. \quad (4.45)$$

Then, in view of (4.45) with  $t = 0$ , (4.44), (4.6) with  $\mu = 1$ , (4.31) and (4.42),

$$\sup_{n \geq 1} \mathbb{E}_{\mathbb{Q}^{\bar{q}}} \left[ \int_0^{\tau_n^0} f^+(s, \tilde{q}_s) ds \right] \leq \mathbb{E}_{\mathbb{Q}^{\bar{q}}} \left[ \int_0^T f^-(s, \tilde{q}_s) ds \right] + |Y_0| + \mathbb{E}_{\mathbb{Q}^{\bar{q}}} \left[ \sup_{t \in [0, T]} |Y_t| \right] < +\infty,$$

which together with Fatou's lemma and (4.44) yields (4.43). Sending  $n$  to infinity in (4.45) and applying Lebesgue's dominated convergence theorem, we get (4.36).

Finally, note that  $U_t(\xi) = Y_t$  for each  $t \in [0, T]$ . According to the existence and uniqueness of BSDE (3.3) and the related comparison theorem, we easily verify that the operator  $\{U_t(\cdot), t \in [0, T]\}$  defined via (3.2) satisfies (i)-(iv) in the introduction and then constitutes a dynamic concave utility function defined on  $\cap_{\mu > 0} \exp(\mu L)$ . See for example the proof of Theorem 2.16 in [21] for more details. The proof is then complete.

#### 4.2. Proof of Assertion (ii) of Theorem 3.1

Assume that the core function  $f$  satisfies (A0) and (A2), and that

$$|\xi| + \int_0^T h_t dt \in \bigcap_{\mu > 0} \exp(\mu L^{\frac{2}{\alpha^*}}). \quad (4.46)$$

First of all, according to assumptions (A0) and (A2) together with (4.27), (4.46) and the fact that  $\cap_{\mu > 0} \exp(\mu L^{\frac{2}{\alpha^*}}) \subset L^2$ , we can deduce that  $(\bar{q}_t)_{t \in [0, T]} \in \mathcal{H}(\xi, f)$ , and then the space  $\mathcal{H}(\xi, f)$  is nonempty. Furthermore, for each  $(q_t)_{t \in [0, T]} \in \mathcal{H}(\xi, f)$ , its stochastic exponential

$$L_t^q := \exp \left( \int_0^t q_s \cdot dB_s - \frac{1}{2} \int_0^t |q_s|^2 ds \right), \quad t \in [0, T] \quad (4.47)$$

is a uniformly integrable martingale. By (A2) and (3.1) we have for each  $t \in [0, T]$ ,

$$\begin{aligned} \gamma^{-\frac{1}{\alpha-1}} \mathbb{E} \left[ \int_t^T \frac{L_s^q}{L_t^q} |q_s|^{\alpha^*} ds \middle| \mathcal{F}_t \right] &= \gamma^{-\frac{1}{\alpha-1}} \mathbb{E}_{\mathbb{Q}^q} \left[ \int_t^T |q_s|^{\alpha^*} ds \middle| \mathcal{F}_t \right] \\ &\leq \mathbb{E}_{\mathbb{Q}^q} \left[ \int_0^T (h_s + |f(s, q_s)|) ds \middle| \mathcal{F}_t \right] < +\infty. \end{aligned} \quad (4.48)$$

It follows from (4.24) that there exists a positive constant  $C_{\alpha, T}$  depending only on  $(\alpha, T)$  such that

$$\mathbb{E} \left[ \frac{L_T^q}{L_t^q} \left[ \ln \left( 1 + \frac{L_T^q}{L_t^q} \right) \right]^{\frac{\alpha^*}{2}} \middle| \mathcal{F}_t \right] \leq C_{\alpha, T} \mathbb{E} \left[ \int_t^T \frac{L_s^q}{L_t^q} |q_s|^{\alpha^*} ds \middle| \mathcal{F}_t \right] + C_{\alpha, T}. \quad (4.49)$$

Then, applying assumption (A2), inequality (4.4) with

$$(\varepsilon, q, \delta, \mu) = (1, 2, \frac{2}{\alpha^*}, C_{\alpha, T}^{-1} \gamma^{-\frac{1}{\alpha-1}})$$

and inequality (4.49), we conclude that there exists a positive constant  $\bar{C}_{\alpha, \gamma, T}$  depending only on  $(\alpha, \gamma, T)$  such that for each  $t \in [0, T]$ ,

$$\begin{aligned} &\mathbb{E}_{\mathbb{Q}^q} \left[ \xi + \int_t^T f(s, q_s) ds \middle| \mathcal{F}_t \right] \\ &\geq -\mathbb{E} \left[ \left( |\xi| + \int_t^T h_s ds \right) \frac{L_T^q}{L_t^q} \middle| \mathcal{F}_t \right] + \gamma^{-\frac{1}{\alpha-1}} \mathbb{E} \left[ \left( \int_t^T |q_s|^{\alpha^*} ds \right) \frac{L_T^q}{L_t^q} \middle| \mathcal{F}_t \right] \end{aligned}$$

$$\begin{aligned}
&\geq -\mathbb{E} \left[ \exp \left\{ 2(\gamma^{\frac{1}{\alpha-1}} C_{\alpha,T})^{\frac{2}{\alpha^*}} \left( |\xi| + \int_t^T h_s ds \right)^{\frac{2}{\alpha^*}} \right\} \middle| \mathcal{F}_t \right] - \bar{C}_{\alpha,\gamma,T} \\
&\quad - \frac{1}{\gamma^{\frac{1}{\alpha-1}} C_{\alpha,T}} \mathbb{E} \left[ \frac{L_T^q}{L_t^q} \left[ \ln \left( 1 + \frac{L_T^q}{L_t^q} \right) \right]^{\frac{\alpha^*}{2}} \middle| \mathcal{F}_t \right] + \gamma^{-\frac{1}{\alpha-1}} \mathbb{E} \left[ \int_t^T \frac{L_s^q}{L_t^q} |q_s|^{\alpha^*} ds \middle| \mathcal{F}_t \right] \\
&\geq -\mathbb{E} \left[ \exp \left\{ 2(\gamma^{\frac{1}{\alpha-1}} C_{\alpha,T})^{\frac{2}{\alpha^*}} \left( |\xi| + \int_t^T h_s ds \right)^{\frac{2}{\alpha^*}} \right\} \middle| \mathcal{F}_t \right] - \bar{C}_{\alpha,\gamma,T} - \gamma^{-\frac{1}{\alpha-1}},
\end{aligned} \tag{4.50}$$

which means, in view of (4.46), that the process  $U_t(\xi)$  in (3.2) is well defined.

In the sequel, by (A2), (3.4) and (4.8) with  $\mu = \gamma$  and  $\delta = \alpha$  we deduce that  $d\mathbb{P} \times dt - a.e., \forall z \in \mathbb{R}^d$ ,

$$g(\omega, t, z) = \sup_{q \in \mathbb{R}^d} \{z \cdot q - f(\omega, t, q)\} \leq \sup_{q \in \mathbb{R}^d} \left\{ z \cdot q - \gamma^{-\frac{1}{\alpha-1}} |q|^{\alpha^*} + h_t(\omega) \right\} \leq \gamma |z|^\alpha + h_t(\omega),$$

which together with (3.5) and (4.8) with  $\mu = \gamma$  yields that the generator  $g$  defined in (3.4) satisfies assumptions (H0) and (H2) with

$$\bar{h}_t = h_t + \gamma^{-\frac{1}{\alpha-1}} k^{\alpha^*}.$$

It then follows from (ii) of Proposition 2.2 and (4.46) that BSDE (3.3) admits a unique adapted solution  $(Y_t, Z_t)_{t \in [0, T]}$  such that

$$\sup_{t \in [0, T]} |Y_t| \in \bigcap_{\mu > 0} \exp(\mu L^{\frac{2}{\alpha^*}}). \tag{4.51}$$

To show the dual representation, we need to further verify that  $U_t(\xi) = Y_t$  for each  $t \in [0, T]$ , where  $U_t(\xi)$  is defined in (3.2). We first prove that for each  $(q_t)_{t \in [0, T]} \in \mathcal{H}(\xi, f)$ , it holds that

$$\mathbb{E}_{\mathbb{Q}^q} \left[ \xi + \int_t^T f(s, q_s) ds \middle| \mathcal{F}_t \right] \geq Y_t, \quad t \in [0, T]. \tag{4.52}$$

In view of (A0), (3.3) and (3.4), a same computation as that from (4.33)-(4.35) yields that for each  $n \geq 1$ ,

$$Y_t \leq \mathbb{E}_{\mathbb{Q}^q} \left[ Y_{\tau_n^t} + \int_t^{\tau_n^t} f(s, q_s) ds \middle| \mathcal{F}_t \right], \quad t \in [0, T], \tag{4.53}$$

where the stopping time  $\tau_n^t$  is defined in (4.34). On the other hand, by (4.48) and (4.49) with  $t = 0$  together with (3.1) we know that

$$\mathbb{E} \left[ L_T^q \left[ \ln(1 + L_T^q) \right]^{\frac{\alpha^*}{2}} \right] < +\infty, \quad \mathbb{E}_{\mathbb{Q}^q} \left[ \int_0^T |f(s, q_s)| ds \right] < +\infty,$$

and then, in view of (4.51) and (4.4) with  $(\varepsilon, q, \mu, \delta) = (1, 2, 1, \frac{2}{\alpha^*})$ ,

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}^q} \left[ \sup_{t \in [0, T]} |Y_t| \right] &= \mathbb{E} \left[ \left( \sup_{t \in [0, T]} |Y_t| \right) L_T^q \right] \\
&\leq \mathbb{E} \left[ \exp \left\{ 2 \left( \sup_{t \in [0, T]} |Y_t| \right)^{\frac{2}{\alpha^*}} \right\} \right] + \mathbb{E} \left[ L_T^q \left[ \ln(1 + L_T^q) \right]^{\frac{\alpha^*}{2}} \right] + \tilde{C}_\alpha < +\infty,
\end{aligned}$$

where  $\tilde{C}_\alpha$  is a positive constant depending only on  $\alpha$ . Thus, the desired assertion (4.52) follows by sending  $n$  to infinity in (4.53) and applying Lebesgue's dominated convergence theorem.

Next, we set  $\tilde{q}_s \in \partial g(s, Z_s)$  for each  $s \in [0, T]$  and prove that  $(\tilde{q}_t)_{t \in [0, T]} \in \mathcal{H}(\xi, f)$  and

$$Y_t = \mathbb{E}_{\mathbb{Q}^{\tilde{q}}} \left[ \xi + \int_t^T f(s, \tilde{q}_s) ds \middle| \mathcal{F}_t \right], \quad t \in [0, T]. \quad (4.54)$$

Since  $f(s, \tilde{q}_s) = Z_s \cdot \tilde{q}_s - g(s, Z_s)$ , by (A2) and (4.8) with  $\mu = 2^{\alpha-1}\gamma$  and  $\delta = \alpha$  we have for each  $s \in [0, T]$ ,

$$g(s, Z_s) \leq Z_s \cdot \tilde{q}_s - \gamma^{-\frac{1}{\alpha-1}} |\tilde{q}_s|^{\alpha^*} + h_s \leq 2^{\alpha-1}\gamma |Z_s|^\alpha + \frac{1}{2}\gamma^{-\frac{1}{\alpha-1}} |\tilde{q}_s|^{\alpha^*} - \gamma^{-\frac{1}{\alpha-1}} |\tilde{q}_s|^{\alpha^*} + h_s,$$

and then, in view of the basic assumption of  $1 < \alpha < 2 < \alpha^*$ ,

$$\int_0^T |\tilde{q}_s|^2 ds \leq \int_0^T (1 + |\tilde{q}_s|^{\alpha^*}) ds \leq T + 2\gamma^{\frac{1}{\alpha-1}} \int_0^T [2^{\alpha-1}\gamma(1 + |Z_s|^2) + h_s - g(s, Z_s)] ds < +\infty.$$

Moreover, by letting  $q_s = \tilde{q}_s \mathbf{1}_{[0, \sigma_n]}(s)$  in (4.13) of Proposition 4.3, we can conclude that there exists a positive constant  $C_{\alpha, T}$  depending only on  $(\alpha, T)$  such that for each  $n \geq 1$ ,

$$\mathbb{E} \left[ L_{\sigma_n}^{\tilde{q}} [\ln(1 + L_{\sigma_n}^{\tilde{q}})]^{\frac{\alpha^*}{2}} \right] \leq C_{\alpha, T} \mathbb{E} \left[ \int_0^{\sigma_n} L_t^{\tilde{q}} |\tilde{q}_t|^{\alpha^*} dt \right] + C_{\alpha, T}, \quad (4.55)$$

where the stopping time  $\sigma_n$  is defined in (4.37). In view of (3.3) together with the fact that  $f(s, \tilde{q}_s) = Z_s \cdot \tilde{q}_s - g(s, Z_s)$ , by an identical analysis to (4.39) we obtain that for each  $n \geq 1$ ,

$$Y_0 = \mathbb{E}_{\mathbb{Q}^{\tilde{q}}} \left[ Y_{\sigma_n} + \int_0^{\sigma_n} f(s, \tilde{q}_s) ds \right]. \quad (4.56)$$

On the other hand, applying assumption (A2), inequality (4.4) with

$$(\varepsilon, q, \delta, \mu) = (1, 2, \frac{2}{\alpha^*}, \frac{1}{2}C_{\alpha, T}^{-1}\gamma^{-\frac{1}{\alpha-1}})$$

and inequality (4.55), we obtain that there exists a positive constant  $\bar{C}_{\alpha, \gamma, T}$  depending only on  $(\alpha, \gamma, T)$  such that

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}^{\tilde{q}}} \left[ \xi + \int_0^{\sigma_n} f(s, \tilde{q}_s) ds \right] \\ & \geq -\mathbb{E} \left[ \left( |\xi| + \int_0^{\sigma_n} h_s ds \right) L_{\sigma_n}^{\tilde{q}} \right] + \gamma^{-\frac{1}{\alpha-1}} \mathbb{E} \left[ \left( \int_0^{\sigma_n} |\tilde{q}_s|^{\alpha^*} ds \right) L_{\sigma_n}^{\tilde{q}} \right] \\ & \geq -\mathbb{E} \left[ \exp \left\{ 2(2\gamma^{\frac{1}{\alpha-1}} C_{\alpha, T})^{\frac{2}{\alpha^*}} \left( |\xi| + \int_0^{\sigma_n} h_s ds \right)^{\frac{2}{\alpha^*}} \right\} \right] - \bar{C}_{\alpha, \gamma, T} \\ & \quad - \frac{1}{2}\gamma^{-\frac{1}{\alpha-1}} C_{\alpha, T}^{-1} \mathbb{E} \left[ L_{\sigma_n}^{\tilde{q}} [\ln(1 + L_{\sigma_n}^{\tilde{q}})]^{\frac{\alpha^*}{2}} \right] + \gamma^{-\frac{1}{\alpha-1}} \mathbb{E} \left[ \int_0^{\sigma_n} L_s^{\tilde{q}} |q_s|^{\alpha^*} ds \right] \\ & \geq -\mathbb{E} \left[ \exp \left\{ 2(2\gamma^{\frac{1}{\alpha-1}} C_{\alpha, T})^{\frac{2}{\alpha^*}} \left( |\xi| + \int_0^T h_s ds \right)^{\frac{2}{\alpha^*}} \right\} \right] \\ & \quad + \frac{1}{2}\gamma^{-\frac{1}{\alpha-1}} \mathbb{E} \left[ \int_0^{\sigma_n} L_s^{\tilde{q}} |q_s|^{\alpha^*} ds \right] - \bar{C}_{\alpha, \gamma, T} - \frac{1}{2}\gamma^{-\frac{1}{\alpha-1}}. \end{aligned} \quad (4.57)$$

In view of (4.46), (4.56) and (4.57), we conclude that there exists a positive constant  $C$  independent of  $n$  such that

$$\sup_{n \geq 1} \mathbb{E} \left[ \int_0^{\sigma_n} L_s^{\tilde{q}} |\tilde{q}_s|^{\alpha^*} ds \right] \leq C$$

and then, in view of (4.55),

$$\sup_{n \geq 1} \mathbb{E} \left[ L_{\sigma_n}^{\tilde{q}} [\ln(1 + L_{\sigma_n}^{\tilde{q}})]^{\frac{\alpha^*}{2}} \right] \leq C_{\alpha, T}(C + 1) < +\infty. \quad (4.58)$$



From the de La Vallée Poussin lemma, we deduce that the sequence of random variables  $(L_{\sigma_n}^{\tilde{q}})_{n=1}^{\infty}$  is uniformly integrable, and then  $\mathbb{E}[L_T^{\tilde{q}}] = 1$ , so the process  $(L_t^{\tilde{q}})_{t \in [0, T]}$  is a uniformly integrable martingale. Furthermore, by applying Fatou's lemma and (4.58), we obtain

$$\mathbb{E} \left[ L_T^{\tilde{q}} \left[ \ln(1 + L_T^{\tilde{q}}) \right]^{\frac{\alpha^*}{2}} \right] \leq \liminf_{n \rightarrow \infty} \mathbb{E} \left[ L_{\sigma_n}^{\tilde{q}} \left[ \ln(1 + L_{\sigma_n}^{\tilde{q}}) \right]^{\frac{\alpha^*}{2}} \right] < +\infty. \quad (4.59)$$

To show that  $(\tilde{q}_t)_{t \in [0, T]} \in \mathcal{H}(\xi, f)$ , we have to prove that

$$\mathbb{E}_{\mathbb{Q}^{\tilde{q}}} \left[ |\xi| + \int_0^T (h_s + |f(s, \tilde{q}_s)|) ds \right] < +\infty. \quad (4.60)$$

Indeed, by (A2), (4.4) with  $(\varepsilon, q, \mu, \delta) = (1, 2, 1, \frac{2}{\alpha^*})$ , (4.46) and (4.59), we have

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}^{\tilde{q}}} \left[ |\xi| + \int_0^T (h_s + f^-(s, \tilde{q}_s)) ds \right] \\ & \leq 2\mathbb{E}_{\mathbb{Q}^{\tilde{q}}} \left[ |\xi| + \int_0^T h_s ds \right] = 2\mathbb{E} \left[ \left( |\xi| + \int_0^T h_s ds \right) L_T^{\tilde{q}} \right] \\ & < 2\mathbb{E} \left[ \exp \left\{ 2 \left( |\xi| + \int_0^T h_s ds \right)^{\frac{2}{\alpha^*}} \right\} \right] + 2\mathbb{E} \left[ L_T^{\tilde{q}} \left[ \ln(1 + L_T^{\tilde{q}}) \right]^{\frac{\alpha^*}{2}} \right] + 2\tilde{C}_\alpha < +\infty, \end{aligned} \quad (4.61)$$

where  $\tilde{C}_\alpha$  is a positive constant depending only on  $\alpha$ . By a similar computation to (4.56), we deduce that for each  $n \geq 1$ ,

$$Y_t = \mathbb{E}_{\mathbb{Q}^{\tilde{q}}} \left[ Y_{\tau_n^t} + \int_t^{\tau_n^t} f(s, \tilde{q}_s) ds \middle| \mathcal{F}_t \right], \quad t \in [0, T]. \quad (4.62)$$

Then, in view of (4.62) with  $t = 0$ , (4.4) with  $(\varepsilon, q, \mu, \delta) = (1, 2, 1, \frac{2}{\alpha^*})$ , (4.51) and (4.59),

$$\sup_{n \geq 1} \mathbb{E}_{\mathbb{Q}^{\tilde{q}}} \left[ \int_0^{\tau_n^0} f^+(s, \tilde{q}_s) ds \right] \leq \mathbb{E}_{\mathbb{Q}^{\tilde{q}}} \left[ \int_0^T f^-(s, \tilde{q}_s) ds \right] + |Y_0| + \mathbb{E}_{\mathbb{Q}^{\tilde{q}}} \left[ \sup_{t \in [0, T]} |Y_t| \right] < +\infty,$$

which together with Fatou's lemma and (4.61) yields (4.60). Sending  $n$  to infinity in (4.62) and applying Lebesgue's dominated convergence theorem, we get (4.54).

Finally, note that  $U_t(\xi) = Y_t$  for each  $t \in [0, T]$ . According to the existence and uniqueness of BSDE (3.3) and the related comparison theorem, we easily verify that the operator  $\{U_t(\cdot), t \in [0, T]\}$  defined via (3.2) satisfies (i)-(iv) in the introduction and then constitutes a dynamic concave utility function defined on  $\cap_{\mu > 0} \exp(\mu L^{\frac{2}{\alpha^*}})$ . The proof is then complete.

#### 4.3. Proof of Assertion (iii) of Theorem 3.1

Assume that the core function  $f$  satisfies (A0) and (A3), and that there exists a positive constant  $\bar{\mu} > 0$  such that

$$|\xi| + \int_0^T h_t dt \in L \exp[\bar{\mu}(\ln L)^{1+2\lambda}]. \quad (4.63)$$

First of all, according to assumptions (A0) and (A3) together with (4.27), (4.63) and the fact that  $L \exp[\bar{\mu}(\ln L)^{1+2\lambda}] \subset L^2$ , we can deduce that  $(\bar{q}_t)_{t \in [0, T]} \in \mathcal{H}(\xi, f)$ , and then the space  $\mathcal{H}(\xi, f)$  is nonempty. Furthermore, for each  $(q_t)_{t \in [0, T]} \in \mathcal{H}(\xi, f)$ , its stochastic exponential

$$L_t^q := \exp \left( \int_0^t q_s \cdot dB_s - \frac{1}{2} \int_0^t |q_s|^2 ds \right), \quad t \in [0, T] \quad (4.64)$$

is a uniformly integrable martingale. By (A3) and (3.1), we have for each  $t \in [0, T]$ ,

$$\begin{aligned} c\mathbb{E} \left[ \int_t^T \frac{L_s^q}{L_t^q} \exp \left( 2\gamma^{-\frac{1}{\lambda}} |q_s|^{\frac{1}{\lambda}} \right) ds \middle| \mathcal{F}_t \right] &= c\mathbb{E}_{\mathbb{Q}^q} \left[ \int_t^T \exp \left( 2\gamma^{-\frac{1}{\lambda}} |q_s|^{\frac{1}{\lambda}} \right) ds \middle| \mathcal{F}_t \right] \\ &\leq c\mathbb{E}_{\mathbb{Q}^q} \left[ \int_0^T (h_s + |f(s, q_s)|) ds \middle| \mathcal{F}_t \right] < +\infty. \end{aligned} \quad (4.65)$$

It follows from (4.25) with  $\mu = \bar{\mu}^{-\frac{1}{1+2\lambda}}$  that for each  $\varepsilon > 0$ , there exists a positive constant  $\tilde{C}_{\varepsilon, \bar{\mu}, \gamma, \lambda, T}$  depending only on  $(\varepsilon, \bar{\mu}, \gamma, \lambda, T)$  such that for each  $t \in [0, T]$ ,

$$\begin{aligned} &\mathbb{E} \left[ \frac{L_T^q}{L_t^q} \exp \left\{ \bar{\mu}^{-\frac{1}{1+2\lambda}} \left[ \ln \left( 1 + \frac{L_T^q}{L_t^q} \right) \right]^{\frac{1}{1+2\lambda}} \right\} \middle| \mathcal{F}_t \right] \\ &\leq \varepsilon \mathbb{E} \left[ \int_t^T \frac{L_s^q}{L_t^q} \exp \left( 2\gamma^{-\frac{1}{\lambda}} |q_s|^{\frac{1}{\lambda}} \right) ds \middle| \mathcal{F}_t \right] + \tilde{C}_{\varepsilon, \bar{\mu}, \gamma, \lambda, T}. \end{aligned} \quad (4.66)$$

Then, applying assumption (A3), inequality (4.3) with  $\mu = \bar{\mu}$  and  $\delta = \frac{1}{1+2\lambda}$ , and inequality (4.66) with  $\varepsilon = c$ , we conclude that for each  $t \in [0, T]$ ,

$$\begin{aligned} &\mathbb{E}_{\mathbb{Q}^q} \left[ \xi + \int_t^T f(s, q_s) ds \middle| \mathcal{F}_t \right] \\ &\geq -\mathbb{E} \left[ \left( |\xi| + \int_t^T h_s ds \right) \frac{L_T^q}{L_t^q} \middle| \mathcal{F}_t \right] + c\mathbb{E} \left[ \left( \int_t^T \exp \left( 2\gamma^{-\frac{1}{\lambda}} |q_s|^{\frac{1}{\lambda}} \right) ds \right) \frac{L_T^q}{L_t^q} \middle| \mathcal{F}_t \right] \\ &\geq -\mathbb{E} \left[ \left( |\xi| + \int_t^T h_s ds \right) \exp \left\{ \bar{\mu} \left[ \ln \left( 1 + |\xi| + \int_t^T h_s ds \right) \right]^{1+2\lambda} \right\} \middle| \mathcal{F}_t \right] \\ &\quad - \mathbb{E} \left[ \frac{L_T^q}{L_t^q} \exp \left\{ \left[ \bar{\mu}^{-\frac{1}{1+2\lambda}} \ln \left( 1 + \frac{L_T^q}{L_t^q} \right) \right]^{\frac{1}{1+2\lambda}} \right\} \middle| \mathcal{F}_t \right] + c\mathbb{E} \left[ \int_t^T \frac{L_s^q}{L_t^q} \exp \left( 2\gamma^{-\frac{1}{\lambda}} |q_s|^{\frac{1}{\lambda}} \right) ds \middle| \mathcal{F}_t \right] \\ &\geq -\mathbb{E} \left[ \left( |\xi| + \int_t^T h_s ds \right) \exp \left\{ \bar{\mu} \left[ \ln \left( 1 + |\xi| + \int_t^T h_s ds \right) \right]^{1+2\lambda} \right\} \middle| \mathcal{F}_t \right] - \tilde{C}_{c, \bar{\mu}, \gamma, \lambda, T}, \end{aligned} \quad (4.67)$$

which means, in view of (4.63), that the process  $U_t(\xi)$  in (3.2) is well defined.

In the sequel, by (A3), (3.4) and (4.4) with  $(\varepsilon, q, \mu, \delta) = (c, 2, \gamma, \frac{1}{\lambda})$  we can deduce that there exists a constant  $C_{c, \gamma, \lambda} > 0$  depending only on  $(c, \gamma, \lambda)$  such that  $d\mathbb{P} \times dt - a.e.$ , for each  $z \in \mathbb{R}^d$ ,

$$\begin{aligned} g(\omega, t, z) &= \sup_{q \in \mathbb{R}^d} \{ z \cdot q - f(\omega, t, q) \} \leq \sup_{q \in \mathbb{R}^d} \left\{ z \cdot q - c \exp \left( 2\gamma^{-\frac{1}{\lambda}} |q|^{\frac{1}{\lambda}} \right) + h_t(\omega) \right\} \\ &\leq \gamma |z| (\ln(1 + |z|))^\lambda + h_t(\omega) + C_{c, \gamma, \lambda}, \end{aligned}$$

which together with (3.5) and (4.4) with  $(\varepsilon, q, \mu, \delta) = (1, 2, \gamma, \frac{1}{\lambda})$  yields that the generator  $g$  defined in (3.4) satisfies assumptions (H0) and (H3) with

$$\bar{h}_t = h_t + \exp \left( 2\gamma^{-\frac{1}{\lambda}} |k|^{\frac{1}{\lambda}} \right) + C_{c, \gamma, \lambda}.$$

It then follows from (iii) of Proposition 2.2 and (4.63) that BSDE (3.3) admits a unique adapted solution  $(Y_t, Z_t)_{t \in [0, T]}$  such that for some  $\tilde{\mu} > 0$ ,

$$\sup_{t \in [0, T]} |Y_t| \in L \exp[\tilde{\mu} (\ln L)^{1+2\lambda}]. \quad (4.68)$$

To show the dual representation, we need to further verify that  $U_t(\xi) = Y_t$  for each  $t \in [0, T]$ , where  $U_t(\xi)$  is defined in (3.2). We first prove that for each  $(q_t)_{t \in [0, T]} \in \mathcal{H}(\xi, f)$ , it holds that

$$\mathbb{E}_{\mathbb{Q}^q} \left[ \xi + \int_t^T f(s, q_s) ds \middle| \mathcal{F}_t \right] \geq Y_t, \quad t \in [0, T]. \quad (4.69)$$

In view of (A0), (3.3) and (3.4), a same computation as that from (4.33)-(4.35) yields that for each  $n \geq 1$ ,

$$Y_t \leq \mathbb{E}_{\mathbb{Q}^q} \left[ Y_{\tau_n^t} + \int_t^{\tau_n^t} f(s, q_s) ds \middle| \mathcal{F}_t \right], \quad t \in [0, T], \quad (4.70)$$

where the stopping time  $\tau_n^t$  is defined in (4.34). On the other hand, by (4.65) and (4.66) with  $t = 0$  together with (3.1), we know that

$$\mathbb{E} \left[ L_T^q \exp \left\{ \tilde{\mu}^{-\frac{1}{1+2\lambda}} [\ln(1 + L_T^q)]^{\frac{1}{1+2\lambda}} \right\} \right] < +\infty, \quad \mathbb{E}_{\mathbb{Q}^q} \left[ \int_0^T |f(s, q_s)| ds \right] < +\infty,$$

and then, in view of (4.68) and (4.3) with  $\mu = \tilde{\mu}$  and  $\delta = \frac{1}{1+2\lambda}$ ,

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}^q} \left[ \sup_{t \in [0, T]} |Y_t| \right] &= \mathbb{E} \left[ \left( \sup_{t \in [0, T]} |Y_t| \right) L_T^q \right] \\ &\leq \mathbb{E} \left[ L_T^q \exp \left\{ \tilde{\mu}^{-\frac{1}{1+2\lambda}} [\ln(1 + L_T^q)]^{\frac{1}{1+2\lambda}} \right\} \right] \\ &\quad + \mathbb{E} \left[ \left( \sup_{t \in [0, T]} |Y_t| \right) \exp \left\{ \tilde{\mu} \left[ \ln \left( 1 + \sup_{t \in [0, T]} |Y_t| \right) \right]^{1+2\lambda} \right\} \right] < +\infty. \end{aligned}$$

Thus, the desired assertion (4.69) follows by sending  $n$  to infinity in (4.70) and applying Lebesgue's dominated convergence theorem.

Next, we set  $\tilde{q}_s \in \partial g(s, Z_s)$  for each  $s \in [0, T]$  and prove that  $(\tilde{q}_t)_{t \in [0, T]} \in \mathcal{H}(\xi, f)$  and

$$Y_t = \mathbb{E}_{\mathbb{Q}^{\tilde{q}}} \left[ \xi + \int_t^T f(s, \tilde{q}_s) ds \middle| \mathcal{F}_t \right], \quad t \in [0, T]. \quad (4.71)$$

Since  $f(s, \tilde{q}_s) = Z_s \cdot \tilde{q}_s - g(s, Z_s)$ , by (A3) and (4.4) with  $(\varepsilon, q, \mu, \delta) = (\frac{c}{2}, 2, \gamma, \frac{1}{\lambda})$  we deduce that there exists a constant  $\tilde{C}_{c, \gamma, \lambda} > 0$  depending only on  $(c, \gamma, \lambda)$  such that for each  $s \in [0, T]$ ,

$$g(s, Z_s) \leq Z_s \cdot \tilde{q}_s - c \exp \left( 2\gamma^{-\frac{1}{\lambda}} |\tilde{q}_s|^{\frac{1}{\lambda}} \right) + h_s \leq \gamma |Z_s| (\ln(1 + |Z_s|))^\lambda - \frac{c}{2} \exp \left( 2\gamma^{-\frac{1}{\lambda}} |\tilde{q}_s|^{\frac{1}{\lambda}} \right) + h_s + \tilde{C}_{c, \gamma, \lambda},$$

and then, in view of the fact that  $x^2 \leq \frac{c}{2} \exp \left( 2\gamma^{-\frac{1}{\lambda}} x^{\frac{1}{\lambda}} \right) + \tilde{C}_{c, \gamma, \lambda}$  for each  $x \in \mathbb{R}_+$  and some constant  $\tilde{C}_{c, \gamma, \lambda} > 0$  depending only on  $(c, \gamma, \lambda)$ , and the fact that  $x(\ln(1 + x))^\lambda \leq K_\lambda + x^2$  for each  $x \in \mathbb{R}_+$  and some constant  $K_\lambda > 0$  depending only on  $\lambda$ ,

$$\begin{aligned} \int_0^T |\tilde{q}_s|^2 ds &\leq \frac{c}{2} \int_0^T \exp \left( 2\gamma^{-\frac{1}{\lambda}} |\tilde{q}_s|^{\frac{1}{\lambda}} \right) ds + T \tilde{C}_{c, \gamma, \lambda} \\ &\leq \int_0^T [\gamma(K_\lambda + |Z_s|^2) + h_s - g(s, Z_s)] ds + T \tilde{C}_{c, \gamma, \lambda} < +\infty. \end{aligned}$$

Moreover, letting  $q_s = \tilde{q}_s \mathbf{1}_{[0, \sigma_n]}(s)$  in (4.14) of Proposition 4.3, we conclude that for each  $\varepsilon > 0$ , there exists a positive constant  $C_{\varepsilon, \tilde{\mu}, \gamma, \lambda, T}$  depending only on  $(\varepsilon, \tilde{\mu}, \gamma, \lambda, T)$  such that for each  $n \geq 1$ ,

$$\mathbb{E} \left[ L_{\sigma_n}^{\tilde{q}} \exp \left( \tilde{\mu}^{-\frac{1}{1+2\lambda}} [\ln(1 + L_{\sigma_n}^{\tilde{q}})]^{\frac{1}{1+2\lambda}} \right) \right] \leq \varepsilon \mathbb{E} \left[ \int_0^{\sigma_n} L_s^{\tilde{q}} \exp \left( 2\gamma^{-\frac{1}{\lambda}} |\tilde{q}_s|^{\frac{1}{\lambda}} \right) ds \right] + C_{\varepsilon, \tilde{\mu}, \gamma, \lambda, T}, \quad (4.72)$$

where the stopping time  $\sigma_n$  is defined in (4.37). In view of (3.3) together with the fact that  $f(s, \tilde{q}_s) = Z_s \cdot \tilde{q}_s - g(s, Z_s)$ , by an identical analysis to (4.39) we obtain that for each  $n \geq 1$ ,

$$Y_0 = \mathbb{E}_{\mathbb{Q}^{\tilde{q}}} \left[ Y_{\sigma_n} + \int_0^{\sigma_n} f(s, \tilde{q}_s) ds \right]. \quad (4.73)$$

On the other hand, by applying assumption (A3), inequality (4.3) with  $\mu = \bar{\mu}$  and  $\delta = \frac{1}{1+2\lambda}$ , and inequality (4.72) with  $\varepsilon = \frac{\varepsilon}{2}$ , we can conclude that

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}^{\tilde{q}}} \left[ \xi + \int_0^{\sigma_n} f(s, \tilde{q}_s) ds \right] \\ & \geq -\mathbb{E} \left[ \left( |\xi| + \int_0^{\sigma_n} h_s ds \right) L_{\sigma_n}^{\tilde{q}} \right] + c \mathbb{E} \left[ \left( \int_0^{\sigma_n} \exp \left( 2\gamma^{-\frac{1}{\lambda}} |\tilde{q}_s|^{\frac{1}{\lambda}} \right) ds \right) L_{\sigma_n}^{\tilde{q}} \right] \\ & \geq -\mathbb{E} \left[ \left( |\xi| + \int_0^{\sigma_n} h_s ds \right) \exp \left\{ \bar{\mu} \left[ \ln \left( 1 + \int_0^{\sigma_n} h_s ds \right) \right]^{1+2\lambda} \right\} \right] \\ & \quad - \mathbb{E} \left[ L_{\sigma_n}^{\tilde{q}} \exp \left\{ \left[ \bar{\mu}^{-\frac{1}{1+2\lambda}} \ln \left( 1 + L_{\sigma_n}^{\tilde{q}} \right) \right]^{\frac{1}{1+2\lambda}} \right\} \right] + c \mathbb{E} \left[ \int_0^{\sigma_n} L_s^{\tilde{q}} \exp \left( 2\gamma^{-\frac{1}{\lambda}} |\tilde{q}_s|^{\frac{1}{\lambda}} \right) ds \right] \\ & \geq -\mathbb{E} \left[ \left( |\xi| + \int_0^T h_s ds \right) \exp \left\{ \bar{\mu} \left[ \ln \left( 1 + \int_0^T h_s ds \right) \right]^{1+2\lambda} \right\} \right] \\ & \quad + \frac{c}{2} \mathbb{E} \left[ \int_0^{\sigma_n} L_s^{\tilde{q}} \exp \left( 2\gamma^{-\frac{1}{\lambda}} |\tilde{q}_s|^{\frac{1}{\lambda}} \right) ds \right] - C_{c/2, \bar{\mu}, \gamma, \lambda, T}. \end{aligned} \quad (4.74)$$

In view of (4.63), (4.73) and (4.74), we can conclude that there exists a positive constant  $C > 0$  independent of  $n$  such that

$$\sup_{n \geq 1} \mathbb{E} \left[ \int_0^{\sigma_n} L_s^{\tilde{q}} \exp \left( 2\gamma^{-\frac{1}{\lambda}} |\tilde{q}_s|^{\frac{1}{\lambda}} \right) ds \right] \leq C$$

and then, in view of (4.72),

$$\sup_{n \geq 1} \mathbb{E} \left[ L_{\sigma_n}^{\tilde{q}} \exp \left( \bar{\mu}^{-\frac{1}{1+2\lambda}} \left[ \ln \left( 1 + L_{\sigma_n}^{\tilde{q}} \right) \right]^{\frac{1}{1+2\lambda}} \right) \right] < +\infty. \quad (4.75)$$

According to the de La Vallée Poussin lemma and the last inequality, we deduce that the random variable sequence  $(L_{\sigma_n}^{\tilde{q}})_{n=1}^{\infty}$  is uniformly integrable, and then  $\mathbb{E}[L_T^{\tilde{q}}] = 1$ , so  $(L_t^{\tilde{q}})_{t \in [0, T]}$  is a uniformly integrable martingale. Furthermore, applying Fatou's lemma and (4.75), we obtain

$$\begin{aligned} & \mathbb{E} \left[ L_T^{\tilde{q}} \exp \left( \bar{\mu}^{-\frac{1}{1+2\lambda}} \left[ \ln \left( 1 + L_T^{\tilde{q}} \right) \right]^{\frac{1}{1+2\lambda}} \right) \right] \\ & \leq \liminf_{n \rightarrow \infty} \mathbb{E} \left[ L_{\sigma_n}^{\tilde{q}} \exp \left( \bar{\mu}^{-\frac{1}{1+2\lambda}} \left[ \ln \left( 1 + L_{\sigma_n}^{\tilde{q}} \right) \right]^{\frac{1}{1+2\lambda}} \right) \right] < +\infty. \end{aligned} \quad (4.76)$$

To show that  $(\tilde{q}_t)_{t \in [0, T]} \in \mathcal{H}(\xi, f)$ , we have to prove that

$$\mathbb{E}_{\mathbb{Q}^{\tilde{q}}} \left[ |\xi| + \int_0^T (h_s + |f(s, \tilde{q}_s)|) ds \right] < +\infty. \quad (4.77)$$

Indeed, by (A3), (4.3) with  $\mu = \bar{\mu}$  and  $\delta = \frac{1}{1+2\lambda}$ , (4.63) and (4.76), we have

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}^{\tilde{q}}} \left[ |\xi| + \int_0^T (h_s + f^-(s, \tilde{q}_s)) ds \right] \\ & \leq 2\mathbb{E}_{\mathbb{Q}^{\tilde{q}}} \left[ |\xi| + \int_0^T h_s ds \right] = 2\mathbb{E} \left[ \left( |\xi| + \int_0^T h_s ds \right) L_T^{\tilde{q}} \right] \end{aligned}$$

$$\begin{aligned}
&\leq \mathbb{E} \left[ L_T^{\tilde{q}} \exp \left( \bar{\mu}^{-\frac{1}{1+2\lambda}} \left[ \ln(1 + L_T^{\tilde{q}}) \right]^{\frac{1}{1+2\lambda}} \right) \right] \\
&\quad + \mathbb{E} \left[ \left( |\xi| + \int_0^{\sigma_n} h_s ds \right) \exp \left\{ \bar{\mu} \left[ \ln \left( 1 + |\xi| + \int_0^{\sigma_n} h_s ds \right) \right]^{1+2\lambda} \right\} \right] < +\infty.
\end{aligned} \tag{4.78}$$

By a similar computation to (4.56), we deduce that for each  $n \geq 1$ ,

$$Y_t = \mathbb{E}_{\mathbb{Q}^{\tilde{q}}} \left[ Y_{\tau_n^t} + \int_t^{\tau_n^t} f(s, \tilde{q}_s) ds \middle| \mathcal{F}_t \right], \quad t \in [0, T]. \tag{4.79}$$

Then, in view of (4.79) with  $t = 0$ , (4.3) with  $\mu = \tilde{\mu}$  and  $\delta = \frac{1}{1+2\lambda}$ , (4.68) and (4.76),

$$\sup_{n \geq 1} \mathbb{E}_{\mathbb{Q}^{\tilde{q}}} \left[ \int_0^{\tau_n^0} f^+(s, \tilde{q}_s) ds \right] \leq \mathbb{E}_{\mathbb{Q}^{\tilde{q}}} \left[ \int_0^T f^-(s, \tilde{q}_s) ds \right] + |Y_0| + \mathbb{E}_{\mathbb{Q}^{\tilde{q}}} \left[ \sup_{t \in [0, T]} |Y_t| \right] < +\infty,$$

which together with Fatou's lemma and (4.78) yields (4.77). Sending  $n$  to infinity in (4.79) and applying Lebesgue's dominated convergence theorem, we get (4.71).

Finally, note that  $U_t(\xi) = Y_t$  for each  $t \in [0, T]$ . According to the existence and uniqueness of BSDE (3.3) and the related comparison theorem, we can easily verify that the operator  $\{U_t(\cdot), t \in [0, T]\}$  defined via (3.2) satisfies (i)-(iv) in the introduction and then constitutes a dynamic concave utility function defined on  $\cup_{\mu > 0} L \exp[\mu(\ln L)^{1+2\lambda}]$ . The proof is then complete.

#### 4.4. Proof of Assertion (iv) of Theorem 3.1

Assume that the core function  $f$  satisfies (A0) with  $k \leq \gamma$  and (A4), and that

$$|\xi| + \int_0^T h_t dt \in \bigcap_{\bar{\mu} > 0} L \exp(\bar{\mu}(\ln L)^{\frac{1}{2}}). \tag{4.80}$$

Given a constant  $\mu > \mu_0 := \gamma\sqrt{2T}$ . First of all, in view of (A4), it is clear from (3.2) that

$$U_t(\xi) := \operatorname{ess\,inf}_{q \in \mathcal{H}(\xi, f)} \mathbb{E}_{\mathbb{Q}^q} \left[ \xi + \int_t^T f(s, q_s) ds \middle| \mathcal{F}_t \right], \quad t \in [0, T], \tag{4.81}$$

where

$$\begin{aligned}
\bar{\mathcal{H}}(\xi, f) := & \left\{ (\mathcal{F}_t)\text{-progressively measurable } \mathbb{R}^d\text{-valued process } (q_t)_{t \in [0, T]} : \right. \\
& d\mathbb{P} \times dt - a.e., |q_t| \leq \gamma \text{ and } \mathbb{E}_{\mathbb{Q}^q} \left[ |\xi| + \int_0^T (h_s + |f(s, q_s)|) ds \right] < +\infty \\
& \left. \text{with } L_t^q := \exp \left( \int_0^t q_s \cdot dB_s - \frac{1}{2} \int_0^t |q_s|^2 ds \right), t \in [0, T] \text{ and } \frac{d\mathbb{Q}^q}{d\mathbb{P}} = L_T^q \right\}.
\end{aligned}$$

Moreover, we see from [27, Lemma 2.6] that for any  $(\mathcal{F}_t)$ -progressively measurable  $\mathbb{R}^d$ -valued process  $(q_t)_{t \in [0, T]}$  such that  $|q_t| \leq \gamma$ ,  $d\mathbb{P} \times dt - a.e.$ , and for each  $\bar{\mu} > \mu_0$ , we have

$$\mathbb{E} \left[ \exp \left( \frac{1}{\bar{\mu}^2} \left| \int_t^T q_s \cdot dB_s \right|^2 \right) \middle| \mathcal{F}_t \right] \leq \frac{1}{\sqrt{1 - \frac{2\gamma^2}{\bar{\mu}^2}(T-t)}} < +\infty, \quad t \in [0, T]. \tag{4.82}$$

Next, we show that  $(\bar{q}_t)_{t \in [0, T]} \in \bar{\mathcal{H}}(\xi, f)$ , and then the space  $\bar{\mathcal{H}}(\xi, f)$  is nonempty. Indeed, from (A0) with  $k \leq \gamma$  and (A4) we can verify that  $d\mathbb{P} \times dt - a.e.$ ,  $|\bar{q}_t| \leq \gamma$  and  $|f(t, \bar{q}_t)| \leq h_t$ , and then according

to (4.7) with some constant  $q := p \in (1, \frac{\mu}{\mu_0})$  and (4.82) with  $\bar{\mu} = \frac{\mu}{p} > \mu_0$  and  $t = 0$  together with (4.80), we deduce that there exists a constant  $C_{\mu,p} > 0$  depending only on  $(\mu, p)$  such that

$$\begin{aligned}
& \mathbb{E}_{\mathbb{Q}^{\bar{q}}} \left[ |\xi| + \int_0^T (h_s + |f(s, \bar{q}_s)|) ds \right] \leq 2 \mathbb{E}_{\mathbb{Q}^{\bar{q}}} \left[ |\xi| + \int_0^T h_s ds \right] \\
& = 2 \mathbb{E} \left[ \left( |\xi| + \int_0^T h_s ds \right) L_T^{\bar{q}} \right] \leq 2 \mathbb{E} \left[ \left( |\xi| + \int_0^T h_s ds \right) \exp \left( \int_0^T \bar{q}_s \cdot dB_s \right) \right] \\
& \leq 2C_{\mu,p} \mathbb{E} \left[ \exp \left( \frac{p}{\mu^2} \left| \int_0^T \bar{q}_s \cdot dB_s \right|^2 \right) \right] \\
& \quad + 2 \mathbb{E} \left[ \left( |\xi| + \int_0^T h_s ds \right) \exp \left\{ \mu \sqrt{\ln \left( 1 + |\xi| + \int_0^T h_s ds \right)} \right\} \right] < +\infty.
\end{aligned} \tag{4.83}$$

Hence,  $(\bar{q}_t)_{t \in [0, T]} \in \bar{\mathcal{H}}(\xi, f)$ . Furthermore, for each  $(q_t)_{t \in [0, T]} \in \bar{\mathcal{H}}(\xi, f)$ , according to assumption (A4), inequality (4.7) with some constant  $q := p \in (1, \frac{\mu}{\mu_0})$  and inequality (4.82) with  $\bar{\mu} = \frac{\mu}{p} > \mu_0$ , we deduce that there exists a constant  $C_{\mu,p} > 0$  depending only on  $(\mu, p)$  such that for each  $t \in [0, T]$ ,

$$\begin{aligned}
& \mathbb{E}_{\mathbb{Q}^q} \left[ \xi + \int_t^T f(s, q_s) ds \middle| \mathcal{F}_t \right] \geq - \mathbb{E}_{\mathbb{Q}^q} \left[ |\xi| + \int_t^T h_s ds \middle| \mathcal{F}_t \right] \\
& = - \mathbb{E} \left[ \left( |\xi| + \int_t^T h_s ds \right) \frac{L_T^q}{L_t^q} \right] \geq - \mathbb{E} \left[ \left( |\xi| + \int_t^T h_s ds \right) \exp \left( \int_t^T q_s \cdot dB_s \right) \middle| \mathcal{F}_t \right] \\
& \geq -C_{\mu,p} \mathbb{E} \left[ \exp \left( \frac{p}{\mu^2} \left| \int_t^T q_s \cdot dB_s \right|^2 \right) \middle| \mathcal{F}_t \right] \\
& \quad - \mathbb{E} \left[ \left( |\xi| + \int_t^T h_s ds \right) \exp \left\{ \mu \sqrt{\ln \left( 1 + |\xi| + \int_t^T h_s ds \right)} \right\} \middle| \mathcal{F}_t \right] \\
& \geq - \mathbb{E} \left[ \left( |\xi| + \int_t^T h_s ds \right) \exp \left\{ \mu \sqrt{\ln \left( 1 + |\xi| + \int_t^T h_s ds \right)} \right\} \middle| \mathcal{F}_t \right] - \frac{C_{\mu,p}}{\sqrt{1 - \frac{2p^2\gamma^2}{\mu^2}(T-t)}}.
\end{aligned} \tag{4.84}$$

which means, in view of (4.80), that the process  $U_t(\xi)$  in (4.81) and (3.2) is well defined.

In the sequel, by (3.4) and (A4) we can deduce that  $d\mathbb{P} \times dt - a.e.$ , for each  $z \in \mathbb{R}^d$ ,

$$g(\omega, t, z) = \sup_{q \in \mathbb{R}^d} \{z \cdot q - f(\omega, t, q)\} \leq \sup_{q \in \mathbb{R}^d, |q| \leq \gamma} \{z \cdot q + h_t(\omega)\} = \gamma|z| + h_t(\omega),$$

which together with (3.5) and the condition of  $k \leq \gamma$  yields that the generator  $g$  defined in (3.4) satisfies assumptions (H0) and (H4) with  $\bar{h}_t = h_t$ . It then follows from (iv) of Proposition 2.2 and (4.80) that BSDE (3.3) admits a unique adapted solution  $(Y_t, Z_t)_{t \in [0, T]}$  such that for each  $\bar{\mu} > 0$ ,

$$\text{the process } \left( |Y_t| \exp \left( \bar{\mu} \sqrt{\ln(1 + |Y_t|)} \right) \right)_{t \in [0, T]} \text{ belongs to class (D)}. \tag{4.85}$$

To show the dual representation, we need to further verify that  $U_t(\xi) = Y_t$  for each  $t \in [0, T]$ , where  $U_t(\xi)$  is defined in (4.81). We first prove that for each  $(q_t)_{t \in [0, T]} \in \bar{\mathcal{H}}(\xi, f)$ , it holds that

$$\mathbb{E}_{\mathbb{Q}^q} \left[ \xi + \int_t^T f(s, q_s) ds \middle| \mathcal{F}_t \right] \geq Y_t, \quad t \in [0, T]. \tag{4.86}$$

In view of (A0), (3.3) and (3.4), a same computation as that from (4.33)-(4.35) yields that for each  $n \geq 1$ ,

$$\begin{aligned} Y_t &\leq \mathbb{E}_{\mathbb{Q}^q} \left[ Y_{\tau_n^t} + \int_t^{\tau_n^t} f(s, q_s) ds \middle| \mathcal{F}_t \right] \\ &= \mathbb{E} \left[ Y_{\tau_n^t} \frac{L_{\tau_n^t}^q}{L_t^q} \middle| \mathcal{F}_t \right] + \mathbb{E}_{\mathbb{Q}^q} \left[ \int_t^{\tau_n^t} f(s, q_s) ds \middle| \mathcal{F}_t \right], \quad t \in [0, T], \end{aligned} \quad (4.87)$$

where the stopping time  $\tau_n^t$  is defined in (4.34). According to (4.7) with some constant  $q := p \in (1, \frac{\mu}{\mu_0})$  and (4.82) with  $\bar{\mu} = \frac{\mu}{p} > \mu_0$ , we can deduce that there exists a constant  $C_{\mu, p} > 0$  depending only on  $(\mu, p)$  such that for each  $n \geq 1$  and  $t \in [0, T]$ ,

$$\begin{aligned} Y_{\tau_n^t} \frac{L_{\tau_n^t}^q}{L_t^q} &\leq |Y_{\tau_n^t}| \exp \left( \int_t^{\tau_n^t} q_s \cdot dB_s \right) \\ &\leq C_{\mu, p} \exp \left( \frac{p}{\mu^2} \left| \int_t^{\tau_n^t} q_s \cdot dB_s \right|^2 \right) + |Y_{\tau_n^t}| \exp \left\{ \mu \sqrt{\ln(1 + |Y_{\tau_n^t}|)} \right\} \end{aligned}$$

and

$$\begin{aligned} \mathbb{E} \left[ \exp \left( \frac{p}{\mu^2} \left| \int_t^{\tau_n^t} q_s \cdot dB_s \right|^2 \right) \right] &= \mathbb{E} \left[ \exp \left( \frac{p^2}{\mu^2} \left| \int_t^{\tau_n^t} q_s \cdot dB_s \right|^2 \right) \right] \\ &\leq \frac{1}{\sqrt{1 - \frac{2p^2\gamma^2}{\mu^2}(T-t)}} < +\infty, \end{aligned}$$

which together with (4.85) yield that for each  $t \in [0, T]$ , the sequence of random variables  $Y_{\tau_n^t} \frac{L_{\tau_n^t}^q}{L_t^q}$  are uniformly integrable. On the other hand, since  $(q_t)_{t \in [0, T]} \in \bar{\mathcal{H}}(\xi, f)$ , we have

$$\mathbb{E}_{\mathbb{Q}^q} \left[ \int_0^T |f(s, q_s)| ds \right] < +\infty.$$

Thus, the desired assertion (4.86) follows immediately by sending  $n$  to infinity in (4.87).

Next, we set  $\tilde{q}_s \in \partial g(s, Z_s)$  for each  $s \in [0, T]$  and prove that  $(\tilde{q}_t)_{t \in [0, T]} \in \bar{\mathcal{H}}(\xi, f)$  and

$$Y_t = \mathbb{E}_{\mathbb{Q}^{\tilde{q}}} \left[ \xi + \int_t^T f(s, \tilde{q}_s) ds \middle| \mathcal{F}_t \right], \quad t \in [0, T]. \quad (4.88)$$

Since  $f(s, \tilde{q}_s) = Z_s \cdot \tilde{q}_s - g(s, Z_s)$ , by (A4) we deduce that  $d\mathbb{P} \times dt - a.e.$ ,  $|\tilde{q}_t| \leq \gamma$ , and by (4.82) we obtain that for some constant  $p \in (1, \frac{\mu}{\mu_0})$ ,

$$\mathbb{E} \left[ \exp \left( \frac{p}{\mu^2} \left| \int_0^T \tilde{q}_s \cdot dB_s \right|^2 \right) \right] \leq \mathbb{E} \left[ \exp \left( \frac{p^2}{\mu^2} \left| \int_0^T \tilde{q}_s \cdot dB_s \right|^2 \right) \right] \leq \frac{1}{\sqrt{1 - \frac{2p^2\gamma^2}{\mu^2}(T-t)}} < +\infty. \quad (4.89)$$

To show that  $(\tilde{q}_t)_{t \in [0, T]} \in \bar{\mathcal{H}}(\xi, f)$ , we have to prove that

$$\mathbb{E}_{\mathbb{Q}^{\tilde{q}}} \left[ |\xi| + \int_0^T (h_s + |f(s, \tilde{q}_s)|) ds \right] < +\infty. \quad (4.90)$$

Indeed, by (A4) and (4.7) with  $q := p$  together with (4.89) and (4.80) we deduce that there exists a constant  $\bar{C}_{\mu, p} > 0$  depending only on  $(\mu, p)$  such that

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}^{\tilde{q}}} \left[ |\xi| + \int_0^T (h_s + f^-(s, \tilde{q}_s)) ds \right] &\leq 2\mathbb{E}_{\mathbb{Q}^{\tilde{q}}} \left[ |\xi| + \int_0^T h_s ds \right] \\ &= 2\mathbb{E} \left[ \left( |\xi| + \int_0^T h_s ds \right) L_T^{\tilde{q}} \right] \leq 2\mathbb{E} \left[ \left( |\xi| + \int_0^T h_s ds \right) \exp \left( \int_0^T \tilde{q}_s \cdot dB_s \right) \right] \end{aligned}$$

$$\begin{aligned}
&\leq 2\bar{C}_{\mu,p}\mathbb{E}\left[\exp\left(\frac{p}{\mu^2}\left|\int_0^T\tilde{q}_s\cdot dB_s\right|^2\right)\right] \\
&\quad +2\mathbb{E}\left[\left(|\xi|+\int_0^T h_s ds\right)\exp\left\{\mu\sqrt{\ln\left(1+|\xi|+\int_0^T h_s ds\right)}\right\}\right]<+\infty.
\end{aligned}
\tag{4.91}$$

By a similar computation to (4.39), we obtain that for each  $n \geq 1$ ,

$$\begin{aligned}
Y_t &= \mathbb{E}_{\mathbb{Q}^{\tilde{q}}}\left[Y_{\tau_n^t}+\int_t^{\tau_n^t}f(s,\tilde{q}_s)ds\middle|\mathcal{F}_t\right] \\
&= \mathbb{E}\left[Y_{\tau_n^t}\frac{L_{\tau_n^t}^{\tilde{q}}}{L_t^{\tilde{q}}}\middle|\mathcal{F}_t\right]+\mathbb{E}_{\mathbb{Q}^{\tilde{q}}}\left[\int_t^{\tau_n^t}f(s,\tilde{q}_s)ds\middle|\mathcal{F}_t\right], \quad t \in [0,T].
\end{aligned}
\tag{4.92}$$

On the other hand, by applying (4.7), (4.85) and (4.89) we deduce that the sequence of random variables  $|Y_{\tau_n^t}|L_{\tau_n^t}^{\tilde{q}}/L_t^{\tilde{q}}$  is uniformly integrable for each  $t \in [0, T]$ . Then, in view of (4.92) with  $t = 0$ ,

$$\sup_{n \geq 1}\mathbb{E}_{\mathbb{Q}^{\tilde{q}}}\left[\int_0^{\tau_n^0}f^+(s,\tilde{q}_s)ds\right]\leq\mathbb{E}_{\mathbb{Q}^{\tilde{q}}}\left[\int_0^Tf^-(s,\tilde{q}_s)ds\right]+|Y_0|+\sup_{n \geq 1}\mathbb{E}\left[|Y_{\tau_n^0}|L_{\tau_n^0}^{\tilde{q}}\right]<+\infty,$$

which together with Fatou's lemma and (4.91) yields (4.90). By letting  $n \rightarrow \infty$  in (4.92), we get (4.88).

Finally, note that  $U_t(\xi) = Y_t$  for each  $t \in [0, T]$ . According to the existence and uniqueness of BSDE (3.3) and the related comparison theorem, we easily verify that the operator  $\{U_t(\cdot), t \in [0, T]\}$  defined via (3.2) satisfies (i)-(iv) in the introduction and then constitutes a dynamic concave utility function defined on  $\cap_{\bar{\mu}>0}L \exp[\bar{\mu}(\ln L)^{\frac{1}{2}}]$ . The proof is then complete.

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