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MEAN VALUES OF ARITHMETIC FUNCTIONS ON A SPARSE SET AND APPLICATIONS

HENGCAI TANG & JIE WU

ABSTRACT. Let f be an arithmetic function satisfying some simple conditions. The aim of this paper is to establish some asymptotic estimates for quantities

$$\psi_f(x) := \sum_{n \leq x} \Lambda(n) f\left(\left[\frac{x}{n}\right]\right), \quad \pi_f(x) := \sum_{p \leq x} f\left(\left[\frac{x}{p}\right]\right)$$

for $x \rightarrow \infty$, where $\Lambda(n)$ is the von Mangoldt function and $[t]$ is the integral part of $t \in \mathbb{R}$. These generalise or sharpen some recent results of Saito-Suzuki-Takeda-Yoshida. As an application, we show that

$$\sum_{p \leq x, \left[\frac{x}{p}\right] \text{ is prime}} 1 \underset{x \rightarrow \infty}{\sim} \left(\sum_p \frac{1}{p(p+1)} \right) \frac{x}{\log x}.$$

1. INTRODUCTION

A fundamental question in the analytic number theory is to study the asymptotic behaviour of counting summatory function of some arithmetic functions f , as $x \rightarrow \infty$. A typical example is the Dirichlet divisor problem : let $\tau(n)$ be the number of divisors of n , then for $x \rightarrow \infty$, we have

$$\sum_{n \leq x} \tau(n) = x \log x + (2\gamma - 1)x + O(x^{1/2}),$$

where γ is the Euler constant. Notice that

$$(1.1) \quad \sum_{n \leq x} \tau(n) = \sum_{n \leq x} \left[\frac{x}{n}\right],$$

where $[t]$ denotes the integral part of $t \in \mathbb{R}$, it seems natural to consider a more general problem

$$S_f(x) := \sum_{n \leq x} f\left(\left[\frac{x}{n}\right]\right),$$

such that (1.1) corresponds the case of $f = \text{id}$ (identical function, i.e., $\text{id}(n) = n$ for all $n \in \mathbb{N}^*$). An interesting aspect of this problem is that the set

$$\mathcal{S}(x) := \left\{ \left[\frac{x}{n}\right] : n \in \mathbb{N} \cap [1, x] \right\}$$

is sparse in $\mathbb{N} \cap [1, x]$:

$$|\mathcal{S}(x)| \underset{x \rightarrow \infty}{\sim} 2\sqrt{x}.$$

Since the Euler function φ is the convolution of id and the Möbius function μ , i.e. $\varphi = \text{id} * \mu$, we could consider that φ is rather close to id . Bordellès, Dai, Heyman, Pan and Shparlinski [2] proposed to investigate the asymptotic behaviour of the summatory function $S_\varphi(x)$, as $x \rightarrow \infty$, and proved the following inequalities

$$(1.2) \quad \left(\frac{2629}{4009} \cdot \frac{6}{\pi^2} + o(1) \right) x \log x \leq S_\varphi(x) \leq \left(\frac{2629}{4009} \cdot \frac{6}{\pi^2} + \frac{1380}{4009} + o(1) \right) x \log x.$$

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They also conjectured that

$$(1.3) \quad S_\varphi(x) \underset{x \rightarrow \infty}{\sim} \frac{6}{\pi^2} x \log x.$$

The bounds in (1.2) have been sharpened by Wu [15]. Recently Zhai [17] resolved the conjecture (1.3) by the Vinogradov's method, and a more general result has been established by Ma, Wu and Zhao [8]. Let r_1, r_2, r_3 be three non decreasing functions defined on $[1, \infty[$ such that

$$(1.4) \quad 1 \leq r_j(x) \ll x^{\eta_j} \quad (x \geq 1, j = 1, 2, 3), \quad r_3(x) \xrightarrow{x \rightarrow \infty} \infty,$$

where $\eta_j \in [0, 1[$ are constants. Let $f = \text{id} * g$ be an arithmetic function satisfying the following conditions:

$$(1.5) \quad |f(n)| \leq nr_1(n) \quad (n \geq 1),$$

$$(1.6) \quad \sum_{n \leq x} |g(n)| \ll xr_2(x) \quad (x \geq 1),$$

$$(1.7) \quad \sum_{n \leq x} g(n) = D_g x + O(x/r_3(x)) \quad (x \geq 1),$$

where D_g is a constant (eventually equal to 0). Ma, Wu and Zhao [8] proved that for any constant $A > 0$,

$$(1.8) \quad S_f(x) = C_f x \log x + O_A(x \mathcal{R}_S(x, z))$$

holds uniformly for $x \geq 3$ and $1 \leq z \leq \exp\{A^{1/3}(\log x)^{2/3}(\log_2 x)^{1/3}\}$, where

$$(1.9) \quad C_f := \sum_{n=1}^{\infty} \frac{g(n)}{n^2}$$

and

$$\mathcal{R}_S(x, z) := (\log x)^{2/3}(\log_2 x)^{1/3}r_1(x) + \frac{r_2(x/z)}{(\log x)^A} + \frac{r_2(x/z) \log x}{z} + \frac{z \log x}{r_3(\sqrt{x}/z)}.$$

When $f = \varphi$, we can take $A = 1$ and choose

$$r_1(x) = r_2(x) = 1, \quad r_3(x) = \exp(-c(\log x)^{3/5}(\log_2 x)^{-1/5}), \quad z = \log x,$$

where \log_k is the k -fold iterated logarithm and $c > 0$ is a positive constant. Thus (1.8) gives Zhai's result (1.3).

For small arithmetic functions, i.e., $f(n) \ll n^\theta$ ($n \geq 1$) with some $\theta \in [0, 1[$, many authors also studied the asymptotic behaviour of $S_f(x)$, see [2, 16, 17, 20, 12, 3]. In particular, Wu [16] and Zhai [17] proved independently

$$S_f(x) = c_f x + O(x^{(1+\theta)/2}) \quad (x \rightarrow \infty),$$

where

$$(1.10) \quad c_f := \sum_{n=1}^{\infty} \frac{f(n)}{n(n+1)}.$$

Very recently Saito, Suzuki, Takeda and Yoshida [11] considered prime-analogue of $S_f(x)$. Let $\Omega(n)$ be the number of all prime factors of n , then it is well known that

$$\sum_{n \leq x} \Omega(n) = \sum_{p \leq x} \left[\frac{x}{p} \right] = x \log_2 x + O(x) \quad (x \rightarrow \infty).$$

Let $\Lambda(n)$ be the von Mangoldt function and define

$$(1.11) \quad \psi_f(x) := \sum_{n \leq x} \Lambda(n) f\left(\left[\frac{x}{n} \right]\right), \quad \pi_f(x) := \sum_{p \leq x} f\left(\left[\frac{x}{p} \right]\right).$$

They [11, Theorems 7 and 9] proved that for $x \rightarrow \infty$,

$$(1.12) \quad \psi_\varphi(x) = \frac{6}{\pi^2} x \log x + O(x(\log x)^{2/3}(\log_2 x)^{1/3})$$

and

$$(1.13) \quad \frac{2}{\pi^2} x \log_2 x + O(x \log_3 x) \leq \pi_\varphi(x) \leq \left(\frac{2}{\pi^2} + \frac{2}{3}\right) x \log_2 x + O(x \log_3 x).$$

In [11], Saito, Suzuki, Takeda and Yoshida also studied the asymptotic behaviour of $\psi_f(x)$ and $\pi_f(x)$ for small arithmetic functions. Suppose that f satisfies the following conditions:

(a) There is a constant $\alpha \geq 0$ such that

$$(1.14) \quad \sum_{n \leq x} \frac{|f(n)|}{n} \ll (\log x)^\alpha \quad (x \geq 2).$$

(b) There is a constant $\theta \in [0, \frac{1}{2}[$ such that

$$(1.15) \quad f(n) \ll n^\theta \quad (n \geq 1).$$

Then they [11, Theorems 5 and 6] established that there is a positive constant $c > 0$ such that for $x \rightarrow \infty$,

$$(1.16) \quad \psi_f(x) = c_f x + O(x \exp(-c\sqrt{\log x}))$$

and for any non-negative integer K ,

$$(1.17) \quad \pi_f(x) = \frac{x}{\log x} \sum_{k=0}^K \frac{a_k(f)}{(\log x)^k} + O_K\left(\frac{x}{(\log x)^{K+2}}\right)$$

where c_f is defined as in (1.10) and

$$(1.18) \quad a_k(f) := \int_1^\infty \frac{f([u])(\log u)^k}{u^2} du.$$

The aim of this paper is to generalise and sharp these results.

Theorem 1.1. (i) *Let f be an arithmetic function satisfying the conditions (1.5), (1.6) and (1.7). Then for any $A \geq 1$, there exists a positive constant $B = B(A)$ such that*

$$(1.19) \quad \psi_f(x) = C_f x \log x + O_A(x \mathcal{R}_\psi(x, z))$$

holds uniformly for $x \geq 3$ and $(\log x)^B \leq z \leq x^{1/12}$, where C_f is defined as in (1.9) and

$$(1.20) \quad \mathcal{R}_\psi(x, z) := (\log x)^{2/3} (\log_2 x)^{1/3} r_1(x) + \frac{r_2(x/z)}{(\log x)^A} + \frac{\log x}{r_3(x^{1/4}/z)} + \frac{z \log x}{r_3(x/z)}.$$

Here the implied constant depends on A only.

(ii) *Let f be an arithmetic function satisfying the condition (1.7) and there is a positive constant $c < 1$ such that one of the following two conditions*

$$(1.21) \quad f(p-1) < cf(1)p \quad \text{or} \quad f(p-1) > c^{-1}f(1)p > 0$$

holds for an infinity of prime numbers p . Then the error term of (1.19) is $\Omega(x)$.

As applications of Theorem 1.1, we consider four special arithmetic functions:

- the Euler function $\varphi(n)$;
- the alternating sum-of-divisors function $\beta(n)$;
- the sum-of-divisors function $\sigma(n)$;
- the Dedekind function $\Psi(n)$.

We have the following relations:

$$(1.22) \quad \varphi = \text{id} * \mu, \quad \beta = \text{id} * \lambda, \quad \sigma = \text{id} * \mathbb{1}, \quad \Psi = \text{id} * \mu^2,$$

where $\lambda(n) := (-1)^{\Omega(n)}$ is the Liouville function and $\mathbb{1}(n) = 1$ for all $n \in \mathbb{N}^*$. It is well known that they verify the conditions (1.5), (1.6) and (1.7).

Corollary 1.2. (i) For $x \rightarrow \infty$, we have

$$(1.23) \quad \psi_\varphi(x) = \frac{6}{\pi^2} x \log x + O(x(\log x)^{2/3}(\log_2 x)^{1/3}),$$

$$(1.24) \quad \psi_\beta(x) = \frac{\pi^2}{15} x \log x + O(x(\log x)^{2/3}(\log_2 x)^{1/3}),$$

$$(1.25) \quad \psi_\sigma(x) = \frac{\pi^2}{6} x \log x + O(x(\log x)^{2/3}(\log_2 x)^{4/3}),$$

$$(1.26) \quad \psi_\Psi(x) = \frac{15}{\pi^2} x \log x + O(x(\log x)^{2/3}(\log_2 x)^{4/3}).$$

(ii) The error terms of (1.23)-(1.26) are $\Omega(x)$.

The next theorem generalises Saito-Suzuki-Takeda-Yoshida's (1.13).

Theorem 1.3. (i) Let f be an arithmetic function satisfying the conditions $0 \leq f(n) \leq Cn$ ($n \geq 1$) and (1.6) and (1.7). Then for any $A \geq 1$, there exists a positive constant $B = B(A)$ such that

$$(1.27) \quad \frac{1}{3}C_f x \log_2 x + O_A(x\mathcal{R}_\pi(x, z)) \leq \pi_f(x) \leq \frac{1}{3}(C_f + 2C)x \log_2 x + O_A(x\mathcal{R}_\pi(x, z))$$

holds uniformly for $x \geq 3$ and $(\log x)^B \leq z \leq x^{1/12}$, where $C_f := \sum_{n=1}^{\infty} g(n)n^{-2}$ and

$$(1.28) \quad \mathcal{R}_\pi(x, z) := \log_3 x + \frac{r_2(x/z)}{(\log x)^A} + \frac{\log x}{r_3(x^{1/4}/z)} + \frac{z \log x}{r_3(x/z)}.$$

Here the implied constant depends on A only.

(ii) In particular, (1.13) is true and

$$(1.29) \quad \frac{\pi^2}{45} x \log_2 x + O(x \log_3 x) \leq \pi_\beta(x) \leq \frac{\pi^2 + 30}{45} x \log_2 x + O(x \log_3 x).$$

For small arithmetic functions, we have the following results which improve Saito-Suzuki-Takeda-Yoshida's (1.16) and (1.17).

Theorem 1.4. Let $f : \mathbb{N} \rightarrow \mathbb{C}$ be an arithmetic function satisfying

$$(1.30) \quad f(n) \ll n^\theta \quad (n \geq 1)$$

with some constant $\theta \in [0, 1[$.

(i) There exists a positive constant $c_1 > 0$ such that for $x \rightarrow \infty$, we have

$$(1.31) \quad \psi_f(x) = c_f x + O(x \exp(-c_1(\log x)^{1/3}(\log_2 x)^{-1/3})),$$

where c_f is defined as in (1.10).

(ii) Furthermore, if f also verifies the condition (1.14), then the error term in (1.31) can be replaced by $O(x \exp(-c_1(\log x)^{3/5}(\log_2 x)^{-1/5}))$.

Theorem 1.5. Let $f : \mathbb{N} \rightarrow \mathbb{C}$ be an arithmetic function satisfying (1.30).

(i) There exists a positive constant $c_2 > 0$ such that for $x \rightarrow \infty$, we have

$$(1.32) \quad \pi_f(x) = x \int_1^{x^{2/7}} \frac{f([u])}{u^2 \log(x/u)} du + O(x \exp(-c_2(\log x)^{1/3}(\log_2 x)^{-1/3})).$$

Furthermore, if f verifies the condition (1.14), then the error term in (1.31) can be replaced by $O(x \exp(-c_2(\log x)^{3/5}(\log_2 x)^{-1/5}))$.

(ii) For any non-negative integer K , the asymptotic formula (1.17) holds.

As an application, we have the following result.

Corollary 1.6. *For $x \rightarrow \infty$, we have*

$$(1.33) \quad \sum_{p \leq x, \lfloor \frac{x}{p} \rfloor \text{ is prime}} 1 = x \int_1^{x^{2/7}} \frac{\mathbb{1}_{\mathbb{P}}([u])}{u^2 \log(x/u)} du + O(x \exp(-c(\log x)^{3/5} (\log_2 x)^{-1/5}))$$

$$= \left(\sum_p \frac{1}{p(p+1)} \right) \frac{x}{\log x} \left\{ 1 + O\left(\frac{1}{\log x}\right) \right\},$$

where $\mathbb{1}_{\mathbb{P}}$ is the characteristic function of the set of prime numbers \mathbb{P} .

In the sum on the right-hand side of (1.33), we count prime numbers p with multiplicity. It seems more interesting to count without multiplicity. For this we introduce

$$\pi_{\mathbb{S}}(x) := \sum_{\substack{p \leq x \\ \exists n \text{ such that } [x/n]=p}} 1 \quad \text{and} \quad \pi_{\mathbb{S},2}(x) := \sum_{\substack{p \leq x \\ \exists p' \text{ such that } [x/p']=p}} 1.$$

According to [6], we have

$$\pi_{\mathbb{S}}(x) \sim 2x^{1/2}/\log x \quad \text{as } x \rightarrow \infty.$$

A stronger result can be found in [9].

Theorem 1.7. (i) *For $x \rightarrow \infty$, we have $\pi_{\mathbb{S},2}(x) \gg x^{19/59}/\log x$.*

(ii) *Let $\pi(x)$ be the number of primes $\leq x$ and let ε be an arbitrarily small positive number. Suppose that for $y = (\log x)^{2+\varepsilon}$, the inequality*

$$(1.34) \quad \pi(x+y) - \pi(x) \gg_{\varepsilon} y/\log x$$

holds for $x \rightarrow \infty$. Then we have $\pi_{\mathbb{S},2}(x) \gg_{\varepsilon} x^{1/2}/(\log x)^{2+\varepsilon}$.

Noticing that n and $[x/n]$ are symmetric with respect to $x^{1/2}$ logarithmically, it seems rather interesting to compare Corollary 1.6 and Theorem 1.7 with Goldbach's conjecture and the twin prime conjecture:

$$\sum_{p \leq N, N-p \text{ is prime}} 1 \underset{N \rightarrow \infty}{\sim} 2 \prod_{2 < p \leq N} \frac{p-1}{p-2} \prod_{p > 2} \left(1 - \frac{1}{(p-1)^2} \right) \frac{N}{(\ln N)^2},$$

$$\sum_{p \leq x, p+2 \text{ is prime}} 1 \underset{x \rightarrow \infty}{\sim} 2 \prod_{p > 2} \left(1 - \frac{1}{(p-1)^2} \right) \frac{x}{(\ln x)^2}.$$

About the hypothesis (1.34), we refer the reader to Granville-Lumley's paper [5].

2. PRELIMINARY LEMMAS

In this section, we shall cite some lemmas, which will be needed later. The first one is the Vaughan identity [14, formula (3)].

Lemma 2.1. *There are six real arithmetical functions $\alpha_k(n)$ verifying*

$$|\alpha_k(n)| \leq \tau(n) \log(2n) \quad (n \geq 1, 1 \leq k \leq 6)$$

such that, for all $N \geq 100$, $N < N' \leq 2N$ and any arithmetical function $a(n)$, we have

$$(2.1) \quad \sum_{N < n \leq N'} \Lambda(n) a(n) = S_1 + S_2 + S_3 + S_4,$$

where $\tau(n)$ is the classical divisor function and

$$\begin{aligned} S_1 &:= \sum_{\ell \leq N^{1/3}} \alpha_1(\ell) \sum_{N < \ell m \leq N'} a(\ell m), \\ S_2 &:= \sum_{\ell \leq N^{1/3}} \alpha_2(\ell) \sum_{N < \ell m \leq N'} (\log m) a(\ell m), \\ S_3 &:= \sum_{\substack{N^{1/3} < \ell, m \leq N^{2/3} \\ N < \ell m \leq N'}} \alpha_3(\ell) \alpha_4(m) a(mn), \\ S_4 &:= \sum_{\substack{N^{1/3} < \ell, m \leq N^{2/3} \\ N < \ell m \leq N'}} \alpha_5(\ell) \alpha_6(n) a(\ell m). \end{aligned}$$

The second lemma is [4, Theorem A.6].

Lemma 2.2. *Let $\psi(t) := t - [t] - \frac{1}{2}$ for $t \in \mathbb{R}$. For $x \geq 1$ and $H \geq 1$, we have*

$$(2.2) \quad \psi(x) = - \sum_{1 \leq |h| \leq H} \Phi\left(\frac{h}{H+1}\right) \frac{e(hx)}{2\pi i h} + R_H(x),$$

where $e(t) := e^{2\pi i t}$, $\Phi(t) := \pi t(1 - |t|) \cot(\pi t) + |t|$ and the error term $R_H(x)$ satisfies

$$(2.3) \quad |R_H(x)| \leq \frac{1}{2H+2} \sum_{0 < |h| \leq H} \left(1 - \frac{|h|}{H+1}\right) e(hx).$$

The third lemma is an estimate of exponential sum of Vinogradov type.

Lemma 2.3. *For any $A \geq 1$, there is a positive constant $B = B(A) > 0$ such that*

$$(2.4) \quad \sum_{N \leq n < N'} e(T/n) \ll N(\log T)^{-A}$$

holds uniformly for

$$(2.5) \quad \exp(B(\log T)^{2/3}(\log_2 T)^{1/3}) \leq N \leq T(\log T)^{-B} \quad \text{and} \quad N < N' \leq 2N,$$

where the implied constant depends on A only.

Proof. According to [19, Lemma 2.2], there is a positive constant c_1 such that we have

$$\sum_{N \leq n < N'} e(T/n) \ll N \exp(-c_1(\log N)^3/(\log T)^2)$$

uniformly for $N \leq T^{2/3}$. This implies (2.4) provided that $\exp(B(\log T)^{2/3}(\log_2 T)^{1/3}) \leq N \leq T^{2/3}$ with $B = (A/c_1)^{1/3}$.

Next we suppose that $T^{2/3} \leq N \leq T(\log T)^{-B}$ with $B = A$. Applying the exponent pair $(\frac{1}{6}, \frac{4}{6})$ (see [4]), we get

$$\sum_{N \leq n < N'} e(T/n) \ll (T/N)^{1/6} N^{4/6} + (T/N)^{-1} N \ll (TN^3)^{1/6} + T^{-1} N^2 \ll N(\log T)^{-A}.$$

Combining these, we obtain the required result with $B = \max(A, (A/c_1)^{1/3})$. \square

The fourth lemma is [19, Lemma 2.4].

Lemma 2.4. *Assume that the arithmetic function f satisfies the conditions (1.6) and (1.7). Then*

$$(2.6) \quad \sum_{n \leq x} f(n) = \frac{1}{2} C_f x^2 - D_g x \frac{(z - [z])^2 + [z]}{2z} + O\left(\frac{x r_2(x/z)}{z} + \frac{xz}{r_3(x/z)}\right) - \Delta_g(x, z)$$

holds uniformly for $x \geq 2$ and $1 \leq z \leq x^{1/3}$, where the constant C_f is defined as in (1.9) and

$$(2.7) \quad \Delta_g(x, z) := x \sum_{d \leq x/z} \frac{g(d)}{d} \psi\left(\frac{x}{d}\right).$$

Furthermore, for $x \geq 2$ we have

$$(2.8) \quad \sum_{n \leq x} f(n) = \frac{1}{2} C_f x^2 + O(x(\log x) r_2(x)).$$

3. BOUND ON AN AVERAGE OF Δ_g

The aim of this section is to prove the following Proposition 3.1, which will play a key role in the proof of Theorems 1.1 and 1.3.

Proposition 3.1. *Assume that the arithmetic function f satisfies the hypothesis (1.6) and (1.7). Let $\delta \in \{0, 1\}$, $\eta > \frac{12}{11}$, $A \geq 1$ and let B be the constant determined by Lemma 3.2 below. Define*

$$(3.1) \quad N_0 := \exp(B(\log x)^{2/3}(\log_2 x)^{1/3}).$$

Let $\Delta_g(x, z)$ be defined by (2.7) of Lemma 2.4. Then we have

$$(3.2) \quad \Delta_{g,\delta}^\psi(x, z) := \sum_{N_0 < n \leq x^{1/\eta}} \Lambda(n) \Delta_g\left(\frac{x}{n} - \delta, z\right) \ll_{A,\eta} \frac{x r_2(x/z)}{(\log x)^A} + \frac{x \log x}{r_3(x^{1-1/\eta}/z)}$$

and

$$(3.3) \quad \Delta_{g,\delta}^\pi(x, z) := \sum_{N_0 < p \leq x^{1/\eta}} \Delta_g\left(\frac{x}{p} - \delta, z\right) \ll_{A,\eta} \frac{x r_2(x/z)}{(\log x)^A} + \frac{x \log x}{r_3(x^{1-1/\eta}/z)}$$

uniformly for $x \geq 10$ and $(\log x)^B \leq z \leq x^{1/12}$.

In order to prove Proposition 3.1, we need to establish the following lemma. This lemma is essentially due to Liu [7] (see also [13, Lemmas 6, 7, 12] and [11, Lemma 22]). For convenience, we give a complete (and simpler) proof here.

Lemma 3.2. *Let $\psi(t) := t - [t] - \frac{1}{2}$ for $t \in \mathbb{R}$ and $\iota \in \mathbb{R}$. Then for any $A > 1$, there is a positive constant $B = B(A) \geq 2A + 8$ such that the inequalities*

$$(3.4) \quad \Xi(x; N, N') := \sum_{N \leq n < N'} \Lambda(n) \psi\left(\frac{x}{n} - \iota\right) \ll_A N(\log x)^{-(A+1)}$$

and

$$(3.5) \quad \sum_{N \leq n < N'} \frac{\Lambda(n)}{n} \psi\left(\frac{x}{n} - \iota\right) \ll_A (\log x)^{-A}$$

$$(3.6) \quad \sum_{N \leq p < N'} \frac{1}{p} \psi\left(\frac{x}{p} - \iota\right) \ll_A (\log x)^{-A}$$

hold uniformly for

$$(3.7) \quad x \geq 10, \quad \exp(B(\log x)^{2/3}(\log_2 x)^{1/3}) \leq N \leq x(\log x)^{-B} \quad \text{and} \quad N < N' \leq 2N.$$

The implied constants depend on A and are independent of ι .

Proof. Firstly we prove (3.4). Using (2.2) of Lemma 2.2, for any $H \geq 1$ we can write

$$\begin{aligned} \sum_{N \leq n < N'} \Lambda(n) \psi\left(\frac{x}{n} - \iota\right) &= - \sum_{1 \leq |h| \leq H} \Phi\left(\frac{h}{H+1}\right) \frac{1}{2\pi i h} \sum_{N \leq n < N'} \Lambda(n) e\left(\frac{hx}{n} - h\iota\right) \\ &\quad + \sum_{N \leq n < N'} \Lambda(n) R\left(\frac{x}{n} - \iota\right). \end{aligned}$$

In view of (2.3) of Lemma 2.2, it follows that

$$\begin{aligned} |\Xi(x; N, N')| \leq & \left| \sum_{1 \leq |h| \leq H} \Phi\left(\frac{h}{H+1}\right) \frac{e(-h\iota)}{2\pi i h} \sum_{N \leq n < N'} \Lambda(n) e\left(\frac{hx}{n}\right) \right| \\ & + \frac{1}{2H+2} \sum_{0 \leq |h| \leq H} \left(1 - \frac{|h|}{H+1}\right) e(-h\iota) \sum_{N \leq n < N'} \Lambda(n) e\left(\frac{hx}{n}\right). \end{aligned}$$

Noticing the fact that $0 < \Phi(t) < 1$ ($0 < |t| < 1$) and using the prime number theorem:

$$(3.8) \quad \sum_{n \leq x} \Lambda(n) = x + O(x \exp(-c(\log x)^{3/5}(\log_2 x)^{-1/5})) \quad (x \geq 3),$$

where $c > 0$ is a positive constant, we can derive that, for any $H \in [1, N]$,

$$(3.9) \quad \Xi(x; N, N') \ll \frac{N}{H} + \sum_{1 \leq h \leq H} \frac{1}{h} \left| \sum_{N \leq n < N'} \Lambda(n) e\left(\frac{hx}{n}\right) \right|.$$

For the last inner sum, we apply the Vaughan identity (2.1) to write

$$(3.10) \quad \sum_{N \leq n < N'} \Lambda(n) e\left(\frac{hx}{n}\right) = \mathfrak{S}_1 + \mathfrak{S}_2 + \mathfrak{S}_3 + \mathfrak{S}_4,$$

where

$$\begin{aligned} \mathfrak{S}_1 &:= \sum_{d \leq N^{1/3}} \alpha_1(d) \sum_{N < d\ell \leq N'} e\left(\frac{hx}{d\ell}\right), \\ \mathfrak{S}_2 &:= \sum_{d \leq N^{1/3}} \alpha_2(d) \sum_{N < d\ell \leq N'} (\log \ell) e\left(\frac{hx}{d\ell}\right), \\ \mathfrak{S}_3 &:= \sum_{\substack{N^{1/3} < d, \ell \leq N^{2/3} \\ N < d\ell \leq N'}} \alpha_3(d) \alpha_4(\ell) e\left(\frac{hx}{d\ell}\right), \\ \mathfrak{S}_4 &:= \sum_{\substack{N^{1/3} < d, \ell \leq N^{2/3} \\ N < d\ell \leq N'}} \alpha_5(d) \alpha_6(\ell) e\left(\frac{hx}{d\ell}\right). \end{aligned}$$

The sums \mathfrak{S}_1 and \mathfrak{S}_2 are called as type I, \mathfrak{S}_3 and \mathfrak{S}_4 as type II.

A. Estimates of \mathfrak{S}_1 and \mathfrak{S}_2

Since $N \leq x/(\log x)^B$ and $d \leq N^{1/3} \leq x^{1/3}$, then

$$hx/d \geq x/d \geq (N/d)(\log x)^B \geq (N/d)(\log(x/d))^B \quad \text{and} \quad hx/d \geq x^{2/3}.$$

Thus, we can apply Lemma 2.3 and the inequality $\alpha_1(\ell) \ll \tau(\ell) \log x$ to derive that

$$\begin{aligned} \mathfrak{S}_1 &\ll \sum_{d \leq N^{1/3}} |\alpha_1(d)| (N/d) (\log(hx/d))^{-(A+5)} \\ (3.11) \quad &\ll N (\log x)^{-(A+4)} \sum_{\ell \leq N^{1/3}} \tau(d)/d \\ &\ll N (\log x)^{-(A+2)}. \end{aligned}$$

The same bound also holds for \mathfrak{S}_2 , since the factor $\log m$ in \mathfrak{S}_2 can be removed by a simple partial integration.

B. Estimates of \mathfrak{S}_3 and \mathfrak{S}_4

We shall only bound \mathfrak{S}_3 , since \mathfrak{S}_4 can be treated in the same way. For this, we shall use two different methods according to the size of N :

$$\exp(B(\log x)^{2/3}(\log_2 x)^{1/3}) \leq N \leq x^{1/2} \quad \text{or} \quad x^{1/2} \leq N \leq x(\log x)^{-B}.$$

Firstly we consider the first case. In view of the symmetry of the variables d and ℓ , we can suppose that $N^{1/2} \leq d \leq N^{2/3}$. Thus using the bound $\alpha_3(\ell) \ll \tau(\ell) \log x$, we can derive that

$$\begin{aligned} \mathfrak{S}_3 &\ll (\log x) \sum_{N^{1/2} \leq d \leq N^{2/3}} \tau(d) \left| \sum_{N/d < \ell \leq N'/d} \alpha_4(\ell) e\left(\frac{hx}{d\ell}\right) \right| \\ &\ll (\log x)^2 \max_{N^{1/2} \leq D \leq N^{2/3}} \sum_{d \sim D} \tau(d) \left| \sum_{N/d < \ell \leq N'/d} \alpha_4(\ell) e\left(\frac{hx}{d\ell}\right) \right|, \end{aligned}$$

where the symbol $d \sim D$ means that $D < d \leq 2D$. By the Cauchy's inequality, it follows that

$$\begin{aligned} |\mathfrak{S}_3|^2 &\ll (\log x)^5 \max_{N^{1/2} \leq D \leq N^{2/3}} D \sum_{d \sim D} \tau(d) \left| \sum_{N/d < \ell \leq N'/d} \alpha_4(d) e\left(\frac{hx}{d\ell}\right) \right|^2 \\ &\ll (\log x)^7 \max_{N^{1/2} \leq D \leq N^{2/3}} D \sum_{\ell \sim N/D} \sum_{\ell' \sim N/D} \tau(\ell) \tau(\ell') \left| \sum_{d \sim D} \tau(d) e\left(\frac{h(\ell' - \ell)x}{d\ell\ell'}\right) \right|. \end{aligned}$$

Separating the contribution of $\ell = \ell'$ and $\ell \neq \ell'$, we can find that

$$(3.12) \quad |\mathfrak{S}_3|^2 \ll (\log x)^{10} \max_{N^{1/2} \leq D \leq N^{2/3}} D \left(N + \sum_{\ell \sim N/D} \sum_{\substack{\ell' \sim N/D \\ \ell' \neq \ell}} \tau(\ell) \tau(\ell') \left| \sum_{d \sim D} \tau(d) e\left(\frac{h(\ell' - \ell)x}{d\ell\ell'}\right) \right| \right).$$

Noticing that $\tau(d) = \sum_{d_1 d_2 = d} 1$ and using the symmetry of d_1 and d_2 , we can suppose that $d_1 \leq d_2$ and write

$$(3.13) \quad \begin{aligned} \sum_{d \sim D} \tau(d) e\left(\frac{h(\ell' - \ell)x}{d\ell\ell'}\right) &= \sum_{D < d_1 d_2 \leq 2D} e\left(\frac{h(\ell' - \ell)x}{d_1 d_2 \ell \ell'}\right) \\ &\ll (\log x) \max_{1 \leq D_1 \leq D^{1/2}} \sum_{d_1 \sim D_1} \left| \sum_{d_2 \sim D/d_1} e\left(\frac{h(\ell' - \ell)x}{d_1 d_2 \ell \ell'}\right) \right|. \end{aligned}$$

Using the hypothesis $N \leq x^{1/2}$ and $\ell \neq \ell'$, it is easy to verify that

$$x^{4/3} \geq Dx \geq \left| \frac{h(\ell' - \ell)x}{d_1 \ell \ell'} \right| \geq \frac{x}{d_1 (N/D)^2} \geq \left(\frac{D}{d_1}\right) x^{1/4}.$$

Thus we can apply Lemma 2.3 to get that

$$\begin{aligned} \sum_{d \sim D} \tau(d) e\left(\frac{h(\ell' - \ell)x}{d\ell\ell'}\right) &\ll (\log x) \max_{1 \leq D_1 \leq D^{1/2}} \sum_{d_1 \sim D_1} \frac{D}{d_1} (\log x)^{-(2A+30)} \\ &\ll D (\log x)^{-(2A+28)}. \end{aligned}$$

Inserting this bound into (3.12), we easily derive that

$$(3.14) \quad \mathfrak{S}_3 \ll N (\log x)^{-(A+3)}.$$

The same bound holds for \mathfrak{S}_4 . Combining (3.11) and (3.14) with (3.10), it follows that

$$\sum_{N \leq n < N'} \Lambda(n) e\left(\frac{hx}{n}\right) \ll N (\log x)^{-(A+3)}.$$

Inserting it into (3.9) and taking $H = (\log x)^{A+1}$, we obtain the required result (3.4).

Next we suppose that $x^{1/2} \leq N \leq x(\log x)^{-B}$. In (3.13), applying the exponent pair (κ, λ) to the inner sum over d and using the fact that $\ell, \ell' \sim N/D$ with $\ell \neq \ell'$, we obtain that

$$\begin{aligned} \sum_{d \sim D} \tau(d) e\left(\frac{h(\ell' - \ell)x}{d\ell\ell'}\right) &\ll (\log x) \max_{1 \leq D_1 \leq D^{1/2}} \sum_{d_1 \sim D_1} \left(\left(\frac{h|\ell' - \ell|x}{D(D/d_1)\ell\ell'}\right)^\kappa \left(\frac{D}{d_1}\right)^\lambda + \left(\frac{h|\ell' - \ell|x}{D(D/d_1)\ell\ell'}\right)^{-1} \right) \\ &\ll (\log x)^2 \left(\left(\frac{h|\ell' - \ell|x}{N^2}\right)^\kappa D^{(1+\kappa+\lambda)/2} + \left(\frac{h|\ell' - \ell|x}{N^2}\right)^{-1} \right). \end{aligned}$$

From this, we deduce that

$$\begin{aligned} \max_{N^{1/2} \leq D \leq N^{2/3}} D \sum_{\substack{\ell \sim N/D \\ \ell' \sim N/D \\ \ell \neq \ell'}} \tau(\ell)\tau(\ell') \left| \sum_{d \sim D} \tau(d) e\left(\frac{h(\ell' - \ell)x}{d\ell\ell'}\right) \right| \\ \ll (\log x)^6 \left((hx)^\kappa N^{(7-5\kappa+\lambda)/4} + (hx)^{-1} N^3 \right). \end{aligned}$$

Inserting it into (3.12), it follows that

$$(3.15) \quad \mathfrak{S}_3 \ll (\log x)^8 \left(N^{5/6} + (hx)^{\kappa/2} N^{(7-5\kappa+\lambda)/8} + (hx)^{-1/2} N^{3/2} \right).$$

The same bound also holds for \mathfrak{S}_4 . Combining (3.11), (3.15) with (3.9), it follows that

$$\Xi(x; N, N') \ll (\log x)^8 \left(N^{5/6} + NH^{-1} + (x^{4\kappa} N^{7-5\kappa+\lambda} H^{4\kappa})^{1/8} + (x^{-1} N^3)^{1/2} \right)$$

for all $H \in [1, N]$. Optimizing H over $[1, N]$ and using the hypothesis $x^{1/2} \leq N \leq x(\log x)^{-B}$, we find that

$$\begin{aligned} \Xi(x; N, N') &\ll (\log x)^8 \left(N^{5/6} + (x^{4\kappa} N^{7-\kappa+\lambda})^{1/(8+4\kappa)} + (x^{4\kappa} N^{7-5\kappa+\lambda})^{1/8} + (x^{-1} N^3)^{1/2} \right) \\ &\ll (\log x)^8 \left(N^{5/6} + N^{(7+7\kappa+\lambda)/(8+4\kappa)} + N^{(7+3\kappa+\lambda)/8} + N(\log x)^{-B/2} \right). \end{aligned}$$

Taking $(\kappa, \lambda) = (\frac{1}{30}, \frac{26}{30})$, we obtain the required inequality (3.4). Then, a simple partial integration implies that (3.5).

Finally we prove (3.6). Under the condtion (3.7), we have that

$$\begin{aligned} \left| \sum_{N \leq p^\nu < N', \nu \geq 2} \frac{\log p}{p^\nu} \psi\left(\frac{x}{p^\nu} - \iota\right) \right| &\leq \sum_{N \leq p^\nu < N', \nu \geq 2} \frac{\log p}{p^\nu} \ll \frac{(\log N)^2}{N} \\ &\ll \exp(-(B/2)(\log x)^{2/3}). \end{aligned}$$

Thus, under the same condition, we have

$$\begin{aligned} \sum_{N \leq p < N'} \frac{\log p}{p} \psi\left(\frac{x}{p} - \iota\right) &= \sum_{N \leq n < N'} \frac{\Lambda(n)}{n} \psi\left(\frac{x}{n} - \iota\right) + (\exp(-(B/2)(\log x)^{2/3})) \\ &\ll_A (\log x)^{-A}. \end{aligned}$$

From this, a simple partial integration gives us (3.6). □

Now we are ready to prove Proposition 3.1.

Proof of Proposition 3.1. By (2.7) of Lemma 2.3, we can write

$$\begin{aligned} \Delta_{g,\delta}^\psi(x, z) &= \sum_{N_0 < n \leq x^{1/\eta}} \Lambda(n) \left(\frac{x}{n} - \delta\right) \sum_{d \leq (x/n-\delta)/z} \frac{g(d)}{d} \psi\left(\frac{x}{dn} - \frac{\delta}{d}\right) \\ (3.16) \quad &= x \sum_{N_0 < n \leq x^{1/\eta}} \frac{\Lambda(n)}{n} \sum_{d \leq x/(nz)} \frac{g(d)}{d} \psi\left(\frac{x}{dn} - \frac{\delta}{d}\right) + O(|E_1| + |E_2|), \end{aligned}$$

where the implied constant is absolute and

$$E_1 := \sum_{N_0 < n \leq x^{1/\eta}} \Lambda(n) \sum_{d \leq (x/n-\delta)/z} \frac{g(d)}{d} \psi\left(\frac{x}{dn} - \frac{\delta}{d}\right),$$

$$E_2 := x \sum_{N_0 < n \leq x^{1/\eta}} \frac{\Lambda(n)}{n} \sum_{(x/n-\delta)/z < d \leq x/(nz)} \frac{g(d)}{d} \psi\left(\frac{x}{dn} - \frac{\delta}{d}\right).$$

In view of the prime number theorem and the hypothesis (1.6), we have

$$E_1 \ll \sum_{n \leq x^{1/\eta}} \Lambda(n) \sum_{d \leq x/z} \frac{|g(d)|}{d} \ll x^{1/\eta} r_2(x/z) \log x.$$

Since $\delta/z < 1$, the hypothesis (1.7) and the prime number theorem allow us to derive that

$$E_2 \leq x \sum_{n \leq x^{1/\eta}} \frac{\Lambda(n)}{n} \frac{|g([x/(nz)])|}{[x/(nz)]} \ll x \sum_{n \leq x^{1/\eta}} \frac{\Lambda(n)}{n} \frac{1}{r_3([x/(nz)])} \ll \frac{x \log x}{r_3(x^{1-1/\eta}/z)}.$$

Inserting these into (3.16) and inverting the order of summations, we get

$$(3.17) \quad \Delta_{g,\delta}^\psi(x, z) = \Delta_{g,\delta}^{\psi,\dagger}(x, z) + O\left(x^{1/\eta} r_2(x/z) \log x + \frac{x \log x}{r_3(x^{1-1/\eta}/z)}\right).$$

where

$$\Delta_{g,\delta}^{\psi,\dagger}(x, z) := x \sum_{d \leq x/(N_0 z)} \frac{g(d)}{d} \sum_{N_0 < n \leq \min\{x^{1/\eta}, x/(dz)\}} \frac{\Lambda(n)}{n} \psi\left(\frac{x}{dn} - \frac{\delta}{d}\right).$$

For $0 \leq k \leq (\log(\min\{x^{1/\eta}, x/(dz)\}/N_0))/\log 2$, let $N_k := 2^k N_0$ and define

$$\mathfrak{S}_k(d) := \sum_{N_k < n \leq 2N_k} \frac{\Lambda(n)}{n} \psi\left(\frac{x}{dn} - \frac{\delta}{d}\right).$$

Noticing that $N_0 \leq N_k \leq x/(dz) \leq (x/d)(\log x)^{-B} \leq (x/d)(\log(x/d))^{-B}$, we can apply (3.5) of Lemma 3.2 with $\iota = \delta/d$ to derive that

$$\mathfrak{S}_k(d) \ll (\log(x/d))^{-B} \ll (\log x)^{-A-2},$$

since $\log(x/d) \geq \log N_0 \geq B(\log x)^{2/3}$. Note that the implied constant is independent from δ/d . Inserting this into the expression of $\Delta_{g,\delta}^{\psi,\dagger}(x, z)$ and using the condition (1.6), we can deduce that

$$(3.18) \quad \Delta_{g,\delta}^{\psi,\dagger}(x, z) \ll x \sum_{d \leq x/(N_0 z)} \frac{|g(d)|}{d} \sum_{2^k N_0 \leq x/(dz)} |\mathfrak{S}_k(d)|$$

$$\ll \frac{x}{(\log x)^{A+1}} \sum_{d \leq x/z} \frac{|g(d)|}{d} \ll \frac{x r_2(x/z)}{(\log x)^A}.$$

Inserting (3.18) into (3.17), we obtain (3.2).

The inequality (3.3) can be proved in the same way (only difference is to apply (3.6) in place of (3.5)) and we omit it. \square

4. PROOF OF THEOREM 1.1

Firstly, we divide the sum over n into three parts :

$$(4.1) \quad \psi_f(x) = \psi_{f,1}(x) + \psi_{f,2}(x) + \psi_{f,3}(x),$$

where

$$\begin{aligned}\psi_{f,1}(x) &:= \sum_{n \leq N_0} \Lambda(n) f\left(\left[\frac{x}{n}\right]\right), \\ \psi_{f,2}(x) &:= \sum_{N_0 < n \leq x^{3/4}} \Lambda(n) f\left(\left[\frac{x}{n}\right]\right), \\ \psi_{f,3}(x) &:= \sum_{x^{3/4} < n \leq x} \Lambda(n) f\left(\left[\frac{x}{n}\right]\right).\end{aligned}$$

Here $N_0 := \exp(B(\log x)^{2/3}(\log_2 x)^{1/3})$ is defined as in (3.1).

In view of the hypothesis (1.5) and the prime number theorem (3.8), we have

$$(4.2) \quad \psi_{f,1}(x) \leq \sum_{n \leq N_0} \Lambda(n) \frac{x}{n} r_1(x) \ll x r_1(x) \log N_0.$$

Noticing that $d = [x/n]$ is equivalent to $x/n - 1 < d \leq x/n$ and using Lemma 2.4, for any $(\log x)^B \leq z \leq x^{1/12}$ ($\leq (x/n)^{1/3}$), we can write

$$\begin{aligned}\psi_{f,2}(x) &= \sum_{N_0 < n \leq x^{3/4}} \Lambda(n) \sum_{x/n-1 < d \leq x/n} f(d) \\ &= \sum_{N_0 < n \leq x^{3/4}} \Lambda(n) \left\{ \frac{1}{2} C_f \left(\frac{2x}{n} - 1 \right) - D_g \frac{(z - [z])^2 + [z]}{2z} \right. \\ &\quad \left. + O\left(\frac{xz}{nr_3(x/z)} + \frac{xr_2(x/z)}{nz} \right) - \Delta_g\left(\frac{x}{n}, z\right) + \Delta_g\left(\frac{x}{n} - 1, z\right) \right\},\end{aligned}$$

where $\Delta_g(x, z)$ is defined by (2.7) of Lemma 2.4. With the help of the prime number theorem (3.8) and (3.2) of Proposition 3.1 with $\eta = \frac{4}{3}$, we easily derive that

$$(4.3) \quad \psi_{f,2}(x) = \frac{3}{4} C_f x \log x + O\left(\frac{xr_2(x/z)}{(\log x)^A} + \frac{xz \log x}{r_3(x/z)} + \frac{x \log x}{r_3(x^{1/4}/z)} \right),$$

where we have removed the term $\frac{xr_2(x/z) \log x}{z}$ (since it can be bounded by $\frac{xr_2(x/z)}{(\log x)^A}$).

Finally we evaluate $\psi_{f,3}(x)$. Write

$$\psi_{f,3}(x) = \sum_{d \leq x^{1/4}} f(d) \sum_{\max(x/(d+1), x^{3/4}) < n \leq x/d} \Lambda(n) = \psi_{f,3}^\dagger(x) + \psi_{f,3}^\sharp(x),$$

where

$$\begin{aligned}\psi_{f,3}^\dagger(x) &:= \sum_{d \leq x^{1/4}} f(d) \sum_{x/(d+1) < n \leq x/d} \Lambda(n), \\ \psi_{f,3}^\sharp(x) &:= \sum_{x^{1/4}-1 < d \leq x^{1/4}} f(d) \sum_{x/(d+1) < n \leq x/d} \Lambda(n).\end{aligned}$$

It is easy to bound $\psi_{f,3}^\sharp(x)$. In view of (1.5), we have

$$\psi_{f,3}^\sharp(x) \ll x(\log x) \sum_{x^{1/4}-1 < d \leq x^{1/4}} \frac{f(d)}{d^2} \ll x(\log x) \frac{f([x^{1/4}])}{[x^{1/4}]^2} \ll x^{3/4} r_1(x) \log x.$$

In order to estimate $\psi_{f,3}^\dagger(x)$, firstly we recall Huxley's prime number theorem in short intervals (see [10]): For any $\varepsilon > 0$,

$$(4.4) \quad \sum_{x < n \leq x+y} \Lambda(n) = y \left\{ 1 + O\left(\exp(-c(\log x)^{1/3}(\log_2 x)^{-1/3})\right) \right\},$$

$$(4.5) \quad \sum_{x < p \leq x+y} 1 = \int_x^{x+y} \frac{dt}{\log t} \left\{ 1 + O\left(\exp(-c(\log x)^{1/3}(\log_2 x)^{-1/3})\right) \right\}$$

hold uniformly for $x \geq 3$ and $x^{7/12+\varepsilon} \leq y \leq x$. When $d \leq x^{1/4}$, we have $x/d \geq x^{3/4}$ and $x/(d(d+1)) \geq (x/(d+1))^{3/5}$. Thus we can apply (4.4) to derive that

$$\psi_{f,3}^\dagger(x) = \sum_{d \leq x^{1/4}} f(d) \frac{x}{d(d+1)} \left\{ 1 + O\left(\exp(-c(\log x)^{1/3}(\log_2 x)^{-1/3})\right) \right\}.$$

On the other hand, by (2.8) of Lemma 2.4, it follows that

$$\begin{aligned} \sum_{d \leq x^{1/4}} \frac{f(d)}{d(d+1)} &= \sum_{d \leq x^{1/4}} \frac{f(d)}{d^2} - \sum_{d \leq x^{1/4}} \frac{f(d)}{d^2(d+1)} \\ &= \int_{1-}^{x^{1/4}} t^{-2} d\left(\frac{1}{2}C_f t^2 + O(tr_2(t) \log t)\right) + O(1) \\ &= \frac{1}{4}C_f \log x + O(1). \end{aligned}$$

Combining these estimates, we find that

$$(4.6) \quad \psi_{f,3}(x) = \frac{1}{4}C_f x \log x + O(x + x^{3/4}r_1(x) \log x).$$

Now the required result (1.19) follows from (6.1), (4.2), (4.3) and (4.6).

Finally we prove the second assertion. Let $E(x)$ be the error term of (1.19), i.e.

$$\psi_f(x) = C_f x \log x + E(x),$$

and define $E^*(x) := \max\{|E(x)|, |E(x-1)|\}$.

Putting $d = [x/n]$, we have $x/n - 1 < d \leq x/n$ and $x/(d+1) < n \leq x/d$. Thus,

$$\psi_f(x) = \sum_{d \leq x} f(d) \sum_{x/(d+1) < n \leq x/d} \Lambda(n) = \sum_{dn \leq x} f(d)\Lambda(n) - \sum_{(d+1)n \leq x} f(d)\Lambda(n).$$

Setting $f(0) = 0$, we can write

$$(4.7) \quad \psi_f(x) = \sum_{dn \leq x} (f(d) - f(d-1))\Lambda(n) = \sum_{m \leq x} \sum_{d|m} (f(d) - f(d-1))\Lambda\left(\frac{m}{d}\right).$$

Firstly we suppose that

$$(4.8) \quad f(p-1) < cf(1)p$$

holds for an infinity of primes p . In view of (4.7), for each prime p we can write

$$\begin{aligned} \sum_{d|2p} (f(d) - f(d-1))\Lambda\left(\frac{2p}{d}\right) &= \psi_f(2p) - \psi_f(2p-1) \\ &= C_f(2p \log(2p) - (2p-1) \log(2p-1)) + E(2p) - E(2p-1) \\ &\leq 2E^*(2p) + O(\log p). \end{aligned}$$

On the other hand, our hypothesis (4.8) and (1.7) allow us to deduce that

$$\begin{aligned} \sum_{d|2p} (f(d) - f(d-1)) \Lambda\left(\frac{2p}{d}\right) &= (f(p) - f(p-1)) \log 2 + (f(2) - f(1)) \log p \\ &= (g(1)p + g(p)f(1) - f(p-1)) \log 2 + (f(2) - f(1)) \log p \\ &\geq (1-c)f(1)p \log 2 + O(p/r_3(p)) \\ &\geq \frac{1}{2}(1-c)f(1)p \end{aligned}$$

for an infinity of primes p . Combining these, we find that

$$E^*(2p) \geq \frac{1}{5}(1-c)f(1)p$$

for an infinity of primes p .

Next we suppose that

$$(4.9) \quad f(p-1) > c^{-1}f(1)p > 0$$

holds for an infinity of primes p . As before, for each prime p we can write

$$\sum_{d|2p} (f(d) - f(d-1)) \geq -2E^*(p) + O(\log p).$$

On the other hand, our hypothesis (4.9) and (1.7) allow us to deduce that

$$\sum_{d|2p} (f(d) - f(d-1)) \leq -\frac{1}{2}(c^{-1} - 1)f(1)p$$

for an infinity of primes p . Combining these, we find that

$$E^*(p) \geq \frac{1}{5}(c^{-1} - 1)f(1)p > 0$$

for an infinity of primes p . □

5. PROOF OF COROLLARY 1.2

5.1. Proof of (1.23) and (1.24).

Since $\varphi = \text{id} * \mu$ and $\beta = \text{id} * \lambda$, we have $g = \mu$ or $g = \lambda$ and the following well-known bound

$$\sum_{n \leq x} g(n) \ll x \exp(-c(\log x)^{3/5}(\log_2 x)^{-1/5}) \quad (x \geq 2).$$

Thus φ and β verify the conditions (1.5), (1.6) and (1.7) with $D_g = 0$ and

$$r_1(x) = 1, \quad r_2(x) = 1, \quad r_3(x) = \exp(-c_4(\log x)^{3/5}(\log_2 x)^{-1/5}).$$

The asymptotic formulas (1.23)-(1.24) follow from Theorem 1.1 with the choice of $A = 1$ and $z = \log^{B(1)}(3x)$.

5.2. Proof of (1.25) and (1.26).

In this case, we have $g = \mathbb{1}$ or μ^2 and

$$\sum_{n \leq x} g(n) = \begin{cases} x + O(1) & \text{if } g = \mathbb{1}, \\ (6/\pi^2)x + O(\sqrt{x}) & \text{if } g = \mu^2. \end{cases}$$

Thus the function $\sigma(n)$ and $\Psi(n)$ verify the conditions (1.5), (1.6) and (1.7) with $D_{\mathbb{1}} = 1$ or $D_{\Psi} = 6/\pi^2$ and

$$r_1(x) = C \log_2(3x), \quad r_2(x) = 1, \quad r_3(x) = \sqrt{x}.$$

where $C > 0$ is a positive constant. The asymptotic formulas (1.25) and (1.26) follow from Theorem 1.1 with the choice of $A = 1$ and $z = \log^{B(1)}(3x)$.

5.3. Proof of the second assertion.

Firstly we verify that functions φ, σ, β and Ψ satisfy the condition (1.21) of Theorem 1.1.

For each odd prime p , we can write $p - 1 = 2^\nu m$ with $2 \nmid m$. Thus

$$\varphi(p - 1) = \varphi(2^\nu)\varphi(m) \leq 2^{\nu-1}m < \frac{1}{2}p$$

for all odd primes p .

For all primes p and integers $\nu \geq 1$, we have

$$\beta(p^\nu) = \sum_{0 \leq j \leq \nu} (-1)^{\nu-j} p^j = \frac{p^{\nu+1} + (-1)^\nu}{p+1} \leq \begin{cases} \frac{3}{4}2^\nu & \text{if } p = 2, \\ p^\nu & \text{otherwise.} \end{cases}$$

For each odd prime p , we can write $p - 1 = 2^\nu m$ with $2 \nmid m$. Thus

$$\beta(p - 1) = \beta(2^\nu)\beta(m) \leq \frac{3}{4}(p - 1) \leq \frac{3}{4}p$$

for all odd primes p .

For all odd primes p , we have

$$\sigma(p - 1) \geq (p - 1) + \frac{1}{2}(p - 1) + 1 > \frac{5}{4}p.$$

For all primes p and integers $\nu \geq 1$, we have

$$\Psi(p^\nu) = p^\nu + p^{\nu-1} \geq \begin{cases} \frac{3}{2}2^\nu & \text{if } p = 2, \\ p^\nu & \text{otherwise.} \end{cases}$$

For each odd prime p , we can write $p - 1 = 2^\nu m$ with $2 \nmid m$. Thus

$$\Psi(p - 1) = \Psi(2^\nu)\Psi(m) \geq \frac{3}{2}(p - 1) \geq \frac{5}{4}p$$

for all odd primes $p \geq 7$.

Thus Theorem 1.1(ii) implies that the error terms of (1.23)-(1.26) are $\Omega(x)$. \square

6. PROOF OF THEOREM 1.3

Firstly we divide the sum over n into three parts :

$$(6.1) \quad \pi_f(x) = \pi_{f,1}(x) + \pi_{f,2}(x) + \pi_{f,3}(x),$$

where

$$\pi_{f,1}(x) := \sum_{p \leq N_0} f\left(\left[\frac{x}{p}\right]\right), \quad \pi_{f,2}(x) := \sum_{N_0 < p \leq x^{3/4}} f\left(\left[\frac{x}{p}\right]\right), \quad \pi_{f,3}(x) := \sum_{x^{3/4} < p \leq x} f\left(\left[\frac{x}{p}\right]\right).$$

Since $0 \leq f(n) \leq Cn$ ($n \geq 1$), the prime number theorem allows us to derive that

$$(6.2) \quad 0 \leq \pi_{f,1}(x) \leq C \sum_{p \leq N_0} \frac{x}{p} = \frac{2}{3}Cx \log_2 x + O(x \log_3 x),$$

$$(6.3) \quad \pi_{f,3}(x) \leq C \sum_{x^{3/4} < p \leq x} \frac{x}{p} \ll x.$$

Noticing that $d = [x/p]$ is equivalent to $x/p - 1 < d \leq x/p$, for $(\log x)^B \leq z \leq x^{1/12}$ ($\leq (x/p)^{1/3}$) we can apply Lemma 2.4 to write

$$\begin{aligned} \pi_{f,2}(x) &= \sum_{N_0 < p \leq x^{3/4}} \sum_{x/p-1 < d \leq x/p} f(d) \\ &= \sum_{N_0 < p \leq x^{3/4}} \left\{ \frac{1}{2}C_f \left(\frac{2x}{p} - 1 \right) - D_g \frac{(z - [z])^2 + [z]}{2z} \right. \\ &\quad \left. + O\left(\frac{xz}{pr_3(x/z)} + \frac{xr_2(x/z)}{pz} \right) - \Delta_g \left(\frac{x}{p}, z \right) + \Delta_g \left(\frac{x}{p} - 1, z \right) \right\}, \end{aligned}$$

where $\Delta_g(x, z)$ is defined by (2.7) of Lemma 2.4. By the prime number theorem and (3.3) of Proposition 3.1 with $\eta = \frac{4}{3}$, we easily derive that

$$(6.4) \quad \pi_{f,2}(x) = \frac{1}{3}C_f x \log_2 x + O\left(\frac{xr_2(x/z)}{(\log x)^A} + \frac{xz \log x}{r_3(x/z)} + \frac{x \log x}{r_3(x^{1/4}/z)} + x\right),$$

where we have removed the term $\frac{xr_2(x/z) \log x}{z}$ (since it can be bounded by $\frac{xr_2(x/z)}{(\log x)^A}$). Now the required result (1.27) follows from (6.1), (6.2), (6.3) and (6.4).

Next we prove the second assertion. In the proof of Corollary 1.2, we have seen that φ and β verify the conditions (1.6) and (1.7) with $D_g = 0$, $r_2(x) = 1$ and $r_3(x) = \exp(-c(\log x)^{3/5}(\log_2 x)^{-1/5})$. On the other hand, we have trivially $\varphi(n) \leq n$ and $\beta(n) \leq n$ for all $n \geq 1$. Thus the first assertion gives immediately the second. \square

7. PROOF OF THEOREM 1.7

7.1. Proof of the first assertion.

Write

$$(7.1) \quad \psi_f(x) = \psi_f^\dagger(x) + \psi_f^\sharp(x),$$

where

$$\psi_f^\dagger(x) := \sum_{n \leq x^{5/7}} \Lambda(n) f\left(\left[\frac{x}{n}\right]\right), \quad \psi_f^\sharp(x) := \sum_{x^{5/7} < n \leq x} \Lambda(n) f\left(\left[\frac{x}{n}\right]\right).$$

In view of the hypothesis (1.30) and the prime number theorem (3.8), we have

$$(7.2) \quad \psi_f^\dagger(x) \ll x^\theta \sum_{n \leq x^{5/7}} \frac{\Lambda(n)}{n^\theta} \ll x^{(5+2\theta)/7}.$$

On the other hand, noticing that $m = [x/n] \Leftrightarrow x/(m+1) < n \leq x/m$, we can write

$$\begin{aligned} \psi_f^\sharp(x) &= \sum_{m \leq x^{2/7}} f(m) \sum_{\max(x^{5/7}, x/(m+1)) < n \leq x/m} \Lambda(n) \\ &= \sum_{m \leq x^{2/7}} f(m) \sum_{x/(m+1) < n \leq x/m} \Lambda(n) + O(x^{(5+2\theta)/7}), \end{aligned}$$

where we have used the following bound

$$\begin{aligned} \sum_{x^{2/7}-1 < m \leq x^{2/7}} f(m) \sum_{x^{5/7} < n \leq x/m} \Lambda(n) &\ll \sum_{n \leq 2x^{5/7}} \Lambda(n) \sum_{x^{2/7}-1 < m \leq x^{2/7}} f(m) \\ &\ll x^{2\theta/7} \sum_{n \leq 2x^{5/7}} \Lambda(n) \ll x^{(5+2\theta)/7}. \end{aligned}$$

On the other hand, it is easy to verify that for $m \leq x^{2/7}$ we have $x/m^2 \geq (x/m)^{3/5}$. Applying Huxley's prime number theorem in short intervals of the form (4.4), we can derive that

$$(7.3) \quad \psi_f^\sharp(x) = \sum_{m \leq x^{2/7}} f(m) \frac{x}{m(m+1)} \left\{1 + O\left(\exp(-c(\log x)^{1/3}(\log_2 x)^{-1/3})\right)\right\} + O(x^{(5+2\theta)/7}).$$

Using the hypothesis (1.30), we have

$$\begin{aligned} \sum_{m \leq x^{2/7}} \frac{|f(m)|}{m(m+1)} &\ll \sum_{m=1}^{\infty} \frac{1}{m^{2-\theta}} \ll 1, \\ x \sum_{m \leq x^{2/7}} \frac{f(m)}{m(m+1)} &= c_f x + O(x^{(5+2\theta)/7}). \end{aligned}$$

Inserting these into (7.3), it follows that

$$(7.4) \quad \psi_f^\sharp(x) = c_f x + O\left(x \exp(-c_1(\log x)^{1/3}(\log_2 x)^{-1/3})\right).$$

Now the required result follows from (7.1), (7.2) and (7.4).

7.2. Proof of the second assertion.

Now suppose that f also verifies the hypothesis (1.30). In place of (7.3), we have, thanks to the prime number theorem (3.8) and (1.30),

$$\begin{aligned} \psi_f^\sharp(x) &= \sum_{m \leq x^{2/7}} f(m) \left\{ \frac{x}{m(m+1)} + O\left(\frac{x}{m} \exp(-c(\log x)^{3/5}(\log_2 x)^{-1/5})\right) \right\} + O(x^{(5+2\theta)/7}) \\ &= c_f x + O\left(x \exp(-c_1(\log x)^{3/5}(\log_2 x)^{-1/5})\right). \end{aligned}$$

Now the required result follows from this, (7.1) and (7.2). \square

8. PROOF OF THEOREM 1.5

8.1. Proof of the first assertion.

In view of the hypothesis (1.30), we have

$$\sum_{p \leq x^{5/7}} f\left(\left[\frac{x}{p}\right]\right) \ll x^\theta \sum_{p \leq x^{5/7}} \frac{1}{p^\theta} \ll \frac{x^{(5+2\theta)/7}}{\log x}.$$

Thus, we can write that

$$(8.1) \quad \pi_f(x) = \pi_f^\sharp(x) + O(x^{(5+2\theta)/7}/\log x),$$

where

$$\pi_f^\sharp(x) := \sum_{x^{5/7} < p \leq x} f\left(\left[\frac{x}{p}\right]\right).$$

Using Huxley's prime number theorem in short intervals of the form (4.5), we have

$$\begin{aligned} (8.2) \quad \pi_f^\sharp(x) &= \sum_{m \leq x^{2/7}} f(m) \sum_{\max(x^{5/7}, x/(m+1)) < p \leq x/m} 1 \\ &= \sum_{m \leq x^{2/7}} f(m) \sum_{x/(m+1) < p \leq x/m} 1 + O\left(\frac{x^{(5+2\theta)/7}}{\log x}\right) \\ &= \sum_{m \leq x^{2/7}} f(m) \int_{x/(m+1)}^{x/m} \frac{dt}{\log t} \left\{ 1 + O\left(e^{-c(\log x)^{1/3}(\log_2 x)^{-1/3}}\right) \right\} + O\left(\frac{x^{(5+2\theta)/7}}{\log x}\right). \end{aligned}$$

Inverting the order of summation, it follows that

$$\begin{aligned} \sum_{m \leq x^{2/7}} f(m) \int_{x/(m+1)}^{x/m} \frac{dt}{\log t} &= \int_{x/(x^{2/7+1})}^x \sum_{x/t-1 < m \leq \min(x^{2/7}, x/t)} f(m) \frac{dt}{\log t} \\ &= \int_{x^{5/7}}^x \sum_{x/t-1 < m \leq x/t} f(m) \frac{dt}{\log t} + O\left(\frac{x^{(3+2\theta)/7}}{\log x}\right), \\ &= \int_{x^{5/7}}^x \frac{f([x/t])}{\log t} dt + O\left(\frac{x^{(3+2\theta)/7}}{\log x}\right). \end{aligned}$$

Inserting this into (8.2) and making the changement of variables $u = x/t$, it follows that

$$(8.3) \quad \pi_f^\sharp(x) = x \int_1^{x^{2/7}} \frac{f([u])}{u^2 \log(x/u)} du + O\left(x \exp(-c(\log x)^{1/3}(\log_2 x)^{-1/3})\right).$$

Now the required result (1.32) follows from (8.1) and (8.3).

Next we suppose that f also verifies the hypothesis (1.14). As before, in place of (8.2) and (8.3), we have

$$(8.4) \quad \begin{aligned} \pi_f^\sharp(x) &= \sum_{m \leq x^{2/7}} f(m) \left\{ \int_{x/(m+1)}^{x/m} \frac{dt}{\log t} + O\left(\frac{x}{m} e^{-c(\log x)^{3/5}(\log_2 x)^{-1/5}}\right) \right\} + O\left(\frac{x^{(5+2\theta)/7}}{\log x}\right). \\ &= x \int_1^{x^{2/7}} \frac{f([u])}{u^2 \log(x/u)} du + O\left(x e^{-c_2(\log x)^{3/5}(\log_2 x)^{-1/5}}\right). \end{aligned}$$

From (8.1) and (8.4), we can obtain the required result.

8.2. Proof of the second assertion.

We have

$$\begin{aligned} x \int_1^{x^{2/7}} \frac{f([u])}{u^2 \log(x/u)} du &= \frac{x}{\log x} \int_1^{x^{2/7}} \frac{f([u]) du}{u^2(1 - (\log u)/\log x)} \\ &= \frac{x}{\log x} \int_1^{x^{2/7}} \left\{ \sum_{k=0}^K \frac{(\log u)^k}{(\log x)^k} + O\left(\frac{(\log u)^{K+1}}{(\log x)^{K+1}}\right) \right\} \frac{f([u])}{u^2} du \\ &= \frac{x}{\log x} \left\{ \sum_{k=0}^K \frac{a_k(f)}{(\log x)^k} + O\left(\frac{1}{(\log x)^{K+1}}\right) \right\}, \end{aligned}$$

where we have used the hypothesis (1.30) to get

$$\begin{aligned} \int_{x^{2/7}}^{\infty} \frac{f([u])(\log u)^k}{u^2} du &\ll \int_{x^{2/7}}^{\infty} \frac{(\log u)^k}{u^{2-\theta}} du \ll \frac{(\log x)^k}{x^{2(1-\theta)/7}}, \\ \int_1^{x^{2/7}} \frac{f([u])(\log u)^{K+1}}{u^2} du &\ll \int_1^{\infty} \frac{(\log u)^{K+1}}{u^{2-\theta}} du \ll 1. \end{aligned}$$

Combining these we can obtain that

$$(8.5) \quad \pi_f^\sharp(x) = \frac{x}{\log x} \left\{ \sum_{k=0}^K \frac{a_k(f)}{(\log x)^k} + O\left(\frac{1}{(\log x)^{K+1}}\right) \right\}.$$

Inserting (8.5) into (8.1), we obtain the desired result. \square

9. PROOF OF COROLLARY 1.6

Clearly the characteristic function of \mathbb{P} verifies the hypothesis (1.30) with $\theta = 0$ and (1.14) with $\alpha = 1$. Thus Theorem 1.7 gives us the required result by noticing that

$$a_0 = \int_1^{\infty} \frac{\mathbb{1}_{\mathbb{P}}([u])}{u^2} du = \sum_p \int_p^{p+1} \frac{1}{u^2} du = \sum_p \frac{1}{p(p+1)}.$$

10. PROOF OF THEOREM 1.7

Firstly we note that

$$\mathcal{S}(x) = \left\{ p \in \mathbb{P} : \exists n \in [1, x] \text{ such that } \left\lfloor \frac{x}{n} \right\rfloor = p \right\}.$$

Further, if $\left\lfloor \frac{x}{p'} \right\rfloor = p \in \mathbb{P}$, then $x/(p+1) < p' \leq x/p$. Thus we can write

$$\pi_{\mathcal{S},2}(x) = \sum_{p \leq x} \mathbb{1}(\left([x/p] - [x/(p+1)]) \cap \mathbb{P} \neq \emptyset\right).$$

For $p \leq x^{19/59}$, we have

$$\left[\frac{x}{p}\right] - \left[\frac{x}{p+1}\right] > \frac{x}{p(p+1)} - 1 \geq \frac{1}{2} \left(\frac{x}{p}\right)^{21/40}.$$

According to Baker-Harman [1, Theorem 1], for $p \leq x^{19/59}$ we have

$$([x/p] - [x/(p+1)]) \cap \mathbb{P} \neq \emptyset.$$

Thus

$$\pi_{\mathfrak{S},2}(x) \geq \pi(x^{19/59}).$$

For $p \leq x^{1/2}/(\log x)^{1+\varepsilon}$, we have $x/p(p+1) \geq (\log(x/p))^{2+\varepsilon}$. Thus under the hypothesis (1.34), we have that

$$\pi_{\mathfrak{S},2}(x) \geq \pi(x^{1/2}/(\log x)^{1+\varepsilon}). \quad \square$$

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