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White noise driven SPDEs with mean reflection

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Abstract

In this paper, we study a new type of SPDEs with reflection (called mean reflected SPDEs), where the compensating reflection part depends not on the paths but on the law of the solution. Focusing on solutions (u, K) with deterministic K , we obtain the well-posedness of such SPDE. Utilising the weak convergence approach, we then establish the large deviation principles for this type of SPDEs.

Keywords: SPDEs; Mean reflection; Large deviation principle; Weak convergence method

Mathematics Subject Classification (2010): 60H15, 60F10

1 Introduction

The aim of this paper is to study mean reflected SPDEs driven by a space-time white noise, which are reflected SPDEs with a constraint on the law of the process u rather than on its paths. More precisely, we mainly consider the following parabolic SPDE:

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} - \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t;u) = \sigma(x,t;u) \dot{W}(x,t) + \dot{K}(x,t), & x \in [0,1], t \geq 0; \\ u(x,0) = u_0(x), & x \in [0,1]; \\ u(0,t) = u(1,t) = 0, & t \geq 0, \end{cases} \quad (1.1)$$

satisfying

$$E[h(x,t,u(x,t))] \geq 0, \quad \forall x \in [0,1], t \geq 0. \quad (1.2)$$

Here \dot{W} denotes the space-time white noise defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Set $\mathcal{F}_t := \sigma\{W(A) : A \in \mathcal{B}([0,1] \times [0,t])\} \vee \mathcal{N}$, where \mathcal{N} is the class of \mathbb{P} -null sets of \mathcal{F} . The initial condition $u_0(x)$ is a non-negative continuous function, which vanishes at points 0 and 1. The coefficients f and σ are measurable mappings from $[0,1] \times \mathbb{R}_+ \times C([0,1] \times \mathbb{R}_+)$ into \mathbb{R} . Moreover, K is a measure which compensates reflections of the functional of u of the form $E[h(x,t,u(x,t))] \geq 0$ for a given continuous function h .

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Reflection problems for SPDEs have been widely studied in recent years. In [26], Nualart and Pardoux first established the existence and uniqueness of solution to the obstacle problem for stochastic heat equation, driven by a space-time white noise with a constant diffusion coefficient. This result was generalized by Donati-Martin and Pardoux in [11], in which both the drift and the diffusion coefficients are nonlinear. They used penalization method to prove the existence of the solution but did not obtain the uniqueness result. And then in 2009, Xu and Zhang [29] finally established the uniqueness. Moreover, the reflected system of SPDEs have also been studied in [25, 30, 31]. In all the above mentioned papers, they do not consider the case when the reflection is on the law of the solution.

The notion of mean reflection is motivated by the mean field game theory. In this new type of equations, the compensating reflection part depends not on the paths but on the law of the solution. In [1], Briand, Elie and Hu studied a system being reflected according to the mean of the process and called it a mean reflected Backward Stochastic Differential Equation (MR BSDE). With quadratic generator and bounded or unbounded terminal condition, the well-posedness of MR BSDE was generalized in [16, 17]. And then, [2] focused on the forward version (MR SDE) as well as its approximation by an appropriate interacting particle system and numerical schemes. In [4], the results of [1, 2] was enlarged to the multi-dimensional case and for rather general constraint sets on the law in the backward and forward cases. MR SDE with jumps can be found in [3] and MR BSDE by allowing a random measure in the diffusion part is considered in [15]. Recently, mean reflection with two constraints for BSDEs and SDEs are solved in [13] and [12], respectively.

However, to the best of our knowledge, mean reflected SPDEs have not yet been considered in the literature. The purpose of this paper is twofold. In the first part, we aim to establish the existence and uniqueness of the deterministic flat solution, defined in Definition 2.3, for mean reflected SPDEs (1.1). In our paper, the constraint depends on the distribution of u , and not the pointwise values of the process u . For this reason, we have no comparison theorem and cannot obtain the corresponding estimates as in penalized classical SPDEs. Our proof is mainly based on the method of Picard iteration. Inspired by thoughts of [26, 29], it is possible to reduce the problem to a completely deterministic obstacle problem (see (3.2) for linear mean reflection and (4.1) for general mean reflection), for which one has existence and uniqueness for a large class of driving functions. More importantly, we can control differences in the uniform norm of solution to obstacle problem (3.2) by differences in the uniform norm of obstacles (see Lemma 3.2). Moreover, based on the property (4.3) of operator $L_{x,t}$ (defined in (4.2)) and the equivalence relation (4.4), the differences in the uniform norm of solution to obstacle problem (4.1) can also be controlled by the expectation of differences in the uniform norm of obstacles (see Lemma 4.3). In the second part of the paper, we consider the small noise perturbation of (1.1):

$$\begin{cases} \frac{\partial u^\varepsilon(x,t)}{\partial t} - \frac{\partial^2 u^\varepsilon(x,t)}{\partial x^2} + f(x,t;u^\varepsilon) = \sqrt{\varepsilon}\sigma(x,t;u^\varepsilon)\dot{W}(x,t) + \dot{K}^\varepsilon(x,t); \\ u^\varepsilon(x,0) = u_0(x); \\ u^\varepsilon(0,t) = u^\varepsilon(1,t) = 0, \end{cases}$$

satisfying $E[h(x,t,u^\varepsilon(x,t))] \geq 0$ for all $x \in [0,1]$, $t \geq 0$ and $\int_0^\infty \int_0^1 E[h(x,t,u^\varepsilon(x,t))] K^\varepsilon(dx,dt) = 0$. We devote to establishing a large deviation principle (LDP) for $\{u^\varepsilon(\cdot, \cdot)\}_{\varepsilon>0}$ in $C([0,1] \times [0,T])$, the space of continuous functions in $[0,1] \times [0,T]$ endowed with the uniform topology. Our proof is based on the weak convergence method, which provides a new perspective on this topic. The key idea of this approach is to prove a certain variational representation formula about the Laplace

transform of bounded continuous functionals, which then leads to the verification of the equivalence between the LDP and the Laplace principle. In this paper, we use a new sufficient condition (see Lemma 5.3) proposed in [24], which is a generalized version of classical weak convergence criteria by Budhiraja et al. [6, 7]. This new sufficient condition has been widely applied (see [10, 14, 18–20, 28] and so on).

The organization of this paper is as follows. In Section 2, we first give Example 2.1 which shows that for the mean reflected SPDEs, it will lead to an infinite number of solutions if the reflection term is allowed to be random. To ensure the uniqueness of the solution, we introduce the notion of deterministic flat solution (see Definition 2.3). The main assumptions of the paper are stated at the end of Section 2. Inspired by thoughts of [26, 29] and applying the method of Picard iteration, we study the SPDE with linear mean reflection in Section 3 and extend the result to general mean reflection in Section 4. In Section 5, we are concerned about the LDP for $\{u^\varepsilon(\cdot, \cdot)\}_{\varepsilon>0}$, which is proved in Theorem 5.7.

In this paper, we use the following notation. Denote by $G_t(x, y)$ is the fundamental solution of the heat equation with Dirichlet boundary condition. $\langle \cdot, \cdot \rangle$ is the scalar product in $L^2(0, 1)$ and $\|\cdot\|_\infty$ denotes the uniform norm in $C([0, 1] \times [0, t])$. i.e. $\|\varphi\|_\infty^t := \sup_{x \in [0, 1], s \in [0, t]} |\varphi(x, s)|$, $\varphi \in C([0, 1] \times [0, t])$. Let $B((x_0, t_0), r)$ be the open ball, which is centered at (x_0, t_0) with radius $r > 0$.

2 Preliminaries and hypotheses

In order to establish the well-posedness of mean reflected SPDEs, we consider only deterministic flat solution, which means that the process K is restricted to be non-random. This is because if the reflection term is allowed to be random, it will lead to an infinite number of solutions to the SPDE (see Example 2.1). Furthermore, we give the definition of deterministic flat solution to problem (1.1). At the end of this section, some of the main assumptions of the paper are stated.

Example 2.1 Let us consider a special case where the coefficient σ is a positive constant and the mean reflection is linear. Namely,

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} - \frac{\partial^2 u(x,t)}{\partial x^2} = \sigma \dot{W}(x,t) + \dot{K}(x,t), & x \in [0, 1], t \geq 0; \\ u(x,0) = u_0(x), & x \in [0, 1]; \\ u(0,t) = u(1,t) = 0, & t \geq 0; \\ E[u(x,t)] \geq y(x,t), & x \in [0, 1], t \geq 0, \end{cases} \quad (2.1)$$

with the Skorokhod condition

$$\int_0^\infty \int_0^1 (E[u(x,t)] - y(x,t)) K(dx, dt) = 0. \quad (2.2)$$

It is well known that (2.2) makes it possible to qualify the solution as “flat” when satisfied. In (2.1), y is a deterministic continuous map from $[0, 1] \times \mathbb{R}_+$ to \mathbb{R} , that is, a deterministic lower bound on the expected value of the solution. Moreover, $y(x, 0) = 0$ for all $x \in [0, 1]$ and $y(0, t) = y(1, t) = 0$. The solution to (2.1) is the couple of (u, K) , in which the process K need to ensure that the constraint is satisfied in a minimal way according to the Skorokhod condition (2.2).

Now, we are going to prove that the mean reflected SPDE (2.1) exists a deterministic flat solution. In fact, several authors including Walsh [27] have shown that

$$z(x, t) := \int_0^1 G_t(x, y) u_0(y) dy + \sigma \int_0^t \int_0^1 G_{t-s}(x, y) W(dy, ds),$$

is the weak solution of the parabolic SPDE:

$$\begin{cases} \frac{\partial z(x, t)}{\partial t} - \frac{\partial^2 z(x, t)}{\partial x^2} = \sigma \dot{W}(x, t), & x \in [0, 1], t \geq 0; \\ z(x, 0) = u_0(x), & x \in [0, 1]; \\ z(0, t) = z(1, t) = 0, & t \geq 0. \end{cases}$$

Define $\bar{z} := u - z$ and $\nu(x, t) := E[z(x, t)] - y(x, t)$, then \bar{z} satisfies the following parabolic PDE

$$\begin{cases} \frac{\partial \bar{z}(x, t)}{\partial t} - \frac{\partial^2 \bar{z}(x, t)}{\partial x^2} = \dot{K}(x, t), & x \in [0, 1], t \geq 0; \\ \bar{z}(x, 0) = 0, & x \in [0, 1]; \\ \bar{z}(0, t) = \bar{z}(1, t) = 0, & t \geq 0. \end{cases} \quad (2.3)$$

It is worth pointing out that this process K is deterministic, which can be constructed by penalization method (see (2.6) in Remark 2.2). This implies \bar{z} is also deterministic. Hence, we have

$$\bar{z}(x, t) = E[\bar{z}(x, t)] \geq -\nu(x, t), \quad \forall x \in [0, 1], t \geq 0, \quad (2.4)$$

and

$$\int_0^\infty \int_0^1 (\bar{z}(x, t) + \nu(x, t)) K(dx, dt) = 0. \quad (2.5)$$

From [29, Proposition 2.1], it follows that the parabolic PDE (2.3) admits a unique solution (\bar{z}, K) satisfying (2.4) and (2.5). Based on the above observation, we finally deduce that the mean reflected SPDE (2.1) has a unique deterministic flat solution with $(u = \bar{z} + z, K)$.

Remark 2.2 *Since the solution \bar{z} of (2.3)-(2.5) is deterministic, then by the penalization method as in [26, Theorem 1.4], we can construct a sequence \bar{z}^n to approximate \bar{z} as follows.*

$$\begin{cases} \frac{\partial \bar{z}^n(x, t)}{\partial t} - \frac{\partial^2 \bar{z}^n(x, t)}{\partial x^2} = n(\bar{z}^n(x, t) + \nu(x, t))^- , \\ \bar{z}^n(x, 0) = 0, \\ \bar{z}^n(0, t) = \bar{z}^n(1, t) = 0. \end{cases} \quad (2.6)$$

It is easy to see that \bar{z}^n increases as n tends to infinity. However, when the coefficient σ is not constant or the function $f \not\equiv 0$ in (1.1), \bar{z} is random in these cases. Note that the constraint depends on the distribution of u , and not the pointwise values of the process u . Hence, we cannot ensure the penalized sequence \bar{z}^n will be increasing at this time and this method fails. As can be seen below, the existence of the solution can also be proved by iteration, and holds even for more general SPDEs where the coefficients depend on the past.

If the reflection term is allowed to be random, then below we show that mean reflected SPDE (2.1) may have an infinite number of flat solutions (u^α, K^α) . Indeed, for any real $\alpha \neq 0$, let $M_t^\alpha := E[\mathbf{1}_{\{Y > 0\}} | \mathcal{F}_t]$ with $Y = \int_0^\infty e^{-\alpha^2 s} dB_s$, then it is easy to see that every M_t^α is a bounded positive martingale. Moreover, according to [22, Example 7.5], we have

$$M_t^\alpha = \Phi \left(\frac{\int_0^t e^{-\alpha^2 s} dB_s}{\sqrt{\int_t^\infty e^{-2\alpha^2 s} ds}} \right) = \Phi \left(\sqrt{2}|\alpha| e^{\alpha^2 t} \left(\int_0^t e^{-\alpha^2 s} dB_s \right) \right),$$

and $E[M_t^\alpha] = 1/2$, where Φ is the cumulative distribution function of the standard normal distribution. Define the random process $K^\alpha(x, t) := \int_0^t \int_0^x (M_s^\alpha + \frac{1}{2}) K(dy, ds)$, where K is just the deterministic reflection obtained above. Fix α and then given K^α , we consider

$$\bar{z}^\alpha(x, t) := \int_0^t \int_0^1 G_{t-s}(x, y) dK^\alpha(y, s).$$

Since $G_t(x, y)$ is the fundamental solution of the heat equation with Dirichlet boundary condition, then we have \bar{z}^α satisfies the following parabolic equation

$$\begin{cases} \frac{\partial \bar{z}^\alpha(x, t)}{\partial t} = \frac{\partial^2 \bar{z}^\alpha(x, t)}{\partial x^2} + (M_t^\alpha + \frac{1}{2}) \dot{K}(x, t), \\ \bar{z}^\alpha(x, 0) = 0, \quad \bar{z}^\alpha(0, t) = \bar{z}^\alpha(1, t) = 0. \end{cases} \quad (2.7)$$

Using the facts that $E[M_t^\alpha + \frac{1}{2}] = 1$ for all $0 \leq t \leq T$ and K is deterministic, we deduce

$$\begin{aligned} E[\bar{z}^\alpha(x, t)] &= \int_0^t \int_0^1 G_{t-s}(x, y) \left(E[M_s^\alpha] + \frac{1}{2} \right) K(dy, ds) \\ &= \int_0^t \int_0^1 G_{t-s}(x, y) K(dy, ds) \\ &= \bar{z}(x, t), \end{aligned}$$

which implies that $u^\alpha := \bar{z}^\alpha + z$ satisfies $E[u^\alpha(x, t)] = E[u(x, t)]$ for any real $\alpha \neq 0$. Consequently, from $E[u(x, t)] \geq y(x, t), \forall (x, t) \in [0, 1] \times [0, T]$ and $E[u(x, t)] - y(x, t) = 0, dK(x, t)$ -a.e., we conclude that for any $T > 0$,

$$\int_0^T \int_0^1 (E[u^\alpha(x, t)] - y(x, t)) dK^\alpha(x, t) = \int_0^T \int_0^1 (E[u(x, t)] - y(x, t)) (M_t^\alpha + 1/2) K(dx, dt) = 0.$$

Thus for any real $\alpha \neq 0$, (u^α, K^α) is also a flat solution to mean reflected SPDE (2.1).

One way to overcome this difficulty is to turn to the consideration of minimal solution. Assume that (\hat{u}, \hat{K}) is a minimal flat solution of mean reflected SPDE (2.1), that is $\hat{u} \leq u^\alpha$ for any real $\alpha \neq 0$, where u^α are just defined above. Clearly, there must exist a $\alpha \neq 0$ such that $\hat{u}(x_0, t_0) < u^\alpha(x_0, t_0)$ for some $(x_0, t_0) \in [0, 1] \times [0, T]$. Moreover, we claim that $E[u^\alpha(x_0, t_0)] = y(x_0, t_0)$. In fact, if not, then we have $E[u^\alpha(x_0, t_0)] > E[\hat{u}(x_0, t_0)] \geq y(x_0, t_0)$. Since

$$E[u^\alpha(x, t)] = E[u(x, t)] = E[z(x, t)] + \bar{z}(x, t), \forall (x, t) \in [0, 1] \times [0, T],$$

it yields $E[z(x_0, t_0)] > y(x_0, t_0)$ and $\bar{z}(x_0, t_0) = 0$. Meanwhile, we obtain $E[(\hat{u} - z)(x_0, t_0)] < 0$, which is a contraction with $E[(\hat{u} - z)(x, t)] \geq 0$ for any $(x, t) \in [0, 1] \times [0, T]$. Consequently, the

equality $E[u^\alpha(x_0, t_0)] = y(x_0, t_0)$ does hold. However, $E[\widehat{u}(x_0, t_0)] < E[u^\alpha(x_0, t_0)] = y(x_0, t_0)$ at this time and thus the constraint is not satisfied. Hence, we conclude from above displays that the mean reflected SPDE (2.1) has no minimal solution.

Based on Example 2.1, in order to establish the well-posedness of mean reflected SPDEs, we consider only the case where the reflection term is deterministic. Now, we give the definition of the deterministic flat solution of mean reflected SPDE (1.1).

Definition 2.3 *A pair (u, K) is said to be a deterministic flat solution of mean reflected SPDE (1.1) if the process K is deterministic and*

- (i) u is a continuous random process on $[0, 1] \times \mathbb{R}_+$; $u(x, t)$ is \mathcal{F}_t measurable and satisfies the constraint (1.2).
- (ii) K is a measure on $(0, 1) \times \mathbb{R}_+$ such that $K((\varepsilon, 1 - \varepsilon) \times [0, T]) < \infty$ for every small $\varepsilon > 0$ and $T > 0$.
- (iii) (u, K) solves the parabolic SPDE in the following sense: $\forall t \in \mathbb{R}_+, \varphi \in C_0^2([0, 1])$ with $\varphi(0) = \varphi(1) = 0$,

$$\begin{aligned} & \langle u(t), \varphi \rangle - \int_0^t \langle u(s), \varphi'' \rangle ds + \int_0^t \int_0^1 f(x, s; u) \varphi(x) dx ds \\ &= \langle u_0, \varphi \rangle + \int_0^t \int_0^1 \sigma(x, s; u) \varphi(x) W(dx, ds) + \int_0^t \int_0^1 \varphi(x) K(dx, ds) \quad a.s. \end{aligned}$$

- (iv) K increases only when needed. i.e. $\int_0^\infty \int_0^1 E[h(x, s, u(x, s))] K(dx, ds) = 0$.

We next introduce the precise assumptions on the coefficients. f, σ are two measurable mappings,

$$f, \sigma : [0, 1] \times \mathbb{R}_+ \times C([0, 1] \times \mathbb{R}_+) \rightarrow \mathbb{R},$$

which satisfies

- (I) for any $u, v \in C([0, 1] \times \mathbb{R}_+)$, $(x, t) \in [0, 1] \times \mathbb{R}_+$ such that $u^t = v^t$,

$$\begin{aligned} f(x, t; u) &= f(x, t; v) \\ \sigma(x, t; u) &= \sigma(x, t; v), \end{aligned}$$

where u^t, v^t denote the restriction of u, v to $[0, 1] \times [0, t]$ respectively.

- (II) for any $T > 0$, there exist constants $C(T)$ and $M(T)$ depending only on T such that for any $x \in [0, 1], t \in [0, T], u, v \in C([0, 1] \times \mathbb{R}_+)$,

$$|f(x, t; u) - f(x, t; v)| + |\sigma(x, t; u) - \sigma(x, t; v)| \leq C(T) |u - v|_\infty^t, \quad (2.8)$$

$$|f(x, t; u)| + |\sigma(x, t; u)| \leq M(T) (1 + |u|_\infty^t). \quad (2.9)$$

In addition, we consider some assumptions on the constraint function h .

(H) The continuous function $h : [0, 1] \times \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ is a $\mathcal{B}([0, 1]) \times \mathcal{B}(\mathbb{R}_+) \times \mathcal{B}(\mathbb{R})$ -measurable map satisfying

1. $\forall t \geq 0$, $h(0, t, 0) = h(1, t, 0) = 0$ and $y \mapsto h(x, t, y)$ is strictly increasing.
2. $\forall x \in [0, 1]$, $h(x, 0, u_0(x)) \geq 0$ and $h(x, t, \infty) > 0$ for any $(x, t) \in [0, 1] \times \mathbb{R}_+$.
3. $\forall (x, t) \in [0, 1] \times \mathbb{R}_+$, $\forall y \in \mathbb{R}$, $|h(x, t, y)| \leq C_1(1 + |y|)$ for some positive constant C_1 .
4. $\forall (x, t) \in [0, 1] \times \mathbb{R}_+$, h is a bi-Lipschitz function in y : there exist positive constants $0 < c_h \leq C_h$ such that

$$c_h|y - z| \leq |h(x, t, y) - h(x, t, z)| \leq C_h|y - z|, \quad y, z \in \mathbb{R}. \quad (2.10)$$

It is worth noticing that the assumption (2.10) will imply a Lipschitz property of the operator $L_{x,t}$ (defined in (4.2)) for the L^1 -norm, uniformly in x and t (see Lemma 4.1). This plays a crucial role in the proof of SPDE (1.1) with general mean reflection (1.2), which can be seen from the detailed proof in Section 4. **Some common functions satisfying these conditions include $2y - y^+$, $\sin y + 2y$ and $x(1-x)(-\sqrt{y^+ + 1} + y + 1)$.**

3 SPDE with linear mean reflection

In this section, we mainly study a special case of (1.2) where the mean reflection is linear. i.e. $h : (x, t, Y) \mapsto Y - y(x, t)$. More precisely, the constraint condition is

$$E[u(x, t)] \geq y(x, t), \quad (x, t) \in [0, 1] \times \mathbb{R}_+, \quad (3.1)$$

where y is a deterministic continuous map from $[0, 1] \times \mathbb{R}_+$ to \mathbb{R} . Throughout this section, we assume $y(x, 0) = 0$ and $y(0, t) = y(1, t) = 0$. The main result is Theorem 3.2, which states that mean reflected SPDE (1.1) with constraint condition (3.1) admits a unique deterministic flat solution (u, K) .

Before giving the main result, we firstly consider the following deterministic parabolic obstacle problem

$$\begin{cases} \frac{\partial \bar{z}(x, t)}{\partial t} - \frac{\partial^2 \bar{z}(x, t)}{\partial x^2} = \dot{K}(x, t); \\ \bar{z}(x, t) \geq -v(x, t); \\ \int_0^T \int_0^1 (z + v)(x, t) dK(x, t) = 0, \quad \text{for all } T > 0. \end{cases} \quad (3.2)$$

Here, the function v belongs to $C([0, 1] \times [0, T])$ and satisfies $v(x, 0) = u_0(x)$.

Definition 3.1 *A pair (\bar{z}, K) is called a solution to problem (3.2) if it satisfies the following conditions*

- (1) \bar{z} is a continuous function on $[0, 1] \times [0, T]$ satisfying $\bar{z}(x, 0) = 0$ and $\bar{z}(0, t) = \bar{z}(1, t) = 0$.
- (2) K is a measure on $(0, 1) \times \mathbb{R}_+$ such that $K((\varepsilon, 1 - \varepsilon) \times [0, T]) < \infty$ for every small $\varepsilon > 0$ and $T > 0$.

(3) \bar{z} weakly solves the PDE

$$\frac{\partial \bar{z}}{\partial t} = \frac{\partial^2 \bar{z}}{\partial x^2} + \dot{K}.$$

That is, for all $t \geq 0$ and $\phi \in C_0^2((0, 1))$,

$$\int_0^1 \bar{z}(x, t) \phi(x) dx = \int_0^t \int_0^1 \bar{z}(x, s) \phi''(x) dx ds + \int_0^t \int_0^1 \phi(x) K(dx, ds).$$

$$(4) \int_0^t \int_0^1 (\bar{z}(x, s) + v(x, s)) K(dx, ds) = 0.$$

The above obstacle problem on the compact interval $[0, 1]$ was originally discussed in [26, Theorem 1.4]. Based on thoughts of it, the following lemma has been given in [29, Proposition 2.1]. This will be very helpful when proving estimates later, as it will allow us to control our mean reflected SPDE by a SPDE without mean reflection.

Lemma 3.2 *If $v(x, 0) = u_0(x)$ for all $x \in [0, 1]$, $v(0, t) = v(1, t) = 0$ for all $t \geq 0$, the obstacle problem (3.2) admits a unique solution. Moreover, $|\bar{z}|_\infty^T \leq |v|_\infty^T$.*

Now, the main result of this section is given as follows.

Theorem 3.3 *Under the assumptions (I) and (II), the parabolic SPDE (1.1) with linear mean reflection (3.1) admits a unique deterministic flat solution. Moreover, $E[|u|_\infty^T]^p < \infty$ for all $p \geq 1$.*

Proof: Let us first prove the existence of deterministic flat solution to mean reflected SPDE (1.1). In view of the globally Lipschitz condition of the coefficients f and σ , it follows from [27] that

$$\begin{aligned} z_1(x, t) &:= \int_0^1 G_t(x, y) u_0(y) dy - \int_0^t \int_0^1 G_{t-s}(x, y) f(y, s; u_0) dy ds \\ &\quad + \int_0^t \int_0^1 G_{t-s}(x, y) \sigma(y, s; u_0) W(dy, ds), \end{aligned}$$

satisfying the SPDE

$$\begin{cases} \frac{\partial z_1(x, t)}{\partial t} - \frac{\partial^2 z_1(x, t)}{\partial x^2} + f(x, t; u_0) = \sigma(x, t; u_0) \dot{W}(x, t); \\ z_1(x, 0) = u_0(x); \\ z_1(0, t) = z_1(1, t) = 0. \end{cases}$$

Denote by (\bar{z}_1, K_1) be the unique deterministic solution of obstacle problem (3.2) with $v(x, t) = E[z_1(x, t)] - y(x, t)$. Set $u_1 := z_1 + \bar{z}_1$, then it is easy to verify that (u_1, K_1) is the unique solution of the following mean reflected SPDE

$$\begin{cases} \frac{\partial u_1(x, t)}{\partial t} - \frac{\partial^2 u_1(x, t)}{\partial x^2} + f(x, t; u_0) = \sigma(x, t; u_0) \dot{W}(x, t) + \dot{K}_1; \\ u_1(x, 0) = u_0(x); \\ u_1(0, t) = u_1(1, t) = 0, \end{cases}$$

with $E[u_1(x, t)] \geq y(x, t)$ for any $(x, t) \in [0, 1] \times \mathbb{R}_+$. We iterate this procedure and suppose u_{n-1} has been defined. Let

$$\begin{aligned} z_n(x, t) &:= \int_0^1 G_t(x, y) u_0(y) dy - \int_0^t \int_0^1 G_{t-s}(x, y) f(y, s; u_{n-1}) dy ds \\ &\quad + \int_0^t \int_0^1 G_{t-s}(x, y) \sigma(y, s; u_{n-1}) W(dy, ds), \end{aligned}$$

and (\bar{z}_n, K_n) be the unique deterministic flat solution of obstacle problem (3.2) with $v(x, t) = E[z_n(x, t)] - y(x, t)$. Define $u_n := z_n + \bar{z}_n$, then (u_n, K_n) is the unique solution of the following mean reflected SPDE

$$\begin{cases} \frac{\partial u_n(x, t)}{\partial t} - \frac{\partial^2 u_n(x, t)}{\partial x^2} + f(x, t; u_{n-1}) = \sigma(x, t; u_{n-1}) \dot{W}(x, t) + \dot{K}_n; \\ u_n(x, 0) = u_0(x); \\ u_n(0, t) = u_n(1, t) = 0, \end{cases} \quad (3.3)$$

with $E[u_n(x, t)] \geq y(x, t)$ for any $(x, t) \in [0, 1] \times \mathbb{R}_+$. Moreover, using the fact that \bar{z}_n is deterministic, it yields

$$E[u_n(x, t)] = E[z_n(x, t)] + \bar{z}_n(x, t) \geq y(x, t),$$

with the Skorokhod condition $\int_0^t \int_0^1 (E[u_n(x, t)] - y(x, t)) K_n(dx, dt) = 0$.

The existence part is divided into two steps.

Step 1. Now, we prove the sequence $u_n(x, t)$ converges in L^p , uniformly with respect to (x, t) . Indeed, by a similar argument as in [26, Theorem 1.4] and Jensen's inequality, we have

$$\begin{aligned} |\bar{z}_n - \bar{z}_{n-1}|_\infty^T &\leq \sup_{x \in [0, 1], t \in [0, T]} |(E[z_n(x, t)] - y(x, t)) - (E[z_{n-1}(x, t)] - y(x, t))| \\ &\leq \sup_{x \in [0, 1], t \in [0, T]} E[|z_n(x, t) - z_{n-1}(x, t)|] \\ &\leq E[|z_n - z_{n-1}|_\infty^T], \end{aligned}$$

which implies

$$\begin{aligned} |u_n - u_{n-1}|_\infty^T &\leq |z_n - z_{n-1}|_\infty^T + |\bar{z}_n - \bar{z}_{n-1}|_\infty^T \\ &\leq |z_n - z_{n-1}|_\infty^T + E[|z_n - z_{n-1}|_\infty^T], \end{aligned} \quad (3.4)$$

almost surely. On the other hand, we obtain from Lemma 3.2 that

$$|\bar{z}_n|_\infty^T \leq |E[z_n] - y|_\infty^T \leq |y|_\infty^T + E[|z_n|_\infty^T].$$

Hence,

$$|u_n|_\infty^T = |z_n + \bar{z}_n|_\infty^T \leq |y|_\infty^T + |z_n|_\infty^T + E[|z_n|_\infty^T], \quad (3.5)$$

almost surely. Combined with the first step in the proof of [11, Theorem 3.1], there exists a constant $c(T, p, |u_0|_\infty^T)$ such that

$$\begin{aligned} E[(|u_1|_\infty^T)^p] &\leq 3^{p-1} \cdot E\left[(|y|_\infty^T)^p + (|z_1|_\infty^T)^p + (E[|z_1|_\infty^T])^p \right] \\ &\leq 3^{p-1} \left\{ (|y|_\infty^T)^p + 2E[(|z_1|_\infty^T)^p] \right\} \\ &\leq c(T, p, |u_0|_\infty^T). \end{aligned}$$

Then if $E[(|u_{n-1}|_\infty^T)^p] < \infty$, the same arguments as above can also yield

$$\begin{aligned} E[(|u_n|_\infty^T)^p] &\leq 3^{p-1} \{ (|y|_\infty^T)^p + 2E[(|z_{n-1}|_\infty^T)^p] \} \\ &\leq c(T, p, |u_0|_\infty^T, E[(|z_{n-1}|_\infty^T)^p]) \\ &< \infty. \end{aligned} \tag{3.6}$$

In view of (3.4) and by Jensen's inequality, we deduce that

$$\begin{aligned} &E[(|u_n - u_{n-1}|_\infty^T)^p] \\ &\leq E\left[\left(|z_n - z_{n-1}|_\infty^T + E[|z_n - z_{n-1}|_\infty^T]\right)^p\right] \\ &\leq 2^{p-1} \cdot E[(|z_n - z_{n-1}|_\infty^T)^p] + (E[|z_n - z_{n-1}|_\infty^T])^p \\ &\leq 2^p E[(|z_n - z_{n-1}|_\infty^T)^p] \\ &\leq c^p E\left[\left(\sup_{x \in [0,1], t \in [0,T]} \left| \int_0^t \int_0^1 G_{t-s}(x, y) [f(y, s; u_{n-1}) - f(y, s; u_{n-2})] dy ds \right|\right)^p\right] \\ &\quad + c^p E\left[\left(\sup_{x \in [0,1], t \in [0,T]} \left| \int_0^t \int_0^1 G_{t-s}(x, y) [\sigma(y, s; u_{n-1}) - \sigma(y, s; u_{n-2})] W(dy, ds) \right|\right)^p\right] \\ &=: c^p E[(|I_1|_\infty^T)^p] + c^p E[(|I_2|_\infty^T)^p]. \end{aligned} \tag{3.7}$$

By (2.8) and Hölder's inequality, it shows for any $p > 6$

$$\begin{aligned} E[(|I_1|_\infty^T)^p] &\leq \left(\sup_{x \in [0,1], t \in [0,T]} \int_0^t \int_0^1 G_{t-s}^q(x, y) dy ds \right)^{\frac{p}{q}} \\ &\quad \times C^p(T) E\left[\int_0^T (|u_{n-1} - u_{n-2}|_\infty^t)^p dt\right], \end{aligned}$$

where $q := p/(p-1) < 3$. Since function G has the property (6.1) in Appendix, then it yields

$$E[(|I_1|_\infty^T)^p] \leq c(T, p) E\left[\int_0^T (|u_{n-1} - u_{n-2}|_\infty^t)^p dt\right]. \tag{3.8}$$

Meanwhile,

$$\begin{aligned} E[|I_2|^p] &= E\left[\left|\int_0^t \int_0^1 G_{t-s}(x, y) [\sigma(y, s; u_{n-1}) - \sigma(y, s; u_{n-2})] W(dy, ds)\right|^p\right] \\ &\leq C(T)^p E\left[\left(\int_0^t \int_0^1 G_{t-s}^2(x, y) \cdot (|u_{n-1} - u_{n-2}|_\infty^s)^2 dy ds\right)^{\frac{p}{2}}\right] \\ &\leq C(T)^p \left(\sup_{x \in [0,1], t \in [0,T]} \int_0^t \int_0^1 G_{t-s}^{\frac{2p}{p-2}}(x, y) dy ds \right)^{\frac{p-2}{2}} \\ &\quad \times E\left[\int_0^t \int_0^1 (|u_{n-1} - u_{n-2}|_\infty^s)^p dy ds\right] \\ &\leq c(T, p) E\left[\int_0^t \int_0^1 (|u_{n-1} - u_{n-2}|_\infty^s)^p dy ds\right], \end{aligned}$$

where the property (6.1) in Appendix and the fact that $2p/(p-2) < 3$ have been used. So from the same calculations as in [27, Corollary 3.4] or [11, Theorem 3.1], we have the following estimate, again with $p > 6$ and $s, t \leq T$,

$$E[|I_2(x, t) - I_2(y, s)|^p] \leq c(T) |(x, t) - (y, s)|^{\frac{p}{4}-3} \cdot E\left[\int_0^{(t \vee s)} (|u_{n-1} - u_{n-2}|_\infty^r)^p dr\right].$$

Choosing $p > 20$ and by a version of the famous Kolmogorov Lemma [11, Lemma 3.1], then there exists a random variable $Y(\omega)$ satisfying

$$E[Y^p] \leq a c(T) \left(E\left[\int_0^{(t \vee s)} (|u_{n-1} - u_{n-2}|_\infty^r)^p dr\right] \right), \quad (3.9)$$

such that

$$|I_2(x, t) - I_2(y, s)|^p \leq Y^p(\omega) \cdot |(x, t) - (y, s)|^{\frac{p}{4}-5} \left(\log \left(\frac{\gamma}{|(x, t) - (y, s)|} \right) \right)^2, \quad (3.10)$$

where $a, \gamma, c(T)$ are constants depending only on p and T . Note that $I_2(y, 0) = 0$ and $I_2(x, t) = 0$ once x takes the boundary point 0 or 1, then we have for any $x, y \in (0, 1)$ and $t \in (0, T]$,

$$0 < t \leq |(x, t) - (y, 0)| \leq \sqrt{x^2 + t^2} < (1 + T^2)^{\frac{1}{2}},$$

which implies

$$\begin{aligned} & \sup_{x \in (0, 1), t \in (0, T]} |(x, t) - (y, 0)|^{\frac{p}{4}-5} \left(\log \left(\frac{\gamma}{|(x, t) - (y, 0)|} \right) \right)^2 \\ & \leq (1 + T^2)^{\frac{1}{2}(\frac{p}{4}-5)} \cdot 2 \left[(\log \gamma)^2 + \left(\frac{\log(1 + T^2)}{2} \right)^2 \right] \\ & \leq c(T, p). \end{aligned}$$

Consequently, it yields from (3.9) and (3.10) that

$$E[(|I_2|_\infty^T)^p] \leq c(T, p) E\left[\int_0^T (|u_{n-1} - u_{n-2}|_\infty^t)^p dt\right]. \quad (3.11)$$

Combining (3.7), (3.8) and (3.11), we conclude that

$$\begin{aligned} & E[(|u_n - u_{n-1}|_\infty^T)^p] \\ & \leq c(p, C, T) E\left[\int_0^T (|u_{n-1} - u_{n-2}|_\infty^t)^p dt\right] \\ & \leq c^2(p, C, T) \cdot \left(\int_0^T dt \int_0^t E[(|u_{n-2} - u_{n-3}|_\infty^s)^p] ds \right) \\ & = c^2(p, C, T) \cdot \left(\int_0^T E[(|u_{n-2} - u_{n-3}|_\infty^s)^p] (T - s) ds \right) \end{aligned}$$

$$\begin{aligned}
&\leq \cdots \leq c^{n-1}(p, C, T) \cdot E \left[\int_0^T (|u_1 - u_0|_\infty^s)^p \cdot \frac{(T-s)^{n-2}}{(n-2)!} ds \right] \\
&\leq c^{n-1}(p, C, T) \cdot E[(|u_1 - u_0|_\infty^T)^p] \cdot \left(\int_0^T \frac{(T-s)^{n-2}}{(n-2)!} ds \right) \\
&= \frac{(cT)^{n-1}}{(n-1)!} \cdot E[(|u_1 - u_0|_\infty^T)^p],
\end{aligned}$$

where C is the parameter in (2.8) and the third step follows from changing the order of integration. Therefore, for any $m \geq n \geq 1$,

$$E[(|u_m - u_n|_\infty^T)^p] \leq E[(|u_1 - u_0|_\infty^T)^p] \cdot \left(\sum_{k=n}^{m-1} \frac{(cT)^k}{k!} \right) \rightarrow 0, \quad (3.12)$$

as n and m tends to ∞ . Thus there exists a random field $u(x, t) \in C([0, 1] \times [0, T])$ satisfying $E[(|u|_\infty^T)^p] < \infty$ such that

$$\lim_{n \rightarrow \infty} E[(|u_n - u|_\infty^T)^p] = 0. \quad (3.13)$$

Step 2. We next show that u satisfies the parabolic SPDE (1.1) with linear mean reflection (3.1). Clearly, it is sufficient to verify the conditions (i)-(iv) of Definition 2.3 hold for $u(x, t)$.

In view of (3.6), $E[u_n(x, t)] \geq y(x, t)$ and by the dominated convergence theorem, we have $E[u(x, t)] \geq y(x, t)$, $(x, t) \in [0, 1] \times \mathbb{R}_+$. For any $\varphi \in C_0^\infty((0, 1) \times \mathbb{R}_+)$, let $\varphi(s)$ denote the function $\varphi(\cdot, s)$ and $f(s, u)$ denote the function $f(\cdot, s; u)$. Then it follows from (3.3) that for all $t \geq 0$,

$$\begin{aligned}
&\langle u_n(t), \varphi(t) \rangle - \int_0^t \left\langle u_n(s), \frac{\partial \varphi(s)}{\partial s} \right\rangle ds - \int_0^t \langle u_n(s), \varphi''(s) \rangle ds + \int_0^t \langle f(s, u_{n-1}), \varphi(s) \rangle ds \\
&= \langle u_0, \varphi(0) \rangle + \int_0^t \int_0^1 \sigma(x, s; u_{n-1}) \varphi(x, s) W(dx, ds) + \int_0^t \int_0^1 \varphi(x, s) K_n(dx, ds). \quad (3.14)
\end{aligned}$$

By (2.8), (3.13) and the dominated convergence theorem, we can obtain

$$\int_0^t \int_0^1 \sigma(x, s; u_{n-1}) \varphi(x, s) W(dx, ds) \xrightarrow[n \rightarrow \infty]{L^2} \int_0^t \int_0^1 \sigma(x, s; u) \varphi(x, s) W(dx, ds).$$

All the terms on the left-hand side of (3.14) converge a.s. to the corresponding terms with u_n replaced by u (taking a subsequence, if necessary), as n tends to ∞ . Hence, K_n converges in the distributional sense to some positive distribution K on $(0, 1) \times [0, \infty)$. It follows from its positivity that K is a measure on $(0, 1) \times [0, \infty)$. Taking $n \rightarrow \infty$ in (3.14), then we have

$$\begin{aligned}
&\langle u(t), \varphi(t) \rangle - \int_0^t \left\langle u(s), \frac{\partial \varphi(s)}{\partial s} \right\rangle ds - \int_0^t \langle u(s), \varphi''(s) \rangle ds + \int_0^t \langle f(s, u), \varphi(s) \rangle ds \\
&= \langle u_0, \varphi(0) \rangle + \int_0^t \int_0^1 \sigma(x, s; u) \varphi(x, s) W(dx, ds) + \int_0^t \int_0^1 \varphi(x, s) K(dx, ds).
\end{aligned}$$

Clearly, taking a non-negative function $\tilde{\varphi} \in C_0^\infty((0, 1) \times \mathbb{R}_+)$ satisfying $\tilde{\varphi}(x, t) = 1$ on $(\text{supp } K) \cap ([\varepsilon, 1 - \varepsilon] \times [0, T])$ for every small $\varepsilon > 0$, it yields from above equality that $K([\varepsilon, 1 - \varepsilon] \times [0, T]) < \infty$ for every small $\varepsilon > 0$ and $T > 0$.

Let $\{\tilde{u}_n > 0\} := \{(x, t) \in [0, 1] \times [0, T] : (E[u_n(x, t)] - y(x, t)) > 0\}$ and $\{\tilde{u} > 0\}$ similarly. To prove condition (iv) of Definition 2.3, it is sufficient to prove for any $T > 0$,

$$\{\tilde{u} > 0\} \subset (\text{supp } K)^c. \quad (3.15)$$

Indeed, for any $(x_0, t_0) \in \{\tilde{u} > 0\}$, there exists a open ball $B((x_0, t_0), r)$ and some positive constant α such that $\tilde{u}(x, t) \geq 2\alpha > 0$, $\forall (x, t) \in B((x_0, t_0), r)$. Since

$$\begin{aligned} \lim_{n \rightarrow \infty} |\tilde{u}_n - \tilde{u}|_\infty^T &= \lim_{n \rightarrow \infty} \left(\sup_{x \in [0, 1], t \in [0, T]} |E[u_n(x, t)] - E[u(x, t)]| \right) \\ &\leq \lim_{n \rightarrow \infty} E[|u_n - u|_\infty^T] \\ &= 0, \end{aligned}$$

there exists a N_0 such that $\tilde{u}_n(x, t) \geq \alpha > 0$, $(x, t) \in B((x_0, t_0), r)$ holds for any $n \geq N_0$. Choose a function $\phi \in C_0^\infty([0, 1] \times [0, T])$, which satisfies $0 \leq \phi \leq 1$ and

$$\phi(x, t) = \begin{cases} 1, & (x, t) \in B((x_0, t_0), r/2), \\ 0, & (x, t) \in B((x_0, t_0), r)^c, \end{cases}$$

then

$$\begin{aligned} 0 \leq \int_0^T \int_0^1 \phi(x, t) K_n(dx, dt) &\leq \int_{B((x_0, t_0), r)} 1 K_n(dx, dt) \\ &\leq \frac{1}{\alpha} \int_0^T \int_0^1 \tilde{u}_n(x, t) K_n(dx, dt) \\ &= 0. \end{aligned}$$

This yields $\int_0^T \int_0^1 \phi(x, t) K_n(dx, dt) = 0$. Hence,

$$K\left(B((x_0, t_0), r/2)\right) \leq \int_0^T \int_0^1 \phi(x, t) K(dx, dt) = \lim_{n \rightarrow \infty} \int_0^T \int_0^1 \phi(x, t) K_n(dx, dt) = 0.$$

Consequently, $K(B((x_0, t_0), r/2)) = 0$ for any $(x_0, t_0) \in \{\tilde{u} > 0\}$, which implies (3.15).

It only remains to check conditions (iii) of Definition 2.3. We first consider a continuous stochastic process v defined by

$$v(x, t) := \int_0^1 G_t(x, y) u_0(y) dy + \int_0^t \int_0^1 G_{t-s}(x, y) \sigma(x, s; u) W(dy, ds).$$

That means v satisfies

$$\langle v(t), \psi \rangle - \int_0^t \langle v(s), \psi'' \rangle ds = \langle u_0, \psi \rangle + \int_0^t \int_0^1 \sigma(x, s; u) \psi(x) W(dx, ds),$$

for any C^∞ function ψ with compact support in $(0, 1)$. Thus, (u, K) is also the solution to a problem similar to (3.2), which satisfies $u_0 = v_0$, $(u - v)(0, t) = (u - v)(1, t) = 0$ and for any $t \geq 0$,

$$\langle (u - v)(t), \psi \rangle - \int_0^t \langle (u - v)(s), \psi'' \rangle ds + \int_0^t \langle f(s, u), \psi \rangle ds = \int_0^t \int_0^1 \psi(x) K(dx, ds) \quad \text{a.s.}$$

Note that, Nualart and Pardoux in [26] have shown this problem has a unique solution for the given function v . Hence applying their results, we can obtain for all $\varphi \in C^2([0, 1])$ with $\text{supp } \varphi \subset (0, 1)$,

$$\begin{aligned} & \langle u(t), \varphi \rangle - \int_0^t \langle u(s), \varphi'' \rangle ds + \int_0^t \int_0^1 f(x, s; u) \varphi(x) dx ds \\ = & \langle u_0, \varphi \rangle + \int_0^t \int_0^1 \sigma(x, s; u) \varphi(x) W(dx, ds) + \int_0^t \int_0^1 \varphi(x) K(dx, ds) \quad a.s.. \end{aligned}$$

The proof of the existence is now complete.

Now we turn to the proof of uniqueness. Suppose that $(u_{(1)}, K_{(1)})$ and $(u_{(2)}, K_{(2)})$ are two solutions of mean reflected SPDE (1.1) with linear constraint (3.1) and the associated Skorokhod condition. Let

$$\begin{aligned} v_{(i)}(x, t) & := \int_0^1 G_t(x, y) u_0(y) dy - \int_0^t \int_0^1 G_{t-s}(x, y) f(y, s; u_{(i)}) dy ds \\ & \quad + \int_0^t \int_0^1 G_{t-s}(x, y) \sigma(y, s; u_{(i)}) W(dy, ds), \end{aligned} \quad (3.16)$$

and $\bar{z}_{(i)} := u_{(i)} - v_{(i)}$, then $(\bar{z}_{(i)}, K_{(i)})$ is the unique solution of obstacle problem (3.2) with $v = (E[v_{(i)}] - y)$, $i = 1, 2$, respectively. Similar to (3.4), it yields $|u_{(1)} - u_{(2)}|_\infty^T \leq E[|v_{(1)} - v_{(2)}|_\infty^T] + |v_{(1)} - v_{(2)}|_\infty^T$. Define a stopping time by $\tau_N := \inf\{t \geq 0 : \sup_{x \in [0, 1]} |u_{(1)}(x, t)| \vee \sup_{x \in [0, 1]} |u_{(2)}(x, t)| > N\}$, then τ_N tends to ∞ as $N \rightarrow \infty$. Similar to the arguments of (3.7), (3.8) and (3.11), we can also deduce that

$$\begin{aligned} E\left[\left(|u_{(1)} - u_{(2)}|_\infty^{T \wedge \tau_N}\right)^p\right] & \leq 2^p \cdot E\left[\left(|v_{(1)} - v_{(2)}|_\infty^{T \wedge \tau_N}\right)^p\right] \\ & \leq c(p, C, T, N) E\left[\int_0^T \left(|u_{(1)} - u_{(2)}|_\infty^{t \wedge \tau_N}\right)^p dt\right], \end{aligned}$$

where C is the parameter in (2.8). This implies $E[|u_{(1)} - u_{(2)}|_\infty^{T \wedge \tau_N}] = 0$. Taking $N \rightarrow \infty$, then $E[|u_{(1)} - u_{(2)}|_\infty^T] = 0$. Hence, we have $u_{(1)} = u_{(2)}$, a.s., and therefore $K_{(1)} = K_{(2)}$. This ends the proof of the theorem. \square

4 SPDE with general mean reflection

For general mean reflection, we first consider the following parabolic obstacle problem

$$\begin{cases} \frac{\partial \bar{z}(x, t)}{\partial t} - \frac{\partial^2 \bar{z}(x, t)}{\partial x^2} = \dot{K}(x, t); \\ E[h(x, t, (\bar{z} + v)(x, t))] \geq 0; \\ \int_0^T \int_0^1 E[h(x, t, (\bar{z} + v)(x, t))] dK(x, t) = 0, \quad \text{for all } T > 0, \end{cases} \quad (4.1)$$

where v is a continuous \mathcal{F}_t -adapted process taking values in $C([0, 1] \times [0, T])$ and $E[|v|_\infty^T] < \infty$ for any $T > 0$. Moreover, $v(x, 0) \geq 0$ and $v(0, t) = v(1, t) = 0$ for all $x \in [0, 1], t \geq 0$. Firstly, we focus on the existence and uniqueness of obstacle problem (4.1) (Theorem 4.2). It should be

pointed that, with the help of the operator $L_{x,t}$ (defined in (4.2)), the relation (4.4) holds and then we can obtain the equivalence between obstacle problems (4.1) and (4.5). In addition, under the assumption (H), the differences in the uniform norm of solution to obstacle problem (4.1) can be controlled by the expectation of differences in the uniform norm of obstacles (Lemma 4.3), which plays a crucial role in SPDEs with general mean reflection. Based on them, we finally show that for general mean reflection, SPDE (1.1) admits a unique deterministic flat solution (Theorem 4.4).

Let \mathcal{S}^1 be the set of continuous \mathcal{F}_t -adapted process on $[0, 1] \times [0, T]$ with $E[|v|_\infty^T] < \infty$ for any $T > 0$ and $L^1(\mathcal{F}_t)$ denote the space of real valued \mathcal{F}_t -measurable integrable random variables X , for any $(x, t) \in [0, 1] \times [0, T]$. Define an operator

$$L_{x,t} : L^1(\mathcal{F}_t) \rightarrow [0, \infty), \quad X \mapsto \inf\{m \geq 0 : E[h(x, t, m + X)] \geq 0\}, \quad (4.2)$$

for any $(x, t) \in [0, 1] \times [0, T]$. Under the assumption (H), it is easy to see that $L_{x,t}$ is well defined and equals to zero for any random variable $X \geq 0$, *a.s.*. And more importantly, the operator $L_{x,t}$ has the following properties, which are highly crucial to establish inequality (4.7).

Lemma 4.1 *Under the assumption (H), the operator $L_{x,t}$ is Lipschitz continuous for the L^1 -norm, uniformly in x and t . Namely, there exists a constant $l \geq 0$ such that for any $x \in [0, 1]$ and $t \in [0, T]$,*

$$|L_{x,t}(X) - L_{x,t}(Y)| \leq lE[|X - Y|], \quad X, Y \in L^1(\mathcal{F}_t). \quad (4.3)$$

Moreover, for the given random process $v \in \mathcal{S}^1$, $L_{x,t}(v(x, t))$ is continuous with respect to (x, t) .

Proof: By an argument as in [1, Lemma 8], it is clear that (4.3) is satisfied as soon as the constraint function h is a bi-Lipschitz function in y . Next, let us show that the operator $L_{x,t}$ is continuous with respect to (x, t) .

In fact, observe first that the map $m \mapsto E[h(x, t, m + X)]$ is continuous and strictly increasing. Moreover, since the constraint function h is continuous and has linear growth, then

$$E[h(x, t, m + X)] \leq C_1(1 + |m| + E[|X|]) < \infty,$$

for any $x \in [0, 1]$ and $t \in [0, T]$, where the facts that m is finite and $X \in \mathcal{S}^1$ have been used. If $E[h(x, t, v(x, t))] \leq 0$, we have $L_{x,t}(v(x, t)) \geq 0$. Thus for any $m_1 < L_{x,t}(v(x, t)) < m_2$, the dominated convergence theorem yields

$$\begin{aligned} \lim_{(y,s) \rightarrow (x,t)} E[h(y, s, m_1 + v(y, s))] &= E[h(x, t, m_1 + v(x, t))] < E[h(x, t, L_{x,t}(v(x, t)) + v(x, t))] = 0 \\ &< E[h(x, t, m_2 + v(x, t))] = \lim_{(y,s) \rightarrow (x,t)} E[h(y, s, m_2 + v(y, s))]. \end{aligned}$$

Consequently, when $|(y, s) - (x, t)|$ is small enough, we obtain

$$E[h(y, s, m_1 + v(y, s))] < 0 \quad \text{and} \quad E[h(y, s, m_2 + v(y, s))] > 0,$$

which implies $m_1 < L_{y,s}(v(y, s)) < m_2$. On the other hand, if $E[h(x, t, v(x, t))] > 0$, then in this case $L_{x,t}(v(x, t)) = 0$. Since $\lim_{(y,s) \rightarrow (x,t)} E[h(y, s, v(y, s))] = E[h(x, t, v(x, t))] > 0$ and as long as $|(y, s) - (x, t)|$ is sufficiently small, we have $E[h(y, s, v(y, s))] > 0$ and $L_{y,s}(v(y, s)) = 0$. \square

Furthermore, it is worth noticing that the following equivalence relation

$$E[h(x, t, (\bar{z} + v)(x, t))] \geq 0 \quad \Leftrightarrow \quad \bar{z}(x, t) \geq L_{x,t}(v(x, t)), \quad (4.4)$$

holds, where $L_{x,t}$ is the operator defined in (4.2). Indeed, since the constraint function h is strictly increasing in y , then $\bar{z}(x, t) \geq L_{x,t}(v(x, t))$ gives

$$E[h(x, t, (\bar{z} + v)(x, t))] \geq E[h(x, t, L_{x,t}(v(x, t)) + v(x, t))] \geq 0.$$

Meanwhile, if $E[h(x, t, (\bar{z} + v)(x, t))] \geq 0$ holds, then we obviously have $\bar{z}(x, t) \geq L_{x,t}(v(x, t))$ for any $x \in [0, 1]$ and $t \in [0, T]$ by the definition of operator $L_{x,t}$. This equivalence relation will play a crucial role in Theorem 4.2 and Lemma 4.3.

Theorem 4.2 *For the given random process $v \in \mathcal{S}^1$, if Assumption (H) about the constraint function h holds, then similar to (3.2), the obstacle problem (4.1) exists a unique pair (\bar{z}, K) , which satisfies*

$$\begin{cases} \frac{\partial \bar{z}(x, t)}{\partial t} - \frac{\partial^2 \bar{z}(x, t)}{\partial x^2} = \dot{K}(x, t); \\ \bar{z}(x, t) \geq L_{x,t}(v(x, t)); \\ \int_0^T \int_0^1 (\bar{z}(x, t) - L_{x,t}(v(x, t))) dK(x, t) = 0, \quad \text{for all } T > 0. \end{cases} \quad (4.5)$$

Proof: We construct a solution by means of the well-known penalization method. Based on the equivalence relation (4.4), it inspires us to take the penalty term to be $n(\bar{z}(x, t) - L_{x,t}(v(x, t)))^-$ instead of $n(E[h(x, t, (\bar{z} + v)(x, t))])^-$, which is novel. Firstly, fix n and denote by \bar{z}^n the solution of the following equation

$$\begin{cases} \frac{\partial \bar{z}^n(x, t)}{\partial t} - \frac{\partial^2 \bar{z}^n(x, t)}{\partial x^2} = n(\bar{z}^n(x, t) - L_{x,t}(v(x, t)))^-, \\ \bar{z}^n(x, 0) = 0, \quad \bar{z}^n(0, t) = \bar{z}^n(1, t) = 0. \end{cases} \quad (4.6)$$

Applying the inequality $|a^- - b^-| \leq |a - b|$, we can obtain

$$\left| (\bar{z}^1(x, t) - L_{x,t}(v(x, t)))^- - (\bar{z}^2(x, t) - L_{x,t}(v(x, t)))^- \right| \leq |\bar{z}^1 - \bar{z}^2|_{\infty}^T.$$

Thus for each n , (4.6) admits a unique continuous solution \bar{z}^n belonging to $L^2((0, T); H^1(0, 1)) \cap C([0, 1] \times \mathbb{R}_+)$.

Here, we give an outline only and the further details of the proof can be seen in Appendix 6. Based on thoughts of [26] and Assumption (H), we firstly devote to proving that the solution \bar{z}^n of PDE (4.6) increases as n tends to infinity. Define $\bar{z}(x, t) := \sup_n \bar{z}^n(x, t)$ for any $(x, t) \in [0, 1] \times [0, \infty)$ and

$$K^n(dx, dt) := n(\bar{z}^n(x, t) - L_{x,t}(v(x, t)))^- dx dt.$$

Letting $n \rightarrow \infty$, we obtain that K^n converges in distribution to some positive distribution K on $(0, 1) \times [0, T]$. It follows from its positivity that K is a measure. Combined with the equivalence relation (4.4) and by a similar arguments as in [26, Theorem 1.4] or [21, Theorem 2.6], one can derive that (\bar{z}, K) satisfies conditions (1)-(3) and (4.5), which implies that it is a deterministic flat solution of obstacle problem (4.1). To prove the uniqueness, the equivalence relation (4.4) is very important in the proof and more details can be referenced in Appendix 6. \square

Lemma 4.3 Let \bar{z}_i be the first component of the solution to obstacle problem (4.1) corresponding to process $v_i \in \mathcal{S}^1, i = 1, 2$, respectively. Then under the Assumption (H), we have for any $T > 0$,

$$|\bar{z}_1 - \bar{z}_2|_\infty^T \leq lE[|v_1 - v_2|_\infty^T] \quad (4.7)$$

where l is the constant in (4.3).

Proof: Let $\bar{z}^{1,n}$ and $\bar{z}^{2,n}$ be the solutions of (4.6) corresponding to v_1 and v_2 , respectively. To prove (4.7), it is sufficient to show that:

$$|\bar{z}^{1,n} - \bar{z}^{2,n}|_\infty^T \leq lE[|v_1 - v_2|_\infty^T], \quad (4.8)$$

In fact, for notational simplicity, let $\Psi := \bar{z}^{1,n} - \bar{z}^{2,n} - \sup_{x \in [0,1], t \in [0,T]} |L_{x,t}(v_1(x,t)) - L_{x,t}(v_2(x,t))|$. Testing the equation with Ψ^+ , then we obtain

$$\begin{aligned} & \int_0^T \left\langle \frac{\partial \Psi_t}{\partial t}, \Psi_t^+ \right\rangle dt - \int_0^T \left\langle \frac{\partial^2 \Psi_t}{\partial x^2}, \Psi_t^+ \right\rangle dt + \int_0^T \int_0^1 n \left((\bar{z}^{2,n}(x,t) - L_{x,t}(v_2(x,t)))^- \right. \\ & \left. - (\bar{z}^{1,n}(x,t) - L_{x,t}(v_1(x,t)))^- \right) \Psi^+(x,t) dx dt = 0. \end{aligned}$$

In view of the inequality $a^- - b^- \geq -(b - a)^-$, we have

$$\begin{aligned} & \left(\bar{z}^{2,n}(x,t) - L_{x,t}(v_2(x,t)) \right)^- - \left(\bar{z}^{1,n}(x,t) - L_{x,t}(v_1(x,t)) \right)^- \\ & \geq - \left((\bar{z}^{1,n}(x,t) - L_{x,t}(v_1(x,t))) - (\bar{z}^{2,n}(x,t) - L_{x,t}(v_2(x,t))) \right)^- \\ & = - \left((\bar{z}^{1,n} - \bar{z}^{2,n})(x,t) - (L_{x,t}(v_1(x,t)) - L_{x,t}(v_2(x,t))) \right)^-. \end{aligned} \quad (4.9)$$

Since

$$(\bar{z}^{1,n} - \bar{z}^{2,n})(x,t) > \sup_{x \in [0,1], t \in [0,T]} |L_{x,t}(v_1(x,t)) - L_{x,t}(v_2(x,t))| \geq L_{x,t}(v_1(x,t)) - L_{x,t}(v_2(x,t)),$$

holds for any $(x,t) \in \{(x,t) : \Psi(x,t) > 0\}$, it yields

$$\left((\bar{z}^{1,n} - \bar{z}^{2,n})(x,t) - (L_{x,t}(v_1(x,t)) - L_{x,t}(v_2(x,t))) \right)^- = 0.$$

Consequently, it follows from (4.9) that

$$\left(\bar{z}^{2,n}(x,t) - L_{x,t}(v_2(x,t)) \right)^- - \left(\bar{z}^{1,n}(x,t) - L_{x,t}(v_1(x,t)) \right)^- \geq 0.$$

Thus the same calculation as in (6.5) implies $\Psi^+ = 0$. Namely,

$$(\bar{z}^{1,n} - \bar{z}^{2,n})(x,t) \leq \sup_{x \in [0,1], t \in [0,T]} |L_{x,t}(v_1(x,t)) - L_{x,t}(v_2(x,t))|.$$

Interchanging $\bar{z}^{1,n}$ and $\bar{z}^{2,n}$, then $(\bar{z}^{2,n} - \bar{z}^{1,n})(x,t) \leq \sup_{x \in [0,1], t \in [0,T]} |L_{x,t}(v_2(x,t)) - L_{x,t}(v_1(x,t))|$ is actually true. Hence, we have

$$|(\bar{z}^{1,n} - \bar{z}^{2,n})(x,t)| \leq \sup_{x \in [0,1], t \in [0,T]} |L_{x,t}(v_1(x,t)) - L_{x,t}(v_2(x,t))|. \quad (4.10)$$

Combined with (4.3) and by Jensen's inequality, we therefore obtain

$$|\bar{z}^{1,n} - \bar{z}^{2,n}|_\infty^T \leq l \sup_{x \in [0,1], t \in [0,T]} E[|(v_1 - v_2)(x, t)|] \leq lE[|v_1 - v_2|_\infty^T],$$

which implies the inequality (4.8). \square

Based on Theorem 4.2 and Lemma 4.3, by the method of Picard iteration, we can deduce the following theorem for SPDE (1.1) with general mean reflection.

Theorem 4.4 *Suppose Assumptions (I), (II) and (H) hold, then there exists a unique deterministic flat solution of mean reflected SPDE (1.1). Moreover, $E[(|u|_\infty^T)^p] < \infty$ for all $p \geq 1$.*

Proof: Following the same proof as Theorem 3.3, we will also use a Picard iteration to obtain existence. However, in this case, it is noted that \bar{z}_i should be taken as the first component of the solution to obstacle problem (4.1) with $v = z_i$. That is,

$$\begin{aligned} z_n(x, t) &= \int_0^1 G_t(x, y) u_0(y) dy - \int_0^t \int_0^1 G_{t-s}(x, y) f(y, s; u_{n-1}) dy ds \\ &\quad + \int_0^t \int_0^1 G_{t-s}(x, y) \sigma(y, s; u_{n-1}) W(dy, ds), \quad n \in \mathbb{N}_+, \end{aligned}$$

and (\bar{z}_n, K_n) is the unique deterministic solution of obstacle problem (4.1) with $v = z_n$. Set $u_n := z_n + \bar{z}_n$, then (u_n, K_n) is the deterministic flat solution of the following SPDE

$$\begin{cases} \frac{\partial u_n(x, t)}{\partial t} - \frac{\partial^2 u_n(x, t)}{\partial x^2} + f(x, t; u_{n-1}) = \sigma(x, t; u_{n-1}) \dot{W}(x, t) + \dot{K}_n; \\ u_n(x, 0) = u_0(x); \\ u_n(0, t) = u_n(1, t) = 0, \end{cases}$$

with the constraint (1.2). From the proof of [11, Theorem 3.1] (refer to the estimate (18) in [11]), it follows that $E[(|z_1|_\infty^T)^p] < \infty, \forall p \geq 1$. In view of (4.7), then we have $E[(|u_1|_\infty^T)^p] \leq c(T, l, p) E[(|z_1|_\infty^T)^p] < \infty$, where $c(T, l, p)$ is some constant related to constants T, l and p . Without loss of generality, we assume $E[(|u_{n-1}|_\infty^T)^p] < \infty$ for any $p \geq 1$. Let $p > 6$ and $1/p + 1/q = 1$, we deduce that

$$\begin{aligned} E[(|z_n|_\infty^T)^p] &\leq c \left\{ \left| \int_0^1 G_t(x, y) u_0(y) dy \right|^p \right. \\ &\quad + \left(\sup_{x \in [0,1], t \in [0,T]} \int_0^t \int_0^1 G_{t-s}^q(x, y) dy ds \right)^{\frac{p}{q}} E \left[\int_0^T \int_0^1 |f(y, s; u_{n-1})|^p dx ds \right] \\ &\quad + \left(\sup_{x \in [0,1], t \in [0,T]} \int_0^t \int_0^1 G_{t-s}^{\frac{2p}{p-2}}(x, y) dy ds \right)^{\frac{p-2}{2}} E \left[\int_0^T \int_0^1 |\sigma(y, s; u_{n-1})|^p dx ds \right] \Big\} \\ &\leq c(T, M, p) \left(1 + \int_0^T E \left[\sup_{y \in [0,1], s \leq t} |u_{n-1}(y, s)|^p \right] dt \right), \end{aligned}$$

where (2.9) and (6.1) have been used in the last step. Combined with Lemma 4.3, we claim that the inequality (3.4) also holds. Hence, we can obtain the conclusion as (3.12): $E[(|u_m - u_n|_\infty^T)^p] \rightarrow 0$ as

n, m tends to ∞ . Thus there exists a random field $u(x, t) \in C([0, 1] \times [0, T])$ satisfying $E[|u|_\infty^p] < \infty$ such that $\lim_{n \rightarrow \infty} E[|u_n - u|_\infty^p] = 0$. It is easy to see that u satisfies the SPDE (1.1) with mean reflection (1.2) in full generality. Besides, following the similar argument as in the proof of Theorem 3.3 and combining the uniqueness of obstacle problem (4.1), it implies the deterministic flat solution is unique and we therefore omit the proof here. \square

5 Large deviation principles

This section is concerned with the following small noise perturbation of (1.1):

$$\begin{cases} \frac{\partial u^\varepsilon(x, t)}{\partial t} - \frac{\partial^2 u^\varepsilon(x, t)}{\partial x^2} + f(x, t; u^\varepsilon) = \sqrt{\varepsilon} \sigma(x, t; u^\varepsilon) \dot{W}(x, t) + \dot{K}^\varepsilon(x, t); \\ u^\varepsilon(x, 0) = u_0(x); \\ u^\varepsilon(0, t) = u^\varepsilon(1, t) = 0, \end{cases} \quad (5.1)$$

satisfying $E[h(x, t, u^\varepsilon(x, t))] \geq 0$ for all $x \in [0, 1]$, $t \geq 0$ and $\int_0^\infty \int_0^1 E[h(x, t, u^\varepsilon(x, t))] K^\varepsilon(dx, dt) = 0$. We devote to establishing a large deviation principle for $\{u^\varepsilon(\cdot, \cdot)\}_{\varepsilon > 0}$ in $C([0, 1] \times [0, T])$, the space of continuous functions in $[0, 1] \times [0, T]$ endowed with the uniform topology. In Subsection 5.1, we introduce the weak convergence method. Subsection 5.2 mainly study the associated skeleton equations. Finally, the large deviation principle is proved in Subsection 5.3 (Theorem 5.7).

5.1 Weak convergence approach

Now, we briefly recall the notions of large deviations. Let $\{X^\varepsilon\}_{\varepsilon > 0}$ be a family of random variables defined on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$ taking values in some Polish space \mathcal{E} .

Definition 5.1 (Rate function) *A function $I : \mathcal{E} \rightarrow [0, \infty]$ is called a rate function if I is lower semi-continuous. Moreover, a rate function I is called a good rate function if the level set $\{x \in \mathcal{E} : I(x) \leq M\}$ is compact for each $M < \infty$.*

Definition 5.2 (Large deviation principle) *The sequence $\{X^\varepsilon\}_{\varepsilon > 0}$ is said to satisfy the large deviation principle with rate function I if for each Borel subset A of \mathcal{E}*

$$-\inf_{x \in A^\circ} I(x) \leq \liminf_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P}(X^\varepsilon \in A) \leq \limsup_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P}(X^\varepsilon \in A) \leq -\inf_{x \in \bar{A}} I(x),$$

where A° and \bar{A} denote the interior and closure of A in \mathcal{E} , respectively.

The Cameron-Martin space associated with the Brownian sheet $W(x, t)$, $x \in [0, 1]$, $t \in [0, T]$ is given by

$$\mathcal{H} = \left\{ g = \int_0^\cdot \int_0^\cdot \dot{g}(x, s) dx ds : \int_0^T \int_0^1 \dot{g}^2(x, s) dx ds < \infty \right\}.$$

Let $\|g\|_{\mathcal{H}} := (\int_0^T \int_0^1 \dot{g}^2(x, s) dx ds)^{1/2}$ and $B_N := \{g \in \mathcal{H} : \|g\|_{\mathcal{H}} \leq N\}$, then the set B_N is a compact Polish space endowed with the weak topology of \mathcal{H} . Define

$$\mathcal{A} := \{ \phi : \phi \text{ is an } \mathcal{H}\text{-valued } \mathcal{F}_t\text{-predictable processes} \},$$

and

$$\mathcal{A}_N := \{\phi \in \mathcal{A} : \phi(\omega) \in B_N, \mathbb{P}\text{-a.s.}\}.$$

A sufficient condition (see Lemma 5.3) for the LDP of the sequence X^ε as $\varepsilon \rightarrow 0$, has been formulated recently in [24, Theorem 3.2]. It is a generalized version of classical weak convergence criteria by Budhiraja et al.[6, 7] and turns out to be more convenient in some applications. Intuitively, as the parameter $\varepsilon \rightarrow 0$, the disappearance of the small noise in (5.1) makes condition (II) of the following Lemma 5.3 not difficult to check. This new sufficient condition is widely applied to prove the LDP (refer to [10, 14, 18–20, 28] and so on).

Lemma 5.3 *For each $\varepsilon > 0$, $\Gamma^\varepsilon : C([0, 1] \times [0, T]) \rightarrow \mathcal{E}$ is a measurable map and let $X^\varepsilon := \Gamma^\varepsilon(W)$. Suppose that there exists a measurable mapping $\Gamma^0 : C([0, 1] \times [0, T]) \rightarrow \mathcal{E}$ such that the following two items hold:*

- (I) *For every $N < +\infty$ and any family $\{g^\varepsilon\}_{\varepsilon>0} \subset B_N$ that g^ε converges to some element g in B_N as $\varepsilon \rightarrow 0$, $\Gamma^0(\int_0^1 \int_0^T \dot{g}^\varepsilon(x, s) dx ds)$ converges to $\Gamma^0(\int_0^1 \int_0^T \dot{g}(x, s) dx ds)$ in the space \mathcal{E} .*
- (II) *For every $N < +\infty$, any family $\{g^\varepsilon\}_{\varepsilon>0} \subset \mathcal{A}_N$ and $\delta > 0$,*

$$\lim_{\varepsilon \rightarrow 0} \mathbb{P}(d(Y^\varepsilon, Z^\varepsilon) > \delta) = 0,$$

where $Y^\varepsilon := \Gamma^\varepsilon(W(\cdot, \cdot) + \frac{1}{\sqrt{\varepsilon}} \int_0^1 \int_0^T \dot{g}^\varepsilon(x, s) dx ds)$, $Z^\varepsilon := \Gamma^0(\int_0^1 \int_0^T \dot{g}^\varepsilon(x, s) dx ds)$ and $d(\cdot, \cdot)$ stands for the metric in the space \mathcal{E} .

Then the family $\{X^\varepsilon\}_{\varepsilon>0}$ satisfies a large deviation principle in \mathcal{E} with the following good rate function I defined by

$$I(\varphi) := \inf_{\{g \in \mathcal{H}; \varphi = \Gamma^0(\int_0^1 \int_0^T \dot{g}(x, s) dx ds)\}} \left\{ \frac{1}{2} \int_0^T \int_0^1 \dot{g}^2(x, s) dx ds \right\}, \quad \forall \varphi \in \mathcal{E}.$$

By convention, $I(\varphi) = \infty$, if $\{g \in \mathcal{H}; \varphi = \Gamma^0(\int_0^1 \int_0^T \dot{g}(x, s) dx ds)\} = \emptyset$.

5.2 Skeleton equation

Fix $g \in \mathcal{H}$, we consider the skeleton equation associated with (5.1), that is the following deterministic PDE:

$$\begin{cases} \frac{\partial u^g(x, t)}{\partial t} - \frac{\partial^2 u^g(x, t)}{\partial x^2} + f(x, t; u^g) = \sigma(x, t; u^g) \dot{g}(x, t) + \dot{K}^g(x, t), & x \in [0, 1], t \geq 0; \\ u^g(x, 0) = u_0(x), & x \in [0, 1]; \\ u^g(0, t) = u^g(1, t) = 0, & t \geq 0, \end{cases} \quad (5.2)$$

satisfying $h(x, t, u^\varepsilon(x, t)) \geq 0$ for all $x \in [0, 1]$, $t \geq 0$ and $\int_0^\infty \int_0^1 h(x, t, u^\varepsilon(x, t)) K^\varepsilon(dx, dt) = 0$. In this subsection, the main purpose is to show that the skeleton equation (5.2) admits a unique solution for every $g \in B_N$ (Theorem 5.4) and the continuity of the skeleton equation (Theorem 5.5).

Analogously to Definition 2.3, a pair of (u^g, K^g) is said to be a solution of problem (5.2), if it satisfies

- (i) u^g is a continuous process on $[0, 1] \times \mathbb{R}_+$ and $h(x, t, u^g(x, t)) \geq 0$.
- (ii) K^g is a measure on $(0, 1) \times \mathbb{R}_+$ such that $\int_0^t \int_0^1 h(x, t, u^g(x, t)) K^g(dx, ds) = 0$.
- (iii) (u^g, K^g) solves the parabolic PDE in the following sense: for every $t \in [0, \infty)$ and $\varphi \in C^2([0, 1])$ with $\varphi(0) = \varphi(1) = 0$,

$$\begin{aligned} & \langle u^g(t), \varphi \rangle - \int_0^t \langle u^g(s), \varphi'' \rangle ds + \int_0^t \int_0^1 f(x, s; u^g) \varphi(x) dx ds \\ &= \langle u_0, \varphi \rangle + \int_0^t \int_0^1 \sigma(x, s; u^g) \varphi(x) \dot{g}(x, s) dx ds + \int_0^t \int_0^1 \varphi(x) K^g(dx, ds). \end{aligned}$$

Firstly, we will prove the existence and uniqueness of solutions to problem (5.2) for any $g \in B_N$. The proof is similar to Theorem 3.3. But it should be noted that when dealing with the heat kernel G , we try to obtain the forms (6.2)-(6.4) in Lemma 6.1 instead of form (6.1) in Appendix.

Theorem 5.4 *Assume that the assumption (H) holds and f, σ satisfy the conditions (I), (II), then for any $g \in B_N$, problem (5.2) admits a unique solution.*

Proof: Let us first prove the existence of problem (5.2). Similar to the proof of Theorem 4.4, we use a Picard argument to prove the existence of the solution to (5.2). Let

$$\begin{aligned} z_1^g(x, t) &:= \int_0^1 G_t(x, y) u_0(y) dy - \int_0^t \int_0^1 G_{t-s}(x, y) f(y, s; u_0) dy ds \\ &\quad + \int_0^t \int_0^1 G_{t-s}(x, y) \sigma(y, s; u_0) \dot{g}(y, s) dy ds, \end{aligned}$$

and (\bar{z}_1^g, K_1^g) be the solution of (4.1) with $v = z_1^g$. We denote $u_1^g := z_1^g + \bar{z}_1^g$, then (u_1^g, K_1^g) is a solution of the following PDE

$$\begin{cases} \frac{\partial u_1^g(x, t)}{\partial t} - \frac{\partial^2 u_1^g(x, t)}{\partial x^2} + f(x, t; u_0) = \sigma(x, t; u_0) \dot{g}(x, t) + \dot{K}_1^g(x, t), \\ u_1^g(x, 0) = u_0(x), \\ u_1^g(0, t) = u_1^g(1, t) = 0, \end{cases}$$

satisfying $h(x, t, u_1^\varepsilon(x, t)) \geq 0$ for all $(x, t) \in [0, 1] \times \mathbb{R}_+$ and $\int_0^\infty \int_0^1 h(x, t, u_1^\varepsilon(x, t)) K_1^\varepsilon(dx, dt) = 0$. Generally, set

$$\begin{aligned} z_n^g(x, t) &:= \int_0^1 G_t(x, y) u_0(y) dy - \int_0^t \int_0^1 G_{t-s}(x, y) f(y, s; u_{n-1}^g) dy ds \\ &\quad + \int_0^t \int_0^1 G_{t-s}(x, y) \sigma(y, s; u_{n-1}^g) \dot{g}(y, s) dy ds, \end{aligned}$$

and define (\bar{z}_n^g, K_n^g) as the solution of (4.1) with $v = z_{n-1}^g$. Hence, we deduce that $(u_n^g := z_n^g + \bar{z}_n^g, K_n^g)$ is the solution of the following PDE

$$\begin{cases} \frac{\partial u_n^g(x, t)}{\partial t} - \frac{\partial^2 u_n^g(x, t)}{\partial x^2} + f(x, t; u_{n-1}^g) = \sigma(x, t; u_{n-1}^g) \dot{g}(x, t) + \dot{K}_n^g(x, t), \\ u_n^g(x, 0) = u_0(x), \\ u_n^g(0, t) = u_n^g(1, t) = 0, \end{cases}$$

satisfying $h(x, t, u_n^\varepsilon(x, t)) \geq 0$ for all $(x, t) \in [0, 1] \times \mathbb{R}_+$ and $\int_0^\infty \int_0^1 h(x, t, u_n^\varepsilon(x, t)) K_n^\varepsilon(dx, dt) = 0$.

In view of (4.7), then we can obtain

$$\begin{aligned}
& |u_n^g - u_{n-1}^g|_\infty^T \\
&= |(z_n^g + \bar{z}_n^g) - (z_{n-1}^g + \bar{z}_{n-1}^g)|_\infty^T \\
&\leq (1+l) |z_n^g - z_{n-1}^g|_\infty^T \\
&\leq (1+l) \sup_{x \in [0,1], t \in [0,T]} \left| \int_0^t \int_0^1 G_{t-s}(x, y) (f(y, s; u_{n-1}^g) - f(y, s; u_{n-2}^g)) dy ds \right| \\
&\quad + (1+l) \sup_{x \in [0,1], t \in [0,T]} \left| \int_0^t \int_0^1 G_{t-s}(x, y) (\sigma(y, s; u_{n-1}^g) - \sigma(y, s; u_{n-2}^g)) \dot{g}(y, s) dy ds \right| \\
&=: (1+l) |I_1|_\infty^T + (1+l) |I_2|_\infty^T,
\end{aligned}$$

where l is the constant in (4.3). By (2.8) and Hölder's inequality, it yields for any $p > 4$,

$$\begin{aligned}
(I_1(x, t))^p &\leq C(T) \left| \int_0^t |u_{n-1}^g - u_{n-2}^g|_\infty^s \left(\int_0^1 G_{t-s}(x, y) dy \right) ds \right|^p \\
&\leq C(T) \cdot t^{\frac{p}{2}} \left[\int_0^t \left(|u_{n-1}^g - u_{n-2}^g|_\infty^s \right)^2 \left(\int_0^1 G_{t-s}^2(x, y) dy \right) ds \right]^{\frac{p}{2}} \\
&\leq C(T) \cdot t^{\frac{p}{2}} \left[\int_0^t \left(\int_0^1 G_{t-s}^2(x, y) dy \right)^{\frac{p-2}{2}} ds \right]^{\frac{p-2}{2}} \cdot \left[\int_0^t \left(|u_{n-1}^g - u_{n-2}^g|_\infty^s \right)^p ds \right].
\end{aligned}$$

Combined with the estimate (6.2) for function G , we have

$$\begin{aligned}
& (|I_1|_\infty^T)^p \\
&\leq C(T, p) \left\{ \sup_{x \in [0,1]} \left[\int_0^T \left(\int_0^1 G_{t-s}^2(x, y) dy \right)^{\frac{p-2}{2}} ds \right]^{\frac{p-2}{2}} \right\} \cdot \left[\int_0^T \left(|u_{n-1}^g - u_{n-2}^g|_\infty^s \right)^p ds \right] \\
&\leq C(T, p) \cdot T^{\frac{p-4}{4}} \left[\int_0^T \left(|u_{n-1}^g - u_{n-2}^g|_\infty^s \right)^p ds \right], \tag{5.3}
\end{aligned}$$

where $C(T, p)$ is a constant depending on T and p . Meanwhile, in view of $\|g\|_{\mathcal{H}} \leq N$ and from above similar arguments, it follows that

$$\begin{aligned}
& (|I_2|_\infty^T)^p \\
&\leq C(T) \sup_{x \in [0,1], t \in [0,T]} \left| \int_0^t \int_0^1 G_{t-s}^2(x, y) \left(|u_{n-1}^g - u_{n-2}^g|_\infty^s \right)^2 dy ds \right|^{\frac{p}{2}} \cdot \left| \int_0^t \int_0^1 \dot{g}^2(y, s) dy ds \right|^{\frac{p}{2}} \\
&\leq C(T) \left\{ \sup_{x \in [0,1]} \left[\int_0^T \left(\int_0^1 G_{t-s}^2(x, y) dy \right)^{\frac{p-2}{2}} ds \right]^{\frac{p-2}{2}} \right\} \cdot \left| \int_0^t \int_0^1 \dot{g}^2(y, s) dy ds \right|^{\frac{p}{2}} \\
&\quad \cdot \left[\int_0^T \left(|u_{n-1}^g - u_{n-2}^g|_\infty^s \right)^p ds \right] \\
&\leq C(T, p) \cdot N^p T^{\frac{p-4}{4}} \left[\int_0^T \left(|u_{n-1}^g - u_{n-2}^g|_\infty^s \right)^p ds \right]. \tag{5.4}
\end{aligned}$$

Therefore, we conclude from above displays that

$$\begin{aligned}
\left(|u_n^g - u_{n-1}^g|_\infty^T\right)^p &\leq C(T, p, l, N) \cdot \int_0^T \left(|u_{n-1}^g - u_{n-2}^g|_\infty^s\right)^p ds \\
&\leq C^{m-1}(T, p, l, N) \int_0^T \int_0^{t_1} \cdots \int_0^{t_{n-2}} \left(|u_1^g - u_0|_\infty^{t_{n-1}}\right)^p dt_{n-1} \cdots dt_2 dt_1 \\
&\leq C^{m-1}(T, p, l, N) \frac{T^{n-1}}{(n-1)!} \cdot (|u_1^g - u_0|_\infty^T)^p,
\end{aligned} \tag{5.5}$$

where $C(T, p, l, N)$ is some constant depending on T, p, l and N . Then it implies for any $m \geq n \geq 1$,

$$|u_m^g - u_n^g|_\infty^T \leq |u_1^g - u_0|_\infty^T \cdot \sum_{k=n}^{m-1} \left(\frac{C^k(T, p, l, N) T^k}{k!} \right)^{\frac{1}{p}} \rightarrow 0,$$

as $m, n \rightarrow \infty$. Therefore, $\{u_n^g\}_{n \geq 1}$ is a Cauchy sequence in $C([0, 1] \times [0, T])$ and denote its limit by $u^g(x, t)$.

By Assumptions (H), then we have $|h(x, t, u_n^g(x, t))| \leq C(1 + |u_n^g|_\infty^T) < \infty$, which implies $\lim_{n \rightarrow \infty} h(x, t, u_n^g(x, t)) = h(x, t, u^g(x, t))$. On the other hand, in view of $h(x, t, u_n^g(x, t)) \geq 0$, we can obtain that $h(x, t, u^g(x, t)) \geq 0$. Let

$$\begin{aligned}
z^g(x, t) &:= \int_0^1 G_t(x, y) u_0(y) dy - \int_0^t \int_0^1 G_{t-s}(x, y) f(y, s; u^g) dy ds \\
&\quad + \int_0^t \int_0^1 G_{t-s}(x, y) \sigma(y, s; u^g) \dot{g}(y, s) dy ds,
\end{aligned}$$

and $\tilde{u} := z^g + \bar{z}^g$, where (\bar{z}^g, K^g) solves the parabolic obstacle problem (4.1) with $v = z^g$. Consequently,

$$\begin{aligned}
&\langle \tilde{u}^g(t), \varphi \rangle - \int_0^t \langle \tilde{u}^g(s), \varphi'' \rangle ds + \int_0^t \int_0^1 f(x, s; \tilde{u}^g) \varphi(x) dx ds \\
&= \langle u_0, \varphi \rangle + \int_0^t \int_0^1 \sigma(x, s; \tilde{u}^g) \varphi(x) \dot{g}(y, s) dy ds + \int_0^t \int_0^1 \varphi(x) K^g(dx, ds),
\end{aligned}$$

holds for any $\varphi \in C^2([0, 1])$ with $\varphi(0) = \varphi(1) = 0$. From a similar arguments as in (5.5), it yields that

$$\left(|u_n^g - \tilde{u}^g|_\infty^T\right)^p \leq C(T, l, p) \left(|u_{n-1}^g - u^g|_\infty^T\right)^p \rightarrow 0,$$

as n tends to infinity. Hence we finally have $\tilde{u}^g = u^g$. Just as the proof in Theorem 3.3, one can similarly prove the uniqueness of solution to (5.2) and we therefore omit it here. The proof of this is complete. \square

In view of Theorem 5.4, we can define a mapping $\Gamma^0 : C([0, 1] \times [0, T]) \rightarrow C([0, 1] \times [0, T])$ by

$$\Gamma^0(g) := \begin{cases} u^g & \text{if } g \in \mathcal{H}, \\ 0 & \text{otherwise,} \end{cases} \tag{5.6}$$

where u^g is the first component of the solution to equation (5.2). Next, we are going to prove the continuity of the mapping Γ^0 . Namely, let u^{g_n} denote the solution of (5.2) with \dot{g} replaced by \dot{g}_n and we will show that u^{g_n} converges to the solution u^g of the skeleton equation (5.2) in $C([0, 1] \times [0, T])$, if $g_n \rightarrow g$ weakly in B_N .

Theorem 5.5 *Under assumptions in Theorem 5.4, u^g is a continuous mapping from $g \in B_N$ into $C([0, 1] \times [0, T])$.*

Proof: Let $g_n, g \in B_N$ be such that g_n converges weakly to g as n tends to infinity. To prove the theorem, it is sufficient to show that

$$\lim_{n \rightarrow \infty} |u^{g_n} - u^g|_{\infty}^T = 0, \quad (5.7)$$

where u^{g_n} and u^g are the first component of the solutions to (5.2) associated with g_n and g , respectively. Define

$$\begin{aligned} z^{g_n}(x, t) &:= \int_0^1 G_t(x, y) u_0(y) dy - \int_0^t \int_0^1 G_{t-s}(x, y) f(y, s; u^{g_n}) dy ds \\ &\quad + \int_0^t \int_0^1 G_{t-s}(x, y) \sigma(y, s; u^{g_n}) \dot{g}_n(y, s) dy ds, \end{aligned}$$

and

$$\begin{aligned} z^g(x, t) &:= \int_0^1 G_t(x, y) u_0(y) dy - \int_0^t \int_0^1 G_{t-s}(x, y) f(y, s; u^g) dy ds \\ &\quad + \int_0^t \int_0^1 G_{t-s}(x, y) \sigma(y, s; u^g) \dot{g}(y, s) dy ds, \end{aligned}$$

then it is easy to see that $(u^{g_n} - z^{g_n})$ and $(u^g - z^g)$ are the solutions of obstacle problem (4.1) associated with z^{g_n} and z^g , respectively. Using the estimate (4.7) and by the fact that z^{g_n} and z^g are deterministic, then it yields

$$\begin{aligned} |u^{g_n} - u^g|_{\infty}^T &\leq (1+l) |z^{g_n} - z^g|_{\infty}^T \\ &\leq (1+l) \sup_{x \in [0,1], t \in [0,T]} \left| \int_0^t \int_0^1 G_{t-s}(x, y) [f(y, s; u^{g_n}) - f(y, s; u^g)] dy ds \right| \\ &\quad + (1+l) \sup_{x \in [0,1], t \in [0,T]} \left| \int_0^t \int_0^1 G_{t-s}(x, y) [\sigma(y, s; u^{g_n}) - \sigma(y, s; u^g)] \dot{g}_n(y, s) dy ds \right| \\ &\quad + (1+l) \sup_{x \in [0,1], t \in [0,T]} \left| \int_0^t \int_0^1 G_{t-s}(x, y) \sigma(y, s; u^g) [\dot{g}_n(y, s) - \dot{g}(y, s)] dy ds \right| \\ &=: (1+l) [|I_1|_{\infty}^T + |I_2|_{\infty}^T + |I_3|_{\infty}^T], \end{aligned}$$

where l is the constant in (4.3). Similar to (5.3) and (5.4), we can obtain that for any $p > 4$, $(|I_1|_{\infty}^T)^p \leq C(T, p) \int_0^T (|u^{g_n} - u^g|_{\infty}^s)^p ds$ and $(|I_2|_{\infty}^T)^p \leq C(T, p, N) \int_0^T (|u^{g_n} - u^g|_{\infty}^s)^p ds$. Hence,

$$(|u^{g_n} - u^g|_{\infty}^T)^p \leq C \int_0^T (|u^{g_n} - u^g|_{\infty}^s)^p ds + 3^{p-1} \cdot [((1+l)|I_3|_{\infty}^T)]^p.$$

Denote

$$F_n(x, t) := \int_0^t \int_0^1 G_{t-s}(x, y) \sigma(y, s; u^g) [\dot{g}_n(y, s) - \dot{g}(y, s)] dy ds.$$

Combined with Gronwall's inequality, (5.7) will be satisfied if we can prove

$$\lim_{n \rightarrow \infty} |F_n|_\infty^T = 0. \quad (5.8)$$

The rest of the proof is to establish (5.8). Note that for any fixed $x \in [0, 1]$ and $t \in [0, T]$,

$$\begin{aligned} & \int_0^t \int_0^1 G_{t-s}^2(x, y) \sigma^2(y, s; u^g) dy ds \\ & \leq M^2(T) (1 + |u^g|_\infty^T)^2 \left(\int_0^t \int_0^1 G_{t-s}^2(x, y) dy ds \right) \\ & < \infty. \end{aligned}$$

Since g_n converges weakly to g in \mathcal{H} , we have $\lim_{n \rightarrow \infty} F_n(x, t) = 0$ for any fixed $x \in [0, 1]$ and $t \in [0, T]$. Therefore, we just remain to prove that $\{F_n(x, t)\}_{n \geq 1}$ is relatively compact in $C([0, 1] \times [0, T])$. In fact,

$$\begin{aligned} \sup_n |F_n|_\infty^T & \leq M(T) (1 + |u^g|_\infty^T) \cdot \left(\sup_{x \in [0, 1], t \in [0, T]} \left| \int_0^t \int_0^1 G_s^2(x, y) dy ds \right| \right)^{\frac{1}{2}} \\ & \quad \cdot \left(\int_0^T \int_0^1 |\dot{g}_n(y, s) - \dot{g}(y, s)|^2 dy ds \right)^{\frac{1}{2}} \\ & < \infty, \end{aligned}$$

where the fact that $\|g_n\|_{\mathcal{H}}$ and $\|g\|_{\mathcal{H}}$ are bounded by N has been used. On the other hand, we claim that $\{F_n(x, t)\}_{n \geq 1}$ is also equi-continuous. Indeed, for every $0 \leq s \leq t \leq T$ and $x, y \in [0, 1]$, we have for any $p > 4$,

$$\begin{aligned} & |F_n(x, t) - F_n(s, y)|^p \\ = & \left| \int_s^t \int_0^1 G_{t-r}(x, z) \sigma(z, r; u^g) (\dot{g}_n(z, r) - \dot{g}(z, r)) dz dr \right. \\ & \left. + \int_0^s \int_0^1 (G_{t-r}(x, z) - G_{s-r}(y, z)) \sigma(z, r; u^g) (\dot{g}_n(z, r) - \dot{g}(z, r)) dz dr \right|^p \\ \leq & 3^{p-1} \cdot \left| \int_s^t \int_0^1 G_{t-r}(x, z) \sigma(z, r; u^g) (\dot{g}_n(z, r) - \dot{g}(z, r)) dz dr \right|^p \\ & + 3^{p-1} \cdot \left| \int_0^s \int_0^1 (G_{t-r}(x, z) - G_{s-r}(x, z)) \sigma(z, r; u^g) (\dot{g}_n(z, r) - \dot{g}(z, r)) dz dr \right|^p \\ & + 3^{p-1} \cdot \left| \int_0^s \int_0^1 (G_{s-r}(x, z) - G_{s-r}(y, z)) \sigma(z, r; u^g) (\dot{g}_n(z, r) - \dot{g}(z, r)) dz dr \right|^p. \end{aligned}$$

By (2.8), Hölder's inequality and the estimate (6.2), it yields

$$\begin{aligned}
& \left| \int_s^t \int_0^1 G_{t-r}(x, z) \sigma(z, r; u^g) (\dot{g}_n(z, r) - \dot{g}(z, r)) dz dr \right|^p \\
& \leq \left| \int_s^t \int_0^1 G_{t-r}^2(x, z) \sigma^2(z, r; u^g) dz dr \right|^{\frac{p}{2}} \cdot \left| \int_s^t \int_0^1 (\dot{g}_n(z, r) - \dot{g}(z, r))^2 dz dr \right|^{\frac{p}{2}} \\
& \leq (2N)^p M(T)^2 \left| \int_s^t \left(\int_0^1 G_{t-r}^2(x, z) dz \right) (1 + |u^g|_\infty^r)^2 dr \right|^{\frac{p}{2}} \\
& \leq C(T, N, p) \left[\int_s^t \left(\int_0^1 G_{t-r}^2(x, z) dz \right)^{\frac{p}{p-2}} dr \right]^{\frac{p-2}{2}} \cdot \left[\int_s^t (1 + |u^g|_\infty^r)^p dr \right] \\
& \leq C(T, N, p) |t - s|^{\frac{p-4}{4}} (1 + |u^g|_\infty^T)^p.
\end{aligned}$$

Using the estimates (6.3) and (6.4) for function G and by similar arguments as above, we can also obtain

$$\begin{aligned}
& \left| \int_0^s \int_0^1 (G_{t-r}(x, z) - G_{s-r}(x, z)) \sigma(z, r; u^g) (\dot{g}_n(z, r) - \dot{g}(z, r)) dz dr \right|^p \\
& \leq C(T, N, p) |t - s|^{\frac{p-4}{4}} (1 + |u^g|_\infty^T)^p,
\end{aligned}$$

and

$$\begin{aligned}
& \left| \int_0^s \int_0^1 (G_{s-r}(x, z) - G_{s-r}(y, z)) \sigma(z, r; u^g) (\dot{g}_n(z, r) - \dot{g}(z, r)) dz dr \right|^p \\
& \leq C(T, N, p) |x - y|^{\frac{p-4}{2}} (1 + |u^g|_\infty^T)^p.
\end{aligned}$$

Consequently, we conclude that for every $p > 4$,

$$|F_n(x, t) - F_n(s, y)|^p \leq C(T, N, p) (1 + |u^g|_\infty^T)^p (|x - y|^{\frac{p-4}{2}} + |t - s|^{\frac{p-4}{4}}),$$

holds for any $0 \leq s \leq t \leq T$ and $x, y \in [0, 1]$. Applying Arzelà-Ascoli theorem, we finally deduce that $\{F_n(x, t)\}_{n \geq 1}$ is relatively compact in $C([0, 1] \times [0, T])$. The theorem is proved. \square

5.3 Large deviation principles

After the preparations in above subsections, we are ready to state and prove the large deviation result (Theorem 5.7). In this subsection, based on the Lemma 5.3, we mainly apply weak convergence approach to establish a LDP for the solution of mean reflected SPDE (5.1) with the constraint (1.2) and the associated Skorokhod condition.

Under Assumptions (I), (II) and (H), it yields from Theorem 4.4 that for every $\varepsilon > 0$, there exists a unique deterministic flat solution $(u^\varepsilon, K^\varepsilon)$ to mean reflected SPDE (5.1) with the constraint (1.2). Therefore, there exists a Borel measurable map

$$\Gamma^\varepsilon : C([0, 1] \times [0, T]) \rightarrow C([0, 1] \times [0, T]),$$

such that u^ε could be represented by $u^\varepsilon(\cdot, \cdot) = \Gamma^\varepsilon(W(\cdot, \cdot))$. For any $\{g^\varepsilon\}_{\varepsilon>0} \subset \mathcal{A}_N$ and let $\bar{u}^\varepsilon := \Gamma^0(g^\varepsilon)$. According to the definition of Γ^0 in (5.6), we deduce from Theorem 5.4 that \bar{u}^ε solves the following PDE

$$\begin{cases} \frac{\partial \bar{u}^\varepsilon(x,t)}{\partial t} - \frac{\partial^2 \bar{u}^\varepsilon(x,t)}{\partial x^2} + f(x,t; \bar{u}^\varepsilon) = \sigma(x,t; \bar{u}^\varepsilon) \dot{g}^\varepsilon(x,t) + \dot{K}^\varepsilon(x,t); \\ \bar{u}^\varepsilon(x,0) = u_0(x); \\ \bar{u}^\varepsilon(0,t) = \bar{u}^\varepsilon(1,t) = 0, \end{cases} \quad (5.9)$$

which satisfies $h(x,t, \bar{u}^\varepsilon(x,t)) \geq 0$ for all $(x,t) \in [0,1] \times \mathbb{R}_+$ and $\int_0^\infty \int_0^1 h(x,t, \bar{u}^\varepsilon(x,t)) \bar{K}^\varepsilon(dx, dt) = 0$. Fixed $g^\varepsilon \in \mathcal{A}_N$, set

$$\frac{d\mathcal{Q}^{g,\varepsilon}}{d\mathcal{P}} := \exp \left\{ -\frac{1}{\sqrt{\varepsilon}} \int_0^T \int_0^1 \dot{g}^\varepsilon(x,s) W(dx, ds) - \frac{1}{2\varepsilon} \int_0^T \int_0^1 (\dot{g}^\varepsilon)^2(x,s) dx ds \right\}.$$

It is well known that $\mathcal{Q}^{g,\varepsilon}$ is obviously a probability measure, which is mutually absolutely continuous with respect to \mathcal{P} . Applying Girsanov's theorem, the process

$$\widetilde{W} := W + \frac{1}{\sqrt{\varepsilon}} \int_0^\cdot \int_0^\cdot \dot{g}^\varepsilon(x,s) dx ds,$$

is a Brownian sheet under the measure $\mathcal{Q}^{g,\varepsilon}$. Let $(\tilde{u}^\varepsilon(\cdot, \cdot) := \Gamma^\varepsilon(\widetilde{W}(\cdot, \cdot)), \tilde{K}^\varepsilon)$ be the unique solution of (5.1) with \widetilde{W} under the measure $\mathcal{Q}^{g,\varepsilon}$. Then it is easy to see that $(\tilde{u}^\varepsilon, \tilde{K}^\varepsilon)$ is the unique deterministic flat solution of the mean reflected SPDE

$$\begin{cases} \frac{\partial \tilde{u}^\varepsilon(x,t)}{\partial t} - \frac{\partial^2 \tilde{u}^\varepsilon(x,t)}{\partial x^2} + f(x,t; \tilde{u}^\varepsilon) = \sqrt{\varepsilon} \sigma(x,t; \tilde{u}^\varepsilon) \dot{W}(x,t) + \sigma(x,t; \tilde{u}^\varepsilon) \dot{g}^\varepsilon(x,t) + \dot{K}^\varepsilon(x,t); \\ \tilde{u}^\varepsilon(x,0) = u_0(x); \\ \tilde{u}^\varepsilon(0,t) = \tilde{u}^\varepsilon(1,t) = 0, \end{cases} \quad (5.10)$$

with $E[h(x,t, \tilde{u}^\varepsilon(x,t))] \geq 0$ for any $(x,t) \in [0,1] \times \mathbb{R}_+$.

According to Lemma 5.3 (a sufficient condition for the LDP) and Theorem 5.5, we only need to prove the following result to establish the large deviation principle for $\{u^\varepsilon(\cdot, \cdot)\}_{\varepsilon>0}$ in $C([0,1] \times [0,T])$.

Theorem 5.6 *For every $N < \infty$ and let $\{g^\varepsilon\}_{\varepsilon>0} \subset \mathcal{A}_N$, then*

$$\left| \Gamma^\varepsilon \left(W(\cdot, \cdot) + \frac{1}{\sqrt{\varepsilon}} \int_0^\cdot \int_0^\cdot \dot{g}^\varepsilon(x,s) dx ds \right) - \Gamma^0 \left(\int_0^\cdot \int_0^\cdot \dot{g}^\varepsilon(x,s) dx ds \right) \right|_\infty^T \rightarrow 0 \quad \text{in probability.}$$

Proof: Recall that $\tilde{u}^\varepsilon(\cdot, \cdot) := \Gamma^\varepsilon(W(\cdot, \cdot) + \frac{1}{\sqrt{\varepsilon}} \int_0^\cdot \int_0^\cdot \dot{g}^\varepsilon(x,s) dx ds)$ is the first component of solution of the mean reflected SPDE (5.10). Moreover, $\bar{u}^\varepsilon := \Gamma^0(\int_0^\cdot \int_0^\cdot \dot{g}^\varepsilon(x,s) dx ds)$ is the the first component of solution of the skeleton equation (5.9). In the following, we aim to prove $|\tilde{u}^\varepsilon - \bar{u}^\varepsilon|_\infty^T \rightarrow 0$ in probability as $\varepsilon \rightarrow 0$. To achieve that, we define

$$\begin{aligned} \tilde{z}^\varepsilon(x,t) &:= \int_0^1 G_t(x,y) u_0(y) dy - \int_0^t \int_0^1 G_{t-s}(x,y) f(y,s; \tilde{u}^\varepsilon) dy ds \\ &\quad + \sqrt{\varepsilon} \int_0^t \int_0^1 G_{t-s}(x,y) \sigma(y,s; \tilde{u}^\varepsilon) W(dx, ds) \\ &\quad + \int_0^t \int_0^1 G_{t-s}(x,y) \sigma(y,s; \tilde{u}^\varepsilon) \dot{g}^\varepsilon(y,s) dy ds, \end{aligned}$$

and

$$\begin{aligned}\bar{z}^\varepsilon(x, t) &:= \int_0^1 G_t(x, y) u_0(y) dy - \int_0^t \int_0^1 G_{t-s}(x, y) f(y, s; \bar{u}^\varepsilon) dy ds \\ &\quad + \int_0^t \int_0^1 G_{t-s}(x, y) \sigma(y, s; \bar{u}^\varepsilon) \dot{g}^\varepsilon(y, s) dy ds.\end{aligned}$$

Then, it is noted that $(\tilde{u}^\varepsilon - \tilde{z}^\varepsilon, \tilde{K}^\varepsilon)$ and $(\bar{u}^\varepsilon - \bar{z}^\varepsilon, \bar{K}^\varepsilon)$ are the solutions to obstacle problem (4.1) with $v = \tilde{z}^\varepsilon$ and $v = \bar{z}^\varepsilon$, respectively. From Lemma 4.3, it follows that $|\tilde{u}^\varepsilon - \bar{u}^\varepsilon|_\infty^T \leq |\tilde{z}^\varepsilon - \bar{z}^\varepsilon|_\infty^T + l E[|\tilde{z}^\varepsilon - \bar{z}^\varepsilon|_\infty^T]$ almost surely. Consequently,

$$\begin{aligned}E[|\tilde{u}^\varepsilon - \bar{u}^\varepsilon|_\infty^T] &\leq (1+l)E[|\tilde{z}^\varepsilon - \bar{z}^\varepsilon|_\infty^T] \\ &\leq (1+l)E\left[\left|\int_0^t \int_0^1 G_{t-s}(x, y) [f(y, s; \tilde{u}^\varepsilon) - f(y, s; \bar{u}^\varepsilon)] dy ds\right|_\infty^T\right] \\ &\quad + (1+l)\sqrt{\varepsilon}E\left[\left|\int_0^t \int_0^1 G_{t-s}(x, y) \sigma(y, s; \tilde{u}^\varepsilon) W(dy, ds)\right|_\infty^T\right] \\ &\quad + (1+l)E\left[\left|\int_0^t \int_0^1 G_{t-s}(x, y) [\sigma(y, s; \tilde{u}^\varepsilon) - \sigma(y, s; \bar{u}^\varepsilon)] \dot{g}^\varepsilon(y, s) dy ds\right|_\infty^T\right] \\ &=: (1+l)E[I_{1,\varepsilon} + I_{2,\varepsilon} + I_{3,\varepsilon}].\end{aligned}$$

Similar to the estimate of (5.3) and (5.4), we deduce from Lemma 6.1 that for any $p > 4$,

$$E[(I_{1,\varepsilon})^p] \leq C(T, p) E\left[\int_0^T (|\tilde{u}^\varepsilon - \bar{u}^\varepsilon|_\infty^s)^p ds\right],$$

and

$$E[(I_{3,\varepsilon})^p] \leq C(T, p, N) E\left[\int_0^T (|\tilde{u}^\varepsilon - \bar{u}^\varepsilon|_\infty^s)^p ds\right].$$

Then Gronwall's inequality yields that $E[(|\tilde{u}^\varepsilon - \bar{u}^\varepsilon|_\infty^T)^p] \leq 3^{p-1} (1+l)^p E[(I_{2,\varepsilon})^p] e^{CT}$. On the other hand, applying (2.9) and Burkholder's inequality, we obtain for any $p > 4$,

$$\begin{aligned}E[(I_{2,\varepsilon})^p] &\leq \varepsilon^{\frac{p}{2}} \cdot M^p E\left[\left(\int_0^T \int_0^1 G_{t-s}^2(x, y) (1 + |\tilde{u}^\varepsilon|_\infty^s) dy ds\right)^{\frac{p}{2}}\right] \\ &\leq \varepsilon^{\frac{p}{2}} \cdot M^p \left[\int_0^T \left(\int_0^1 G_{t-s}^2(x, y) dy\right)^{\frac{p-2}{2}} ds\right]^{\frac{p-2}{2}} \cdot E\left[\int_0^T (1 + |\tilde{u}^\varepsilon|_\infty^s)^p ds\right] \\ &\leq C(T, p, M) \varepsilon^{\frac{p}{2}} E\left[\int_0^T (1 + |\tilde{u}^\varepsilon|_\infty^s)^p ds\right] \\ &\rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0,\end{aligned}$$

where the estimate (6.2) and fact that $E[(|\tilde{u}^\varepsilon|_\infty^T)^p] < \infty$ have been used. By Chebyshev's inequality, we have $|\tilde{u}^\varepsilon - \bar{u}^\varepsilon|_\infty^T$ tends to 0 in probability as $\varepsilon \rightarrow 0$. We complete the proof of this theorem. \square

Combined with Theorem 5.5 and Theorem 5.6, we now are in a position to give the main result of Section 5.

Theorem 5.7 *The family $\{u^\varepsilon\}_{\varepsilon>0}$ satisfies a large deviation principle with the good rate function \bar{I} defined by*

$$\bar{I}(\varphi) := \frac{1}{2} \inf_{\{g \in \mathcal{H}; \varphi = \Gamma^0(\int_0^1 \int_0^1 \dot{g}(x,s) dx ds)\}} \left\{ \int_0^T \int_0^1 \dot{g}^2(x,s) dx ds \right\},$$

where infimum over an empty set is taken as ∞ and Γ^0 is the map defined in (5.6).

6 Appendix

Recall that the function G is the Dirichlet heat kernel on $[0, 1]$. By the appendix in [8, 9, 21], it has the following expression

$$G_t(x, y) = \frac{1}{\sqrt{4\pi t}} \sum_{n=-\infty}^{\infty} \left[\exp\left(-\frac{(x-y+2n)^2}{4t}\right) - \exp\left(-\frac{(x+y+2n)^2}{4t}\right) \right].$$

Now, we give some estimates for function G , which have been proved in [21, Proposition A.4]. They play a crucial role in the whole proof of this section.

Lemma 6.1 *Fix $T > 0$, we have that*

(i)

$$\sup_{x \in [0,1], t \in [0,T]} \int_0^t \int_0^1 G_{t-s}^r(x, y) dy ds < \infty, \quad \forall 0 < r < 3, \quad (6.1)$$

(ii) *For every $t, s \in [0, T]$ and for any $p > 4$,*

$$\sup_{x \in [0,1]} \left(\int_s^t \left[\int_0^1 G_{t-r}^2(x, y) dy \right]^{p/(p-2)} dr \right)^{\frac{p-2}{2}} \leq C_p |t-s|^{(p-4)/4}. \quad (6.2)$$

(iii) *For every $t, s \in [0, T]$ and for any $p > 4$,*

$$\sup_{x \in [0,1]} \left(\int_0^s \left[\int_0^1 (G_{t-r}(x, y) - G_{s-r}(x, y))^2 dy \right]^{p/(p-2)} dr \right)^{\frac{p-2}{2}} \leq C_p |t-s|^{(p-4)/4}. \quad (6.3)$$

(iv) *For every $x, y \in [0, 1]$ and for any $p > 4$,*

$$\sup_{s \in [0,T]} \left(\int_0^s \left[\int_0^1 (G_{s-r}(x, z) - G_{s-r}(y, z))^2 dz \right]^{p/(p-2)} dr \right)^{\frac{p-2}{2}} \leq C_p |x-y|^{(p-4)/2}. \quad (6.4)$$

Proof of Theorem 4.4: Define $\psi := \bar{z}^n - \bar{z}^m$, ($m > n$). We are now going to prove the solution \bar{z}^n of PDE (4.6) increases as n tends to infinity. Actually, it is sufficient to show $\psi^+ = 0$. Indeed, since

$$\begin{cases} \frac{\partial \psi}{\partial t} - \frac{\partial^2 \psi}{\partial x^2} + [-n(\bar{z}^n(x, t) - L_{x,t}(v(x, t)))^- + m(\bar{z}^m(x, t) - L_{x,t}(v(x, t)))^-] = 0, \\ \psi(x, 0) = 0, \psi(0, t) = \psi(1, t) = 0, \end{cases}$$

testing above equation with the positive part of ψ implies

$$\begin{aligned} & \int_0^T \left\langle \frac{\partial \psi_t}{\partial t}, \psi_t^+ \right\rangle dt - \int_0^T \left\langle \frac{\partial^2 \psi_t}{\partial x^2}, \psi_t^+ \right\rangle dt + \int_0^T \int_0^1 \left(-n(\bar{z}^n(x, t) - L_{x,t}(v(x, t)))^- \right. \\ & \quad \left. + m(\bar{z}^m(x, t) - L_{x,t}(v(x, t)))^- \right) \psi^+(x, t) dx dt = 0, \end{aligned} \quad (6.5)$$

for all $T > 0$. It should be pointed that the last term on the left hand side of (6.5) is non-negative. In fact, for $(x, t) \in \{(x, t) : \psi(x, t) > 0\}$, we have $\bar{z}^n(x, t) > \bar{z}^m(x, t)$. Since the function x^- is non-increasing, it leads to

$$(\bar{z}^n(x, t) - L_{x,t}(v(x, t)))^- \leq (\bar{z}^m(x, t) - L_{x,t}(v(x, t)))^-,$$

from which it follows that

$$\begin{aligned} & \left(-n(\bar{z}^n(x, t) - L_{x,t}(v(x, t)))^- + m(\bar{z}^m(x, t) - L_{x,t}(v(x, t)))^- \right) \psi^+(x, t) \\ & = -n \left((\bar{z}^n(x, t) - L_{x,t}(v(x, t)))^- - (\bar{z}^m(x, t) - L_{x,t}(v(x, t)))^- \right) \psi^+(x, t) \\ & \quad + (m - n) (\bar{z}^m(x, t) - L_{x,t}(v(x, t)))^- \psi^+(x, t) \\ & \geq 0. \end{aligned}$$

Note that ψ^+ is zero at time $t = 0$ and vanishes at points $x = 0$ and $x = 1$. On the other hand, from integrating by parts, (6.5) can deduce

$$\frac{1}{2} \|(\psi_T)^+\|_{L^2([0,1])}^2 + \int_0^T \left\| \frac{\partial(\psi_t)^+}{\partial x} \right\|_{L^2([0,1])}^2 dt + \text{Non-negative Part} = 0.$$

Based on above displays, we conclude that for all $(x, t) \in [0, 1] \times [0, T]$, $\psi^+ = 0$, i.e. $\bar{z}^n \leq \bar{z}^m$. By choosing $v_2 = 0$ and using the fact that the operator $L_{x,t}$ equals to zero for any random process $X \geq 0$, a.s., we have $\bar{z}^{2,n} = 0$, where $\bar{z}^{2,n}$ is the unique solution of (4.6) corresponding to v_2 . In view of $v \in \mathcal{S}^1$ and by (4.8), then the solution \bar{z}^n of PDE (4.6) is upper bounded.

We next show the uniqueness. Let (\bar{z}_1, K_1) and (\bar{z}_2, K_2) be two solutions to obstacle problem (4.1) with random obstacle $v \in \mathcal{S}^1$. Set $\Phi(x, t) := \bar{z}_1(x, t) - \bar{z}_2(x, t)$ and fix some $\phi(x) \in C_c^\infty((0, 1))$. Since function Φ is not regular enough, we need a smooth approximation and then take a limit. Let ϵ be a symmetric, smooth, non-negative definite function which is supported on $[-1, 1]$ and so that $\int_{-1}^1 \epsilon(x) dx = 1$. Define $\epsilon_{n,m}(x, t) := \epsilon_n(x)\epsilon_m(t)$ with $\epsilon_n(x) := n\epsilon(nx)$, then a smooth approximation of $\Phi(x, t)\phi^2(x)$ can be given by

$$d_{n,m} := [(\Phi\phi) * \epsilon_{n,m}] \phi,$$

where $*$ denotes the convolution on $[0, 1]^2$. That is,

$$\begin{aligned} d_{n,m}(x, t) & = \left(\int_0^\infty \int_0^1 \Phi(y, s)\phi(y)\epsilon_n(x-y)\epsilon_m(t-s) dy ds \right) \phi(x) \\ & = \left(\int_{(t-1/n)^+}^{(t+1/n)} \int_0^1 \Phi(y, s)\phi(y)\epsilon_n(x-y)\epsilon_m(t-s) dy ds \right) \phi(x). \end{aligned}$$

Then, testing the equation for Φ against this function implies

$$\begin{aligned} \int_0^1 d_{n,m}(x, T) \Phi(x, T) dx &= \int_0^T \left\langle \frac{\partial d_{n,m}(t)}{\partial t}, \Phi(t) \right\rangle dt + \int_0^T \left\langle \frac{\partial^2 d_{n,m}(t)}{\partial x^2}, \Phi(t) \right\rangle dt \\ &\quad + \int_0^T \int_0^1 d_{n,m}(x, t) (K_1(dx, dt) - K_2(dx, dt)). \end{aligned} \quad (6.6)$$

We now take the limit for each term of (6.6) as n and m tends to infinity. As $n, m \rightarrow \infty$, applying the dominated convergence theorem yields

$$\int_0^1 d_{n,m}(x, T) \Phi(x, T) dx \rightarrow \int_0^1 \Phi^2(x, T) \phi^2(x) dx.$$

In addition, the limit of the combination of the reflection terms is at most zero. In fact, we have

$$\begin{aligned} \lim_{n, m \rightarrow \infty} \int_0^T \int_0^1 d_{n,m}(x, t) (K_1(dx, dt) - K_2(dx, dt)) \\ = \int_0^T \int_0^1 \Phi(x, t) \phi^2(x) (K_1(dx, dt) - K_2(dx, dt)). \end{aligned} \quad (6.7)$$

Combined with the fact that $\bar{z}_i(x, t) - L_{x,t}(v(x, t))$ is zero on the support of K_i and from (4.4), we deduce that (6.7) is equal to

$$\begin{aligned} & - \int_0^T \int_0^1 \left(\bar{z}_2(x, t) - L_{x,t}(v(x, t)) \right) \phi^2(x) K_1(dx, dt) \\ & - \int_0^T \int_0^1 \left(\bar{z}_1(x, t) - L_{x,t}(v(x, t)) \right) \phi^2(x) K_2(dx, dt) \leq 0, \end{aligned}$$

where the inequalities $\bar{z}_i(x, t) \geq L_{x,t}(v(x, t))$, $i = 1, 2$ have been used. On the other hand, by a same argument as in [21, Lemma B.4], we can also obtain

$$\int_0^T \left\langle \frac{\partial d_{n,m}(t)}{\partial t}, \Phi(t) \right\rangle dt \rightarrow \frac{1}{2} \int_0^1 \Phi^2(x, T) \phi^2(x) dx,$$

and

$$\limsup_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \int_0^T \left\langle \frac{\partial^2 d_{n,m}(t)}{\partial x^2}, \Phi(t) \right\rangle dt \leq \frac{1}{2} \int_0^T \int_0^1 \Phi^2(x, s) (\phi^2)''(x) dx.$$

Consequently, we conclude from above displays that

$$\int_0^1 \Phi^2(x, T) \phi^2(x) dx \leq \int_0^T \int_0^1 \Phi^2(x, s) (\phi^2)''(x) dx, \quad (6.8)$$

for any time $T > 0$ and function $\phi \in C_c^\infty((0, 1))$. Finally, following the proof of [26], we can obtain $\Phi \equiv 0$ and then by (6.6), $K_1 = K_2$. We refer the reader to [21, 26] for more details. This completes the proof of the theorem.

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