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THE EARTHQUAKE METRIC ON TEICHMÜLLER SPACE

YI HUANG, KEN'ICHI OHSHIKA, HUIPING PAN, AND ATHANASE
PAPADOPOULOS

ABSTRACT. This is the first paper to systematically study the earthquake metric, an asymmetric Finsler metric on Teichmüller space introduced by Thurston. We provide proofs for several assertions of Thurston and establish new properties of this metric, among which are incompleteness, asymptotic distance to the boundary and comparisons with the Thurston metric and the Weil–Petersson metric. In doing so, we propose a novel asymmetric generalisation of the notion of completion for symmetric metrics, which we call the FD-completion, and prove that for the earthquake metric the FD-completion and various symmetrised metric completions coincide with the Weil–Petersson completion. We also answer a question of Thurston by giving an interpretation of this metric arising from a global minimisation problem, namely, the earthquake magnitude minimisation problem. At several points of this paper, we formulate a certain number of open problems which will show that the earthquake metric constitutes a promising subject.

Keywords.— hyperbolic surfaces, Teichmüller theory, earthquakes, earthquake metric, Finsler metrics, Thurston metric, Weil–Petersson metric, moduli space, augmented Teichmüller space, Deligne–Mumford compactification.

AMS classification.— 30F60, 32G15, 30F10, 52A21.

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1. INTRODUCTION

Fenchel–Nielsen twists are deformations of hyperbolic metrics on a surface S obtained by cutting the surface along simple closed geodesics and regluing with displacement. They constitute special examples of a general family of hyperbolic surface deformations which cut, twist and reglue with respect to *measured geodesic laminations*. Thurston [59] first defined these generalised metric deformations, and dubbed them *earthquakes*.

The well-known *earthquake theorem* ([32, Theorem 2] and [59, III.1.3.1.]) is Thurston’s first important result on earthquakes. It asserts that for any ordered pair of distinct hyperbolic metrics on S there is a unique (left) earthquake¹ deforming from the initial metric to the terminal metric. This result was famously employed in Kerckhoff’s solution [32] to the Nielsen realisation problem: Kerckhoff showed that any finite subgroup of the mapping class group of S can be realised as an isometry (sub)group of some hyperbolic metric on S .

Kerckhoff’s proof made use of *convexity* (specifically, of geodesic lengths with respect to earthquakes) — a favourite tool in the Thurston school. Another context in which convexity and earthquakes intermingle is described in [61, Theorem 5.2] (also see Theorem 3.3 in the present paper), where Thurston showed that the derivatives of unit-speed earthquakes at each point in Teichmüller space $x \in \mathcal{T}(S)$ prescribes the boundary sphere for a convex body in $T_x\mathcal{T}(S)$ which contains the origin. Using this convex body, he defined a weak norm (Definition 2.15) to take value 1 on this sphere, and dubbed it the *earthquake norm*, which we denote by $\|\cdot\|_e$. The earthquake norm therefore defines a Finsler metric on Teichmüller space, which we call the *earthquake metric* d_e (Definition 1.2).

The earthquake norm has an intriguing interpretation: Barbot and Fillastre spurred by a comment of Seppi, realised that the additive symmetrisation of the earthquake norm measures the volume of the convex core of certain associated quasi-Fuchsian co-Minkowski surfaces, see [5, p. 650]. Yet the earthquake metric remains mysterious. Thurston muses:

Perhaps there is some global minimization problem which the earthquake norm measures, but I don’t know [...] any global interpretation for distances with respect to the earthquake norm.

In response, we show in Theorem 1.7 that the earthquake metric *does* have a minimisation formulation in terms of earthquake *magnitudes* (Definition 1.5). Our nomenclature retroactively reinterprets the following words by Thurston [61, p. 22]):

The distance between two elements of Teichmüller space in the earthquake norm is the infimum of the total magnitude of a sequence of earthquakes transforming one to the other.

In other words, the earthquake metric quantifies the following:

¹We say “earthquake” without qualifying adjectives to mean the left earthquake.

How efficiently can we earthquake from one hyperbolic metric to another?

We establish basic geometric properties of this metric, and discover connections with the Thurston (asymmetric) metric and the Weil–Petersson metric. We believe that these connections will give the earthquake metric a secure footing in Teichmüller and moduli space theory.

We mentioned the paper [5] by Barbot and Fillastre in which the earthquake metric appears in the form of a norm. The two authors write, on p. 650 of their paper:

[...] Note that the earthquake norm also induces an asymmetric distance on Teichmüller space, but, to the best of our knowledge, nothing is known about this distance.

The present paper is the first contribution in this direction.

Throughout this paper, $S = S_{g,n}$ denotes a Riemann surface of genus $g \geq 0$, with $n \geq 0$ cusps, and negative Euler characteristic. Let us state our results in some detail, anticipating some of the notation set out precisely in the next section.

1.1. Earthquake metric. Kerckhoff [34, Proposition 2.6] showed that every tangent vector $v \in T_x\mathcal{T}(S)$ at a point $x \in \mathcal{T}(S)$ can be uniquely expressed as an *infinitesimal earthquake*, i.e. a tangent vector of the form

$$\mathbf{e}_\lambda(x) := \left. \frac{d}{dt} \right|_{t=0} E_{t\lambda}(x)$$

of an earthquake path $E_{t\lambda}(x)$ shearing with respect to a measured lamination $\lambda \in \mathcal{ML}(S)$. Thurston [61, Theorem 5.2] further showed that the set

$$\{v \in T_x\mathcal{T}(S) \mid v = \mathbf{e}_\lambda(x) \text{ for } \ell_\lambda(x) = 1\}$$

is the boundary of a convex ball in $T_x\mathcal{T}(S)$ containing the origin in its interior, and hence we can define a weak norm as follows.

Definition 1.1 (earthquake norm). For every $x \in \mathcal{T}(S)$, the function

$$\|\cdot\|_e : T_x\mathcal{T}(S) \rightarrow \mathbb{R}_{\geq 0}, \quad \|\mathbf{e}_\lambda(x)\|_e := \ell_x(\lambda)$$

defines a weak norm (Definition 2.15) on $T_x\mathcal{T}(S)$ called the *earthquake norm*.

Definition 1.2 (earthquake metric). The earthquake norm induces a notion of length for C^1 paths in $\mathcal{T}(S)$. This determines a path metric on $\mathcal{T}(S)$ (Definition 2.17) where the distance from x to y is defined as the infimum of the length, with respect to the earthquake norm, of piecewise C^1 paths from x to y . We refer to this distance metric as the *earthquake metric* d_e .

Remark 1.3 (left/right earthquake metric). Working with right earthquakes rather than left earthquakes, we get a distance function $d_e^\#$ instead of d_e . This is also a Finsler metric, satisfying $d_e^\#(x, y) := d_e(y, x)$ — it is also called the reverse metric of d_e (Definition 2.7). Its underlying norm is $\|v\|_e^\# = \|-v\|_e$. We refer to d_e and $d_e^\#$ respectively as the left and right earthquake metrics respectively when we need to distinguish between the two, and omit “left” when only dealing with d_e .

The left and right earthquake metrics induce natural topologies on Teichmüller space. The following is an immediate corollary of Proposition A.7:

Proposition 1.4 (usual topology). *The left and right earthquake metrics d_e and $d_e^\#$ both induce the usual topology on Teichmüller space.*

1.2. Magnitude. We propose a natural energy function for quantifying *efficient* earthquaking, and arrive at an alternative formulation of d_e .

Definition 1.5 (magnitude). Consider a piecewise earthquake path consisting of earthquake segments joining x_1, \dots, x_{m+1} such that $x_{i+1} = E_{\lambda_i}(x_i)$, for $i = 1, \dots, m$. We refer to the quantity $\sum_{i=1}^m \ell_{\lambda_i}(x_i)$ as the *magnitude* of the given piecewise earthquake path.

Remark 1.6. The magnitude of an earthquake segment is independent of the choice of homothetic representative. In particular, the magnitude is both

- the time taken for traversing an earthquake segment when the measured lamination λ is normalised to have hyperbolic length 1, and
- the hyperbolic length of the measured lamination λ needed to traverse the given earthquake segment in precisely time 1.

Theorem 1.7 (magnitude minimisation). *For arbitrary $x, y \in \mathcal{T}(S)$, the earthquake distance $d_e(x, y)$ is equal to the infimum of the magnitudes of all piecewise earthquake paths from x to y .*

1.3. Duality to the Thurston metric. In [5, Section 16.3.4.5], Barbot and Fillastre explain a duality, first hinted at by Thurston on the last lines of page 20 and on page 21 of [61]:

Theorem 1.8 (infinitesimal duality). *At each point $x \in \mathcal{T}(S)$, there is a linear isometry between the normed spaces $(T_x \mathcal{T}(S), \|\cdot\|_e)$ and $(T_x^* \mathcal{T}(S), \|\cdot\|_{\text{Th}}^*)$. The isometry is induced by the Weil–Petersson symplectic duality.*

Remark 1.9. Theorem 1.8 entwines the earthquake, the Thurston, and the Weil–Petersson geometries of Teichmüller space. In light of upcoming results such as connecting the earthquake and Weil–Petersson metrics, it is tempting to wonder if this helps to explain the similarities between the two metrics.

For expositional clarity and completeness, we give a proof of Theorem 1.8 via Lemma 3.1. Using Theorem 1.8, we obtain the rigidity theorem.

Theorem 1.10 (mapping class group rigidity). *The isometry group of $(\mathcal{T}(S), d_e)$ is the extended mapping class group, with the usual action. This statement holds except when $S = S_{0,4}, S_{0,5}, S_{1,1}$ or $S_{2,0}$, where the isometry group is the extended mapping class group modulo hyperelliptic involution.*

In fact, we obtain the above rigidity result from an infinitesimal version of this theorem (see Theorem 6.4), which parallels Royden’s infinitesimal rigidity theorem for the Teichmüller metric [54].

Remark 1.11. Lipschitz constants for stretch maps underpinning the Thurston metric and the magnitudes for earthquakes are quantities associated with different deformation phenomena:

- the former measures tensile stress accumulated during elastic deformations of hyperbolic surfaces,
- whereas the latter measures accumulated shear stress which result in brittle deformation of hyperbolic surfaces.

It is tempting, although heavily speculative, to wonder if the Finsler structure of the Thurston and earthquake metrics are consequences of some general physical principle regarding energy minimisation, and further, whether the infinitesimal duality has an energy-based interpretation.

1.4. Earthquake paths and earthquake metric geodesics. Mirzakhani [44] and Calderon–Farre [17] showed that the earthquake flow is a natural conceptual hyperbolic analogue of the Teichmüller horocycle flow. This narrative roughly accords with Thurston’s observation in [61, p. 22] that earthquakes are *not* geodesics for this metric. In particular, he asserts that on the Teichmüller space of 1-cusped tori,

Earthquakes along simple curves are approximately horocycles.

He remarked that the inefficiency of horocycles in navigating Teichmüller space suggests the following statement, of which we provide a proof, for general hyperbolic surfaces:

Theorem 1.12. *Sufficiently long left earthquake paths in $\mathcal{T}(S)$ cannot be geodesics with respect to the left earthquake metric.*

We presently have very little understanding regarding geodesics of the earthquake metric, and much is open for discovery. Here is a sampling of some of the questions we wish to be able to answer:

Question 1.13. Is it possible to give an explicit (i.e. in coordinates) description of even a single geodesic segment for the earthquake metric?

Question 1.14. Can short earthquake segments ever be geodesic?

Question 1.15. Is the reverse-time parameterisation of some geodesic ever also a geodesic for the earthquake metric?

Question 1.16. Is the earthquake metric geodesically convex, i.e. can two arbitrary points in Teichmüller space be joined by a geodesic?

1.5. Norm and metric comparisons. To help familiarising and contextualising the earthquake metric, we first study comparisons among the following Finsler norms on Teichmüller space:

- the earthquake norm $\|\cdot\|_e$,
- the Weil–Petersson norm $\|\cdot\|_{\text{WP}}$,
- the Thurston norm $\|\cdot\|_{\text{Th}}$, and
- the Teichmüller norm $\|\cdot\|_{\text{T}}$.

Before stating the theorem, let us introduce a notation. For any $x \in \mathcal{T}(S)$, let $\ell_{\text{sys}}(x)$ be the *systolic length* of the hyperbolic surface x , that is, the length of a shortest closed geodesic on x . With this notation in preparation, our result is:

Theorem 1.17. *There are positive constants C_0, C_1, C_2 , depending only on the topology of S , such that for any $x \in \mathcal{T}(S)$ and any $v \in T_x\mathcal{T}(S)$,*

$$C_0 \ell_{\text{sys}}(x) \text{Log}\left(\frac{1}{\ell_{\text{sys}}(x)}\right) \|v\|_{\text{Th}} \leq \|v\|_e \leq C_1 \|v\|_{\text{WP}} \leq C_2 \|v\|_{\text{Th}},$$

where $\text{Log}(t) := \max\{1, \log t\}$.

There is one more metric we shall consider on Teichmüller space, namely, the Teichmüller metric; we denote it by d_T , and we denote the norm of a vector v for this metric by $\|v\|_T$. With this notation, combining Theorem 1.17 with Wolpert's [65, Lemma 3.1] and an estimate due to Burns–Masur–Wilkinson [10, Lemma 5.4], we can form the following *cycle* of comparisons:

Corollary 1.18. *There are positive constants C_0, C_1, C_2, C_3 depending only on the topology of S , such that for any $x \in \mathcal{T}(S)$ and any $v \in T_x\mathcal{T}(S)$,*

$$C_0 \ell_{\text{sys}}(x) \text{Log}\frac{1}{\ell_{\text{sys}}(x)} \|v\|_{\text{Th}} \leq \|v\|_e \leq C_1 \|v\|_{\text{WP}} \leq C_2 \|v\|_{\text{Th}} \leq 2C_2 \|v\|_T \leq \frac{C_3 \|v\|_{\text{WP}}}{\ell_{\text{sys}}(x)}.$$

Corollary 1.18 induces the following metric comparisons.

Corollary 1.19. *There are constants C_1, C_2 depending only on the topology of S such that for any two points x, y in $\mathcal{T}(S)$, we have*

$$d_e(x, y) \leq C_1 d_{\text{WP}}(x, y) \leq C_2 d_{\text{Th}}(x, y) \leq 2C_2 d_T(x, y).$$

1.6. Metric incompleteness and completion. Since the Weil–Petersson metric is incomplete, one intuitively expects the uniform domination of d_e by d_{WP} , given in Corollary 1.19, to be an obstruction to any natural notion of completeness for the earthquake metric — this includes a notion called forward *FD-completeness*, defined for asymmetric metrics, which we introduce in Section 10.1:

Proposition 1.20. *The earthquake metric is not forward FD-complete.*

In Section 10.1 and Appendix B we also define the forward *finite distance-series completion (FD-completion)* — a novel metric completion for asymmetric metric spaces. The FD-completion generalises the Cauchy completion for symmetric metric spaces (Theorem B.12), and naturally contains the original metric space as a forward-dense subset (Proposition B.4). Lipschitz maps in the original space can be extended to the FD-completion in a functorial way (Proposition B.17).

Theorem 1.21 (completion). *The forward FD-completions of $\mathcal{T}(S)$ with respect to both the left and right earthquake metrics coincide with the Weil–Petersson metric completion, i.e. the augmented Teichmüller space, obtained*

by adding boundary strata corresponding to Teichmüller spaces of nodal Riemann surfaces.

Combining Theorem 1.21, [27], and [40], we further obtain:

Corollary 1.22 (Deligne–Mumford compactification). *The forward FD-completion of the moduli space $\mathcal{M}(S)$ with respect to the earthquake metric coincides with the Deligne–Mumford compactification.*

Remark 1.23. The fact that the earthquake metric, which is defined purely in terms of hyperbolic geometry, is able to access the Deligne–Mumford compactification with its complex analytic/algebraic geometric roots, may seem unexpected. This is reminiscent of a result of Wolpert in [67], saying that the Weil–Petersson metric, which has complex analytical origins, is equal to a certain Riemannian metric first constructed by Thurston, defined in terms of the behaviour of the hyperbolic lengths of closed geodesics.

Remark 1.24. We shall show in Theorem 10.27 that analogous results concerning augmented Teichmüller space also hold for symmetrisation (Definition 10.22 and Definition 10.24) of the earthquake metric. Among these, we shall highlight the Finsler symmetrisations $D^{(1)}$ and $D^{(\infty)}$ (see Definition 10.24) as potential objects for closer study. The former has a geometric interpretation as its Finsler norm in terms of co-Minkowskian surfaces [5, p. 650], and the latter infimises magnitude without favouring left or right earthquakes.

Theorem 1.25 (metric extension). *The left earthquake metric d_e has a unique continuous extension \bar{d}_e to an asymmetric metric on the forward FD-completion $\overline{\mathcal{T}(S)}$. Likewise, the right earthquake metric $d_e^\#$ uniquely extends to an asymmetric metric $\bar{d}_e^\#$ on the forward FD-completion $\overline{\mathcal{T}(S)}^\#$ of $(\mathcal{T}(S), d_e^\#)$. Both $(\overline{\mathcal{T}(S)}, \bar{d}_e)$ and $(\overline{\mathcal{T}(S)}^\#, \bar{d}_e^\#)$ are Busemannian metric spaces.*

We introduce the term “Busemannian” in Definition 2.12.

Question 1.26. By Corollary 1.22, the completion locus of the earthquake metric consists of strata of lower-dimensional Teichmüller spaces (of possibly disconnected surfaces). Does the extended metric on each stratum coincide with the intrinsic earthquake metric for that Teichmüller space?

1.7. Distance to the boundary. The first inequality in Corollary 1.19 says a great deal more than just the incompleteness of the earthquake metric. It asserts that any point in $\mathcal{T}(S)$ is a bounded distance away from the completion locus.

Corollary 1.27 (Bounded diameter). *The moduli space $\mathcal{M}(S)$ has bounded diameter with respect to the earthquake metric.*

One line of inquiry is to ponder:

Question 1.28. What is the diameter of $\mathcal{M}(S)$ with respect to the earthquake metric?

Another is to study the behaviour of the earthquake metric as one approaches the completion locus:

Theorem 1.29. *Let $\overline{\mathcal{M}(S)}$ denote the completion of the moduli space with respect to the earthquake metric. There exists a constant $C_S > 1$, depending only on the topology of S , such that for any $x \in \mathcal{M}(S)$,*

$$(1) \quad 2\ell_{\text{sys}}(x)\text{Log}\left(\frac{1}{\ell_{\text{sys}}(x)}\right) \leq d_e(x, \partial\overline{\mathcal{M}(S)}) \leq 2C_S \cdot \ell_{\text{sys}}(x)\text{Log}\left(\frac{1}{\ell_{\text{sys}}(x)}\right),$$

where, as before, $\text{Log}(t) := \max\{1, \log(t)\}$. Moreover, we have the following asymptotic behaviour as $\ell_{\text{sys}}(x) \rightarrow 0$

$$(2) \quad \frac{d_e(x, \partial\overline{\mathcal{M}(S)})}{2\ell_{\text{sys}}(x)\text{Log}\left(\frac{1}{\ell_{\text{sys}}(x)}\right)} \rightarrow 1.$$

Both inequalities hold with $d_e(x, \partial\overline{\mathcal{M}(S)})$ replaced by $d_e(\partial\overline{\mathcal{M}(S)}, x)$.

Remark 1.30. Although Theorem 1.29 is phrased in terms of the moduli space, the result is still true if we replace $\mathcal{M}(S)$ by $\mathcal{J}(S)$.

1.8. Further comparisons with the Weil–Petersson metric. Yet another line of inquiry suggested by Corollary 1.19 is whether d_e and d_{WP} are bi-Lipschitz equivalent. We show in Proposition 7.7 that this is false, essentially because d_e and d_{WP} behave very differently in the thin part of Teichmüller space. Such fine structure is ignored by coarse geometry, and we have the following positive result:

Theorem 1.31 (Earthquake vs. Weil–Petersson). *The earthquake metric is quasi-isometric to the Weil–Petersson metric. More precisely, there are constants $D_1, D_2 > 0$ which depend only on the topology of S , such that*

$$\forall x, y \in \mathcal{J}(S), \quad D_1 d_{\text{WP}}(x, y) - D_2 \leq d_e(x, y) \leq C d_{\text{WP}}(x, y)$$

where C is the constant from Corollary 1.19.

The proof of Theorem 1.31 also shows that $(\mathcal{J}(S), d_e)$ is quasi-isometric to the pants graph $\mathcal{P}(S)$ — the infinite graph whose vertices are pants decompositions of S , with an edge between any two pants decompositions if and only if the decompositions are related by an elementary move. Indeed, Brock’s result [9, Theorem 1.1] saying that $(\mathcal{J}(S), d_{\text{WP}})$ is quasi-isometric to $\mathcal{P}(S)$ implies the claim.

Remark 1.32 (navigating Teichmüller space). Theorem 1.31 suggests that (at least) for “long” paths in Teichmüller space endowed with the earthquake metric, a potential strategy for efficiently traversing this metric space is to use the pants complex as a guide, running between regions of Teichmüller space with very short pants decompositions.

1.9. Asymmetry of the earthquake metric. It is known that the Thurston norm is asymmetric at some points in $\mathcal{T}(S)$ (see [61, Figure 1] and [58, Theorem 5.3]). This confers a similar norm-wise asymmetry to the earthquake norm via Theorem 1.8. We shall prove a stronger asymmetry using the infinitesimal rigidity (Theorem 6.4), that is, we shall prove that the earthquake norm, hence the Thurston norm also, is *nowhere* symmetric.

Theorem 1.33 (norm asymmetry). *The earthquake norm is asymmetric at every point of $\mathcal{T}(S)$. That is: for any $x \in \mathcal{T}(S)$, there exists a tangent vector $v \in T_x\mathcal{T}(S)$ such that $\|v\|_e \neq \|-v\|_e$.*

As a direct consequence of Theorem 1.33, we obtain:

Corollary 1.34. *The earthquake metric is not symmetric, that is, there exist x and y in $\mathcal{T}(S)$ such that $d_e(x, y) \neq d_e(y, x)$. Equivalently, the left earthquake metric and the right earthquake metric are not equal.*

Remark 1.35. Curiously, we have not been able to find a direct argument which proves Corollary 1.34.

The following “equivalences” we obtain (often with the Weil–Peterson metric as a bridge) between the left and right earthquake metrics compensate for the asymmetry of the (left) earthquake metric:

- Proposition 1.4 says that they are homeomorphic;
- Theorem 1.31 shows that they are quasi-isometric;
- Theorem 1.21 shows that they have the same forward FD-completion;
- and Theorem 1.29 shows that they have similar behaviour in terms of distances from the completion locus.

And one is tempted to pose the following question.

Question 1.36. Is the left earthquake metric bi-Lipschitz equivalent to the right earthquake metric?

Actually, a positive answer to the above question would suffice to show that the earthquake metric is geodesic. However, there is every likelihood that the two metrics are *not* bi-Lipschitz equivalent.

Organisation of the paper. Our paper is structured as follows.

- In Section 2, we review some of the necessary background Teichmüller theory and a small amount of asymmetric metric theory.
- We investigate the properties of the earthquake norm in Section 3, showing that it defines a Finsler metric.
- In Section 4, we strengthen the known regularity of $E_{t\lambda}(x)$ with respect to both time t and space x derivatives. We use this to show that the earthquake metric is (locally) Lipschitz continuous.
- We then utilise the established regularity to show, in Section 5, an energy-based formulation of the earthquake metric.

- In Section 6, we show that the the isometry group of the earthquake metric is equal to the extended mapping class group. The infinitesimal version of this result can be used to show that the earthquake metric is asymmetric.
- In Section 7, we compare the earthquake metric with the Thurston metric, the Weil–Petersson metric, and the Teichmüller metric on $\mathcal{T}(S)$, both infinitesimally and globally.
- In Section 8, we investigate the geodesic properties of the earthquake metric, showing a negative result, namely, that (long) earthquakes are not geodesics of this metric.
- We then go on to study the behaviour of the earthquake metric in the thin part of Teichmüller space, determining the magnitude of earthquakes needed to deform a marked surface with short systole to a nodal Riemann surface.
- In Section 10, we define FD-completions of asymmetric metric spaces, and show that the FD-completion of Teichmüller space equipped with the earthquake metric is equal to the Weil–Petersson completion.
- Finally, in Section 11, we show that the earthquake metric is coarsely equivalent to the Weil–Petersson metric.

At the end of this paper, we have included two appendices, the first outlining general facts regarding Finsler metrics of low regularity, and the second establishing various foundational properties of the FD-completion of an asymmetric metric space.

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2. BACKGROUND

In this section, in an effort to make the paper self-contained, we have collected a certain number of fundamental ideas, definitions and properties which explain the results of this paper and set them in the appropriate context.

2.1. Teichmüller and moduli space. Given an oriented connected surface of genus $g \geq 0$ with $n \geq 0$ punctures, $S = S_{g,n}$, we let

- $\mathcal{T}(S)$ denote the *Teichmüller space* of S — the space of complete hyperbolic metrics on S up to isometries isotopic to the identity map on S ;
- $\mathcal{M}(S)$ denote the *moduli space* of S — the space of isometry classes of complete hyperbolic metrics on S .

For expositional simplicity, we may refer to $x \in \mathcal{T}(S)$ simply as a marked (hyperbolic) surface, as opposed to an equivalence class of hyperbolic metrics. Likewise, we refer to a point in $\mathcal{M}(S)$ as a (hyperbolic) surface.

The topology of $\mathcal{T}(S)$ is well understood: the space is analytically equivalent to $\mathbb{R}^{6g-6+2n}$. The moduli space $\mathcal{M}(S)$ has greater topological complexity, and one approach to it is to understand it as a quotient of $\mathcal{T}(S)$ by the *mapping class group*

$$\text{Mod}(S) := \frac{\{ \text{orientation-preserving diffeomorphisms of } S \}}{\{ \text{diffeomorphisms of } S \text{ isotopic to the identity} \}}.$$

It is well known that

- the mapping class group $\text{Mod}(S)$ acts (non-freely) properly discontinuously on $\mathcal{T}(S)$,
- the moduli space $\mathcal{M}(S)$, which is the quotient space, is an orbifold,
- and since the Teichmüller space $\mathcal{T}(S)$ is simply connected, it is the orbifold universal cover of $\mathcal{M}(S)$.

There are several modern introductions to these topics, see e.g. [2, 22, 21, 30]. The reader interested in the history of moduli and Teichmüller spaces may refer to [3, 31].

2.2. Lengths of geodesics and measured laminations. The (hyperbolic) length of a non-contractible closed curve γ on S , with respect to a hyperbolic metric $x \in \mathcal{T}(S)$, is defined as the length of the unique shortest closed curve (specifically, a geodesic) in the free homotopy class of γ . From now on, by a closed curve, we always mean a non-contractible one. Linearity gives us a natural extension of the notion of length to weighted curves $c\gamma$ and sums of weighted curves $\sum c_i \gamma_i$. The measured lamination space $\mathcal{ML}(S)$ is, in some precise sense, a completion of the space of weighted *simple* (i.e. non-self-intersecting) closed curves. The elements of $\mathcal{ML}(S)$ are geodesic laminations equipped with transverse measures, see [60, §1].

The hyperbolic length function for weighted simple closed curves extends continuously to a positively homogeneous function on $\mathcal{ML}(S)$. For a general

measured geodesic lamination λ , its length $\ell_x(\lambda)$ is the total mass of the measure on the surface defined as the product of the Lebesgue measure along the leaves of λ and the transverse measure of λ .

Remark 2.1 (length notation). In our most general context, a length function is a map

$$\ell : \mathcal{ML}(S) \times \mathcal{T}(S) \rightarrow \mathbb{R}_{\geq 0},$$

assigning to any pair $(x, \lambda) \in \mathcal{T}(S) \times \mathcal{ML}(S)$ the length of the unique geodesic representative of λ with respect to the metric x . In almost all contexts, however, we fix either the lamination λ or the metric x , and we respectively denote these two functions by:

- $\ell_x : \mathcal{ML}(S) \rightarrow \mathbb{R}_{\geq 0}$, $\ell_x(\lambda) := \ell(\lambda, x)$;
- $\ell_\lambda : \mathcal{T}(S) \rightarrow \mathbb{R}_{\geq 0}$, $\ell_\lambda(x) := \ell(\lambda, x)$.

The hyperbolic length function is continuous, and is positively homogeneous in its first parameter, and the pairing ℓ between Teichmüller space and the measured lamination space induces an embedding of $\mathcal{T}(S)$ into the space $\mathbb{R}_{>0}^{\mathcal{ML}(S)}$ of real functions on $\mathcal{ML}(S)$ by “duality” in the following sense: for each $\lambda \in \mathcal{ML}(S)$, the λ -coordinate for $x \in \mathcal{T}(S)$ is given by $\ell_\lambda(x)$. Thurston showed that the $\mathbb{R}_{>0}$ -projectivisation of this embedding is still an embedding, and that the boundary of Teichmüller space may be interpreted, using this embedding, as the space of projective measured laminations $\mathcal{PML}(S) := (\mathcal{ML}(S) - \{0\})/\mathbb{R}_{>0}$.

2.3. Fenchel–Nielsen twists and earthquakes. The left Fenchel–Nielsen twist with respect to a weighted simple closed curve $c\gamma$ defines a flow on Teichmüller space, whereby the time ℓ_γ map of the flow corresponds to the action of the right Dehn twist along γ (since the action of a mapping class on the Teichmüller space is precomposition by the inverse). Thurston showed that Fenchel–Nielsen twist flows, which shear along simple closed geodesics, generalise to flows with respect to measured laminations in the following way (see [32]):

- extend Fenchel–Nielsen twist flows to flows for (positively) weighted simple closed curves $c\gamma$ via $(t, x) \mapsto E_{tc\gamma}(x)$, for $x \in \mathcal{T}(S)$ and $t \in \mathbb{R}_{>0}$.
- for any sequence $(c_i\gamma_i)$ of weighted simple closed curves converging to a measured lamination λ , show that their corresponding twist flows converge, and that this limit is independent of the choice of the initial sequence $(c_i\gamma_i)$ of weighted simple closed curves.

We refer to any such limiting flow as the *(left) earthquake flow with respect to λ* :

$$(t, x) \mapsto \lim_{i \rightarrow \infty} E_{tc_i\gamma_i}(x) =: E_{t\lambda}(x).$$

Kerckhoff [34] showed that for any $\lambda \in \mathcal{ML}(S)$, the earthquake flow $t \mapsto E_{t\lambda}(x)$ is analytic with respect to t . The infinitesimal earthquake

$$\mathbf{e}(\lambda, x) := \left. \frac{d}{dt} \right|_{t=0} E_{t\lambda}(x)$$

is the basis for defining the earthquake norm (Definition 1.1), and subsequently, the earthquake metric (Definition 1.2).

Remark 2.2 (earthquake vector notation). We adopt notation similar to length function notation (see Remark 2.1) to distinguish situations when we wish to regard earthquake vectors as functions of $\mathcal{ML}(S)$ versus $\mathcal{T}(S)$:

- $\mathbf{e}_x : \mathcal{ML}(S) \rightarrow T_x\mathcal{T}(S)$, $\mathbf{e}_x(\lambda) := \mathbf{e}(\lambda, x)$;
- $\mathbf{e}_\lambda : \mathcal{T}(S) \rightarrow T\mathcal{T}(S)$, $\mathbf{e}_\lambda(x) := \mathbf{e}(\lambda, x)$

Remark 2.3. Thurston provides in [59] a different construction/proof for earthquake flows in greater generality. In particular, this construction works for the universal Teichmüller space.

2.4. Asymmetric and Busemannian metric spaces. Consider the following quote from Gromov in the introduction of [24]:

[...] Besides, one insists that the distance function be symmetric, that is, $d(x, y) = d(y, x)$. (This unpleasantly limits many applications [...])

Many metrics are naturally asymmetric due to global minimisation problems innately bearing such asymmetry. For example, the (minimal) amount of energy that one expends ascending a staircase is (generally) different from the amount needed when descending. Nevertheless, such quantities satisfy the triangle inequality, and thus benefit from the wealth of metric space theory which mathematics has produced. There are various versions of asymmetric metric space theory. For our purposes, we adopt the following:

Definition 2.4 (asymmetric metric space, see, e.g., [43]). A set X endowed with a function $d : X \times X \rightarrow [0, \infty]$ is called an *asymmetric metric space* if for all $x, y, z \in X$

- (1) $d(x, x) = 0$;
- (2) $d(x, y) = d(y, x) = 0 \Rightarrow x = y$;
- (3) $d(x, z) \leq d(x, y) + d(y, z)$.

The adjective “asymmetric” is used by Thurston for a metric which does not satisfy the symmetry axiom in his paper [61], which is where he introduced the Thurston (asymmetric) metric. Before Thurston, asymmetric metrics were used and studied in various contexts. Thurston described his metric as “Finsler asymmetric”. As a matter of terminology, Finsler metrics do not necessarily assume the symmetry axiom. A Finsler metric which is symmetric is usually referred to as a “reversible Finsler metric”.

Remark 2.5. Although semantically confusing, an asymmetric metric is allowed to be symmetric. In any case, we show that the earthquake metric, like the Thurston metric, is non-symmetric.

In this paper, we shall make use of the following notions of symmetrisation of an asymmetric metric:

Definition 2.6 (Symmetrisations). Given an asymmetric metric d , its *sum-symmetrisation* d^{sum} is defined as

$$d^{\text{sum}}(x, y) := \frac{1}{2}(d(x, y) + d(y, x))$$

and its *max-symmetrisation* d^{max} is defined as

$$d^{\text{max}}(x, y) := \max\{d(x, y), d(y, x)\}.$$

Definition 2.7 (reverse metric). Given an asymmetric metric space (X, d) , the function $d^\# : X \times X \rightarrow [0, \infty]$ defined by $d^\#(x, y) := d(y, x)$ is (also) an asymmetric metric on X . We refer to $d^\#$ as the *reverse metric* of d , and call $(X, d^\#)$ the *reverse metric space* of (X, d) .

Definition 2.8 (topologies of asymmetric metric spaces). Asymmetric metrics d induce three natural choices of topologies on X :

- the *forward topology* is generated by *forward open balls*, i.e. sets of the form $B^+(x, \epsilon) := \{y \in X \mid d(x, y) < \epsilon\}$, for arbitrary $\epsilon > 0$;
- the *backward topology* is generated by *backward open balls*, i.e. sets of the form $B^-(x, \epsilon) := \{y \in X \mid d(y, x) < \epsilon\}$, for arbitrary $\epsilon > 0$;
- the *symmetric topology* is generated by the forward and backward open balls.

Remark 2.9. It is fairly straightforward to see that the symmetric topology of an asymmetric metric d agrees with the topology induced by the max-symmetrisation of d .

Definition 2.10 (limits). Given an asymmetric metric space (X, d) , we say that $x \in X$ is a *forward limit* of a sequence (x_n) if $d(x_n, x) \rightarrow 0$. Backward limits are analogously defined.

Remark 2.11. There is an alternative notion of a forward limit stated as requiring that $d(x_n, y) \rightarrow d(x, y)$ for all $y \in X$. This is a stricter condition than what we want.

Asymmetric metrics, including Finsler, were studied extensively by Herbert Busemann, in several papers and books. Busemann is one of the most important promoters of metric geometry. We shall see later that the metric spaces we consider all fall within a strict subclass of asymmetric metric spaces which appear in his work. Busemann refers to metrics which we call asymmetric as “not necessarily symmetric” in [14] — this is semantically clearer, but more cumbersome to say. Nonetheless, it is helpful to have a name for this more general framework. We consider that the following name is deserved:

Definition 2.12 (Busemannian metric space). A set X endowed with a function $d : X \times X \rightarrow [0, \infty)$ is called a *Busemannian metric space* if

- for all $x, y \in X$, $d(x, y) = 0 \Leftrightarrow x = y$;
- for all $x, y, z \in X$, $d(x, z) \leq d(x, y) + d(y, z)$.

- for all $x \in X$ and for all sequences (x_n) in X , we have $d(x_n, x) \rightarrow 0$ if and only if $d(x, x_n) \rightarrow 0$.

Spaces satisfying the three properties of Definition 2.12 are highlighted and studied in [13] (of course, without the name).

Remark 2.13. The concept of Busemannian metric spaces should not be confused with objects such as Busemann (convex) spaces and Busemann G-spaces, which are geodesic metric spaces satisfying additional conditions, and are tied in with different developments in metric geometry.

Remark 2.14. For Busemannian metric spaces, the forward, backward and symmetric topologies all agree. This follows from the fact that these topologies are first countable. We set this topology as the *standard topology* on X . Since d^{\max} is symmetric, continuity of functions defined on Busemannian metric spaces may be verified as usual in the language of limits and/or convergent sequences.

2.5. Finsler geometry of low regularity. Thurston's paper [61] on the Thurston metric, where he defines the earthquake metric, deals with asymmetric Finsler metric structures on Teichmüller space with low regularity (i.e. C^0) on the metric. To begin with, we clarify the manner in which asymmetry manifests itself at the level of the Finsler norm.

Definition 2.15 (weak norm). We define a *weak norm* $\|\cdot\|$ on a vector space V as a function $\|\cdot\| : V \rightarrow [0, \infty)$ satisfying the following properties:

- (1) $\|v\| = 0$ if and only if $v = 0$;
- (2) $\|v_1 + v_2\| \leq \|v_1\| + \|v_2\|$ for all v_1, v_2 in V ;
- (3) $\|\lambda v\| = \lambda\|v\|$ for all v in V and for all $\lambda > 0$.

Remark 2.16. As in the case of a symmetric norm, a weak norm $\|\cdot\|$ on V is completely determined by its unit sphere (for Finsler norms, this is sometimes called the *indicatrix*), that is, the set of vectors $v \in V$ satisfying $\|v\| = 1$. This may be an arbitrary hypersurface bounding a compact convex body in V containing the origin in its interior.

Definition 2.17 (Finsler metric). A *Finsler metric* or a *Finsler norm* on a path-connected C^1 -manifold is a continuous function $\|\cdot\|_X : TX \rightarrow \mathbb{R}_{\geq 0}$ where the restriction of F to each tangent space is a weak norm. For any C^1 path $p : I \rightarrow X$ we may compute its length $L(p)$ with respect to $\|\cdot\|_X$ via $L(p) := \int_I \|p'(s)\|_X ds$. Given any $x, y \in X$, let $\mathbf{P}(x, y)$ be the set of piecewise C^1 paths in X starting at x and ending at y . We define the metric d_X induced by the Finsler metric $\|\cdot\|_X$ as

$$d_X(x, y) := \inf_{p \in \mathbf{P}(x, y)} L(p) = \inf_{p \in \mathbf{P}(x, y)} \int_I \|p'(s)\|_X ds.$$

Remark 2.18. General principles (see Theorem A.8 of the first appendix) tell us that Finsler metrics are always Busemannian, and the respective topologies induced by these two types of metric structures are equivalent.

2.6. The Thurston metric. The *Thurston (asymmetric) metric* measures the logarithmic optimal Lipschitz constant of maps between hyperbolic metrics. Specifically, given any homeomorphism $\varphi : (S, x) \rightarrow (S, y)$, define the *Lipschitz constant of φ* as the quantity

$$\text{Lip}(\varphi) := \sup_{\text{distinct } p, q \in S} \frac{d_y(\varphi(p), \varphi(q))}{d_x(p, q)} \in \mathbb{R} \cup \{\infty\},$$

where d_x and d_y respectively denote distance metrics on S induced by (some fixed choice of isotopy class representative for) the hyperbolic metrics x and y . The Thurston metric is defined by

$$d_{\text{Th}}(x, y) := \inf_{\varphi \sim \text{Id}_S} \log \text{Lip}(\varphi),$$

where the infimum is taken over all homeomorphisms φ isotopic to the identity map Id_S on S . Remarkably, Thurston [61, Theorem 8.5] demonstrated an alternative interpretation for $d_{\text{Th}}(x, y)$ as the supremal ratio of lengths of simple closed curves with respect to x and y . Let \mathcal{S} denote the set of essential simple closed curves on S . Then

$$(3) \quad d_{\text{Th}}(x, y) = \log \sup_{\gamma \in \mathcal{S}} \frac{\ell_\gamma(y)}{\ell_\gamma(x)} = \log \max_{\gamma \in \mathcal{ML}(S) \setminus \{0\}} \frac{\ell_\lambda(y)}{\ell_\lambda(x)}.$$

Thurston further showed that d_{Th} is a Finsler metric on $\mathcal{T}(S)$. First note that the derivative of $\log \ell_\lambda : \mathcal{T}(S) \rightarrow \mathbb{R}$ depends only on the projective class $[\lambda] \in \mathcal{PM}\mathcal{L}(S)$. Thus, for each $x \in \mathcal{T}(S)$, we may define a map $\iota_x : \mathcal{PM}\mathcal{L}(S) \rightarrow T_x^*\mathcal{T}(S)$

$$[\lambda] \mapsto \iota_x([\lambda]) := (d \log \ell_\lambda)_x \in T_x^*\mathcal{T}(S).$$

Theorem 2.19 ([61], Theorem 5.1). *For every $x \in \mathcal{T}(S)$, the map ι_x is an embedding of the projective lamination space $\mathcal{PM}\mathcal{L}(S)$ into the cotangent space $T_x^*\mathcal{T}(S)$. The image set $\iota_x(\mathcal{PM}\mathcal{L}(S))$ is the boundary of a convex open ball containing the origin in $T_x^*\mathcal{T}(S)$.*

By taking $\iota_x(\mathcal{PM}\mathcal{L}(S))$ to be the indicatrix, this defines a (co-)norm on the cotangent space $T_x^*\mathcal{T}(S)$. The *Thurston norm* on the tangent space $T_x\mathcal{T}(S)$ is defined by duality:

$$(4) \quad \|v\|_{\text{Th}} := \sup_{\lambda \in \mathcal{PM}\mathcal{L}(S)} (d \log \ell_\lambda)_x(v) = \sup_{\lambda \in \mathcal{PM}\mathcal{L}(S)} \frac{v(\ell_\lambda)}{\ell_\lambda}.$$

2.7. The Weil–Petersson metric. Teichmüller space is naturally a complex analytic manifold (see, e.g., [30, chapter 7]):

- identify the tangent space $T_x\mathcal{T}(S)$ with the space $HB(x)$ of *harmonic Beltrami differentials* on x , and
- set the complex structure $\sqrt{-1} : T_x\mathcal{T}(S) \rightarrow T_x\mathcal{T}(S)$ on $T_x\mathcal{T}(S)$ to be multiplication by $\sqrt{-1}$ on $HB(x)$.

Definition 2.20 (Weil–Petersson Kähler metric). Let $\rho(z)|dz|^2 = x \in \mathcal{T}(S)$ denote a hyperbolic metric on S , and consider the following Hermitian inner product on $HB(x)$: for any $f_1, f_2 \in HB(x)$,

$$(5) \quad h_{\text{WP}}(f_1, f_2) := \int_S \rho(z)^2 f_1(z) \overline{f_2(z)} dA,$$

where dA is the area form for the metric $\rho(z)|dz|^2$ corresponding to x . It turns out that the collection of such Hermitian inner products varying over the tangent bundle of $\mathcal{T}(S)$ defines a Kähler metric h_{WP} on $\mathcal{T}(S)$; this is the *Weil–Petersson Kähler metric* on $\mathcal{T}(S)$.

- The real component of h_{WP} defines a Riemannian metric $\langle \cdot, \cdot \rangle_{\text{WP}}$, and we refer to it as the *Weil–Petersson metric*.
- We write $\| \cdot \|_{\text{WP}}$ for the Weil–Petersson norm on tangent spaces, and denote the induced distance by d_{WP} .
- The (constant multiple of) the imaginary component, $-2\text{Im}(h_{\text{WP}})$ defines a symplectic form ω_{WP} , which is called the Weil–Petersson symplectic form.

Remark 2.21. We note that Eq. (5) is unconventional in the sense that the Petersson inner product is usually phrased in terms of holomorphic quadratic differentials.

The following lemma is an essential ingredient for the comparison of various norms on the Teichmüller space $\mathcal{T}(S)$.

Lemma 2.22. *For any simple closed curve α on S , let $\nabla \ell_\alpha$ denote the Weil–Petersson gradient field for $\ell_\alpha : \mathcal{T}(S) \rightarrow \mathbb{R}$, then*

$$\|\mathbf{e}_\alpha(x)\|_{\text{WP}} = \frac{1}{2} \|(\nabla \ell_\alpha)_x\|_{\text{WP}}.$$

Proof. By [66, Theorem 2.10], we see that²

$$d\ell_\alpha = \omega_{\text{WP}}(\cdot, \mathbf{e}_\alpha(x)) = -\omega_{\text{WP}}(\mathbf{e}_\alpha(x), \cdot),$$

where ω_{WP} denotes the Weil–Petersson symplectic form. Then, for any tangent vector $v \in T_x \mathcal{T}(S)$,

$$\langle \nabla \ell_\alpha, v \rangle_{\text{WP}} = d\ell_\alpha(v) = -\omega_{\text{WP}}(\mathbf{e}_\alpha(x), v) = 2\langle \sqrt{-1} \mathbf{e}_\alpha(x), v \rangle_{\text{WP}}.$$

Since the pairing is nonsingular, we see that $\mathbf{e}_\alpha(x) = \frac{\sqrt{-1}}{2} (\nabla \ell_\alpha)_x$. To conclude, we write

$$\begin{aligned} \|\mathbf{e}_\alpha(x)\|_{\text{WP}}^2 &= \text{Re}(h_{\text{WP}}(\mathbf{e}_\alpha(x), \mathbf{e}_\alpha(x))) \\ &= \text{Re}(h_{\text{WP}}(\frac{\sqrt{-1}}{2} (\nabla \ell_\alpha)_x, \frac{\sqrt{-1}}{2} (\nabla \ell_\alpha)_x)) \\ &= \text{Re}(h_{\text{WP}}(\frac{1}{2} (\nabla \ell_\alpha)_x, \frac{1}{2} (\nabla \ell_\alpha)_x)) = \|\frac{1}{2} (\nabla \ell_\alpha)_x\|_{\text{WP}}^2. \end{aligned}$$

□

²In this formula, the sign conventions differ from those of Wolpert’s article [66] because we use left earthquakes while Wolpert uses right earthquakes.

2.8. Complexified character varieties. Let $\mathcal{X}_{\text{para}}^{\text{irred}}(\pi_1(S), \text{PSL}_2(\mathbb{C}))$ denote the *character variety* consisting of conjugacy classes of type-preserving (i.e. all punctures, if any, have parabolic holonomy), irreducible representations of $\pi_1(S)$ into $\text{PSL}_2(\mathbb{C})$. Teichmüller space is identified with the subset of $\mathcal{X}_{\text{para}}^{\text{irred}}(\pi_1(S), \text{PSL}_2(\mathbb{C}))$ consisting of discrete faithful representations into $\text{PSL}_2(\mathbb{R}) \subset \text{PSL}_2(\mathbb{C})$. We fix, once and for all, a spin structure on S . This tells us how to lift a representation from $\text{PSL}_2(\mathbb{C})$ to $\text{SL}_2(\mathbb{C})$, and so we have an embedding

$$\mathcal{T}(S) \hookrightarrow V(S) := \mathcal{X}_{\text{para}}^{\text{irred}}(\pi_1(S), \text{SL}_2(\mathbb{C})),$$

where the codomain is the character variety of type-preserving, irreducible representations of $\pi_1(S)$ into $\text{SL}_2(\mathbb{C})$. Faltings [20, Theorem 3] shows that $V(S)$ is a complex manifold³.

2.9. Quasi-Fuchsian space. We make mention of a particular subset $\mathcal{QF}(S)$ of $V(S)$ consisting of all characters of *quasi-Fuchsian representations*. A representation $\rho : \pi_1(S) \rightarrow \text{PSL}_2(\mathbb{C})$ induces a group

$$\rho(\pi_1(S)) \lesssim \text{PSL}_2(\mathbb{C}) = \text{Isom}^+(\mathbb{H}^3),$$

which acts on \mathbb{H}^3 . A quasi-Fuchsian representation is defined as one whose induced group $\rho(\pi_1(S)) \lesssim \text{PSL}_2(\mathbb{C})$ has limit set in $\partial^\infty \mathbb{H}^3 = \mathbb{C}P^1$ which is a quasicircle. Such a quasicircle separates $\mathbb{C}P^1$ into two discs, both admitting proper discontinuous actions of $\pi_1(S)$ via $\rho(\pi_1(S))$. This means that a quasi-Fuchsian representation induces two (marked) Riemann surface structures on S with opposite orientations. Bers' simultaneous uniformisation theorem [6] tells us that the space of such pairs of marked Riemann surfaces precisely parametrises the quasi-Fuchsian space, and

$$\mathcal{QF}(S) = \mathcal{T}(S \cup \bar{S}) = \mathcal{T}(S) \times \mathcal{T}(\bar{S}) = \mathcal{T}(S) \times \overline{\mathcal{T}(S)}.$$

In particular, simultaneous uniformisation tells us that this agreement is a biholomorphism, where the former space is endowed with the complex structure coming from $V(S)$ and the latter is given the usual complex structure on Teichmüller space.

Quasifuchsian representations generalise Fuchsian ones where the limit set circle $\mathbb{R} \cup \{\infty\} = \mathbb{R}P^1 \subset \mathbb{C}^1 = \mathbb{C} \cup \{\infty\}$ has complement naturally regarded as the upper and lower half-planes in \mathbb{C} . The actions of a Fuchsian representation on these two half-planes are related by reflection symmetry, hence the two marked Riemann surfaces parametrising such a Fuchsian representation are antiholomorphic. This gives a diagonal embedding of $\mathcal{T}(S)$ in $\mathcal{QF}(S)$ as a maximal-dimensional totally real submanifold. In particular, this means that the usual complex structure of $\mathcal{T}(S)$, as referenced in the beginning of Section 2.7, is not complex analytically compatible with the natural complex

³Gunning shows this for the $\text{GL}_2(\mathbb{C})$ character variety in earlier work [25, §9], and it seems possible to derive the $\text{SL}_2(\mathbb{C})$ case using the holomorphicity and rank 1-ness of determinant maps on surface group generators.

structure on $\mathcal{QF}(S) \subset V(S)$. However, this diagonal embedding of $\mathcal{T}(S)$ into $\mathcal{QF}(S)$ is real analytic.

2.10. Quakebends. The character variety $V(S) := \mathcal{X}_{\text{para}}^{\text{irred}}(\pi_1(S), \text{SL}_2(\mathbb{C}))$ is a natural parameter space for deformations of Fuchsian representations under *quakebends* — a holonomy-based complexification of earthquakes introduced by Epstein–Marden [19] via the formalism of *quakebend cocycles*. This is shearing with respect to the following “tensor product” of \mathbb{C} and measured lamination space over $\mathbb{R}_{>0}$:

$$\mathcal{ML}_{\mathbb{C}}(S) := \{(z, \lambda) \in \mathbb{C} \times \mathcal{ML}(S)\} / (\forall t \in \mathbb{R}_{>0}, (tz, \lambda) \sim (z, t\lambda)).$$

Conceptually speaking, to quakebend a marked hyperbolic surface $x \in \mathcal{T}(S)$ with respect to

- a real weighted measured lamination (t, λ) for $t \in \mathbb{R}$, is to perform a left earthquake with respect to $t\lambda$ if $t > 0$ and to perform a right earthquake if $t < 0$;
- an imaginary weighted measured lamination $(t\sqrt{-1}, \lambda)$ for $t \in \mathbb{R}$, is to bend the universal cover of x , regarded as a totally geodesic plane in \mathbb{H}^3 , to obtain a $\pi_1(S)$ -equivariant pleated plane;
- a measured lamination with general complex weight, is a combination of earthquaking with respect to the real part of (z, λ) , followed by bending with respect to the imaginary part.

McMullen [42] gave an alternative approach for describing quakebends using intermediary deformations called *complex earthquakes* where one (real) earthquakes and grafts flat cylinders [23, 28, 39], instead of bending by some angle, to produce projective surface structures, and one then goes from a projective structure to a $\text{PSL}_2(\mathbb{C})$ holonomy representation via [28].

Complexifying earthquakes in the form of quakebends affords us simultaneous access to the complex-analytic, as well as to the algebraic structure of the representation varieties. McMullen’s work [42, Theorem 2.5 and Proposition 2.6] clarifies the structure which we need.

Theorem 2.23. *Define the bending holonomy map*

$$\eta : \mathcal{ML}_{\mathbb{C}}(S) \times \mathcal{T}(S) \rightarrow \mathcal{X}_{\text{para}}^{\text{irred}}(\pi_1(S), \text{SL}_2(\mathbb{C}))$$

as the map which takes a complex measured lamination $(z, \lambda) \in \mathcal{ML}_{\mathbb{C}}(S)$ paired with (the character for) a Fuchsian representation $x \in \mathcal{T}(S)$ to (the character for) the induced representation obtained from quakebending x with respect to (z, λ) . The map η is continuous, and it is complex analytic with respect to the z parameter in the complex measured lamination (z, λ) .

2.11. Trace coordinates.

Definition 2.24 (trace functions). For any closed curve α on S , let $\mathbf{a} \in \pi_1(S)$ denote its homotopy class which is determined up to conjugacy. Then the *trace function*

$$\text{tr}_{\alpha} : V(S) \rightarrow \mathbb{C}, \quad \rho \mapsto \text{tr}(\rho(\mathbf{a}))$$

is well defined (i.e. independent of the choice of \mathbf{a}) and holomorphic.

Theorem 2.25 (trace embedding). *The Teichmüller space $\mathcal{T}(S)$ embeds in \mathbb{R}^Γ , for some finite collection of simple closed curves $\Gamma = \gamma_1, \dots, \gamma_D$ on S , via the map*

$$\mathrm{tr}^\Gamma : \mathcal{T}(S) \rightarrow \mathbb{R}^\Gamma, \quad x \mapsto (\mathrm{tr}_{\gamma_1}(x), \dots, \mathrm{tr}_{\gamma_D}(x)).$$

For closed surfaces, Schmutz [56] and Okumura [48] independently show that $D := \dim_{\mathbb{R}} \mathcal{T}(S) + 1$ suffices, and this D is known to be optimal. For punctured surfaces, Hamenstädt [26] shows that $D = \dim_{\mathbb{R}} \mathcal{T}(S) + 1$ still suffices.

Remark 2.26. The above embedding is real analytic. See, e.g., [30, Proposition 6.17] for an argument for closed surfaces which also applies for punctured surfaces.

Remark 2.27 (trace vs. length). Trace functions are more natural when working over $V(S) := \mathfrak{X}_{\mathrm{para}}^{\mathrm{irred}}(\pi_1(S), \mathrm{SL}_2(\mathbb{C}))$ because length functions generally take value in $\mathbb{C}/(2\pi\sqrt{-1}\mathbb{Z})$ rather than \mathbb{C} . However, if one is only working over $\mathcal{QF}(S) \subset V(S)$, the contractibility of $\mathcal{QF}(S)$ gives a canonical extension of $\ell_\gamma : \mathcal{T}(S) \rightarrow \mathbb{R}$ to $\ell_\gamma : \mathcal{QF}(S) \rightarrow \mathbb{C}$ via analytic continuation. This holomorphic function on $\mathcal{QF}(S)$ is called the *complex length* of γ .

2.12. Cosine formula. In our study, we shall often need to know the behaviour of geodesic length functions (or trace functions) with respect to earthquakes. Wolpert's cosine formula [66, Corollary 2.12] describes what happens to geodesic length functions when one performs Fenchel–Nielsen twists, but we shall require Kerckhoff's generalisation. First, we define the total cosine function.

Definition 2.28 (total cosine, see [34]). Given two measured geodesic laminations λ and μ on a hyperbolic surface (S, x) , their *total cosine*, denoted by $\mathrm{Cos}_x(\lambda, \mu)$, is the integral over the surface (S, x)

$$\mathrm{Cos}_x(\lambda, \mu) = \int_{(S, x)} \cos \theta \, d\lambda \times d\mu,$$

of the cosine of the positive angle θ made at each intersection point of the two laminations λ and μ when we look at μ from λ . The aforementioned integral is taken with respect to the product of the two transverse measures.

Lemma 2.29 (Kerckhoff's cosine formula, [34, Proposition 2.5]). *For any measured laminations λ and μ on (S, x) , we have*

$$d\ell_\mu(\mathbf{e}_x(\lambda)) = \mathbf{e}_x(\lambda)(\ell_\mu) = \mathrm{Cos}_x(\lambda, \mu).$$

Corollary 2.30. *Let λ and μ be two measured laminations. Then we have*

$$d\ell_\mu(\mathbf{e}_x(\lambda)) = -d\ell_\lambda(\mathbf{e}_x(\mu)).$$

3. THE EARTHQUAKE NORM

In this section, we first prove that the earthquake metric is a well-defined Finsler metric. This requires the following ingredients:

- the earthquake norm $\|\cdot\|_e$ on each tangent space $T_x\mathcal{T}(S)$ needs to be a weak norm,
- and these norms should vary (at least) continuously with respect to the base point $x \in \mathcal{T}(S)$.

The former may be obtained in several ways. We provide first a proof via an infinitesimal duality between the earthquake norm and the co-norm of the Thurston metric (Theorem 1.8); we then give a more direct proof via Thurston's original strategy, which relies on the convexity of length functions. The latter of the above two requirements implicitly follows from Kerckhoff's proof of the analyticity of earthquakes [34]. We show later in Theorem 4.9 a stronger result, namely, that the earthquake norm is locally Lipschitz continuous.

3.1. Infinitesimal duality to the Thurston metric. Throughout this subsection, although we refer to $\|\cdot\|_e$ as the earthquake *norm*, we only use its invariance with respect to multiplication by non-negative real numbers. This is an important point, as the main result of this subsection will be used to give a proof of the fact that $\|\cdot\|_e$ is a weak norm. The proof of the main result of this subsection depends on the following key lemma in which we use the earthquake vector notation \mathbf{e}_x for the map defined in Remark 2.2 and where $\hat{\mathbf{e}}_x : \mathcal{PM}\mathcal{L}(S) \rightarrow T_x\mathcal{T}(S)$ is the map defined on the quotient space by

$$\hat{\mathbf{e}}_x : \lambda \mapsto \mathbf{e}_x \left(\frac{\lambda}{\ell_x(\lambda)} \right) = \frac{\mathbf{e}_x(\lambda)}{\ell_x(\lambda)}.$$

Lemma 3.1. *Let ω_{WP} be the Weil–Petersson symplectic form on the Teichmüller space $\mathcal{T}(S)$ and consider the linear isomorphism Φ_{WP} defined by*

$$(6) \quad \Phi_{\text{WP}} : T_x\mathcal{T}(S) \rightarrow T_x^*\mathcal{T}(S), \quad v \mapsto \Phi_{\text{WP}}(v) := \omega_{\text{WP}}(\cdot, v).$$

Then, we have

$$(7) \quad \Phi_{\text{WP}}(\hat{\mathbf{e}}_x(\lambda)) = (d \log \ell_\lambda)_x.$$

Proof. For any simple closed curve γ , Wolpert's duality formula for the Weil–Petersson form [66, Theorem 2] tells us that the Fenchel–Nielsen twist vector $\frac{\partial}{\partial \tau_\gamma} \Big|_x$ for γ satisfies $\Phi_{\text{WP}}\left(\frac{\partial}{\partial \tau_\gamma} \Big|_x\right) = (d\ell_\gamma)_x$. Since $\frac{\partial}{\partial \tau_\gamma} \Big|_x$ is precisely the earthquake vector field $\mathbf{e}_x(\gamma)$, normalising by the length of γ , we have:

$$(8) \quad \Phi_{\text{WP}}(\hat{\mathbf{e}}_x(\gamma)) = \Phi_{\text{WP}}\left(\frac{\mathbf{e}_x(\gamma)}{\ell_x(\gamma)}\right) = \frac{(d\ell_\gamma)_x}{\ell_\gamma(x)} = (d \log \ell_\gamma)_x.$$

In the general case where λ is an arbitrary measured geodesic lamination, the continuity of \mathbf{e}_x and ℓ_x ensures that for any sequence of weighted simple closed curves $c_i\gamma_i \rightarrow \lambda$, the sequence $\hat{\mathbf{e}}_x(c_i\gamma_i) = \hat{\mathbf{e}}_x(\gamma_i)$ tends to $\hat{\mathbf{e}}_x(\lambda)$. Furthermore, Kerckhoff's generalised cosine formula for the derivative of general

measured laminations with respect to general earthquakes (Lemma 2.29) asserts that for an arbitrary tangent vector $v = \mathbf{e}_x(\mu) \in T_x\mathcal{J}(S)$,

$$d\ell_{c_i\gamma_i}(v) = \mathbf{e}_x(\mu)(\ell_{c_i\gamma_i}) = \text{Cos}_x(\mu, c_i\gamma_i).$$

The continuity of Cos_x ensures that this converges to $\text{Cos}_x(\mu, \lambda) = d\ell_\lambda(v)$, and thus, $(d\ell_\lambda)_x$ varies continuously with respect to λ , hence

$$\Phi_{\text{WP}}(\hat{\mathbf{e}}_x(\lambda)) = \lim_{i \rightarrow \infty} \Phi_{\text{WP}}\left(\frac{\mathbf{e}_x(\gamma_i)}{\ell_x(\gamma_i)}\right) = \lim_{i \rightarrow \infty} (d \log \ell_{\gamma_i})_x = (d \log \ell_\lambda)_x,$$

which proves the lemma. \square

Recall that the Thurston metric is Finsler. We denote the Thurston conorm on the cotangent space of $\mathcal{J}(S)$ by $\|\cdot\|_{\text{Th}}^*$.

Theorem 1.8 (infinitesimal duality). *At each point $x \in \mathcal{J}(S)$, there is a linear isometry between the normed spaces $(T_x\mathcal{J}(S), \|\cdot\|_e)$ and $(T_x^*\mathcal{J}(S), \|\cdot\|_{\text{Th}}^*)$ induced by Weil–Petersson symplectic duality.*

Proof. The right-hand side in Eq. (7) is precisely Thurston’s expression [61, Theorem 5.1] for the embedding of $\mathcal{PM}\mathcal{L}(S)$ in $T_x^*\mathcal{J}(S)$ as the Thurston co-norm unit sphere. Thus, Φ_{WP} is a linear isomorphism taking the unit sphere in $T_x\mathcal{J}(S)$ with respect to $\|\cdot\|_e$ to that in $T_x^*\mathcal{J}(S)$ with respect to $\|\cdot\|_{\text{Th}}^*$. This induces the required isometry. \square

3.2. Convexity of $\hat{\mathbf{e}}_x(\mathcal{PM}\mathcal{L}(S))$. Kerckhoff uses his work on lines of minima [35, Theorem 2.2] to show, during the proof of [34, Proposition 2.6], the following.

Proposition 3.2. *For every $x \in \mathcal{J}(S)$, the map*

$$\mathbf{e}_x : \mathcal{ML}(S) \rightarrow T_x\mathcal{J}(S), \quad \lambda \mapsto \mathbf{e}_x(\lambda)$$

is a homeomorphism.

Thurston strengthens the above result as follows.

Theorem 3.3 ([61, Theorem 5.2]). *For every $x \in \mathcal{J}(S)$, the map from $\mathcal{ML}(S)$ to $T_x\mathcal{J}(S)$ defined by $\mathbf{e}_x : \mathcal{ML}(S) \rightarrow T_x\mathcal{J}(S)$, $\lambda \mapsto \mathbf{e}_x(\lambda)$ is a homeomorphism onto $T_x\mathcal{J}(S)$. The quotient map $\hat{\mathbf{e}}_x : \mathcal{PM}\mathcal{L}(S) \rightarrow T_x\mathcal{J}(S)$ defined by*

$$\hat{\mathbf{e}}_x : \lambda \mapsto \mathbf{e}_x\left(\frac{\lambda}{\ell_x(\lambda)}\right) = \frac{\mathbf{e}_x(\lambda)}{\ell_x(\lambda)}$$

embeds $\mathcal{PM}\mathcal{L}(S)$ as a hypersurface which is the boundary of a convex subset of $T_x\mathcal{J}(S)$ containing the origin at its interior.

The additional conclusions needed to promote Proposition 3.2 to this theorem are equivalent to asserting that the earthquake norm on the vector space $T_x\mathcal{J}(S)$ is a weak norm (see Definition 2.15). It will turn out that the image of the embedding of $\mathcal{PM}\mathcal{L}(S)$ into $T_x\mathcal{J}(S)$ given by Theorem 3.3 is the unit sphere for the earthquake norm.

We give two proofs for Theorem 3.3.

First proof: The linear isometry Φ_{WP} in Lemma 3.1 and Theorem 1.8 isometrically identifies $\hat{\mathbf{e}}_x(\mathcal{PML}(S))$ with the unit co-vectors in $T_x^*\mathcal{J}(S)$ with respect to the Thurston norm. We see that the unit ball with respect to the Thurston norm in $\|\cdot\|_{\text{Th}}^*$ is the boundary of an open convex ball in $T_x^*\mathcal{J}(S)$ since the Thurston conorm is a weak norm. Therefore $\hat{\mathbf{e}}_x(\mathcal{PML}(S))$ has the same property.

Second proof: We flesh out Thurston's sketch of the proof of Theorem 3.3. This proof does not depend on the isometry described in Theorem 1.8, but does use Kerckhoff's generalisation of Wolpert's cosine formula (Lemma 2.29) in the guise of the following proposition.

Proposition 3.4. *Given an arbitrary $\lambda \in \mathcal{ML}(S)$, let μ be the unique measured geodesic lamination satisfying $\mathbf{e}_x(\mu) = -\mathbf{e}_x(\lambda)$. Then,*

$$(\mathrm{d}\ell_\lambda + \mathrm{d}\ell_\mu)_x = 0 \quad \text{and} \quad \mathrm{Cos}_x(\lambda, \mu) = 0.$$

Proof. By Lemma 3.1, the condition $\mathbf{e}_x(\lambda) + \mathbf{e}_x(\mu) = 0$ is equivalent to

$$(\mathrm{d}\ell_\lambda + \mathrm{d}\ell_\mu)_x = 0.$$

Since earthquakes do not change the length of the sheared lamination, we have $(\mathrm{d}\ell_\lambda)(\mathbf{e}_x(\lambda)) = 0$, and hence $(\mathrm{d}\ell_\mu)_x(\mathbf{e}_x(\lambda)) = 0$. By Lemma 2.29, we have

$$0 = (\mathrm{d}\ell_\mu)_x(\mathbf{e}_x(\lambda)) = \mathbf{e}_x(\lambda)(\ell_\mu) = \mathrm{Cos}_x(\lambda, \mu),$$

hence $\mathrm{Cos}_x(\lambda, \mu) = 0$. □

Second proof of Theorem 3.3. Since $\hat{\mathbf{e}}_x : \mathcal{PML}(S) \rightarrow T_x^*\mathcal{J}(S)$ is a homeomorphism (Proposition 3.2), its image is a hypersurface in $T_x^*\mathcal{J}(S)$. For an arbitrary measured lamination λ on S , the form $(\mathrm{d}\log \ell_\lambda)_x$ is an element of the cotangent space $T_x^*\mathcal{J}(S)$ (see Theorem 2.19). Since $\mathrm{d}\log \ell(\mathcal{PML}(S))$ is the unit sphere with respect to the conorm $\|\cdot\|_{\text{Th}}^*$, it is a convex hypersurface in $T_x^*\mathcal{J}(S)$. Convexity is equivalent to the fact for each λ in $\mathcal{ML}(S)$, there exists a linear function on $T_x^*\mathcal{J}(S)$, i.e. an element v of $T_x\mathcal{J}(S)$, which, when restricted to $\mathrm{d}\log \ell(\mathcal{PML}(S))$, attains its maximum at $(\mathrm{d}\log \ell_\lambda)_x$. By Proposition 3.2, there is a unique measured lamination μ such that $v = \mathbf{e}_x(\mu)$. Then by Corollary 2.30, we have

$$\begin{aligned} & (\mathrm{d}\log \ell_\lambda)(\mathbf{e}_x(\mu)) \\ (9) \quad &= \frac{1}{\ell_x(\lambda)} \mathrm{d}\ell_\lambda(\mathbf{e}_x(\mu)) = \frac{-1}{\ell_x(\lambda)} \mathrm{d}\ell_\mu(\mathbf{e}_x(\lambda)) = -(\mathrm{d}\ell_\mu)(\hat{\mathbf{e}}_x(\lambda)). \end{aligned}$$

Therefore $-(\mathrm{d}\ell_\mu)_x$ attains its maximum on $\hat{\mathbf{e}}_x(\mathcal{PML}(S))$ at $\hat{\mathbf{e}}_x(\lambda)$. The existence of such a linear function implies the convexity of $\hat{\mathbf{e}}(\mathcal{PML}(S))$ at $\hat{\mathbf{e}}_x(\lambda)$. By varying λ over all measured laminations, we see that the image $\hat{\mathbf{e}}_x(\mathcal{PML}(S))$ is a convex hypersurface.

It remains to show that the image of the origin of the vector space $T_x\mathcal{J}(S)$ is contained in the open subset bounded by the image of $\mathcal{PML}(S)$ by the map $\hat{\mathbf{e}}_x$. For an arbitrary $\lambda \in \mathcal{ML}(S) \setminus \{0\}$, by taking μ not disjoint from λ , each of whose leaves is either disjoint from or nearly parallel to λ , we ensure that

$\mathbf{e}_x(\mu)(\ell_\lambda) \neq 0$. By possibly replacing μ with μ' such that $\mathbf{e}_x(\mu) = -\mathbf{e}_x(\mu')$, we obtain that $\mathbf{e}_x(\mu)(\ell_\lambda) > 0$, hence the maximum value of $(d\ell_\mu)_x$ taken at $\hat{\mathbf{e}}_x(\lambda)$ in Eq. (9), which we denote by m_λ , is always positive. Therefore, the hyperplane defined by $(d\ell_\mu)_x = m_\lambda$ has $\hat{\mathbf{e}}(\mathcal{PM}\mathcal{L}(S))$ and the origin $\mathbf{0}$ on the same side. Since this holds for all $\lambda \in \mathcal{ML}(S)$, the interior of the convex ball bounded by $\hat{\mathbf{e}}(\mathcal{PM}\mathcal{L}(S))$ contains $\mathbf{0}$. \square

4. EARTHQUAKE REGULARITY VIA HOLOMORPHICITY

In [34], Kerckhoff shows that earthquakes $E_{t\lambda}(x)$ are real analytic with respect to both the time parameter t and the spatial parameter $x \in \mathcal{T}(S)$. For our purposes, we require continuity of its (higher) derivatives with respect to varying $\lambda \in \mathcal{ML}(S)$. We obtain the following results which will be used later in the paper.

- Proposition 4.1: We derive the Taylor series expansion for earthquakes $E_{t\lambda}(x)$ with respect to time t with coefficients depending continuously on $\lambda \in \mathcal{ML}(S)$ and $x \in \mathcal{T}(S)$, and
- Proposition 4.5: we establish the continuity of the space derivatives, i.e. for x varying in $\mathcal{T}(S)$, of $\mathbf{e}_\lambda(x)$ with respect to λ .

We rely on complex analysis to show both results. We apply the Cauchy integral formula to control the derivatives of complex extensions of earthquakes (complexifying either t or $x \in \mathcal{T}(S)$). In Section 4.3, as an application of this improved understanding of the regularity of earthquakes, we show that the earthquake norm is locally Lipschitz continuous.

4.1. Regularity of time derivatives. We first deal with the time derivative in some detail to illustrate the argument at hand.

Proposition 4.1 (Taylor series for earthquakes). *For any simple closed curve γ on S and for any $x \in \mathcal{T}(S)$, $t \geq 0$, and any $\lambda \in \mathcal{ML}(S)$, we have the following “Taylor series expansion”:*

$$\mathrm{tr}_\gamma \left(E_{\frac{t\lambda}{\mathbf{e}_\lambda(x)}}(x) \right) = \mathrm{tr}_\gamma(x) + \sum_{k=1}^{\infty} \varphi_k^\gamma([\lambda], x) \cdot t^k,$$

where the functions $\varphi_k^\gamma : \mathcal{PM}\mathcal{L}(S) \times \mathcal{T}(S) \rightarrow \mathbb{R}$ are continuous.

Remark 4.2. This proposition ensures that small earthquake segments are uniformly close to “straight line segments” in local trace coordinates. We later use this to C^1 -approximate paths on $\mathcal{T}(S)$ using earthquake segments.

Proof. We know from Theorem 2.23 that the function $F_{\lambda,x}(z) := \eta((z, \lambda), x)$, where $\eta : \mathcal{ML}_\mathbb{C}(S) \times \mathcal{T}(S) \rightarrow \mathcal{X}_{\mathrm{para}}^{\mathrm{irred}}(\pi_1(S), \mathrm{SL}_2(\mathbb{C}))$ denotes the bending holonomy map, is holomorphic. Composed with the trace function tr_γ (Definition 2.24), we have an entire function $\mathrm{Tr}_{\lambda,x} := \mathrm{tr}_\gamma \circ F_{\lambda,x} : \mathbb{C} \rightarrow \mathbb{C}$. Thus, $\mathrm{Tr}_{\lambda,x}$ is equal to its Taylor series (at $z = 0$):

$$\mathrm{Tr}_{\lambda,x}(z) = \mathrm{tr}_\gamma(x) + \sum_{k=1}^{\infty} \phi_{\lambda,x,k} \cdot z^k,$$

where the Cauchy integral formula tells us that

$$(10) \quad \phi_{\lambda,x,k} = \int_{|w|=1} \frac{\mathrm{Tr}_{\lambda,x}(w)}{2\pi\sqrt{-1}w^{k+1}} dw = \int_{|w|=1} \frac{\mathrm{tr}_\gamma \circ \eta((w, \lambda), x)}{2\pi\sqrt{-1}w^{k+1}} dw.$$

Eq. (10) tells us that, for all $k = 1, 2, \dots$, the coefficient $\phi_{\lambda,x,k}$ varies continuously with respect to λ and x .

Going forward, we restrict the z parameter in $\text{Tr}_{\lambda,x}(z)$ to be a positive real parameter $t \geq 0$. In this case, the bending holonomy map $\eta((t, \frac{\lambda}{\ell_\lambda(x)}), x)$ is the left earthquake map $E_{\frac{t\lambda}{\ell_\lambda(x)}}(x)$, and we have:

$$\text{tr}_\gamma \left(E_{\frac{t\lambda}{\ell_\lambda(x)}}(x) \right) = \text{tr}_\gamma(x) + \sum_{k=1}^{\infty} \phi_{\frac{\lambda}{\ell_\lambda(x)}, x, k} \cdot t^k.$$

Setting $\varphi_k^\gamma([\lambda], x) := \phi_{\frac{\lambda}{\ell_\lambda(x)}, x, k}$ defines a continuous function on $\mathcal{PM}\mathcal{L}(S) \times \mathcal{T}(S)$, therefore giving us the desired result. \square

Lemma 4.3. *For any compact $K \subset \mathcal{T}(S)$, there exists $C_{\gamma,K}$ such that for all $t \in [0, 1]$,*

$$\left| \text{tr}_\gamma \left(E_{\frac{t\lambda}{\ell_\lambda(x)}}(x) \right) - \text{tr}_\gamma(x) - \varphi_1^\gamma([\lambda], x)t \right| \leq C_{\gamma,K} t^2.$$

Proof. This essentially follows from Eq. (10), which tells us that

$$\begin{aligned} |\varphi_k^\gamma([\lambda], x)| &= \left| \phi_{\frac{\lambda}{\ell_\lambda(x)}, x, k} \right| = \left| \int_{|w|=2} \frac{|\text{tr}_\gamma \circ \eta((w, \frac{\lambda}{\ell_\lambda(x)}), x)|}{2\pi\sqrt{-1}w^{k+1}} dw \right| \\ &\leq \int_{|w|=2} \frac{|\text{tr}_\gamma \circ \eta((w, \frac{\lambda}{\ell_\lambda(x)}), x)|}{2\pi|\sqrt{-1}w^{k+1}|} d|w| \\ &\leq \int_{|w|=2} \sup_{\substack{|w|=2, \lambda \in \mathcal{ML}(S) \\ x \in K}} \frac{|\text{tr}_\gamma \circ \eta((w, \frac{\lambda}{\ell_\lambda(x)}), x)|}{2\pi \cdot 2^{k+1}} d|w| \\ &= \max_{\substack{|w|=2, [\lambda] \in \mathcal{PM}\mathcal{L}(S) \\ x \in K}} \frac{|\text{tr}_\gamma \circ \eta((w, \frac{\lambda}{\ell_\lambda(x)}), x)|}{2^{k+1}}. \end{aligned}$$

We set

$$c_{\gamma,K} := \max_{\substack{|w|=2, [\lambda] \in \mathcal{PM}\mathcal{L}(S) \\ x \in K}} \left| \text{tr}_\gamma \circ \eta((w, \frac{\lambda}{\ell_\lambda(x)}), x) \right|.$$

We note that the existence of $c_{\gamma,K}$ is due to the compactness of the three spaces $\{|w|=2\} \subset \mathbb{C}$, $\mathcal{PM}\mathcal{L}(S)$ and $K \subset \mathcal{T}(S)$. This implies

$$|\varphi_k^\gamma([\lambda], x)| \leq \frac{c_{\gamma,K}}{2^{k+1}}.$$

Proposition 4.1 then tells us that, for $0 \leq t \leq 1$,

$$\begin{aligned} \left| \text{tr}_\gamma \left(E_{\frac{t\lambda}{\ell_\lambda(x)}}(x) \right) - \text{tr}_\gamma(x) - \varphi_1^\gamma([\lambda], x)t \right| &\leq \sum_{k=2}^{\infty} \frac{c_{\gamma,K}}{2^{k+1}} t^k = \frac{c_{\gamma,K}}{2^3} t^2 \sum_{k=0}^{\infty} \frac{t^k}{2^k} \\ &\leq \frac{c_{\gamma,K}}{2^3} \cdot t^2. \end{aligned}$$

Taking $C_{\gamma,K} := \frac{c_{\gamma,K}}{2^3}$ suffices to give the desired result. \square

Remark 4.4. We can extend the domain of $t \in [0, 1]$ to $[-1, 1]$, where $E_{\frac{t\lambda}{\ell_\lambda(x)}}(x)$ with $t \in [-1, 0)$ represents the surface obtained from x via a *right* earthquake wth respect to $\frac{|t|\lambda}{\ell_\lambda(x)}$.

4.2. Regularity of space derivatives. Having proved the continuity of the time derivatives with respect to $\lambda \in \mathcal{ML}(S)$ in detail, we now appeal to the same arguments, but for the spatial derivatives:

Proposition 4.5. *The earthquake vector field $\mathbf{e}_\lambda : \mathcal{T}(S) \rightarrow T\mathcal{T}(S)$ is real analytic, and its partial derivatives of all orders (including the 0-th) are continuous with respect to $\lambda \in \mathcal{ML}(S)$.*

Proof. We use a part of Kerckhoff’s proof of the analyticity of earthquakes [34, Theorem 1]. For any simple closed curve α , Kerckhoff showed [34, p. 25, 26] that the derivative function $d\ell_\alpha \circ \mathbf{e}_\lambda : \mathcal{T}(S) \rightarrow \mathbb{R}$ complex analytically extends to a neighbourhood \mathcal{N} of the Fuchsian locus $\mathcal{T}(S)$ within the quasi-Fuchsian space $\mathcal{QF}(S)$ (see Section 2.9). The holomorphicity of $\mathbf{e}_\lambda : \mathcal{N} \rightarrow T\mathcal{N} \subset T\mathcal{QF}(S)$ then follows from the following two facts:

- the complex length functions⁴ of simple closed curves are holomorphic functions on $\mathcal{QF}(S)$;
- $\mathcal{QF}(S)$ is parametrised by the complex length functions of finitely many simple closed curves [36, 57].

Applying the Cauchy integral formula to the analytic extension $d\ell_\alpha \circ \mathbf{e}_\lambda : \mathcal{N} \rightarrow \mathbb{C}$, we see that the partial derivatives (of all orders) of $d\ell_\alpha \circ \mathbf{e}_\lambda$ are continuous with respect to λ varying $\mathcal{ML}(S)$. In particular, the restriction of any such derivative to $\mathcal{T}(S)$ is also continuous with respect to λ varying in $\mathcal{ML}(S)$. \square

It is perhaps cleaner to frame Proposition 4.5 via the following:

Corollary 4.6. *The earthquake map $E_\lambda : \mathcal{T}(S) \rightarrow \mathcal{T}(S)$ is real analytic, and its partial derivatives of all orders are continuous with respect to $\lambda \in \mathcal{ML}(S)$.*

Proof. Combine Proposition 4.5 and [34, Theorem 1]. \square

4.3. Local Lipschitz regularity. In Section 6, we make use of Matveev and Troyanov’s work [41] on the Myers–Steenrod theorem for Finsler metrics of low regularity to apply norm-wise results (e.g. asymmetry, rigidity) to obtain information about the earthquake distance metric. To do so, we show that the earthquake norm is locally Lipschitz continuous in the following sense:

Definition 4.7 (Lipschitz continuity of Finsler norms [41, §2.1]). A Finsler norm F on a domain $U \subset \mathbb{R}^N$ is *Lipschitz continuous* if the restriction of $F : TU = U \times \mathbb{R}^N \rightarrow \mathbb{R}_{\geq 0}$ to an *open neighbourhood* of

$$T^1U := \{(x, v) \in U \times \mathbb{R}^N : F(x, v) = 1\}$$

⁴Remark 2.27 explains why complex lengths are well defined over $\mathcal{QF}(S)$.

is a Lipschitz continuous function with respect to the Euclidean metrics on $TU = U \times \mathbb{R}^N$ and $\mathbb{R}_{\geq 0}$.

Remark 4.8 (local Lipschitz continuity). Although the notion of Lipschitz regularity in our context is ill-defined as there is no canonical Euclidean parameterisation of Teichmüller space, *local Lipschitz continuity* is well defined for real analytic, hence locally Lipschitz continuous, coordinate charts on $\mathcal{T}(S)$.

Theorem 4.9 (local Lipschitz continuity of $\|\cdot\|_e$). *The earthquake Finsler norm $\|\cdot\|_e$ is locally Lipschitz continuous.*

Proof. Teichmüller space can be covered by open balls $\{V\}$ where each V is contained in a compact set $K \subset U$ where (U, tr^Γ) is a trace coordinate chart for the set of simple closed curves Γ with $|\Gamma| = \dim \mathcal{T}(S)$. We regard TU as $U \times \mathbb{R}^{6g-6+2n}$, and adopt the following notation:

- let d_1 denote the Euclidean metric on U (pulled back from $\text{tr}^\Gamma(U)$);
- let $\|\cdot\|_2$ denote the Euclidean norm on the parallelised tangent space;
- we often write tangent vectors $\mathbf{e}_x(\lambda) \in T_x U$ in the form $(x, e(x, \lambda)) \in U \times \mathbb{R}^{6g-6+2n}$.

Let A be a bounded open set in $\mathbb{R}^{6g-6+2n}$ whose closure does not contain the origin such that $T^1 K \subset U \times A$. Our goal is to show that the earthquake norm is Lipschitz on $V \times A$, not with respect to the Euclidean metric, but with respect to a distance defined as:

$$d_3((x, v), (y, w)) := d_1(x, y) + \|v - w\|_2.$$

To see that this is sufficient, we observe that d_3 is the Finsler metric for the norm on $U \times \mathbb{R}^{6g-6+2n}$ given by summing the ℓ_1 -norm on the base and the ℓ_2 -norm on the tangent fiber, hence d_3 is uniformly bi-Lipschitz to the Euclidean metric. Alternatively, we can choose V to be a convex domain in U , and observe that TV is convex for both d_3 and the Euclidean metric on TV , and invoke [41, Lemma 2.1].

Ultimately, we wish to show that there exists a constant $C > 0$ such that for any $\mathbf{e}_x(\lambda), \mathbf{e}_y(\mu) \in V \times A$,

$$|\|\mathbf{e}_x(\lambda)\|_e - \|\mathbf{e}_y(\mu)\|_e| \leq C d_3(\mathbf{e}_x(\lambda), \mathbf{e}_y(\mu)).$$

Since by the triangle inequality

$$|\|\mathbf{e}_x(\lambda)\|_e - \|\mathbf{e}_y(\mu)\|_e| \leq |\|\mathbf{e}_x(\lambda)\|_e - \|\mathbf{e}_y(\lambda)\|_e| + |\|\mathbf{e}_y(\lambda)\|_e - \|\mathbf{e}_y(\mu)\|_e|,$$

it suffices to show that for any $\mathbf{e}_x(\lambda), \mathbf{e}_y(\mu) \in V \times A$,

$$(11) \quad |\|\mathbf{e}_x(\lambda)\|_e - \|\mathbf{e}_y(\lambda)\|_e| < C_1 d_1(x, y) \leq C_1 d_3(\mathbf{e}_x(\lambda), \mathbf{e}_y(\mu)), \text{ and}$$

$$(12) \quad |\|\mathbf{e}_y(\lambda)\|_e - \|\mathbf{e}_y(\mu)\|_e| < C_2 \|e(y, \lambda) - e(y, \mu)\|_2 < C_3 d_3(\mathbf{e}_x(\lambda), \mathbf{e}_y(\mu)).$$

We first verify Eq. (11). Invoking the definition of the earthquake norm, we see that

$$(13) \quad \frac{\|\mathbf{e}_x(\lambda)\|_e - \|\mathbf{e}_y(\lambda)\|_e}{d_1(x, y)} = \left| \frac{\ell_\lambda(x) - \ell_\lambda(y)}{d_1(x, y)} \right|.$$

Eq. (11) then follows from the fact that length functions are locally Lipschitz. Indeed, let $d_{Th}(x, y)$ be the Thurston distance from x to y . Without loss of generality, we may assume that $\ell_\lambda(x) \leq \ell_\lambda(y)$. Then

$$\begin{aligned} |\ell_\lambda(x) - \ell_\lambda(y)| &= \ell_\lambda(x) \left(\frac{\ell_\lambda(y)}{\ell_\lambda(x)} - 1 \right) \\ &\leq \ell_\lambda(x) (e^{d_{Th}(x, y)} - 1) \leq C \cdot \ell_\lambda(x) d_{Th}(x, y) \end{aligned}$$

for some constant C depending on K and A . Combined with the fact that $d_{Th}(x, y)$ and $d_1(x, y)$ are locally bi-Lipschitz to each other, this implies Eq. (11).

We next show Eq. (12). The triangle inequality and obtain that

$$\begin{aligned} \|\mathbf{e}_y(\lambda)\|_e &\leq \|\mathbf{e}_y(\mu)\|_e + \|\mathbf{e}_y(\lambda) - \mathbf{e}_y(\mu)\|_e \text{ and} \\ \|\mathbf{e}_y(\mu)\|_e &\leq \|\mathbf{e}_y(\lambda)\|_e + \|\mathbf{e}_y(\mu) - \mathbf{e}_y(\lambda)\|_e, \end{aligned}$$

and hence

$$(14) \quad \|\|\mathbf{e}_y(\lambda)\|_e - \|\mathbf{e}_y(\mu)\|_e\| \leq \max\{\|\mathbf{e}_y(\mu) - \mathbf{e}_y(\lambda)\|_e, \|\mathbf{e}_y(\lambda) - \mathbf{e}_y(\mu)\|_e\}.$$

By the compactness of T^1K , there is a constant C_2 such that

$$(15) \quad \sup_{(y, u) \in TV} \frac{\|(y, u)\|_e}{\|u\|_2} \leq \sup_{(y, u) \in TK} \frac{\|(y, u)\|_e}{\|u\|_2} = \max_{(y, \hat{u}) \in T^1K} \|(y, \hat{u})\|_e = C_2.$$

Combining Eq. (14) and Eq. (15) with $u = e(y, \lambda) - e(y, \mu)$ or $e(y, \mu) - e(y, \lambda)$, we get

$$\begin{aligned} \|\|\mathbf{e}_y(\lambda)\|_e - \|\mathbf{e}_y(\mu)\|_e\| &\leq \max\{\|(y, e(y, \lambda) - e(y, \mu))\|_e, \|(y, e(y, \mu) - e(y, \lambda))\|_e\} \\ &\leq C_2 \|e(y, \lambda) - e(y, \mu)\|_2, \end{aligned}$$

which gives the first inequality in Eq. (12). For the latter inequality, we use the triangle inequality and obtain that

$$\begin{aligned} \|e(y, \lambda) - e(y, \mu)\|_2 &\leq \|e(x, \lambda) - e(y, \mu)\|_2 + \|e(y, \lambda) - e(x, \lambda)\|_2 \\ &\leq d_3(\mathbf{e}_x(\lambda), \mathbf{e}_y(\mu)) + \|e(y, \lambda) - e(x, \lambda)\|_2, \end{aligned}$$

where the second inequality follows from the definition of d_3 . Proposition 4.5 establishes

- the differentiability of the partial derivatives of $\mathbf{e}_\lambda(x)$ with respect to $x \in V \subset K$, and
- the continuity of these derivatives with respect to λ .

Thus, these regularity properties also apply to $e(x, \lambda)$. From this we infer that there exists a constant C'_3 such that for all $x, y \in V \subset K$ and for all $e(x, \lambda), e(y, \lambda) \in A$,

$$(16) \quad \|e(y, \lambda) - e(x, \lambda)\|_2 \leq C'_3 d_1(x, y).$$

Since $d_1(x, y) \leq d_3(\mathbf{e}_x(\lambda), \mathbf{e}_y(\mu))$ (which follows from the definition), we have Eq. (12) with $C_3 = 1 + C'_3$. \square

5. MAGNITUDE MINIMISATION

The main goal of this section is to demonstrate the “magnitude minimisation” interpretation of the earthquake metric (see Definition 1.2) as follows.

Theorem 1.7 (magnitude minimisation). *For any $x, y \in \mathcal{T}(S)$, the earthquake distance $d_e(x, y)$ is equal to the infimal magnitude over the collection of piecewise earthquake paths from x to y .*

5.1. Proof strategy for Theorem 1.7. Given two arbitrary points $x, y \in \mathcal{T}(S)$, the distance $d_e(x, y)$ is realised as the infimum of the earthquake metric lengths of a sequence $(p_j : I \rightarrow \mathcal{T}(S))_{j \in \mathbb{N}}$ of piecewise C^1 -paths starting at x and ending at y . We may assume that p_j is C^1 by approximating each of them by a C^1 -path.

To prove Theorem 1.7, it suffices to take an arbitrary C^1 path $p : [0, L] \rightarrow \mathcal{T}(S)$, and construct a sequence of piecewise earthquake paths which have the same endpoints as p , and whose magnitude tends to the earthquake metric length $\int_0^L \|p'(t)\|_e dt$ of the path $p(t)$. We take the following steps:

- (1) In Lemma 5.2, we show that we may reduce to dealing with a path p with nowhere zero derivative over a local trace coordinate chart (U, tr^Γ) . This is not essential, but simplifies notation and avoids some bi-Lipschitz comparison arguments.
- (2) We break the path into *very* short segments and approximate them (see Fig. 1) first by Euclidean segments (Lemma 5.4) and then by short earthquake segments (Lemma 5.5) sharing the same starting point and initial velocity. We choose the segments to be short enough to ensure that the endpoints of the approximating earthquake segments remain close to $p([0, L])$.
- (3) We show, using Proposition 4.1, that the endpoints of the approximating earthquake segments in the previous step can be linked up with small earthquake segments to form a path of earthquake “sawtooth waves” (see Fig. 2), so that the total contribution of the shorter “vertical segments” make arbitrarily small contributions to the total magnitude as the approximation becomes arbitrarily fine.

Remark 5.1. Since magnitude and path lengths with respect to Finsler metrics are all independent of the parametrisation of a path, we assume henceforth that $p(t)$ is parametrised by earthquake metric arclength, i.e. $\|p'(t)\|_e = 1$. This means that the L in the domain of p is necessarily the earthquake metric length of the path $p(t)$.

5.2. Step 1: reduction to working on charts. Let $\Gamma = \{\gamma_1, \dots, \gamma_D\}$, where $D = \dim_{\mathbb{R}} \mathcal{T}(S) + 1$ be a collection of curves whose holonomy traces specify an embedding of $\mathcal{T}(S)$ as in Theorem 2.25.

Lemma 5.2. *Any C^1 -path p in $\mathcal{T}(S) \hookrightarrow \mathbb{R}^\Gamma$ with nowhere zero derivative can be expressed as a concatenation of finitely many paths p_1, \dots, p_m such*

that there exists a sequence of C^1 -local coordinates $\text{tr}^{\Gamma_j} : V_j \rightarrow \mathbb{R}^{\Gamma_j}$, where $\Gamma_j \subset \Gamma$, such that $\text{tr}^{\Gamma_j} \circ p_j$ defines a path with nowhere zero derivative.

Proof. Define $\Gamma_i := \Gamma - \{\gamma_i\}$. For any point in $\mathcal{J}(S)$, the derivatives $(\text{dtr}^{\Gamma_i})_x$ cannot all be singular, or else the normal vector of $(\text{dtr}^{\Gamma})_x$ would lie in the kernel of every projection map $\mathbb{R}^{\Gamma} \rightarrow \mathbb{R}^{\Gamma_i}$, and hence would be equal to 0. Thus, we can cover $\mathcal{J}(S)$ by C^1 -coordinate charts which map to \mathbb{R}^{Γ_i} , $i = 1, \dots, D$. Now cover $p([0, L])$ by these charts. Thanks to the compactness of $p([0, L])$, there are finitely many charts $(V_j, \text{tr}^{\Gamma_{i_j}})_{j=1, \dots, m}$ and a subdivision $[t_0 = 0, t_1], \dots, [t_{m-1}, t_m = L]$ of $[0, L]$ such that $p([t_{j-1}, t_j])$ is contained in V_j . The result follows from relabelling the Γ_{i_j} as Γ_j . \square

Remark 5.3. Lemma 5.2 allows us to work in local coordinates in $\mathbb{R}^{\dim_{\mathbb{R}} \mathcal{J}(S)}$ to approximate subsegments like $\text{tr}^{\Gamma_j} \circ p_j$ in trace function-based local coordinates. This simplifies notation and allows us to use Proposition 4.1 to understand the behaviour of short earthquake segments.

5.3. Step 2: approximating via earthquake segments. Since we may approximate each path $\text{tr}^{\Gamma_j} \circ p_j$ individually, Lemma 5.2 reduces the problem to approximating a single path $p(t)$ over a single trace coordinate chart (U, tr^{Γ}) . For the remainder of the proof, let

$$\pi : [0, L] \rightarrow \mathbb{R}^{\Gamma}, \quad \pi(t) := \text{tr}^{\Gamma} \circ p(t)$$

denote the corresponding image of the path $p(t)$ in trace coordinates.

Lemma 5.4. *For each $\epsilon > 0$, we can subdivide the interval $[0, L]$ into sub-intervals of the form $[s_i, s_{i+1}]$, where*

$$0 = s_0 < s_1 < s_2 < \dots < s_{m-1} < s_m = L,$$

so that for $i = 0, \dots, m-1$, and for all $s \in [s_i, s_{i+1}]$, we have

$$(17) \quad \|\pi(s) - \pi(s_i) - (s - s_i)\pi'(s_i)\|_{\mathbb{R}^{\Gamma}} < \frac{\epsilon}{2}(s - s_i).$$

Proof. Since $\pi(t)$ is C^1 , its derivative $\pi' : [0, L] \rightarrow \mathbb{R}^{\Gamma}$ is continuous. Thus, we may cover $[0, L]$ with open intervals on which $\pi'(t)$ is nearly constant. The compactness of $[0, L]$ allows us to reduce to a (minimal) finite covering, and choosing $(s_i)_{i=0, \dots, m}$ to lie on overlaps of successive intervals ensures that

$$(18) \quad \text{for all } s \in [s_i, s_{i+1}], \quad \|\pi'(s_i) - \pi'(s)\|_{\mathbb{R}^{\Gamma}} < \frac{\epsilon}{2}.$$

Since $\pi(s) = \pi(s_i) + \int_{s_i}^s \pi'(t) dt$ and $(s - s_i)\pi'(s_i) = \int_{s_i}^s \pi'(s_i) dt$, we have

$$\begin{aligned} \|\pi(s) - \pi(s_i) - (s - s_i)\pi'(s_i)\|_{\mathbb{R}^{\Gamma}} &= \left\| \int_{s_i}^s \pi'(t) dt - \int_{s_i}^s \pi'(s_i) dt \right\|_{\mathbb{R}^{\Gamma}} \\ &\leq \int_{s_i}^s \|\pi'(t) - \pi'(s_i)\|_{\mathbb{R}^{\Gamma}} dt < \frac{\epsilon}{2}(s - s_i). \end{aligned}$$

\square

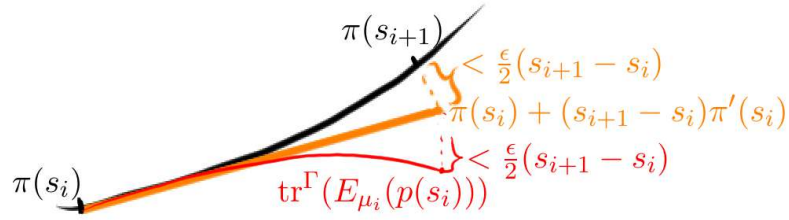


FIGURE 1. A segment of $\pi(t)$ between $t = s_i$ and $t = s_{i+1}$, the approximating linear segment in orange, and the approximating earthquake segment in red.

Lemma 5.5. *For each $\epsilon > 0$, we can subdivide the interval $[0, L]$ into finer intervals of the form $[s_i, s_{i+1}]$, where*

$$0 = s_0 < s_1 < s_2 < \dots < s_{m-1} < s_m = L,$$

so that, for every $i = 0, \dots, m-1$, there is some $\mu_i \in \mathcal{ML}(S)$ such that

- (i) $\ell_{\mu_i}(p(s_i)) = 1$, and
- (ii) $\|\mathrm{tr}^\Gamma(E_{(s_{i+1}-s_i)\mu_i}(p(s_i))) - \pi(s_{i+1})\|_{\mathbb{R}^\Gamma} < \epsilon(s_{i+1} - s_i)$.

Proof. We choose the division points (s_i) to satisfy the following constraints: for all $i = 0, \dots, m$,

- (a) $s_{i+1} - s_i < \min(\frac{\epsilon}{2C_{\Gamma,K}}, 1)$, where $K \subset \mathcal{T}(S)$ is a compact set containing the image of the path $\pi(t)$, and the constant $C_{\Gamma,K}$ is defined to be $\sum_{\gamma \in \Gamma} C_{\gamma,K}$ with $C_{\gamma,K}$ constructed as in Lemma 4.3;
- (b) for any $s \in [s_i, s_{i+1}]$, we have $\|\pi'(s_i) - \pi'(s)\|_{\mathbb{R}^\Gamma} < \frac{\epsilon}{2}$ — this is just Eq. (18).

We already argued in the proof of Lemma 5.4 that condition (b) is possible to enforce, and will yield Eq. (17) as a consequence. The condition (a) is clearly possible.

We now explain the choice of μ_i for each interval $[s_i, s_{i+1}]$: the infinitesimal earthquake map $\mathbf{e}_{(\cdot)}(p(s_i)) : \mathcal{ML}(S) \rightarrow T_{p(s_i)}\mathcal{T}(S)$ is a homeomorphism (see Proposition 3.2, extracted from the proof of [34, Proposition 2.6]), and we choose μ_i to be the unique measured lamination $\mu_i \in \mathcal{ML}(S)$ such that $\mathbf{e}_{\mu_i}(p(s_i)) = p'(s_i)$. Immediate consequences of this choice of μ_i are the following.

- Since $\|p'(s_i)\|_e = 1$ (see Remark 5.1), we have

$$\|\mathbf{e}_{\mu_i}(p(s_i))\|_e = \ell_{\mu_i}(p(s_i)) = 1,$$

which already gives us conclusion (i).

- Moreover,

$$\frac{d}{dt} \Big|_{t=0} \text{tr}^\Gamma(E_{t\mu_i}(p(s_i))) = \text{dtr}^\Gamma(\mathbf{e}_{\mu_i}(p(s_i))) = \text{dtr}^\Gamma(p'(s_i)) = \pi'(s_i),$$

and combining this with Proposition 4.1, we see that

$$\pi'(s_i) = (\varphi_1^\gamma([\mu_i], p(s_i)))_{\gamma \in \Gamma} \in \mathbb{R}^\Gamma.$$

We now prove conclusion (ii) using conditions (a) and (b). By Eq. (17) with $s = s_{i+1}$ and the triangle inequality, it suffices to show that

$$\|\text{tr}^\Gamma(E_{\mu_i}(p(s_i))) - \pi(s_i) - (s_{i+1} - s_i)\pi'(s_i)\|_{\mathbb{R}^\Gamma} < \frac{\epsilon}{2}(s_{i+1} - s_i).$$

The condition (a) means that we may invoke Lemma 4.3 to assert that for all $\gamma \in \Gamma$,

$$|\text{tr}_\gamma(E_{t\mu_i}(p(s_i))) - \text{tr}_\gamma(p(s_i)) - \varphi_1^\gamma([\mu_i], p(s_i)) t| < C_{\gamma,K} t^2.$$

We finish the proof by applying the triangle inequality (see Fig. 1) to isolate each coordinate:

$$\begin{aligned} & \left\| \text{tr}^\Gamma(E_{(s_{i+1}-s_i)\mu_i}(p(s_i))) - \pi(s_i) - \pi'(s_i)(s_{i+1} - s_i) \right\|_{\mathbb{R}^\Gamma} \\ & \leq \sum_{\gamma \in \Gamma} \left| \text{tr}_\gamma(E_{(s_{i+1}-s_i)\mu_i}(p(s_i))) - \text{tr}_\gamma(p(s_i)) - \varphi_1^\gamma([\mu_i], p(s_i))(s_{i+1} - s_i) \right| \\ & < \sum_{\gamma \in \Gamma} C_{\gamma,K} (s_{i+1} - s_i)^2 \leq \left(\sum_{\gamma \in \Gamma} C_{\gamma,K} \right) \cdot \frac{\epsilon}{2C_{\Gamma,K}} \cdot (s_{i+1} - s_i) = \frac{\epsilon}{2}(s_{i+1} - s_i). \end{aligned}$$

□

5.4. Step 3: closing up the earthquake segments.

Lemma 5.6. *There exist constants $\delta > 0$ and M_Γ such that for any $s \in [0, L]$ and any $q \in \overline{B}_{\mathbb{R}^\Gamma}(\pi(s), \delta)$, there exists a unique $\mu \in \mathcal{ML}(S)$ such that for $\tilde{q} := (\text{tr}^\Gamma)^{-1}(q)$, we have*

- (1) $\text{tr}^\Gamma(E_\mu(\tilde{q})) = \pi(s)$, i.e. $E_\mu(\tilde{q}) = p(s)$, and
- (2) $\ell_\mu(\tilde{q}) \leq M_\Gamma \|\pi(s) - q\|_{\mathbb{R}^\Gamma}$.

Proof. We first make the observation that if we use $E_{t\lambda}^\#(\cdot)$ to denote right earthquakes with respect to λ , then whatever statement that we wish to understand regarding left earthquaking from points \tilde{q} to $p(s) = E_{t\lambda}(\tilde{q})$ can be rephrased in terms of right earthquakes with respect to λ deforming from $p(s)$ to $\tilde{q} = E_{t\lambda}^\#(p(s))$. Indeed, since the initial choice of using left earthquakes was arbitrary, we could have developed everything using right earthquakes from the very beginning of this work and obtained a version of the Taylor series Proposition 4.1 for right earthquakes.

We clarify our choice of δ and M_Γ . For $\lambda \in \mathcal{ML}(S)$ normalised so that $\ell_\lambda(p(s)) = 1$. Notice that $E_{t\lambda}^\#(p(s)) = E_{-t\lambda}(p(s))$ for $t > 0$. Hence,

$$\begin{aligned}
& \left\| \text{tr}^\Gamma(E_{t\lambda}^\#(p(s))) - \pi(s) \right\|_{\mathbb{R}^\Gamma} = \left\| \text{tr}^\Gamma(E_{t\lambda}^\#(p(s))) - \text{tr}^\Gamma \gamma(p(s)) \right\|_{\mathbb{R}^\Gamma} \\
& = \left\| \text{tr}^\Gamma(E_{-t\lambda}(p(s))) - \text{tr}^\Gamma \gamma(p(s)) \right\|_{\mathbb{R}^\Gamma} \\
& \geq a_{|\Gamma|} \sum_{\gamma \in \Gamma} \left\| \text{tr}^\gamma(E_{t\lambda}^\#(p(s))) - \text{tr}^\gamma \gamma(p(s)) \right\|_{\mathbb{R}^\Gamma} \\
& \geq a_{|\Gamma|} \sum_{\gamma \in \Gamma} \left(|\varphi_1^\gamma([\lambda], p(s))t| - \left| \text{tr}^\gamma(E_{t\lambda}^\#(p(s))) - \pi(s) - \varphi_1^\gamma([\lambda], p(s))t \right| \right) \\
& \geq a_{|\Gamma|} \sum_{\gamma \in \Gamma} (t|\varphi_1^\gamma([\lambda], p(s))| - t^2 C_{\gamma, K}),
\end{aligned}$$

where $a_{|\Gamma|} > 0$ is a uniform constant used to bound the Euclidean norm below by the L^1 -norm in \mathbb{R}^Γ and where the last inequality follows from Lemma 4.3 and Remark 4.4. Using compactness, we define another constant

$$b_\Gamma := \min_{\substack{[\lambda] \in \mathcal{PM}\mathcal{L}(S) \\ s \in [0, L]}} \sum_{\gamma \in \Gamma} |\varphi_1^\gamma([\lambda], p(s))| > 0,$$

where the positivity follows from Lemma 5.2.

We next choose $\epsilon_0 > 0$ subject to the following conditions:

- $\epsilon_0 \leq \min(1, \frac{b_\Gamma}{2 \sum_{\gamma \in \Gamma} c_{\gamma, K}})$
- ϵ_0 is small enough so that the Euclidean ϵ_0 -neighbourhood $\overline{B}_{\mathbb{R}^\Gamma}(\pi([0, L]), \delta)$ of the image $\pi([0, L])$ of the path π is contained within the image $\text{tr}^\Gamma(U) \subset \mathbb{R}^\Gamma$.

The latter condition is a minor technicality needed for the use of auxiliary coordinate charts in our proof strategy. As a consequence, for every $t \leq \epsilon_0$,

$$(19) \quad t \leq M_\Gamma \left\| \text{tr}^\Gamma(E_{\epsilon\lambda}^\#(p(s))) - \pi(s) \right\|_{\mathbb{R}^\Gamma}, \quad \text{where } M_\Gamma = \frac{2}{a_{|\Gamma|} b_\Gamma}.$$

We are now equipped for the proof of Lemma 5.6. We know from [32, Appendix Theorem 2] that for every point $p(s)$ along the given path p , the right earthquake map

$$E_{(\cdot)}^\#(p(s)) : \mathcal{ML}(S) \rightarrow \mathcal{T}(S), \quad \lambda \mapsto E_\lambda^\#(p(s))$$

is a homeomorphism. We specify that we choose $\delta > 0$ to be any positive constant less than or equal to ϵ_0/M_Γ . By setting $t = M_\Gamma\delta$ in Eq. (19), we see that the boundary sphere of the ball,

$$\mathcal{ML}_{\leq\delta}(p(s)) := \{\mu \in \mathcal{ML}(S) \mid \ell_\mu(p(s)) \leq M_\Gamma\delta\},$$

maps to a sphere lying outside the Euclidean open ball $B_{\mathbb{R}^\Gamma}(\pi(s), \delta)$, and hence the image of $\mathcal{ML}_{\leq\delta}(p(s))$ under $\text{tr}^\Gamma \circ E_{(\cdot)}^\#(p(s))$ contains $\overline{B}_{\mathbb{R}^\Gamma}(\pi(s), \delta)$. This means that for any $s \in [0, L]$ and any $q \in \overline{B}_{\mathbb{R}^\Gamma}(\pi(s), \delta)$, there exists some $\mu \in \mathcal{ML}(S)$ such that $\text{tr}^\Gamma(E_\mu^\#(p(s))) = q$. Moreover, by (19), where λ is assumed to be of unit length, we see that

$$0 \leq \ell_\mu(p(s)) \leq M_\Gamma \|\pi(s) - q\|_{\mathbb{R}^\Gamma}.$$

Equivalently, we have, for $\tilde{q} = (\text{tr}^\Gamma)^{-1}(q)$,

$$\text{tr}^\Gamma(E_\mu(\tilde{q})) = \pi(s) \quad \text{and} \quad \ell_\mu(\tilde{q}) \leq M_\Gamma \|\pi(s) - q\|_{\mathbb{R}^\Gamma}.$$

The uniqueness of μ is due to $E_{(\cdot)}(\tilde{q})$ being a homeomorphism. \square

5.5. Proving the magnitude minimisation interpretation.

Proof of Theorem 1.7. Given $\epsilon > 0$, we approximate $\pi : [0, L] \rightarrow \mathbb{R}^\Gamma$ by forming a sawtooth-like piecewise earthquake path as follows:

- break up the interval $[0, L]$ into intervals $[s_i, s_{i+1}]_{i=0, \dots, m-1}$ subject to Lemma 5.5 with the given ϵ ;
- replace each segment $\pi([s_i, s_{i+1}])$ by two earthquake segments — the first segment is given by (see Lemma 5.5)

$$\text{tr}^\Gamma(E_{t(s_{i+1}-s_i)\mu_i}(p(s_i))), \quad t \in [0, 1],$$

and the second segment joins the endpoint $\text{tr}^\Gamma(E_{(s_{i+1}-s_i)\mu_i}(p(s_i)))$ to $\pi(s_{i+1})$ using Lemma 5.6.

The earthquake metric length of the first segment is $s_{i+1} - s_i$, which is precisely the length of the segment $p([s_i, s_{i+1}])$. Thanks to Lemma 5.5 (ii) and Lemma 5.6 (ii), the length of the second segment is less than



FIGURE 2. Sawtooth waves (red and blue) which approximate the desired C^1 -path $\pi(t)$.

$\epsilon M_\Gamma(s_{i+1} - s_i)$. Therefore, the earthquake metric length of the approximating piecewise earthquake path is $\sum_{i=1}^{m-1} (s_{i+1} - s_i) = L$ up to an error smaller than $\sum_{i=1}^{m-1} \epsilon M_\Gamma(s_{i+1} - s_i) = \epsilon M_\Gamma L$. As $\epsilon \rightarrow 0$, the length of the approximation tends to L , as desired. \square

6. MAPPING CLASS GROUP RIGIDITY AND EARTHQUAKE METRIC
ASYMMETRY

We establish the infinitesimal rigidity of the earthquake metric in two different ways. The first proof relies on Theorem 1.8 and the infinitesimal rigidity of Thurston’s asymmetric metric established in [29, 49]. The second proof makes use of a rigidity result obtained in [47] concerning homeomorphisms of measured laminations space which preserve zero intersection number, which establishes “topological rigidity” (Corollary 6.3). The remainder of the proof follows the third author’s argument in [49, Theorem 3.1]. We then employ infinitesimal rigidity of the earthquake metric to show that d_e is rigid with respect to the mapping class group and that it is asymmetric as a metric.

6.1. Flats in the unit sphere of the earthquake norm. We give a characterisation of pairs of measured laminations with zero intersection number, and show that this also characterises pairs of measured laminations in the unit norm ball of the earthquake Finsler metric which lie on the same flat.

Theorem 6.1 (flatness characterisation). *Let λ and μ be two measured laminations on S . Then the following three conditions are equivalent.*

- (1) *The equality $\|t\mathbf{e}_x(\lambda) + (1-t)\mathbf{e}_x(\mu)\|_e = t\|\mathbf{e}_x(\lambda)\|_e + (1-t)\|\mathbf{e}_x(\mu)\|_e$ holds for every $t \in (0, 1)$.*
- (2) *The same equality holds for some $t \in (0, 1)$.*
- (3) *$i(\lambda, \mu) = 0$.*

Proof. The implication from (1) to (2) is logically trivial.

We show that (3) implies (1). Suppose that $i(\lambda, \mu) = 0$. For every $t \in (0, 1)$, the two measured laminations $t\lambda$ and $(1-t)\mu$ do not intersect transversely, hence $t\lambda + (1-t)\mu$ is a well-defined measured lamination. Indeed, $t\mathbf{e}_x(\lambda) + (1-t)\mathbf{e}_x(\mu) = \mathbf{e}_x(t\lambda + (1-t)\mu)$ because Lemma 2.29 tells us that the two sides of this identity evaluated on ℓ_γ agree for every simple closed curve γ . It follows that

$$\begin{aligned} \|\mathbf{e}_x(t\lambda + (1-t)\mu)\|_e &= \ell_x(t\lambda + (1-t)\mu) \\ (20) \qquad \qquad \qquad &= t\ell_x(\lambda) + (1-t)\ell_x(\mu) = t\|\mathbf{e}_x(\lambda)\|_e + (1-t)\|\mathbf{e}_x(\mu)\|_e, \end{aligned}$$

and we are done.

We next show that (2) implies (3) by proving its contraposition. Suppose that $i(\lambda, \mu) > 0$. Let t be any number in $(0, 1)$. We prove that $\|t\mathbf{e}_x(\lambda) + (1-t)\mathbf{e}_x(\mu)\|_e < t\|\mathbf{e}_x(\lambda)\|_e + (1-t)\|\mathbf{e}_x(\mu)\|_e$ under the assumption that $\|\mathbf{e}_x(\lambda)\|_e = \|\mathbf{e}_x(\mu)\|_e = 1$. By Theorem 3.3, there is a measured lamination ν such that $\mathbf{e}_x(\nu) = t\mathbf{e}_x(\lambda) + (1-t)\mathbf{e}_x(\mu)$. It remains to show that $\|\mathbf{e}_x(\nu)\|_e < 1$.

Suppose, seeking a contradiction, that $\|\mathbf{e}_x(\nu)\|_e = 1$. By Wolpert’s duality, we have

$$(21) \qquad \qquad \qquad t(d\ell_\lambda)_x + (1-t)(d\ell_\mu)_x = (d\ell_\nu)_x.$$

We extend the support $|\nu|$ of ν to a maximal geodesic lamination $\tilde{\nu}$. Since $i(\lambda, \mu) > 0$, either λ or μ intersects $\tilde{\nu}$ transversely. Without loss of generality, we can assume that μ does. Let $w_{\tilde{\nu}}$ be the unit stretch vector along $\tilde{\nu}$. Then, by [29, Lemma 4.2], we have $(d\ell_{\nu})_x(w_{\tilde{\nu}}) = 1$, whereas $(d\ell_{\lambda})_x(w_{\tilde{\nu}}) \leq 1$ and $(d\ell_{\mu})_x(w_{\tilde{\nu}}) < 1$. This shows that the right hand side of Eq. (21) applied to $w_{\tilde{\nu}}$ is 1 whereas the left hand side is less than 1. This is a contradiction, and we have completed the proof. \square

Corollary 6.2. *The unit norm spheres for the earthquake Finsler norm on the tangent spaces of the Teichmüller space $\mathcal{T}(S)$ are strictly convex if and only if $S = S_{0,4}$ or $S_{1,1}$.*

Proof. If $S = S_{0,4}$ or $S_{1,1}$, then S cannot support any pair of distinct measured laminations with null intersection. The result then follows from Theorem 6.1. \square

6.2. Rigidity of the earthquake metric. Theorem 6.1 implies the following topological infinitesimal rigidity property of the earthquake norm. For any $x \in \mathcal{T}(S)$, let $\hat{\mathbf{e}}_x: \mathcal{PM}\mathcal{L}(S) \rightarrow T_x^1\mathcal{T}(S)$ be, as before, the homeomorphism from $\mathcal{PM}\mathcal{L}(S)$ to the unit tangent sphere $T_x^1\mathcal{T}(S)$ which takes $[\lambda] \in \mathcal{PM}\mathcal{L}(S)$ to the unit tangent vector $\frac{1}{\ell_x(\lambda)}\mathbf{e}_x(\lambda)$ with respect to the earthquake norm.

Corollary 6.3 (topological infinitesimal rigidity). *Let $x \in \mathcal{T}(S)$ and $y \in \mathcal{T}(S')$. For any linear isometry $h: (T_x\mathcal{T}(S), \|\cdot\|_e) \rightarrow (T_y\mathcal{T}(S'), \|\cdot\|_e)$ between the tangent spaces with respect to the earthquake norm, there is a homeomorphism $g: S \rightarrow S'$ such that $h(v) = \hat{\mathbf{e}}_y \circ g \circ (\hat{\mathbf{e}}_x)^{-1}(v)$ for any $v \in T_x\mathcal{T}(S)$.*

Proof. This follows from Theorem 1.8 and the topological rigidity of the Thurston norm ([49, Theorem 1.6] or [29, Theorem 1.8]).

If $S \neq S_{0,4}, S_{0,5}, S_{0,6}, S_{1,1}, S_{1,2}, S_{2,0}$, we provide an alternative proof. Since h is an isometry, it sends flats in the unit sphere of $T_x\mathcal{T}(S)$ to flats in the unit sphere of $T_y\mathcal{T}(S')$. Hence, by Theorem 6.1, we see that the composition map $\hat{\mathbf{e}}_y^{-1} \circ h \circ \hat{\mathbf{e}}_x: \mathcal{PM}\mathcal{L}(S) \rightarrow \mathcal{PM}\mathcal{L}(S')$ is a homeomorphism which preserves disjointness. Combined with [47, Theorem 2], this implies that $\hat{\mathbf{e}}_y^{-1} \circ h \circ \hat{\mathbf{e}}_x$ is induced by some homeomorphism $g: S \rightarrow S'$, that is, $\hat{\mathbf{e}}_y^{-1} \circ h \circ \hat{\mathbf{e}}_x([\lambda]) = g([\lambda])$ for any $[\lambda] \in \mathcal{PM}\mathcal{L}(S)$. Hence, $h(v) = \hat{\mathbf{e}}_y \circ g \circ (\hat{\mathbf{e}}_x)^{-1}(v)$ for any $v \in T_x\mathcal{T}(S)$. \square

We now give the statement for the infinitesimal rigidity for the earthquake norm. Let $\mathbf{e}_x: \mathcal{ML}(S) \rightarrow T_x\mathcal{T}(S)$ be the homeomorphism which sends λ to the earthquake vector $\mathbf{e}_x(\lambda)$.

Theorem 6.4 (infinitesimal rigidity). *Any linear isometry $h: (T_x\mathcal{T}(S), \|\cdot\|_e) \rightarrow (T_y\mathcal{T}(S'), \|\cdot\|_e)$ between the tangent spaces at x and y in $\mathcal{T}(S)$ is induced by an extended mapping class, that is, there is a homeomorphism $g: S \rightarrow S'$ with $g^*(y) = x$ such that $h(v) = \mathbf{e}_y \circ g \circ (\mathbf{e}_x)^{-1}(v)$ for any $v \in \mathcal{T}(S)$.*

We provide two different proofs of Theorem 6.4, one proof which works for an arbitrary surface, and the second one which works for $S \neq S_{1,1}, S_{0,4}$.

First proof of Theorem 6.4. By Theorem 1.8, h induces a linear isometry $h^* : T_y^* \rightarrow T_x^*$ with respect to $\|\cdot\|_{\text{Th}}$. The results of [49, Theorem 1.5] or [29, Theorem 1.1 and Theorem 1.15] show that h^* is induced by an extended mapping class. Taking the dual again, we obtain our theorem. \square

Second proof for $S \neq S_{1,1}, S_{0,4}$. Let $g : S \rightarrow S$ be the homeomorphism obtained from Corollary 6.3. Then for any $\lambda \in \mathcal{ML}(S)$, we have

$$h\left(\frac{\mathbf{e}_x(\lambda)}{\ell_x(\lambda)}\right) = \frac{\mathbf{e}_y(g(\lambda))}{\ell_y(g(\lambda))}.$$

We show that there exists a constant K such that for any measured lamination λ , we have $\ell_x(\lambda) = K\ell_y(g(\lambda))$. This implies the conclusion since the length function embeds $\mathcal{J}(S)$ into $P\mathbb{R}_+^{\mathcal{ML}(S)}$ as was shown by Thurston, see [21, 22].

We first note that for any measured lamination λ on S , we have

$$h(\mathbf{e}_x(\lambda)) = \ell_x(\lambda)h\left(\frac{\lambda}{\ell_x(\lambda)}\right) = \ell_x(\lambda)\mathbf{e}_y\left(\frac{g(\lambda)}{\ell_y(g(\lambda))}\right) = \frac{\ell_x(\lambda)}{\ell_y(g(\lambda))}\mathbf{e}_y(g(\lambda)).$$

Suppose that λ and μ are two disjoint measured laminations. Then we have three equalities:

$$\begin{aligned} h(\mathbf{e}_x(\lambda)) &= \frac{\ell_x(\lambda)}{\ell_y(g(\lambda))}\mathbf{e}_y(g(\lambda)), \\ h(\mathbf{e}_x(\mu)) &= \frac{\ell_x(\mu)}{\ell_y(g(\mu))}\mathbf{e}_y(g(\mu)), \text{ and} \\ h(\mathbf{e}_x(\lambda \sqcup \mu)) &= \frac{\ell_x(\lambda) + \ell_x(\mu)}{\ell_y(g(\lambda)) + \ell_y(g(\mu))}\mathbf{e}_y(g(\lambda \sqcup \mu)), \end{aligned}$$

where by $\lambda \sqcup \mu$ we mean the measured lamination obtained from summing two measured laminations λ and μ with disjoint support.

We note that $\mathbf{e}_x(\lambda \sqcup \mu) = \mathbf{e}_x(\lambda) + \mathbf{e}_x(\mu)$, and $\mathbf{e}_y(g(\lambda \sqcup \mu)) = \mathbf{e}_y(g(\lambda)) + \mathbf{e}_y(g(\mu))$. By Theorem 3.3, $\mathbf{e}_y(g(\lambda))$ and $\mathbf{e}_y(g(\mu))$ are linearly independent. Therefore, we have

$$\frac{\ell_x(\lambda)}{\ell_y(g(\lambda))} = \frac{\ell_x(\lambda) + \ell_x(\mu)}{\ell_y(g(\lambda)) + \ell_y(g(\mu))} = \frac{\ell_x(\mu)}{\ell_y(g(\mu))}.$$

This equality holds in particular when λ and μ are disjoint weighted simple closed curves. Recall that two disjoint simple closed curves correspond to the two vertices of an edge in the curve complex. Since the curve complex is connected (for $S \neq S_{1,1}, S_{0,4}$), every pair of simple closed curves can be joined by a sequence of simple closed curves in which each pair of adjacent simple closed curves are disjoint. Therefore, the above equality holds for every pair of weighted simple closed curves even when they are not disjoint. By the density of weighted simple closed curves in $\mathcal{ML}(S)$, the above equality holds for every pair of measured laminations λ and μ , without the assumption that they are disjoint. Thus we have shown that there is a constant K such that $\ell_x(g(\lambda)) = K\ell_y(\lambda)$ for any $\lambda \in \mathcal{ML}(S)$, and we are done. \square

As a corollary, we have the following:

Theorem 1.10 (mapping class group rigidity). The isometry group of $(\mathcal{T}(S), d_e)$ is the extended mapping class group, with the usual action. This statement holds except when $S = S_{0,4}, S_{0,5}, S_{1,1}$ or $S_{2,0}$, for which the isometry group is the extended mapping class group modulo the hyperelliptic involution.

Proof. With Theorem 4.9 at hand, the proof is essentially the same as that of [18, Theorem 1.5] (see also [49, Theorem 1.7 and Theorem 1.8] or [29, Corollary 1.17 and Corollary 1.18]), and here is a sketch: let $f : \mathcal{T}(S) \rightarrow \mathcal{T}(S)$ be an isometry with respect to the earthquake metric. By Theorem 4.9, the earthquake norm is locally Lipschitz. Combined with [41, Theorem A and Theorem B], this implies that the isometry f is of class C^1 and norm-preserving. Combining with Theorem 6.4, we see that at every point $x \in \mathcal{T}(S)$, there exists an extended mapping class g_x of S such that $f(x) = g_x(x)$. The discreteness of the action of the extended mapping class group on $\mathcal{T}(S)$ then enables us to choose a common extended mapping class g such that $f(x) = g(x)$ for all $x \in \mathcal{T}(S)$, and this tells us that every isometry can be realised by the action of some extended mapping class. To finish off the proof, we note that extended mapping classes acts isometrically on $(\mathcal{T}(S), d_e)$, and only hyperelliptic involutions for $S = S_{1,1}, S_{2,0}, S_{0,4}, S_{0,5}$ act trivially. Hence, the isometry group of $(\mathcal{T}(S), d_e)$ is the extended mapping class group, modulo the hyperelliptic involutions for the listed exceptional surfaces. \square

6.3. Asymmetry of the earthquake metric. In this subsection, we use the infinitesimal rigidity to prove the asymmetry of the earthquake norm.

Theorem 1.33 The earthquake norm is asymmetric at every point of $\mathcal{T}(S)$. That is, for any $x \in \mathcal{T}(S)$, there exists a tangent vector $v \in T_x\mathcal{T}(S)$ such that $\|v\|_e \neq \|-v\|_e$.

Proof of Theorem 1.33. Suppose on the contrary that the earthquake norm is symmetric at some point $x \in \mathcal{T}(S)$, which is to say: $\|v\|_e = \|-v\|_e$ for every $v \in T_x\mathcal{T}(S)$. By Theorem 6.4, there is a mapping class g which fixes x such that

$$(22) \quad \mathbf{e}_x \circ g \circ (\mathbf{e}_x)^{-1}(v) = -v, \text{ for all } v \in T_x\mathcal{T}(S).$$

Consider the map $g_* : \mathcal{T}(S) \rightarrow \mathcal{T}(S)$ induced by g , and its induced map on the tangent space, $dg_* : T_x\mathcal{T}(S) \rightarrow T_{g_*(x)}\mathcal{T}(S) = T_x\mathcal{T}(S)$. For any measured lamination λ , the map g sends the earthquake path $E_{t\lambda}(x)$ to the earthquake path $E_{tg(\lambda)}(g(x))$. Therefore,

$$dg_*(\mathbf{e}_x(\lambda)) = \mathbf{e}_{g_*(x)}(g(\lambda)) = \mathbf{e}_x(g(\lambda)),$$

where the second equality follows from the fact that g fixes x . Hence, for any tangent vector $v \in T_x\mathcal{T}(S)$, we have

$$dg_*(v) = \mathbf{e}_x \circ g \circ (\mathbf{e}_x)^{-1}(v).$$

Combined with Eq. (22), this implies that $dg_*(v) = -v$ for every $v \in T_x\mathcal{T}(S)$.

On the other hand, deforming the quotient of (S, x) under the cyclic group generated by g yields a smooth path in $\mathcal{T}(S)$ which passes through x and which is pointwise fixed by g . Hence, the tangent vector $v \neq 0$ to this path at x is fixed by dg_* , contradicting the identity $dg_*(v) = -v$ for all $v \in T_x\mathcal{T}(S)$. This completes the proof. \square

Corollary 6.5. *The Thurston norm is asymmetric at every point of $\mathcal{T}(S)$.*

Proof. Combine Theorem 1.33 with the infinitesimal duality between the earthquake and Thurston metrics (Theorem 1.8). Alternatively, apply the same proof strategy as for Theorem 1.33. \square

Corollary 1.34. *The earthquake metric is not symmetric, that is, there exist x and y in $\mathcal{T}(S)$ such that $d_e(x, y) \neq d_e(y, x)$.*

Proof. We assume, for contradiction, that d_e is symmetric: for all $x, y \in \mathcal{T}(S)$,

$$d_e(x, y) = d_e(y, x) = d_e^\#(x, y).$$

This means that the identity map $\text{Id} : \mathcal{T}(S) \rightarrow \mathcal{T}(S)$ on Teichmüller space induces an isometry between $(\mathcal{T}(S), d_e)$ and $(\mathcal{T}(S), d_e^\#)$. By [41, Theorem B] and Theorem 4.9, the respective Binet–Legendre metrics for the left and right earthquake metrics must be locally Lipschitz continuous, hence [41, Theorem B] ensures that $\|(x, v)\|_e = \text{Id}^*\|(x, v)\|_e^\# = \|(x, v)\|_e^\#$ for all $(x, v) \in T\mathcal{T}(S)$. This contradicts Theorem 1.33, hence there must be points $x, y \in \mathcal{T}(S)$ such that $d_e(x, y) \neq d_e(y, x)$. \square

Remark 6.6. The proof of Corollary 1.34 does not actually need [41, Theorem B] and Theorem 4.9, as those results are used to ensure the regularity of isometries of Finsler metrics with sufficient regularity. Indeed, one needs only apply the arguments outlined on the first page of [41, §3]. We present the above proof for simpler referencing.

7. COMPARISONS WITH OTHER NORMS

The goal of this section is to compare the following Finsler norms on Teichmüller space:

- the earthquake norm $\|\cdot\|_e$,
- the Weil–Petersson norm $\|\cdot\|_{\text{WP}}$,
- the Thurston norm $\|\cdot\|_{\text{Th}}$, and
- the Teichmüller norm $\|\cdot\|_T$.

We prove following Theorem 1.17 mentioned in the introduction.

Theorem 1.17. *There are positive constants C_0, C_1, C_2 , depending only on the topology of S , such that for any $x \in \mathcal{T}(S)$ and any $v \in T_x\mathcal{T}(S)$,*

$$C_0 \ell_{\text{sys}}(x) \text{Log} \frac{1}{\ell_{\text{sys}}(x)} \|v\|_{\text{Th}} \leq \|v\|_e \leq C_1 \|v\|_{\text{WP}} \leq C_2 \|v\|_{\text{Th}},$$

where $\text{Log}(t) = \max\{1, \log t\}$.

We establish Theorem 1.17 by proving each inequality through a series of pairwise norm comparisons: Lemma 7.3, Proposition 7.6, and Theorem 7.8. We strengthen Theorem 1.17 with Wolpert [65, Lemma 3.1] and Burns–Masur–Wilkinson [10, Lemma 5.4] to get Corollary 1.18. This allows us to pairwise compare each of the four norms under consideration, and we state these pairwise comparisons as a corollary for convenient referencing.

Corollary 7.1. *There are positive constants C and C' , depending only on the topology of S , such that for any $x \in \mathcal{T}(S)$ and any $v \in T_x\mathcal{T}(S)$,*

$$(23) \quad C \ell_{\text{sys}}(x) \text{Log} \frac{1}{\ell_{\text{sys}}(x)} \|v\|_{\text{Th}} \leq \|v\|_e \leq C' \|v\|_{\text{Th}}$$

$$(24) \quad C \ell_{\text{sys}}(x) \text{Log} \frac{1}{\ell_{\text{sys}}(x)} \|v\|_{\text{WP}} \leq \|v\|_e \leq C' \|v\|_{\text{WP}}$$

$$(25) \quad C \ell_{\text{sys}}^2(x) \text{Log} \frac{1}{\ell_{\text{sys}}(x)} \|v\|_T \leq \|v\|_e \leq C' \|v\|_T$$

$$(26) \quad C \ell_{\text{sys}}(x) \text{Log} \frac{1}{\ell_{\text{sys}}(x)} \|v\|_{\text{Th}} \leq \|v\|_{\text{WP}} \leq C' \|v\|_{\text{Th}}$$

$$(27) \quad C \ell_{\text{sys}}^2(x) \text{Log} \frac{1}{\ell_{\text{sys}}(x)} \|v\|_T \leq \|v\|_{\text{WP}} \leq C' \|v\|_T$$

$$(28) \quad C \ell_{\text{sys}}(x) \|v\|_T \leq \|v\|_{\text{Th}} \leq C' \|v\|_T.$$

Moreover, we also see that

$$(29) \quad C \ell_{\text{sys}}(x) \|v\|_{\text{Th}} \leq \| -v \|_{\text{Th}} \leq \frac{C'}{\ell_{\text{sys}}(x)} \|v\|_{\text{Th}}$$

$$(30) \quad C \ell_{\text{sys}}^2(x) \text{Log} \frac{1}{\ell_{\text{sys}}(x)} \|v\|_e \leq \| -v \|_e \leq \frac{C'}{\ell_{\text{sys}}(x) \text{Log} \frac{1}{\ell_{\text{sys}}(x)}} \|v\|_e.$$

Remark 7.2. Eq. (28) tells us that the ratio of the Thurston norm over the Teichmüller norm is bounded below by a constant times the (length of the) systole. This resolves [51, Problem 5.3].

7.1. Earthquake vs. Thurston. We start with the comparison between the earthquake norm and the Thurston norm.

Lemma 7.3. *There exists a positive constant C depending only on the topology of S such that for any hyperbolic surface $x \in \mathcal{T}(S)$ and any tangent vector $v \in T_x \mathcal{T}(S)$, we have*

$$(31) \quad C \ell_{\text{sys}}(x) \text{Log}\left(\frac{1}{\ell_{\text{sys}}(x)}\right) \|v\|_{\text{Th}} \leq \|v\|_e,$$

where, as before, $\text{Log}(t) = \max\{1, \log t\}$.

Proof. Let α be a simple closed curve on S . For $x \in \mathcal{T}(S)$, we have $\|\mathbf{e}_x(\alpha)\|_e = \ell_\alpha(x)$, and

$$\|\mathbf{e}_x(\alpha)\|_{\text{Th}} = \sup_{\beta} \frac{\mathbf{e}_x(\alpha)(\ell_\beta)}{\ell_\beta(x)} = \sup_{\beta} \frac{\text{Cos}_x(\alpha, \beta)}{\ell_\beta(x)} \leq \sup_{\beta} \frac{i(\alpha, \beta)}{\ell_\beta(x)},$$

where β ranges over all simple closed curves on x . Therefore,

$$\frac{\|\mathbf{e}_x(\alpha)\|_{\text{Th}}}{\|\mathbf{e}_x(\alpha)\|_e} = \frac{\|\mathbf{e}_x(\alpha)\|_{\text{Th}}}{\ell_\alpha(x)} \leq \sup_{\beta} \frac{i(\alpha, \beta)}{\ell_\alpha(x) \ell_\beta(x)}.$$

Torkaman [62] discovered that as one approaches the thin part of Teichmüller space, i.e. as $\ell_{\text{sys}}(x) \rightarrow 0$, we have

$$(32) \quad \sup_{\alpha, \beta} \frac{i(\alpha, \beta)}{\ell_\alpha(x) \ell_\beta(x)} \sim \frac{1}{2 \ell_{\text{sys}}(x) \log \frac{1}{\ell_{\text{sys}}(x)}}.$$

In particular, there exists a constant C depending only on the topology of S such that

$$(33) \quad \sup_{\alpha, \beta} \frac{i(\alpha, \beta)}{\ell_\alpha(x) \ell_\beta(x)} \leq \frac{C}{\ell_{\text{sys}}(x) \text{Log} \frac{1}{\ell_{\text{sys}}(x)}}.$$

Therefore,

$$\frac{\|\mathbf{e}_x(\alpha)\|_{\text{Th}}}{\|\mathbf{e}_x(\alpha)\|_e} \leq \sup_{\beta} \frac{i(\alpha, \beta)}{\ell_\alpha(x) \ell_\beta(x)} \leq \frac{C}{\ell_{\text{sys}}(x) \text{Log} \frac{1}{\ell_{\text{sys}}(x)}}.$$

□

Note that the estimate in Eq. (33) is weaker than the one in Eq. (32), and that the former estimate can also be proved using the thick-thin decomposition estimate in [38, Proposition 3.1].

7.2. Earthquake vs. Weil–Petersson. We next compare the earthquake norm and the Weil–Petersson norm. We make use of the following estimate for the Weil–Petersson gradient of length functions by Bonsante–Seppi–Tamburelli:

Theorem 7.4 ([8, Theorem H]). *There exists a universal positive constant K such that for any simple closed curve α and any hyperbolic surface $x \in \mathcal{T}(S)$, we have*

$$(34) \quad \|\nabla \ell_\alpha\|_{\text{WP}} \geq \frac{K}{|\chi(S)|} \ell_\alpha,$$

where $\nabla \ell_\alpha$ is the gradient of the length function $\ell_\alpha : \mathcal{T}(S) \rightarrow \mathbb{R}$.

Remark 7.5. Even though the surface S is closed in [8, Theorem H], the same argument holds for punctured surfaces.

Proposition 7.6. *There exists a positive constant C depending only on the topology of S such that for any hyperbolic surface $x \in \mathcal{T}(S)$ and any tangent vector $v \in T_x \mathcal{T}(S)$, we have*

$$(35) \quad \|v\|_e \leq C \|v\|_{\text{WP}},$$

where $C = \frac{2|\chi(S)|}{K}$ depends only on the topological type of S , and $K > 0$ is a universal constant (independent of S) given by Theorem 7.4.

Proof of Proposition 7.6. By Theorem 3.3, the tangent vector v can be represented as the earthquake vector $\mathbf{e}_x(\alpha)$ for some measured lamination α . Since the set of weighted simple closed curves is dense in the space of measured laminations, we have only to consider the case when α is a simple closed curve to get Eq. (34). In this case, by Lemma 2.22, we have

$$\|\mathbf{e}_x(\alpha)\|_{\text{WP}} = \frac{1}{2} \|\nabla \ell_\alpha\|_{\text{WP}}.$$

Combined with Theorem 7.4, this completes the proof. \square

Given Proposition 7.6, one naturally wonders whether the earthquake norm and the Weil–Petersson norm might be bi-Lipschitz equivalent. We show that this is not the case.

Proposition 7.7 (not bi-Lipschitz). *For any simple closed curve α , the earthquake norm and the Weil–Petersson norm for the vectors $\mathbf{e}_x(\alpha)$ ($x \in \mathcal{T}(S)$) are not bi-Lipschitz equivalent. Specifically, there is some uniform constant c such that*

$$\sqrt{\frac{\|\mathbf{e}_x(\alpha)\|_e}{2\pi}} \leq \|\mathbf{e}_x(\alpha)\|_{\text{WP}} \leq \sqrt{2c \|\mathbf{e}_x(\alpha)\|_e}, \text{ as } \ell_\alpha(x) \rightarrow 0.$$

Proof. Given an arbitrary α , Riera’s formula [53, Theorem 2, equation (b)] implies that $\|\nabla \ell_\alpha\|_{\text{WP}} \geq \sqrt{2\ell_\alpha(x)/\pi}$. On the other hand, Wolpert [68, Lemma 3.16] showed that

$$\|\nabla \ell_\alpha\|_{\text{WP}} \leq \sqrt{c(\ell_\alpha(x) + \ell_\alpha^2(x)e^{\ell_\alpha(x)/2})}$$

for some universal constant c . Thus, for small $\ell_\alpha(x)$, we have

$$\sqrt{\frac{\|\mathbf{e}_x(\alpha)\|_e}{2\pi}} = \sqrt{\frac{\ell_\alpha(x)}{2\pi}} \leq \frac{1}{2} \|\nabla \ell_\alpha\|_{\text{WP}} \leq \sqrt{2c \ell_\alpha(x)} = \sqrt{2c \|\mathbf{e}_x(\alpha)\|_e}.$$

Using Lemma 2.22 to replace $\frac{1}{2} \|\nabla \ell_\alpha\|_{\text{WP}}$ with $\|\mathbf{e}_x(\alpha)\|_{\text{WP}}$ yields the result. \square

We now employ Proposition 7.6 to show:

Theorem 7.8. *There exists a constant C depending only on the topology of S , such that for any $x \in \mathcal{T}(S)$ and any tangent vector $v \in T_x \mathcal{T}(S)$,*

$$(36) \quad \frac{\|v\|_{\text{WP}}}{\|v\|_{\text{Th}}} \leq \sup_{v \in T_x \mathcal{T}(S)} \frac{\|v\|_e}{2\|v\|_{\text{WP}}}.$$

In particular,

$$(37) \quad \|v\|_{\text{WP}} \leq C\|v\|_{\text{Th}}.$$

Moreover,

$$(38) \quad \|v\|_e \leq C\|v\|_{\text{Th}}.$$

Proof. By the density of simple closed curves in $\mathcal{PML}(S)$, it suffices to verify Eq. (36) for $v = \mathbf{e}_x(\alpha)$, where α is an arbitrary simple closed curve. For simplicity, we omit the x in $\mathbf{e}_\alpha(x) = v_x(\alpha)$ and write \mathbf{e}_α . Recall from Lemma 2.22 that $\|\mathbf{e}_\alpha\|_{\text{WP}} = \frac{1}{2} \|\nabla \ell_\alpha\|_{\text{WP}}$, hence

$$\begin{aligned} \frac{\|\mathbf{e}_\alpha\|_{\text{WP}}}{\|\mathbf{e}_\alpha\|_{\text{Th}}} &= \frac{\|\nabla \ell_\alpha\|_{\text{WP}}}{2\|\mathbf{e}_\alpha\|_{\text{Th}}} = \frac{1}{2\|\mathbf{e}_\alpha\|_{\text{Th}}} \sup_{\beta} \frac{|\mathrm{d}\ell_\alpha(\mathbf{e}_\beta)|}{\|\mathbf{e}_\beta\|_{\text{WP}}} \\ &= \frac{1}{2\|\mathbf{e}_\alpha\|_{\text{Th}}} \sup_{\beta} \frac{-\mathrm{d}\ell_\alpha(\mathbf{e}_\beta)}{\|-\mathbf{e}_\beta\|_{\text{WP}}} = \frac{1}{2\|\mathbf{e}_\alpha\|_{\text{Th}}} \sup_{\beta} \frac{\mathrm{d}\ell_\beta(\mathbf{e}_\alpha)}{\|\mathbf{e}_\beta\|_{\text{WP}}}, \end{aligned}$$

where the last equality makes use of Corollary 2.30. Hence,

$$\begin{aligned} \frac{\|\mathbf{e}_\alpha\|_{\text{WP}}}{\|\mathbf{e}_\alpha\|_{\text{Th}}} &= \frac{1}{2\|\mathbf{e}_\alpha\|_{\text{Th}}} \sup_{\beta} \frac{[\mathbf{e}_\alpha(\frac{\mathrm{d}\ell_\beta}{\ell_\beta})] \ell_\beta}{\|\mathbf{e}_\beta\|_{\text{WP}}} \leq \frac{1}{2\|\mathbf{e}_\alpha\|_{\text{Th}}} \sup_{\beta} \frac{\|\mathbf{e}_\alpha\|_{\text{Th}} \ell_\beta}{\|\mathbf{e}_\beta\|_{\text{WP}}} \\ &= \frac{1}{2} \sup_{\beta} \frac{\ell_\beta}{\|\mathbf{e}_\beta\|_{\text{WP}}} = \sup_{\beta} \frac{\|\mathbf{e}_\beta\|_e}{2\|\mathbf{e}_\beta\|_{\text{WP}}}. \end{aligned}$$

This proves Eq. (36), which in turn, combined with Proposition 7.6, gives Eq. (37) and Eq. (38). \square

8. LONG EARTHQUAKE PATHS ARE NON-GEODESIC

The main goal of this section is to show the following.

Theorem 1.12. *Sufficiently long left earthquake paths in $\mathcal{T}(S)$ cannot be geodesics with respect to the left earthquake metric.*

8.1. Dehn twists case. We first give an elementary proof for the special case where the earthquake in question is a Fenchel–Nielsen twist, as it tells us something general about the geometry of moduli space. (This kind of argument was originally suggested by Thurston in [61].)

Proof of Theorem 1.12 for Fenchel-Nielsen twists. Let γ be an arbitrary simple closed curve on S , consider an arbitrary point $x \in \mathcal{T}(S)$ and set $x_m := E_{m\gamma}(x) \in \mathcal{T}(S)$ as its m -th right Dehn twist around γ (see Fig. 3). Choose an arbitrary $y \in \mathcal{T}(S)$ such that $\ell_\gamma(y) < \ell_\gamma(x)$, and similarly define $y_m := E_{m\gamma}(y)$. Since d_e is mapping class group invariant, for all $m \in \mathbb{Z}$,

$$d_e(y_m, x_m) = d_e(E_{m\gamma}(y), E_{m\gamma}(x)) = d_e(y, x).$$

Hence, the triangle inequality tells us that

$$d_e(x, x_m) \leq d_e(x, y) + d_e(y, y_m) + d_e(y, x).$$

Since $d_e(y, y_m)$ is bounded above by the length of the earthquake path from y to y_m , we obtain the following.

$$d_e(x, x_m) \leq d_e(x, y) + m\ell_\gamma(y)^2 + d_e(y, x).$$

This inequality ensures that for $m \in \mathbb{N}$ sufficiently large, $d_e(x, x_m)$ cannot be realised by the earthquake path from x to x_m , which has length

$$m\ell_\gamma(x)^2 \gg d_e(x, y) + m\ell_\gamma(y)^2 + d_e(y, x).$$

Since subsegments of geodesics are geodesics, no earthquake path containing $E_{t\gamma}(x)$, $t \in [0, m]$, can be a geodesic. \square

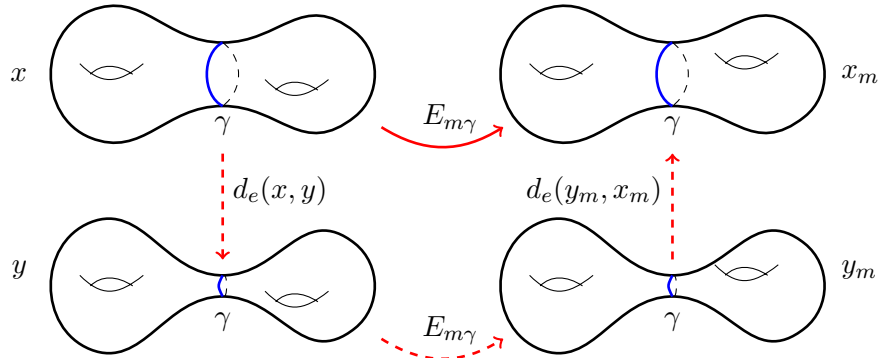


FIGURE 3. Long earthquakes cannot be geodesic.

This proof did not make use of the earthquake metric in a *fundamental* way. The same argument can be used to show that for any mapping class group invariant path metric on Teichmüller space where arbitrarily long earthquakes continue to be geodesics, the distance of Fenchel-Nielsen twists around a fixed simple closed curve with respect to the metric have to be uniform:

Lemma 8.1. *Let d be any mapping class group invariant asymmetric metric d on $\mathcal{T}(S)$ where all left earthquake paths on $\mathcal{T}(S)$ are geodesics. Then for any essential simple closed curve γ on S the following holds.*

$$\text{For any } x, y \in \mathcal{T}(S) \text{ and for any } m \in \mathbb{N}, \quad d(x, E_{m\gamma}(x)) = d(y, E_{m\gamma}(y)).$$

Proof. Since earthquakes for simple closed curves are Fenchel-Nielsen twists, the unique (bi-infinite) left earthquake joining x and $E_\gamma(x)$ contains $E_{m\gamma}(x)$ for all $m \in \mathbb{N}$. In particular,

$$(39) \quad d(x, E_{m\gamma}(x)) = md(x, E_\gamma(x)).$$

By the triangle inequality and the mapping class group invariance of d ,

$$(40) \quad \begin{aligned} d(x, E_{m\gamma}(x)) &\leq d(x, y) + d(y, E_{m\gamma}(y)) + d(E_{m\gamma}(y), E_{m\gamma}(x)) \\ &= d(x, y) + md(y, E_\gamma(y)) + d(y, x). \end{aligned}$$

Combining Eq. (39) and Eq. (40), we obtain

$$d(x, E_\gamma(x)) \leq \frac{1}{m}(d(x, y) + d(y, x)) + d(y, E_\gamma(y)),$$

and letting $m \rightarrow \infty$, we have

$$d(x, E_\gamma(x)) \leq d(y, E_\gamma(y)).$$

By exchanging the roles of x and y , we also have $d(y, E_\gamma(y)) \leq d(x, E_\gamma(x))$, and hence $d(y, E_\gamma(y)) = d(x, E_\gamma(x))$. Again appealing to Eq. (39), we obtain $d(y, E_{m\gamma}(y)) = d(x, E_{m\gamma}(x))$ as required. \square

We suspect that the conclusions for Lemma 8.1 may be too peculiar to hold for general asymmetric metrics on $\mathcal{T}(S)$. Verifying this for the Teichmüller space of flat tori yields Proposition 8.2. To this end, we recall that $\mathcal{T}(S_{1,0}) = \mathbb{H}^2$, that in this case the mapping class group action is that of $\mathrm{PSL}_2(\mathbb{Z})$, and that the earthquake paths in this case are the horocycles of the hyperbolic plane, see [33].

Proposition 8.2. *There is no $\mathrm{PSL}_2(\mathbb{Z})$ -invariant finite asymmetric metric on \mathbb{H}^2 where all anti-clockwise horocycles are geodesics.*

Proof. In $\mathcal{T}(S_{1,0}) = \mathbb{H}^2$, assume that there is an asymmetric metric d with respect to which all earthquake paths, that is, anti-clockwise horocycles, are geodesics. Then, by Lemma 8.1, for any $z \in \mathbb{H}^2$ the horocyclic segment

$$\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \cdot z \quad (t \in [0, 1])$$

is a geodesic (for d) of the same length — we normalise this length to be 1. The oriented segments depicted in Fig. 4, with lengths a, b, c_1, c_2, d_1 and d_2

are all anti-clockwise horocycles, hence they are geodesics by assumption. By the triangle inequality, the lengths of the depicted oriented segments (see Fig. 4) satisfy $a \leq c_1 + d_2$ and $b \leq c_2 + d_1$, hence $a + b \leq (c_1 + c_2) + (d_1 + d_2)$. We note the following.

- The segment of length a is the image of a path of length 1 under $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$:

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{-1} \cdot (1 + i) \quad \text{for } t \in [0, 1],$$

and hence $a = 1$.

- The segment of length b is the concatenation of two paths of length 1:

$$\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \cdot (-1 + i) \quad \text{for } t \in [0, 1] \quad \text{and} \quad \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \cdot i \quad \text{for } t \in [0, 1],$$

and hence $b = 2$.

- the segment of length $c_1 + c_2$ is the image of a subsegment of a length 1 path under $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$:

$$\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}^{-1} \cdot (2 - \sqrt{3})i \quad \text{for } t \in [0, 1],$$

and hence $c_1 + c_2 < 1$;

- the segment of length $d_1 + d_2$ is the image of a subsegment of a length 1 path under $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$:

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} \cdot (2 + \sqrt{3})i \quad \text{for } t \in [0, 1],$$

and hence $d_1 + d_2 < 1$.

This is impossible, for it would cause $3 = a + b \leq c_1 + c_2 + d_1 + d_2 < 2$.

□

Remark 8.3. We finish off this subsection by mentioning the 1932 article [12] by Herbert Busemann (with an English translation in [15]). The setting is that of a metric space to which one adds some very general axioms such as existence of midpoints for any pair of points and extension of geodesics. Busemann calls these spaces “line spaces”. This is the setting in which he placed most of his later works (with different terminology). In the article mentioned, Busemann studies spaces satisfying an additional axiom, namely he considers *geometries where circles of infinite radius are the shortest lines* (this is the title of the paper). The expression “circle of infinite radius” designates a horosphere, that is, the limit of a sequence of geometric circles in a plane passing through a given point with centres going to infinity along a ray starting at this point. Busemann shows that the metrics satisfying the property in the title are generalisations of the Euclidean metric, namely, they are associated with a norm on a vector space. Now the relation between earthquake paths and horocycles which we mentioned in Section 1.4 makes a connection between Busemann’s intuition and the study of a metric on Teichmüller space in which earthquakes are geodesics. Such a metric, if it

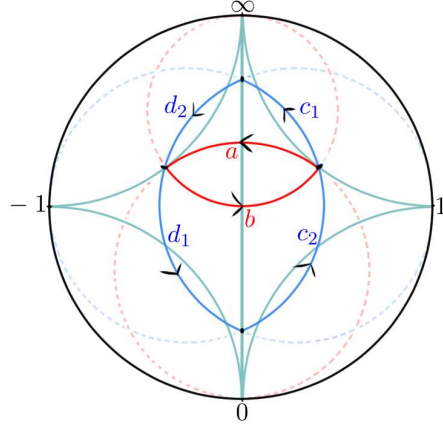


FIGURE 4. Violation of triangle inequality.

exists would have Euclidean behaviour over the whole Teichmüller space. This, together with Proposition 8.2 leads us to the following

Conjecture 8.4. There is no asymmetric metric on Teichmüller space which is mapping class group invariant and for which left earthquakes are geodesics.

8.2. General case.

Proof of Theorem 1.12 for general earthquakes. Let λ be an arbitrary measured lamination and $x \in \mathcal{T}(S)$ an arbitrary point in Teichmüller space. Consider the earthquake ray $E_{t\lambda}(x)$ ($t \geq 0$). By Lemma 2.29, for any measured lamination α , we have

$$\ell_\alpha(E_{t\lambda}(x)) \leq \ell_\alpha(x) + t \cdot i(\alpha, \lambda).$$

By Eq. (3), the Thurston distance $d_{\text{Th}}(x, E_{t\lambda}(x))$ satisfies

$$\begin{aligned} d_{\text{Th}}(x, E_{t\lambda}(x)) &= \log \sup_{\alpha \in \mathcal{ML}(S)} \frac{\ell_\alpha(E_{t\lambda}(x))}{\ell_\alpha(x)} \\ &\leq \log \sup_{\alpha \in \mathcal{ML}(S)} \frac{\ell_\alpha(x) + t \cdot i(\alpha, \lambda)}{\ell_\alpha(x)} \\ &= \log \sup_{\alpha \in \mathcal{ML}} \left(1 + t \frac{i(\alpha, \lambda)}{\ell_\alpha(x)} \right) \\ &= \log \left(1 + t \cdot \max_{\mu \in \mathcal{ML}(S)(\ell_\alpha(x)=1)} i(\mu, \lambda) \right). \end{aligned}$$

Combined with Corollary 1.19, this implies that

(41)

$$d_e(x, E_{t\lambda}(x)) \leq C_2 d_{\text{Th}}(x, E_{t\lambda}(x)) \leq C_2 \log \left(1 + t \cdot \max_{\mu \in \mathcal{ML}(S)(\ell_\alpha(x)=1)} i(\mu, \lambda) \right).$$

We remark that $\max_{\mu \in \mathcal{ML}(S)(\ell_\alpha(x)=1)} i(\mu, \lambda)$ is a constant depending only on x . On the other hand, the earthquake metric length of the earthquake path $\{E_{s\lambda}(x)\}_{s \in [0, t]}$ is $t\ell_\alpha(x)$, which, for large t , is bigger than $d_e(x, E_{t\lambda}(x))$, is bounded above by Eq. (41). Thus, the earthquake path $\{E_{s\lambda}(x)\}_{s \in [0, t]}$ fails to be an earthquake metric geodesic when t is big enough. \square

Remark 8.5. It is possible to push the proof strategy for the Fenchel–Nielsen case to earthquakes with respect to general laminations. Instead of pinching closed geodesics, we consider harmonic stretch lines defined by Pan–Wolf [50]. As is shown in [50, Proposition 13.10], harmonic stretch lines and earthquake paths along the same measured lamination “commute” with each other, and we can also control earthquake distances in terms of Thurston distances by Eq. (38). Therefore we can repeat the same argument as Section 8.1.

Remark 8.6. Theorem 1.12 (at least) for the case of closed surfaces can also be obtained by combining results in [8] and our inequality in Corollary 1.19. For $x, y \in \mathcal{T}(S)$, let λ be the measured lamination such that $E_\lambda(x) = y$. By Theorem C and Theorem F in [8], it follows that

$$\exp \left(\frac{a}{|\chi(S)|} d_{\text{WP}}(x, y) - b|\chi(S)| \right) \leq \frac{1}{4} \ell_\lambda(x) + \frac{\pi^2}{2} |\chi(S)|.$$

Combined with Corollary 1.19, this implies that if $d_e(x, y)$ is sufficiently large, then

$$\begin{aligned} \ell_\lambda(x) &\geq \frac{1}{4} \ell_\lambda(x) + \frac{\pi^2}{2} |\chi(S)| \\ &\geq \exp \left(\frac{a}{|\chi(S)|} d_{\text{WP}}(x, y) - b|\chi(S)| \right) \\ &\geq \exp \left(\frac{a}{|C_1 \chi(S)|} d_e(x, y) - b|\chi(S)| \right) \\ &\geq \exp(C d_e(x, y)) \end{aligned}$$

for some topological constant C . Hence, $d_e(x, y) \leq C^{-1} \log \ell_\lambda(x)$.

8.3. Regularity. We end this section with a *positive* result about the geodesics with respect to the earthquake metric for some special surfaces.

Proposition 8.7. *Geodesics of the earthquake metric on $\mathcal{T}(S_{1,1})$ and $\mathcal{T}(S_{0,4})$ are C^1 .*

Proof. Theorem d of Guillaume Buro's doctoral dissertation [11] asserts that on a two-dimensional differentiable manifold with a Finsler metric which is strictly convex and (locally) Lipschitz, its metric geodesics are C^1 . We previously showed the strict convexity in Corollary 6.2 and the (local) Lipschitzness in Theorem 4.9. \square

9. DISTANCE TO AND FROM THE BOUNDARY

The aim of this section is to show that the earthquake metric is incomplete (see Definition 10.4 for a definition of completeness in the setting of asymmetric metrics), and to quantitatively study this incompleteness. Mumford's compactness theorem [46] and Proposition 1.4 tells us that incompleteness can only happen as the systoles of a family of surfaces geometrically converge to a cusp, and we use the Weil–Petersson completion as a convenient language for discussing sequences which penetrate into arbitrarily thin parts of moduli space.

Theorem 9.1. *Let $\overline{\mathcal{T}(S)}^{\text{WP}}$ denote the completion of Teichmüller space with respect to the Weil–Petersson metric. Then there exists $C > 1$ such that for any $x \in \mathcal{T}(S)$,*

$$2\ell_{\text{sys}}(x)\text{Log}\frac{1}{\ell_{\text{sys}}(x)} \leq d_e(x, \overline{\partial\mathcal{T}(S)}^{\text{WP}}) \leq 2C\ell_{\text{sys}}(x)\text{Log}\frac{1}{\ell_{\text{sys}}(x)}.$$

Moreover, as $\ell_{\text{sys}}(x) \rightarrow 0$ we have

$$\frac{d_e(x, \overline{\partial\mathcal{T}(S)}^{\text{WP}})}{2\ell_{\text{sys}}(x)\text{Log}\frac{1}{\ell_{\text{sys}}(x)}} \rightarrow 1.$$

The same hold for $d_e(\overline{\partial\mathcal{T}(S)}^{\text{WP}}, x)$.

Remark 9.2. As we shall see in the proof, the conclusion also holds if we replace Teichmüller space by the moduli space.

Remark 9.3 (width function). Many of the bounds in this section are obtained by using the fact that geodesic segments must spend some definite time every time they pass through the collar neighbourhood of some simple closed geodesic. We use the following notation:

- let $w(t) = \text{arcsinh}(\text{cosech}\frac{t}{2})$ be the width of a “standard” collar neighbourhood of a simple closed curve with length $t > 0$, and
- we use $\text{Log}(\frac{1}{t}) := \max\{1, \log(\frac{1}{t})\}$ as an approximation for $w(t)$.

We note that both functions are non-increasing in t .

9.1. Lower bound for the earthquake metric.

Lemma 9.4. *For any $x, y \in \mathcal{T}(S)$ and any simple closed curve α , we have*

$$d_e(x, y) \geq 2 \int_{\min\{\ell_\alpha(x), \ell_\alpha(y)\}}^{\max\{\ell_\alpha(x), \ell_\alpha(y)\}} w(\ell) d\ell.$$

Proof. Let $\sigma : [0, 1] \rightarrow \mathcal{T}(S)$ be a C^1 -path connecting x to y . We denote its length with respect to d_e by $L(\sigma)$. By Proposition 4.5, there is a continuous family β_t ($t \in [0, 1]$) of measured laminations such that the tangent vector $\dot{\sigma}(t)$ is equal to the earthquake vector $\mathbf{e}_{\sigma(t)}(\beta_t)$ for every $t \in [0, 1]$. It follows from Lemma 2.29 that

$$\text{for any } t \in [0, 1], \quad \ell_\alpha(\sigma(t')) - \ell_\alpha(\sigma(t)) = \text{Cos}_{\sigma(t)}(\beta_t, \alpha)(t' - t) + o(t' - t)$$

as $t' \rightarrow t$. In particular, for every $\epsilon > 0$, there exists a neighbourhood of t such that

$$|\ell_\alpha(\sigma(t')) - \ell_\alpha(\sigma(t''))| \leq i(\alpha, \beta_t)|t' - t''| + \epsilon|t' - t''|$$

for any t' and t'' in this neighbourhood. We subdivide the interval $[0, 1]$ into finer intervals of the form $[t_i, t_{i+1}]$, where $0 = t_0 < t_1 < \dots < t_n = 1$, so that for every $i = 0, \dots, n-1$,

$$(42) \quad |\ell_\alpha(\sigma(t_{i+1})) - \ell_\alpha(\sigma(t_i))| \leq i(\alpha, \beta_{t_i})(t_{i+1} - t_i) + \epsilon(t_{i+1} - t_i).$$

We next make use of the fact that geodesic segments gain some extra length at least equal to $2w(\ell_\alpha(\sigma))$ each time they pass through the collar neighbourhood of α , to obtain a lower bound of $(t_{i+1} - t_i)\ell_{\beta_{t_i}}(\sigma(t_i))$, which is a constituent of a Riemann sum approximating the earthquake metric length integral for σ . Set $\hat{\ell}_\alpha := \min\{\ell_\alpha(\sigma(t)) \mid t \in [0, 1]\}$. The fact that the function w is monotone non-increasing, combined with Eq. (42), implies that

$$\begin{aligned} (t_{i+1} - t_i)\ell_{\beta_{t_i}}(\sigma(t_i)) &\geq 2(t_{i+1} - t_i)i(\alpha, \beta_{t_i})w(\ell_\alpha(\sigma(t_i))) \\ &\geq 2|\ell_\alpha(\sigma(t_{i+1})) - \ell_\alpha(\sigma(t_i))|w(\ell_\alpha(\sigma(t_i))) - 2\epsilon(t_{i+1} - t_i)w(\hat{\ell}_\alpha). \end{aligned}$$

Hence, a Riemann sum approximating the length $L(\sigma)$ satisfies

$$\sum_{i=0}^{n-1} (t_{i+1} - t_i)\ell_{\beta_{t_i}}(\sigma(t_i)) \geq 2 \sum_i |\ell_\alpha(\sigma(t_{i+1})) - \ell_\alpha(\sigma(t_i))| w(\ell_\alpha(\sigma(t_i))) - 2\epsilon \cdot w(\hat{\ell}_\alpha).$$

Letting $n \rightarrow \infty$ and $\max |t_{i+1} - t_i| \rightarrow 0$, we see that :

$$\begin{aligned} L(\sigma) &= \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} (t_{i+1} - t_i)\ell_{\beta_{t_i}} \\ &\geq \lim_{i \rightarrow \infty} 2 \sum_i |\ell_\alpha(\sigma(t_{i+1})) - \ell_\alpha(\sigma(t_i))| w(\ell_\alpha(\sigma(t_i))) - 2\epsilon \cdot w(\hat{\ell}_\alpha) \\ &= 2 \int_\sigma w(\ell_\alpha(\sigma(t))) |d\ell_\alpha(\sigma(t))| - 2\epsilon \cdot w(\hat{\ell}_\alpha) \\ &\geq 2 \int_{\min\{\ell_\alpha(x), \ell_\alpha(y)\}}^{\max\{\ell_\alpha(x), \ell_\alpha(y)\}} w(\ell) d\ell - 2\epsilon \cdot w(\hat{\ell}_\alpha). \end{aligned}$$

Taking $\epsilon \rightarrow 0$ yields

$$L(\sigma) \geq 2 \int_{\min\{\ell_\alpha(x), \ell_\alpha(y)\}}^{\max\{\ell_\alpha(x), \ell_\alpha(y)\}} w(\ell) d\ell.$$

Since $d_e(x, y)$ is defined as the infimal length $L(\sigma)$ over all C^1 paths σ joining x and y , this gives the requisite lower bound for $d_e(x, y)$. \square

9.2. Upper bound of distance to and from the boundary. We next establish the following upper bound for distances to and from the boundary.

Theorem 9.5 (upper bound). *For all $\epsilon > 0$, there exists $\delta > 0$ such that for any $x \in \mathcal{T}(S)$ with $\ell_{\text{sys}}(x) < \delta$, we have*

$$(43) \quad d_e(x, \overline{\partial\mathcal{T}(S)}^{\text{WP}}) \leq 2(1 + \epsilon)\ell_{\text{sys}}(x) \log \frac{1}{\ell_{\text{sys}}(x)}, \text{ and}$$

$$(44) \quad d_e(\overline{\partial\mathcal{T}(S)}^{\text{WP}}, x) \leq 2(1 + \epsilon)\ell_{\text{sys}}(x) \log \frac{1}{\ell_{\text{sys}}(x)}.$$

We establish the upper bound constructively. We shall consider Fenchel-Nielsen twists around simple closed curves β which wind around some short curve α sufficiently many times to make the angles of intersection between α and β close to 0, and we show that this process causes the length of α to converge to 0 in finite time. For the remainder of this subsection, we adopt the following notation. Given an arbitrary $x \in \mathcal{T}(S)$,

- let Γ be a short pants decomposition for x , i.e. Γ consists of curves shorter than the Bers constant (see, for instance, [16, Chapter 5]);
- let $\alpha \in \Gamma$ denote a systole of x , whose length is shorter than some δ (which we choose so that α is shorter than the width of its collar neighbourhood);
- let S_α be the connected component of $S \setminus (\Gamma \setminus \{\alpha\})$ which is not a pair of pants.

Lemma 9.6. *Let α and β be two simple closed geodesics on $x \in \mathcal{T}(S)$ intersecting at more than one point. For any two intersection points $p_1, p_2 \in \alpha \cap \beta$, their respective intersection angles $\theta_1, \theta_2 \in (0, \pi)$ measured counterclockwise from α to β satisfy*

$$e^{-\ell_\alpha(x)/2} < \frac{\sin \theta_1}{\sin \theta_2} < e^{\ell_\alpha(x)/2}.$$

Proof. Position a lift of α in the upper half plane \mathbb{H}^2 to the geodesic $\overline{0\infty}$ joining 0 and ∞ . Further set a lift of β to be the geodesic $\overline{-1r}$ joining -1 and $r > 0$, so that $\overline{0\infty} \cap \overline{-1r}$ is a lift of p_1 and hence the angle from $\overline{0\infty}$ to $\overline{-1r}$ is θ_1 . By a standard formula relating cross-ratios and intersection angles (see, e.g., [32, Equation (1)]), we have

$$(45) \quad \cos \theta_1 = 2(0, r; -1, \infty) - 1 = \frac{1 - r}{1 + r}.$$

One nearest translate, with respect to α , of $\overline{-1r}$ is the geodesic $\overline{(-e^{\ell_\alpha(x)})(re^{\ell_\alpha(x)})}$ joining $-e^{\ell_\alpha(x)}$ and $re^{\ell_\alpha(x)}$. Since the segment on $\overline{0\infty}$ between its intersections with $\overline{-1r}$ and $\overline{(-e^{\ell_\alpha(x)})(re^{\ell_\alpha(x)})}$ is a fundamental domain, there must be a lift \overline{uv} of β , with end points $u \in (-e^{\ell_\alpha(x)}, -1)$ and $v \in (r, re^{\ell_\alpha(x)})$, such that $\overline{0\infty} \cap \overline{uv}$ is a lift of p_2 . Note that the angle from

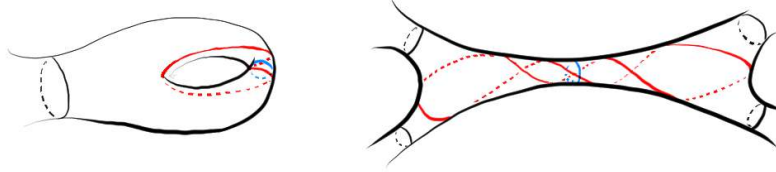


FIGURE 5. The only two possible configurations for α (short curve) and β (long curve) on S_α , up to homeomorphism.

$\overline{0\infty}$ to \overline{uv} is θ_2 , and hence

$$(46) \quad \cos \theta_2 = 2(0, v; u, \infty) - 1 = \frac{1 + \frac{v}{u}}{1 - \frac{v}{u}} = \frac{1 - \frac{v}{|u|}}{1 + \frac{v}{|u|}}.$$

Since $\theta_i \in (0, \pi)$, the quantities $\sin \theta_i$ are positive and Eq. (45) and Eq. (46) mean that

$$(47) \quad \sin \theta_1 = \frac{2}{\sqrt{r} + \sqrt{r^{-1}}}, \quad \text{and} \quad \sin \theta_2 = \frac{2}{\sqrt{v/|u|} + \sqrt{|u|/v}}.$$

The latter quantity $\sin \theta_2$ is infimised by supremising the denominator, which is given by the following choices:

- $u = -1$ and $v = re^{\ell_\alpha}$ when $r \geq 1$, or
- $u = -e^{\ell_\alpha}$ and $v = r$ when $r \leq 1$.

In either case,

$$\sin \theta_2 > \frac{2}{\sqrt{re^{\ell_\alpha}} + \sqrt{r^{-1}e^{\ell_\alpha}}} = \frac{\sin \theta_1}{e^{\ell_\alpha/2}},$$

and hence $\frac{\sin \theta_1}{\sin \theta_2} < e^{\ell_\alpha/2}$. By symmetry, $\frac{\sin \theta_2}{\sin \theta_1} < e^{\ell_\alpha/2}$, and the result follows. \square

Definition 9.7 (minimal angle of intersection). For any two intersecting simple closed geodesics γ_1, γ_2 on (S, x) , we define $\theta_x(\gamma_1, \gamma_2) \in [0, \pi)$ as the angle, measured anti-clockwise from γ_1 to γ_2 , which is minimal among all the intersection points in $\gamma_1 \cap \gamma_2$.

Lemma 9.8. *Let $x \in \mathcal{T}(S)$ be a hyperbolic surface with a shortest pants decomposition Γ . Suppose that β is a simple closed curve which intersects $\alpha \in \Gamma$ minimally and which is disjoint from other curves in Γ . Then there exists a constant C which only depends on the topology of S such that*

$$(48) \quad \frac{\ell_\beta(x)}{2i(\alpha, \beta)} \leq \log \frac{1}{\ell_\alpha(x)} + \log \frac{1}{\sin \theta} + C$$

where θ is the intersection angle from α to β at any point of $\alpha \cap \beta$.

Proof. The connected component S_α is either a one-holed torus or a four-holed sphere (Fig. 5), possibly with cusps in the latter case. We prove the claim for the four-holed sphere; the proof for the one-holed torus case can be similarly derived, or one can also quotient a one-holed torus by its hyperelliptic involution and then lift the resulting orbifold to a four-holed sphere to derive the result for the one-holed torus via the result for the four-holed sphere.

Consider a pair of pants P constituting a component of $S_\alpha \setminus \alpha$. Let

- b denote the length of a component ρ of $\beta \cap P$, and
- d denote the length of the unique orthogeodesic ϱ on P with both endpoints on α . In particular, ρ and ϱ are freely homotopic, with endpoints allowed to move on α .

Take lifts $\tilde{\varrho}, \tilde{\rho}$ of ϱ, ρ which intersect on the universal cover \mathbb{H}^2 (see Fig. 6). They stretch between two lifts $\tilde{\alpha}_1, \tilde{\alpha}_2$ of α . For $i = 1, 2$, let h_i denote the length, along $\tilde{\alpha}_i$, of the segments joining the endpoints of the lifts of $\tilde{\varrho}$ and $\tilde{\rho}$, and let ϕ_i denote the anti-clockwise angle from $\tilde{\alpha}_i$ to $\tilde{\rho}$. Depending on whether $\tilde{\rho}$ and $\tilde{\varrho}$ intersect or not, there are two cases which we should consider.

Case (1): when $\tilde{\rho}$ and $\tilde{\varrho}$ intersect each other.

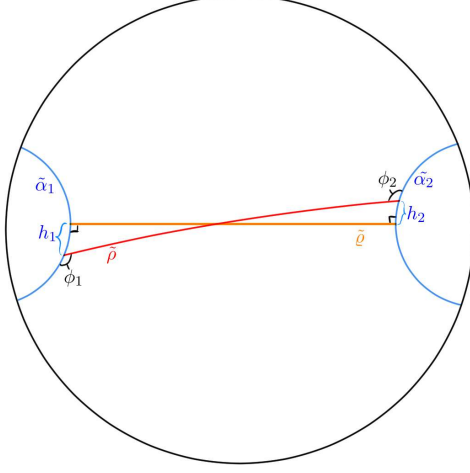


FIGURE 6. Depiction of how \tilde{q} , $\tilde{\rho}$, $\tilde{\alpha}_1$, and $\tilde{\alpha}_2$ are configured.

By [16, Eq. (2.3.2) in Ch. 2],

$$(49) \quad \begin{aligned} \cosh b &= \cosh h_1 \cosh h_2 \cosh d + \sinh h_1 \sinh h_2 \\ &\leq \cosh h_1 \cosh h_2 (\cosh d + 1). \end{aligned}$$

By [16, Formula glossary, 2.2.2 (v)], we have

$$\cosh h_i < \frac{1}{\sin \phi_i}, \quad i = 1, 2.$$

Let θ be the angle from α to β at any point of $\alpha \cap \beta$. By Lemma 9.6, we see that

$$e^{-B/2} \leq \frac{\sin \phi_i}{\sin \theta} \leq e^{B/2}$$

where B is the Bers constant, which depends only on the topological type of S . Hence,

$$\cosh h_i < \frac{1}{\sin \phi_i} \leq \frac{e^{B/2}}{\sin \theta}.$$

Combined with (49), this implies

$$(50) \quad \cosh b < \frac{e^B (\cosh d + 1)}{\sin^2 \theta}.$$

Case (2): when $\tilde{\rho}$ and \tilde{q} are disjoint.

By [16, Eq. (2.3.2) in Ch. 2], we have

$$(51) \quad \begin{aligned} \cosh b &= \cosh h_1 \cosh h_2 \cosh d - \sinh h_1 \sinh h_2 \\ &\leq \cosh h_1 \cosh h_2 \cosh d. \end{aligned}$$

Let $\tilde{\eta}$ be the geodesic segment perpendicular to both $\tilde{\rho}$ and $\tilde{\varrho}$. This segment cuts $\tilde{\rho}$ (resp. $\tilde{\varrho}$) into two subsegments of length b_1 and b_2 (resp. d_1 and d_2) respectively. In particular, $b = b_1 + b_2$ and $d = d_1 + d_2$. By [16, Formula Glossary, 2.3.1 (iii)],

$$\begin{aligned} \cosh d_1 &= \cosh b_1 \sin \phi_1 \\ \cosh d_2 &= \cosh b_2 \sin \phi_2. \end{aligned}$$

Hence,

$$\cosh b \leq 2 \cosh b_1 \cosh b_2 = \frac{2 \cosh d_1 \cosh d_2}{\sin \phi_1 \sin \phi_2} \leq \frac{2 \cosh d}{\sin \phi_1 \sin \phi_2}.$$

Combined with Lemma 9.6, this implies that

$$(52) \quad \cosh b \leq \frac{2e^B \cosh d}{\sin^2 \theta},$$

where B is the Bers constant.

Combining the conclusions Eq. (50) and Eq. (52) from the two cases:

$$(53) \quad \cosh b \leq \frac{2e^B \cosh d}{\sin^2 \theta}.$$

Next, we estimate d from above, which in turn gives the desired upper bound for b . Denote the lengths of the non- α boundaries of P by L_1, L_2 , and let a_i be the length of the subsegment of α between the orthogeodesic ϱ and the orthogeodesic joining α to the length L_i side of P (see Fig. 7).

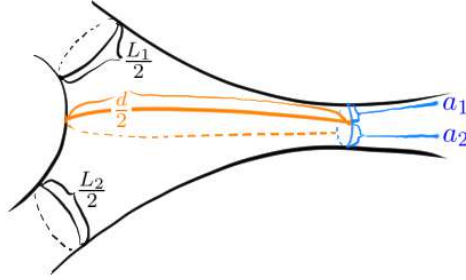
Since $a_1 + a_2 = \frac{\ell_\alpha(x)}{2}$, one of these two segments has length at least $\frac{\ell_\alpha(x)}{4}$. Combine this with [16, Formula glossary, 2.3.4 (i)], and we get

$$(54) \quad \sinh\left(\frac{d}{2}\right) = \frac{\cosh \frac{L_i}{2}}{\sinh a_i} \leq \frac{\cosh B}{\sinh \frac{\ell_\alpha(x)}{4}},$$

where B is the Bers constant for S . Algebraically combining Eq. (50) and Eq. (54), we have

$$(55) \quad b \leq 2 \left(\log \frac{1}{\ell_\alpha(x)} + \log \frac{1}{\sin \theta} + C \right)$$

for a constant C which depends only on the topological type of S . Note that Γ cuts β into $i(\alpha, \beta)$ segments, each of which satisfies (55). The lemma follows. \square

FIGURE 7. Geodesic segments on P .

Proof of Theorem 9.5. We only need to establish Eq. (43), as Eq. (44) follows by doing the same proof by for the right earthquake metric. We also note that if Eq. (43) is true for all $\epsilon > 0$ smaller than some constant, then Theorem 9.5 holds true.

We prove Eq. (43) by explicitly constructing a path x_t of controlled earthquake metric length starting at an arbitrary $x_0 = x$ with small systole α , so that $\ell_\alpha(x_t) \rightarrow 0$ along the path. Fix an angle $\vartheta \in (\frac{\pi}{2}, \pi)$. Let β_0 be a simple closed geodesic on S_α that minimally intersects α and has $\theta_x(\alpha, \beta) > \vartheta$ (see Definition 9.7 for the definition of $\theta_x(\alpha, \beta)$). We produce a piecewise earthquake path x_t in $\mathcal{T}(S)$ as follows.

- We proceed along $x_s := E_{s\beta_0}(x_0)$. By [32, Proposition 3.5], the minimal intersecting angle $\theta_0(s) := \theta_{x_s}(\alpha, \beta_0)$ monotonically decreases, and we stop when the angle reaches ϑ . Denote the time this occurs by t_1 , and let β_1 be the left Dehn twist⁵ of β_0 with respect to α , [32, Proposition 3.5] then ensures that $\theta_1(t_1) := \theta_{x_{t_1}}(\alpha, \beta_1)$ is greater than ϑ .
- We continue along $x_{s+t_1} := E_{s\beta_1}(x_{t_1})$ until $\theta_{x_{s+t_1}}(\alpha, \beta_1) = \vartheta$. Denote the stopping time by t_2 , and let β_2 be the left Dehn twist of β_1 with respect to α . Again, $\theta_2(t_2) := \theta_{x_{t_2}}(\alpha, \beta_2)$ is greater than ϑ .

⁵We remind the reader that the action of left Dehn twists on the Teichmüller space corresponds to right Fenchel-Nielsen twists.

- We iterate the above process indefinitely.

This procedure yields a path where we are always earthquaking with respect to some β_i that intersects α at angle

$$(56) \quad \theta_i(s) \in (\vartheta, \pi) \text{ for } s \in [t_i, t_{i+1}), \text{ and } \theta_i(t_{i+1}) = \vartheta.$$

In particular, $\theta_{x_{t_i}}(\alpha, \beta_{i-1}) = \theta_{i-1}(t_i) = \vartheta$. Hence, by Lemma 9.8 we obtain,

$$\ell_{\beta_{i-1}}(x_{t_i}) \leq 2i(\alpha, \beta_{i-1}) \left(\log \frac{1}{\ell_\alpha(x_{t_i})} + \log \frac{1}{\sin \vartheta} + C \right).$$

Recall that β_i is a left Dehn twist of β_{i-1} and that $\ell_\alpha(x_{t_i})$ is decreasing. Therefore, for any $s \in [t_i, t_{i+1})$,

$$\begin{aligned} \ell_{\beta_i}(x_s) &= \ell_{\beta_i}(x_{t_i}) \leq \ell_{\beta_{i-1}}(x_{t_i}) + i(\alpha, \beta_{i-1})\ell_\alpha(x_{t_i}) \\ &\leq 2i(\alpha, \beta_{i-1}) \left(\log \frac{1}{\ell_\alpha(x_{t_i})} + \log \frac{1}{\sin \vartheta} + C + \ell_\alpha(x_{t_i}) \right) \\ &\leq 2i(\alpha, \beta_{i-1}) \left(\log \frac{1}{\ell_\alpha(x_s)} + \log \frac{1}{\sin \vartheta} + C + \ell_\alpha(x_0) \right) \\ &\leq 2i(\alpha, \beta_i) \left(\log \frac{1}{\ell_\alpha(x_s)} + \log \frac{1}{\sin \vartheta} + C + \ell_\alpha(x_0) \right). \end{aligned}$$

By Wolpert's cosine formula [66, Corollary 2.12], at each interval $[t_i, t_{i+1})$, the length of α changes at the rate of $\frac{d\ell_\alpha}{ds} < i(\alpha, \beta_i) \cos(\vartheta) < 0$ for $s \in [t_i, t_{i+1})$, hence is monotone decreasing, and we have moreover

$$\left| \frac{d\ell_\alpha}{ds} \right| \geq i(\alpha, \beta_i) |\cos(\vartheta)| > 0.$$

We bound the length of the piecewise earthquake path (x_s) as follows:

$$\begin{aligned} &\sum_{i=0}^{\infty} \int_{t_i}^{t_{i+1}} \|\dot{x}_s\|_e ds = \sum_{i=0}^{\infty} \int_{t_i}^{t_{i+1}} \|\mathbf{e}_{\beta_i}(x_s)\|_e ds = \sum_{i=0}^{\infty} \int_{t_i}^{t_{i+1}} \ell_{\beta_i}(x_s) ds \\ &= \sum_{i=0}^{\infty} \int_{\ell_\alpha(t_i)}^{\ell_\alpha(t_{i+1})} \ell_{\beta_i}(x_s) \left(\frac{d\ell_\alpha}{ds} \right)^{-1} d\ell_\alpha = \sum_{i=0}^{\infty} \int_{\ell_\alpha(t_{i+1})}^{\ell_\alpha(t_i)} \ell_{\beta_i}(x_s) \left| \left(\frac{d\ell_\alpha}{ds} \right)^{-1} \right| d\ell_\alpha \\ &\leq \sum_{i=0}^{\infty} \int_{\ell_\alpha(x_{t_{i+1}})}^{\ell_\alpha(x_{t_i})} \frac{2i(\alpha, \beta_i)}{i(\alpha, \beta_i) |\cos \vartheta|} \left(\log \frac{1}{\ell_\alpha(x_{t_s})} + \log \frac{1}{\sin \vartheta} + C + \ell_\alpha(x_0) \right) d\ell_\alpha \\ &= \sum_{i=0}^{\infty} \int_{\ell_\alpha(x_{t_{i+1}})}^{\ell_\alpha(x_{t_i})} \frac{2}{|\cos \vartheta|} \left(\log \frac{1}{\ell_\alpha} + \log \frac{1}{\sin \vartheta} + C + \ell_\alpha(x_0) \right) d\ell_\alpha \\ &\leq \int_0^{\ell_\alpha(x_0)} \frac{2}{|\cos \vartheta|} \left(\log \frac{1}{\ell_\alpha} + \log \frac{1}{\sin \vartheta} + C + \ell_\alpha(x_0) \right) d\ell_\alpha \\ &\leq \frac{2}{|\cos \vartheta|} \ell_\alpha(x_0) \log \frac{1}{\ell_\alpha(x_0)} + \frac{2}{|\cos \vartheta|} \left(\log \frac{1}{\sin \vartheta} + 1 + C + \ell_\alpha(x_0) \right) \ell_\alpha(x_0). \end{aligned}$$

By choosing ϑ close to π and $\delta > \ell_\alpha(x_0)$ sufficiently small ($\log \frac{1}{\delta} \gg \log \frac{1}{\sin \vartheta}$), we can bring the length of the path x_t to the form of the desired expression, as per Eq. (43).

All that remains is to show that the path x_t tends to the boundary locus of Teichmüller space. To this end, showing $\ell_\alpha(x_t) \rightarrow 0$ suffices. Assume otherwise, that $\ell_\alpha(x_t)$ monotonically decreases to some constant $A > 0$. Since only the geometry of S_α changes along the path x_t , the path itself lies within a 2-dimensional subspace of $\mathcal{T}(S)$ parameterised by the length and twist parameters ℓ_α and τ_α . In particular, since $\ell_\alpha(x_t)$ is decreasing and is bounded below by A , the path x_t must converge in $\mathcal{T}(S)$ unless $\tau_\alpha(x_t)$ is unbounded. The latter is impossible, as the length of x_t would be then comparable to the infinite length path formed by joining arbitrarily many α^j left Dehn twists of some point in this region. Thus, the sequence (x_{t_i}) has an accumulation point $\bar{x} \in \mathcal{T}(S)$. We derive a contradiction as follows. Recall that β_j is the j -times iterated Dehn twist of β with respect to α and that $\theta_j(x) := \theta_x(\alpha, \beta_j)$. By [32, Proposition 3.5], we see that $\theta_i(x) < \theta_{i+1}(x)$ for every $x \in \mathcal{T}(S)$. By the continuity of the angle function,

$$\text{for all } j \in \mathbb{N}, \quad \theta_j(\bar{x}) = \lim_{i \rightarrow \infty} \theta_j(x_{t_{i+1}}) \leq \lim_{i \rightarrow \infty} \theta_i(x_{t_{i+1}}) = \vartheta,$$

where the last equality is due to Eq. (56). However, $(\beta_j)_{j \in \mathbb{N}}$ is a sequence of simple closed curves which is obtained by performing the left Dehn twist (i.e. right earthquake) arbitrarily many times with respect to α , and hence $\lim_{j \rightarrow \infty} \theta_j(\bar{x}) = \pi$, and hence $\vartheta = \pi$. This contradicts the initial choice of $\vartheta \in (\frac{\pi}{2}, \pi)$, and thus ℓ_α tends to 0 along x_t . \square

Remark 9.9. It is possible to obtain the above result more directly by concretely specifying how long one should twist along a given curve β_i before switching to β_{i+1} .

9.3. Proof of Theorem 9.1. The upper bound in the statement of Theorem 9.1 follows from the first inequality in Theorem 9.5. For the lower bound, let $\sigma : [0, 1) \rightarrow \mathcal{T}(S)$ be an arbitrary piecewise C^1 -path connecting x to the boundary $\partial \overline{\mathcal{T}(S)}^{\text{WP}}$. We subdivide $[0, 1)$ into segments $[t_i, t_{i+1}]$ with $0 = t_0 < t_1 < \dots < t_n = 1$ such that within each segment, the surfaces $\sigma(t)$, $t \in [t_i, t_{i+1}]$, share a common systole, which we refer to as α_i . By

Lemma 9.4, the length of σ with respect to the earthquake path is at least

$$\begin{aligned}
& 2 \sum_i \int_{\min\{\ell_{\alpha_i}(\sigma(t_i)), \ell_{\alpha_i}(\sigma(t_i))\}}^{\max\{\ell_{\alpha_i}(\sigma(t_i)), \ell_{\alpha_i}(\sigma(t_i))\}} w(\ell) d\ell \\
&= 2 \sum_i \int_{\min\{\ell_{\text{sys}}(\sigma(t_i)), \ell_{\text{sys}}(\sigma(t_i))\}}^{\max\{\ell_{\text{sys}}(\sigma(t_i)), \ell_{\text{sys}}(\sigma(t_i))\}} w(\ell) d\ell \\
&\geq 2 \int_0^{\ell_{\text{sys}}(x)} \log \frac{1}{\ell} d\ell \\
&= 2\ell_{\text{sys}}(x) \log \frac{1}{\ell_{\text{sys}}(x)} + 2\ell_{\text{sys}}(x) \\
&\geq 2\ell_{\text{sys}}(x) \log \frac{1}{\ell_{\text{sys}}(x)}.
\end{aligned}$$

Finally, by Theorem 9.5, we see that

$$\lim_{\ell_{\text{sys}}(x) \rightarrow 0} \frac{d_e(x, \overline{\partial\mathcal{J}(S)})}{2\ell_{\text{sys}}(x) \log \frac{1}{\ell_{\text{sys}}(x)}} \leq 1.$$

When combined with the lower bound established above, this yields the desired asymptotic distance, and this completes the proof. \square

10. EARTHQUAKE METRIC COMPLETION

The incompleteness of the earthquake metric naturally leads one to draw conceptual comparisons with the Weil–Petersson metric. We recall that the completion of the Weil–Petersson metric of Teichmüller space has a rich associated theory: Teichmüller space completes to the augmented Teichmüller space [1], and correspondingly, the moduli space completes to its Deligne–Mumford compactification [27, 40]. What happens when one “completes” the earthquake metric? To rigorously answer this, we first need to explain what it means to complete a nonproper asymmetric metric. We give two approaches:

- (1) we define a wholly novel notion of the completion of asymmetric metric spaces based on sequences which have finite “*distance-series*”;
- (2) we take the completion with respect to a symmetrisation of the earthquake metric.

The first approach is more intrinsic to the metric, but requires the development of a sufficiently geometrically motivated theory of completions for asymmetric metric spaces. The second approach is readily accessible using tools and language at our disposal, but is at the cost of the completion so-produced being more inherently tied to the symmetrised metric rather than the earthquake metric itself, and a priori it may suffer from the non-uniqueness of the choice of symmetrisation. We show that both methods yield the *same* earthquake metric completion of Teichmüller space, and indeed that our two approaches produce completions which (topologically) coincide with the Weil–Petersson completion (Theorem 10.27).

10.1. FD-completions of asymmetric metric spaces. The standard theory of metric completion is founded on Cauchy sequences. We observe that any Cauchy sequence (x_n) in a (symmetric) metric space (X, d) necessarily contains a subsequence (x_{n_i}) such that the series $\sum d(x_{n_i}, x_{n_{i+1}})$, which we call a *distance-series*, is finite. We develop a novel theory of metric completion based on equivalence classes of sequences with finite *distance-series*, instead of Cauchy sequences.

Definition 10.1 (distance-series). Given a sequence (x_n) in an asymmetric metric space (X, d) , we call

$$\sum_{i=1}^{\infty} d(x_i, x_{i+1})$$

the *forward distance-series* for (x_n) . We likewise refer to $\sum_{i=1}^{\infty} d(x_{i+1}, x_i)$ as the *backward distance-series* for (x_n) .

Definition 10.2 (FD-sequence). A sequence in (X, d) is a *forward FD-sequence* if its forward distance-series is finite. Define backward FD-sequences analogously.

Remark 10.3. Unless otherwise specified, any FD-sequence mentioned henceforth without a specifying adjective should be taken as a forward FD-sequence.

Definition 10.4 (FD-completeness for asymmetric metrics). We say that an asymmetric metric space (X, d) is *forward FD-complete* if for every forward FD-sequence (x_n) , there exists some $x \in X$ such that

$$\lim_{n \rightarrow \infty} d(x_n, x) = 0.$$

We next specify an equivalence relation on FD-sequences.

Definition 10.5 (interlacing). Given two sequences (x_i) and (x'_j) , we call a sequence (\hat{x}_k) an *interlacing* of (x_i) and (x'_j) if

- the subsequence of (\hat{x}_k) comprised of elements from (x_i) is infinite and has strictly increasing indices (as elements of (x_i)); and
- the subsequence of (\hat{x}_k) comprised of elements from (x'_j) is also infinite and has strictly increasing indices (as elements of (x'_j)).

Remark 10.6. We can equivalently define interlacings to only admit subsequences which alternate between elements in (x_i) and in (x'_j) . This might be a simpler definition, but at the cost of occasional increase in the technicality of proofs (see, e.g., the proof of Lemma B.2).

Definition 10.7 (FD-equivalence). Two forward FD-sequences (x_i) and (x'_j) are *forward FD-equivalent* if (x_i) and (x'_j) have an interlacing (\hat{x}_k) which is a forward FD-sequence. We similarly define backward FD-equivalence.

Remark 10.8. As a small but oft-used observation, any FD-sequence (x_n) is FD-equivalent to all of its subsequences via the interlacing $(\hat{x}_n) = (x_n)$.

Lemma 10.9. *The forward FD-equivalence is an equivalence relation on the set of forward FD-sequences. Likewise, the backward FD-equivalence is an equivalence relation on the set of backward FD-sequences.*

Proof. Reflexivity and symmetry are clear. For transitivity, let (x_i) , (x'_i) , and (x''_i) be arbitrary forward FD-sequences such that (x_i) and (x'_i) are forward FD-equivalent, and (x'_i) and (x''_i) are forward FD-equivalent. Therefore, there exist subindices i_k and j_k satisfying

$$i_1 < i_3 < i_5 < \cdots; \quad i_2 < i_4 < i_6 < \cdots; \quad j_1 < j_3 < j_5 < \cdots; \quad j_2 < j_4 < j_6 < \cdots.$$

such that both

$$x_{i_1}, x'_{i_2}, x_{i_3}, x'_{i_4}, \cdots \quad \text{and} \quad x'_{j_1}, x''_{j_2}, x'_{j_3}, x''_{j_4}, \cdots$$

are forward FD-sequences. We construct a forward FD-interlacing of (x_i) and (x''_i) as follows. Start with $\hat{x}_1 = x_{i_1}$. Take an odd index k_1 such that $j_{k_1} > i_2$ and set $\hat{x}_2 := x''_{j_{k_1+1}}$. Then

$$d(\hat{x}_1, \hat{x}_2) \leq d(x_{i_1}, x'_{i_2}) + d(x'_{i_2}, x'_{j_{k_1}}) + d(x'_{j_{k_1}}, x''_{j_{k_1+1}}).$$

Next, let k_2 be an even index such that $i_{k_2} > j_{k_1+2}$, and set $\hat{x}_3 := x_{i_{k_2+1}}$. Then

$$d(\hat{x}_2, \hat{x}_3) \leq d_e(x''_{j_{k_1+1}}, x'_{j_{k_1+2}}) + d(x'_{j_{k_1+2}}, x'_{i_{k_2}}) + d(x'_{i_{k_2}}, x_{i_{k_2+1}}).$$

Inductively, we construct an interlacing \hat{x}_i of (x_i) and (x''_i) such that

$$\begin{aligned} \sum_{i \geq 1} d(\hat{x}_i, \hat{x}_{i+1}) &\leq \sum_{i \geq 1} d(x'_i, x'_{i+1}) + \sum_{k \geq 1} [d(x_{i_{2k-1}}, x'_{i_{2k}}) + d(x'_{i_{2k}}, x_{i_{2k+1}})] \\ &\quad + \sum_{k \geq 1} [d(x'_{j_{2k-1}}, x''_{j_{2k}}) + d(x''_{j_{2k}}, x'_{j_{2k+1}})] < \infty, \end{aligned}$$

which means that it is forward FD. Hence (x_i) and (x''_i) are forward FD-equivalent. This shows that forward FD-equivalence is an equivalence relation. The proof for backward FD-equivalence being an equivalence relation is essentially the same. \square

Definition 10.10 (FD-completion of asymmetric metric spaces). The *forward FD-completion* of an asymmetric metric space (X, d) is defined as the set of forward FD-equivalence classes of forward FD-sequences. We denote this set by \overline{X} , and write each FD-equivalence class of an FD-sequence (x_n) in the form $[x_n]$. We define the backward FD-completion analogously, and denote it by $\overline{X}^\#$.

Remark 10.11. At present, \overline{X} and $\overline{X}^\#$ are merely sets. We (asymmetrically) metrize them in Appendix B, Theorem B.6.

Proposition 10.12 (natural inclusion). *The map $\iota : X \rightarrow \overline{X}$ sending $x \in X$ to the FD-equivalence class represented by the constant FD-sequence $(x)_{n \in \mathbb{N}}$ is an injection. In other words, \overline{X} contains a copy of X as the set of FD-equivalence classes represented by constant FD-sequences. This is also true for $\overline{X}^\#$, and we denote the inclusion by $\iota^\#$.*

Proof. The map ι is well defined because constant sequences are FD-sequences. For the injectivity, consider two constant FD-sequences (x) and (y) which are FD-equivalent. The existence of an interlacing means that

$$0 = \lim_{n \rightarrow \infty} d(x, y) = d(x, y) \quad \text{and} \quad 0 = \lim_{n \rightarrow \infty} d(y, x) = d(y, x).$$

By the axioms of asymmetric metric spaces, we see that $x = y$, which is the required injectivity. \square

We use Proposition 10.12 to regard X as a subset of \overline{X} , and this allows us to speak of extensions of functions and metrics.

Definition 10.13 (extensions of functions and metrics). Consider a function $f : (X, d_X) \rightarrow (Y, d_Y)$, and let $\iota_X : X \rightarrow \overline{X}$ and $\iota_Y : Y \rightarrow \overline{Y}$ denote the respective natural inclusion maps for X and Y . We say that ϕ is an *extension* of f if $\iota_Y \circ f = \phi \circ \iota_X$, that is, if it satisfies the following commutative diagram:

$$\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow \iota_X & & \downarrow \iota_Y \\
\overline{X} & \xrightarrow{\phi} & \overline{Y}
\end{array}$$

Correspondingly, an asymmetric metric D on \overline{X} or $\overline{X}^\#$ is a *metric extension* of d_X if $D \circ (\iota_X \times \iota_X) = d_X$.

Having only looked at the FD-completion from a purely set-theoretic point of view, there are indubitably many questions as to why this construction is “natural” and if it merits being dubbed a “completion”. In response, we show in Appendix B that the FD-completions \overline{X} and $\overline{X}^\#$ of an asymmetric metric space (X, d) satisfy the following:

- (1) \overline{X} is naturally metrised via an asymmetric metric \bar{d} which extends the asymmetric metric d (Theorem B.1 and Proposition B.4). Similarly, $\overline{X}^\#$ is naturally metrised via an asymmetric metric $\bar{d}^\#$ which extends the reverse metric $\bar{d}^\#$ on X (Remark B.3);
- (2) the natural inclusion map $\iota : (X, d) \rightarrow (\overline{X}, \bar{d})$ is an isometric embedding with forward-dense image (Proposition B.4);
- (3) forward FD-completions are forward FD-complete (Theorem B.6);
- (4) FD-completions generalise the notion of Cauchy completion when d is symmetric (Theorem B.12);
- (5) the process of taking the forward FD-completion of an asymmetric metric space can be naturally promoted to an endofunctor on the category of asymmetric metric spaces with Lipschitz morphisms (see Definition B.18).

10.2. FD-sequences in the earthquake metric. Having defined FD-completions, we move on to establish the properties of the forward and backward FD-completions of $(\mathcal{T}(S), d_e)$. We employ the following notation throughout:

- $\overline{\mathcal{T}(S)}^{\text{WP}}$ denotes the Weil–Petersson completion of $\mathcal{T}(S)$;
- $\overline{\mathcal{T}(S)}$ denotes the forward FD-completion of $\mathcal{T}(S)$ with respect to the left earthquake metric;
- $\overline{\mathcal{T}(S)}^\#$ denotes the backward FD-completion of $\mathcal{T}(S)$ with respect to the left earthquake metric.

Remark 10.14. Thanks to Remark 1.3, $\overline{\mathcal{T}(S)}^\#$ (regarded purely as a set) is identical to the forward FD-completion of $\mathcal{T}(S)$ with respect to the right earthquake metric. In fact, (see Remark B.3) this identity applies also at the level of the metric extension: the backward metric extension of the left earthquake metric is definitionally equal to the forward metric extension of the right earthquake metric.

Many of the arguments in the remainder of this section depend on understanding the limiting behaviour of FD-sequences in $(\mathcal{T}(S), d_e)$. Crucially, the

length spectrum for an FD-sequence with respect to d_e stabilises, and this can be characterised in terms of convergence in the Weil–Petersson metric.

Lemma 10.15. *Any (forward or backward) FD-sequence (x_n) with respect to the earthquake metric satisfies the following:*

- (i) *for every simple closed curve α , $(\ell_\alpha(x_n))_{n \in \mathbb{N}}$ converges in $[0, \infty]$;*
- (ii) *(x_n) converges in $\overline{\mathcal{T}(S)}^{\text{WP}}$, and hence is a Cauchy sequence with respect to the Weil–Petersson metric.*

Proof. We first note that it suffices to only deal with forward FD-sequences.

(i) We assume that (x_n) is a forward FD-sequence, and verify the convergence of $\ell_\alpha(x_n)$. Consider two subsequences

- (x_{n_i}) where $\ell_\alpha(x_{n_i}) \rightarrow \delta = \liminf_n \ell_\alpha(x_n) \in [0, \infty]$, and
- $(x_{n'_j})$ where $\ell_\alpha(x_{n'_j}) \rightarrow \delta' = \limsup_n \ell_\alpha(x_n) \in [0, \infty]$.

If the sequence $(\ell_\alpha(x_n))$ diverges in $[0, \infty]$, then δ is strictly smaller than δ' , hence is finite. Let $[a, b]$ be any nontrivial closed interval contained in (δ, δ') . By Lemma 9.4, there exists M such that for any $i, j > M$,

$$(57) \quad d_e(x_{n_i}, x_{n'_j}) \geq 2 \int_a^b w(\ell) \, d\ell > 0,$$

On the other hand, the assumption that (x_n) is an FD-sequence implies that for any $m > n$,

$$d_e(x_n, x_m) \leq \sum_{k=n}^{\infty} d_e(x_k, x_{k+1}) \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

This contradicts (57), and hence $\ell_\alpha(x_n)$ converges for every simple closed curve α .

(ii) We now show that (x_n) is a Cauchy sequence with respect to the Weil–Petersson metric. Let B be the Bers constant, which depends only on the topological type of S , and let N_1 be a natural number such that

$$(58) \quad \sum_{k \geq N_1} d_e(x_k, x_{k+1}) < \int_B^{2B} w(\ell) \, d\ell.$$

Consider the surface x_{N_1} , and let Γ be a pants decomposition of x_{N_1} comprised of geodesics of length at most B . For any $\gamma \in \Gamma$, we can see that

$$(59) \quad \ell_\gamma(x_n) < 2B, \quad \text{for all } n > N_1.$$

Otherwise, by Lemma 9.4,

$$d_e(x_{N_1}, x_n) \geq 2 \int_{\ell_\gamma(x_{N_1})}^{\ell_\gamma(x_n)} w(\ell) \, d\ell \geq 2 \int_B^{2B} w(\ell) \, d\ell,$$

which contradicts Eq. (58). Combining (59) with part (i) of this lemma, we see that $\lim_{n \rightarrow \infty} \ell_\gamma(x_n) \in [0, 2B)$ exists for every $\gamma \in \Gamma$. In particular, Γ must contain all the *pinching curves* of (x_n) , whose collection we denote by P . If $\Gamma = P$, the sequence (x_n) converges to a single point in the augmented

Teichmüller space with respect to the Weil–Petersson metric and we are done.

Suppose otherwise that Γ properly contains P . Define $\mathbf{s} := \inf\{\ell_\gamma(x_i) \mid \gamma \in \Gamma \setminus P, i \in \mathbb{N}\}$. We see that $\mathbf{s} > 0$ because there are only finitely many curves in $\Gamma \setminus P$ and none of them are pinched to length 0 as $i \rightarrow \infty$. Any two shortest curves on x_i which minimally intersect γ without intersecting any other curve in Γ are necessarily related by a Dehn twist or a half-Dehn twist with respect to γ . An upper bound for the lengths of such pairs of curves for all $\gamma \in \Gamma \setminus P$ on all $x_i, i \in \mathbb{N}$, exists since $\mathbf{s} > 0$ and Eq. (59). Let N_2 be a constant such that

$$\sum_{k \geq N_2} d_e(x_k, x_{k+1}) < \int_L^{2L} w(\ell) \, d\ell.$$

Given an arbitrary $\gamma \in \Gamma \setminus P$, let α and β denote two shortest curves on x_{N_2} which minimally intersect γ and are disjoint from other curves in Γ . We know that $\ell_\alpha(x_{N_2}), \ell_\beta(x_{N_2}) \leq L$ by our definition of L . Applying the same argument as for the proof of Eq. (59), we see that

$$\ell_\alpha(x_i), \ell_\beta(x_i) \leq 2L, \quad \text{for all } i > N_2.$$

Combined with part (i) of this lemma, the lengths of α and β converge in $(0, 2L)$. This also shows that Fenchel–Nielsen twist parameter for γ also converges as can be seen by considering the Fricke–Klein embedding of Teichmüller space (see, e.g., [30, Theorem 3.12]). Since this holds for all $\gamma \in \Gamma \setminus P$, the criterion of convergence in the Weil–Petersson completion (see, e.g., the characterisation immediately prior to Remark 5.1 in [45]) tells us that (x_n) is a convergent sequence in $\overline{\mathcal{T}(S)}^{\text{WP}}$, and hence is Cauchy. \square

Corollary 10.16. *Let (x_n) and (x'_n) be two FD-equivalent FD-sequences in $(\mathcal{T}(S), d_e)$. Then they are Cauchy equivalent with respect to the Weil–Petersson metric, and moreover we have $\lim_{n \rightarrow \infty} d_e(x_n, x'_n) \rightarrow 0$.*

Proof. By Lemma 10.15, both (x_n) and (x'_n) converge in $\overline{\mathcal{T}(S)}^{\text{WP}}$, whose limits we denote by x and x' respectively. Since (x_n) and (x'_n) are FD-equivalent, there is an FD-interlacing (y_n) between them. Since (y_n) itself is also an FD-sequence, it converges to a point in y in $\overline{\mathcal{T}(S)}^{\text{WP}}$. Since (y_n) contains infinitely many terms in both (x_n) and (x'_n) , we have $x = y = x'$. Therefore, (x_n) and (x'_n) are Cauchy equivalent, and $\lim_{n \rightarrow \infty} d_{\text{WP}}(x_n, x'_n) = 0$. By Corollary 1.19, this implies also $\lim_{n \rightarrow \infty} d_e(x_n, x'_n) = 0$ \square

Lemma 10.17. *Any two Weil–Petersson metric Cauchy sequences (x_n) and (y_n) satisfying $\lim_{n \rightarrow \infty} d_e(x_n, y_n) = 0$ must be Cauchy equivalent with respect to the Weil–Petersson metric.*

Proof. Since (x_n) and (y_n) are Cauchy sequences with respect to the Weil-Petersson metric, they respectively converge to points $x, y \in \overline{\mathcal{T}(S)}^{\text{WP}}$. Assume $x \neq y$. Then, there must be some simple closed curve γ on S such that $|\ell_\gamma(x) - \ell_\gamma(y)| > \delta$ for some $\delta \in (0, \infty]$, where $\delta = \infty$ means that exactly one of $\ell_\gamma(x)$ and $\ell_\gamma(y)$ is infinity. Hence, by Lemma 9.4,

$$\limsup_{n \rightarrow \infty} d_e(x_n, y_n) \geq \int_{\min\{\ell_\gamma(x), \ell_\gamma(y)\}}^{\min\{\ell_\gamma(x), \ell_\gamma(y)\} + \delta} w(\ell) d\ell > 0,$$

which contradicts the assumption that $\lim_{n \rightarrow \infty} d_e(x_n, y_n) = 0$. Thus, the limits x and y must be equal, therefore (x_n) and (y_n) must be Cauchy equivalent. \square

10.3. The metric geometry of $\overline{\mathcal{T}(S)}$. The goal of this subsection is to show that $\overline{\mathcal{T}(S)}$ inherits a unique continuous extension of d_e which is also a Busemannian metric (see Theorem 1.25).

Theorem 10.18. *The function $\bar{d}_e : \overline{\mathcal{T}(S)} \times \overline{\mathcal{T}(S)} \rightarrow [0, \infty)$ given by*

$$\bar{d}_e([x_n], [y_n]) := \lim_{n \rightarrow \infty} d_e(x_n, y_n),$$

where (x_n) and (y_n) are arbitrary FD-sequences respectively representing $[x_n]$ and $[y_n]$, is well defined and defines an asymmetric metric on $\overline{\mathcal{T}(S)}$.

Proof. We first show that \bar{d}_e is well defined. Let $(x_n), (x'_n)$ be FD-sequences representing $[x_n]$ and let $(y_n), (y'_n)$ be FD-sequences representing $[y_n]$. The triangle inequality tells us that

$$d_e(x_n, y_n) \leq d_e(x_n, x'_n) + d_e(x'_n, y'_n) + d_e(y'_n, y_n).$$

Corollary 10.16 tells us that $d_e(x_n, x'_n), d_e(y'_n, y_n) \rightarrow 0$, and hence $\bar{d}_e([x_n], [y_n])$ is independent of the FD-sequences representatives chosen to define the limit. Moreover, the fact that (x_n) and (y_n) are Cauchy sequences in d_{WP} proved in Lemma 10.17 combined with Corollary 1.19 means that $d_e(x_n, y_n) \leq C_1 d_{\text{WP}}(x_n, y_n)$ remains bounded, hence $\bar{d}_e : \overline{\mathcal{T}(S)} \times \overline{\mathcal{T}(S)} \rightarrow [0, \infty)$ is well defined. Note that the expression given for \bar{d}_e agrees with the expression for the standard metric extension of d_e to its completion, as given in Theorem B.1, hence \bar{d}_e is an asymmetric metric. \square

Theorem 10.19. *The function \bar{d}_e defined in Theorem 10.18 is a Busemannian metric on the forward FD-completion $\overline{\mathcal{T}(S)}$.*

Proof. We first note that the triangle inequality holds, and $\bar{d}_e(\xi, \xi) = 0$ is clear. Thus, we need only verify that

- (1) for any $\xi, \eta \in \overline{\mathcal{T}(S)}$, $\bar{d}_e(\xi, \eta) = 0 \Rightarrow \xi = \eta$; and
- (2) for any $\eta \in \overline{\mathcal{T}(S)}$ and any (ξ_k) in $\overline{\mathcal{T}(S)}$, we have $\bar{d}_e(\xi_k, \eta) \rightarrow 0 \Leftrightarrow \bar{d}_e(\eta, \xi_k) \rightarrow 0$.

For (1), suppose that $\bar{d}_e(\xi, \eta) = \lim_{n \rightarrow \infty} d_e(x_n, y_n) = 0$. Let (x_n) and (y_n) be FD-sequences respectively representing ξ and η . Then Lemma 10.17 asserts that (x_n) and (y_n) are equivalent Cauchy sequences with respect to d_{WP} . Since d_{WP} is symmetric, this means that

$$0 = \lim_{n \rightarrow \infty} C_1 d_{\text{WP}}(y_n, x_n) \geq \lim_{n \rightarrow \infty} d_e(y_n, x_n) = \bar{d}_e(\eta, \xi).$$

Since we showed above that \bar{d}_e is an asymmetric metric, the fact that $\bar{d}_e(\xi, \eta) = \bar{d}_e(\eta, \xi) = 0$ implies $\xi = \eta$.

For (2), we shall only show that $\bar{d}_e(\xi_k, \eta) \rightarrow 0$ implies $\bar{d}_e(\eta, \xi_k) \rightarrow 0$; the proof of the converse is essentially identical. For each k , let $(x_{n,k})_{n \in \mathbb{N}}$ denote an arbitrary FD-sequence representative for the FD-equivalence class ξ_k . Likewise let (y_n) be an FD-sequence with respect to d_e which represents the FD-equivalence class $\eta \in \overline{\mathcal{T}(S)}$. By Lemma 10.15, for every k , the sequence $(x_{n,k})_{n \in \mathbb{N}}$ is a Cauchy sequence in $(\mathcal{T}(S), d_{\text{WP}})$, as is (y_n) . We choose a sequence of increasing indices $(m_k)_{k \in \mathbb{N}}$ so that for each $k \in \mathbb{N}$,

$$(60) \quad d_{\text{WP}}(x_{m_k, k}, x_{n, k}) < 2^{-k}, \text{ for every } n > m_k,$$

and

$$|\bar{d}_e(\xi_k, \eta) - d_e(x_{m_k, k}, y_{m_k})| < 2^{-k}.$$

Hence

$$(61) \quad \lim_{k \rightarrow \infty} d_e(x_{m_k, k}, y_{m_k}) = 0.$$

Since $(y_{m_k})_{k \in \mathbb{N}}$ is a Cauchy sequence in $(\mathcal{T}(S), d_{\text{WP}})$ by Lemma 10.17 and Eq. (61), we conclude that $(x_{m_k, k})_{k \in \mathbb{N}}$ is also a Cauchy sequence in $(\mathcal{T}(S), d_{\text{WP}})$ and that $(x_{m_k, k})_{k \in \mathbb{N}}$ and $(y_{m_k})_{k \in \mathbb{N}}$ are Cauchy equivalent in $(\mathcal{T}(S), d_{\text{WP}})$. Therefore,

$$(62) \quad 0 \leq \lim_{k \rightarrow \infty} d_e(y_{m_k}, x_{m_k, k}) \leq \lim_{k \rightarrow \infty} C_1 d_{\text{WP}}(y_{m_k}, x_{m_k, k}) = 0,$$

where the inequality and the constant C_1 both come from Corollary 1.19.

Finally, we make use of Eq. (60), Eq. (62) and the Cauchy convergence of $(y_k)_{k \in \mathbb{N}}$ in $(\mathcal{T}(S), d_{\text{WP}})$ to show:

$$\begin{aligned} & \lim_{k \rightarrow \infty} \bar{d}_e(\eta, \xi_k) \\ &= \lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} d_e(y_n, x_{n, k}) \\ &\leq \lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} (d_e(y_n, y_{m_k}) + d_e(y_{m_k}, x_{m_k, k}) + d_e(x_{m_k, k}, x_{n, k})) \\ &\leq \lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} C_1 (d_{\text{WP}}(y_n, y_{m_k}) + d_{\text{WP}}(y_{m_k}, x_{m_k, k}) + d_{\text{WP}}(x_{m_k, k}, x_{n, k})) = 0. \end{aligned}$$

We have thus verified that \bar{d}_e satisfies all of the axioms for being a Busemannian metric on $(\overline{\mathcal{T}(S)}, \bar{d}_e)$. \square

Remark 10.20. The analogous statements (and proofs) of Theorem 10.18 and Theorem 10.19 also hold for the backward FD-completion $(\overline{\mathcal{T}(S)}^\#, \bar{d}_e^\#)$.

10.4. Homeomorphic extension maps. One important consequence of \bar{d}_e being Busemannian (Theorem 10.19) is that the forward, backward and symmetric topologies on $(\overline{\mathcal{T}(S)}, \bar{d}_e)$ all agree, and we set it as the standard topology on $\overline{\mathcal{T}(S)}$. Likewise, $\bar{d}_e^\#$ is also Busemannian and defines a standard topology on $\overline{\mathcal{T}(S)}^\#$. This allows us to speak of continuous maps between these spaces.

Theorem 10.21. *The identity map of Teichmüller space $\mathcal{T}(S)$ admits a unique homeomorphic extension between any pair of the following three:*

- the forward FD-completion $(\overline{\mathcal{T}(S)}, \bar{d}_e)$,
- the forward FD-completion $(\overline{\mathcal{T}(S)}^\#, \bar{d}_e^\#)$,
- and the Weil–Petersson completion $(\overline{\mathcal{T}(S)}^{\text{WP}}, \bar{d}_{\text{WP}})$.

Proof. We only show that $\text{id} : (\mathcal{T}(S), d_e) \rightarrow (\mathcal{T}(S), d_{\text{WP}})$ has a unique homeomorphic extension. The proof for the backward FD-completion is identical, and the uniqueness of these two homeomorphisms then asserts the uniqueness of the requisite extension homeomorphism between $(\overline{\mathcal{T}(S)}, \bar{d}_e)$ and $(\overline{\mathcal{T}(S)}^\#, \bar{d}_e^\#)$. The proof will be divided into the following steps: we

- define $\Pi : \overline{\mathcal{T}(S)} \rightarrow \overline{\mathcal{T}(S)}^{\text{WP}}$;
- show that the map Π is surjective;
- show that the map Π is injective;
- show that the map Π is a homeomorphism; and
- show that Π is the unique homeomorphic extension.

STEP 1: CONSTRUCTING THE MAP $\Pi : \overline{\mathcal{T}(S)} \rightarrow \overline{\mathcal{T}(S)}^{\text{WP}}$. By Lemma 10.15 any FD-sequence (x_n) in $(\mathcal{T}(S), d_e)$ is a Cauchy sequence in $(\mathcal{T}(S), d_{\text{WP}})$, and hence we define Π by sending an FD-equivalence class represented by an FD-sequence (x_n) with respect to d_e to the Cauchy equivalence class represented by (x_n) with respect to d_{WP} . This is well defined thanks to the last sentence of Lemma 10.17.

STEP 2: ESTABLISHING SURJECTIVITY. Consider an arbitrary element of $\overline{\mathcal{T}(S)}^{\text{WP}}$ and let it be represented by some Cauchy sequence (x_n) with respect to d_{WP} . Then (x_n) contains a subsequence $(x_{n_k})_{k \in \mathbb{N}}$ which is an FD-sequence with respect to d_{WP} . By Proposition 7.6, there exists a constant C_1 such that

$$\sum_{k=1}^{\infty} d_e(x_{n_k}, x_{n_{k+1}}) \leq \sum_{k=1}^{\infty} C_1 d_{\text{WP}}(x_{n_k}, x_{n_{k+1}}) < \infty.$$

Therefore, $(x_{n_k})_{k \in \mathbb{N}}$ is also an FD-sequence with respect to d_e , and its FD-equivalence class maps to the same Cauchy equivalence class as $(x_n)_{n \in \mathbb{N}}$ since $(x_{n_k})_{k \in \mathbb{N}}$ is a subsequence of (x_n) .

STEP 3: ESTABLISHING INJECTIVITY. Consider two (forward) FD-sequences (x_n) and (y_n) in $(\mathcal{T}(S), d_e)$ which respectively represent two arbitrary points

$[x_n], [y_n] \in \overline{\mathcal{T}(S)}$. By Lemma 10.15, these two sequences are Cauchy sequences with respect to d_{WP} . Now suppose that they are Cauchy equivalent in $(\mathcal{T}(S), d_{\text{WP}})$. Then, there are subsequences $(x_{n_k})_{k \in \mathbb{N}}$ and $(y_{n_k})_{k \in \mathbb{N}}$ which are FD-sequences with respect to d_{WP} . The Cauchy equivalence ensures that $(x_{n_k})_{k \in \mathbb{N}}$ and $(y_{n_k})_{k \in \mathbb{N}}$ are FD-equivalent (Lemma B.10) with respect to d_{WP} , and hence there is an interlacing with respect to d_{WP} between them. Thanks to Corollary 1.19, such an interlacing is also an interlacing for d_e , and hence $(x_{n_k})_{k \in \mathbb{N}}$ and $(y_{n_k})_{k \in \mathbb{N}}$ are FD-equivalent with respect to d_e . Therefore, (x_n) , $(x_{n_k})_{k \in \mathbb{N}}$, $(y_{n_k})_{k \in \mathbb{N}}$ and (y_n) are all FD-equivalent, and $[x_n] = [y_n]$. This gives the requisite injectivity.

STEP 4: ESTABLISHING HOMEOMORPHICITY. Steps 2 and 3 show that Π is a bijection. By Corollary 1.19, its inverse map is Lipschitz hence is also continuous (since both metrics are Busemannian). We only need to show that Π is continuous. Fortunately, as both metrics are Busemannian, continuity is the same as the convergence of limits (Remark 2.14). Specifically, we need to verify the following: given an arbitrary sequence (ξ_k) in $(\overline{\mathcal{T}(S)}, \bar{d}_e)$ that converges to some point $\xi \in \overline{\mathcal{T}(S)}$, i.e. $\bar{d}_e(\xi_k, \xi) \rightarrow 0$, then

$$\lim_{k \rightarrow \infty} \bar{d}_{\text{WP}}(\Pi(\xi_k), \Pi(\xi)) = 0.$$

For each k , let $(x_{n,k})_{n \in \mathbb{N}}$ be an arbitrary FD-sequence in $(\mathcal{T}(S), d_e)$ representing ξ_k . By Lemma 10.15, it is also a Cauchy sequence with respect to d_{WP} , and we note that $\Pi(\xi_k)$ is represented by $(\text{id}(x_{n,k}))_{n \in \mathbb{N}} = (x_{n,k})_{n \in \mathbb{N}}$. Let (x_n) be an arbitrary FD-sequence in $(\mathcal{T}(S), d_e)$ representing ξ ; it is likewise a Cauchy sequence with respect to d_{WP} , and we note that $\Pi(\xi)$ is represented by (x_n) .

Since $(x_{n,k})_{n \in \mathbb{N}}$ is a Cauchy sequence for every k , we may choose an increasing sequence $(m_k)_{k \in \mathbb{N}}$ of indices so that for all $n \geq m_k$, $d_{\text{WP}}(x_{m_k,k}, x_{n,k}) < 2^{-k}$. We claim that $(x_{m_k,k})_{k \in \mathbb{N}}$ is an FD-sequence in $(\mathcal{T}(S), d_e)$. This follows from

$$\begin{aligned} & d_e(x_{m_k,k}, x_{m_{k+1},k+1}) \leq C_1 d_{\text{WP}}(x_{m_k,k}, x_{m_{k+1},k+1}) \\ & \leq \lim_{n \rightarrow \infty} C_1 (d_{\text{WP}}(x_{m_k,k}, x_{n,k}) + d_{\text{WP}}(x_{n,k}, x_{n,k+1}) + d_{\text{WP}}(x_{n,k+1}, x_{m_{k+1},k+1})) \\ & \leq C_1 (2^{-k} + \bar{d}_{\text{WP}}(\xi_k, \xi_{k+1}) + 2^{-k-1}), \end{aligned}$$

in conjunction with (ξ_k) being a Cauchy sequence with respect to \bar{d}_{WP} .

Next observe that

$$\begin{aligned} & \lim_{k \rightarrow \infty} d_e(x_{m_k,k}, x_k) \\ & \leq \lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} (d_e(x_{m_k,k}, x_{n,k}) + d_e(x_{n,k}, x_n) + d_e(x_n, x_k)) \\ & \leq \lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} (C_1 d_{\text{WP}}(x_{m_k,k}, x_{n,k}) + d_e(x_{n,k}, x_n) + C_1 d_{\text{WP}}(x_n, x_k)) \\ & = \lim_{k \rightarrow \infty} (C_1 2^{-k} + \bar{d}_e(\xi_k, \xi)) + \lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} C_1 d_{\text{WP}}(x_n, x_k) = 0. \end{aligned}$$

Since the FD-sequences $(x_{m_k, k})_{k \in \mathbb{N}}$ and $(x_k)_{k \in \mathbb{N}}$ satisfy $d_e(x_{m_k, k}, x_k) \rightarrow 0$, Lemmas 10.15 and 10.17 tell us that these two sequences are also equivalent Cauchy sequences in the Weil–Petersson metric. Hence,

$$\begin{aligned} \lim_{k \rightarrow \infty} \bar{d}_{\text{WP}}(\Pi(\xi_k), \Pi(\xi)) &= \lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} d_{\text{WP}}(x_{n, k}, x_n) \\ &\leq \lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} (d_{\text{WP}}(x_{n, k}, x_{m_k, k}) + d_{\text{WP}}(x_{m_k, k}, x_k) + d_{\text{WP}}(x_k, x_n)) = 0. \end{aligned}$$

Thus, we have shown that Π is continuous, hence is a homeomorphism.

STEP 5: ESTABLISHING UNIQUENESS. Consider a homeomorphic extension Π' of $\text{id} : (\mathcal{T}(S), d_e) \rightarrow (\mathcal{T}(S), d_{\text{WP}})$. Then $\Pi' \circ \Pi^{-1}$ gives a homeomorphic extension of $\text{id} : (\mathcal{T}(S), d_{\text{WP}}) \rightarrow (\mathcal{T}(S), d_{\text{WP}})$, which we know, from the uniqueness of continuous extensions of Lipschitz maps to maps between Cauchy completions, must be the identity map on $(\overline{\mathcal{T}(S)})^{\text{WP}}, \bar{d}_{\text{WP}}$. Therefore, Π' is Π . \square

10.5. Completion via symmetrisation. We end this section by explaining what happens when we complete a symmetrisation of the earthquake metric. We first note that there is no canonical choice of symmetrisation for an asymmetric metric d . The most obvious examples are the *sum* and *max* symmetrisations, defined by

$$\begin{aligned} d^{(1)}(x, y) &:= d^{\text{sum}}(x, y) = \frac{1}{2}(d(x, y) + d(y, x)) \quad \text{and} \\ d^{(\infty)}(x, y) &= d^{\text{max}}(x, y) := \max\{d(x, y), d(y, x)\}, \end{aligned}$$

but there are also other symmetrisations.

Definition 10.22 (metric symmetrisation). Indeed, we can interpolate between them via the following family of metrics

$$(63) \quad d^{(p)}(x, y) := \begin{cases} \sqrt[p]{\frac{1}{2}(d(x, y)^p + d(y, x)^p)}, & \text{for } p \in [1, \infty), \\ \max\{d(x, y), d(y, x)\}, & \text{for } p = \infty. \end{cases}$$

We can also symmetrise within the Finsler category by symmetrising the Finsler norm of the metric.

Proposition 10.23. *Given a Finsler metric space (X, d) where d is the path metric induced by the Finsler norm $\|\cdot\|$. For every $p \in [1, \infty]$, the function $\|\cdot\|^{(p)} : T_x X \rightarrow [0, \infty)$,*

$$\|v\|^{(p)} := \begin{cases} \sqrt[p]{\frac{1}{2}(\|v\|^p + \|-v\|^p)}, & \text{for } p \in [1, \infty), \\ \max\{\|v\|, \|-v\|\}, & \text{for } p = \infty. \end{cases}$$

defines a (symmetric) norm on $T_x X$ for every $x \in X$.

Proof. We first note that the properties

- $\|v\|^{(p)} = 0$ if and only if $v = 0$, and
- $\|\lambda v\|^{(p)} = |\lambda| \|v\|^{(p)}$ for all $v \in T_x X$ and for all $\lambda \in \mathbb{R}$

trivially follow from $\|\cdot\|$ being a weak norm. Since $\|\cdot\|^{(p)}$ is positively homogeneous, verifying that it satisfies the triangle inequality is equivalent to showing that $\|\cdot\|^{(p)}$ is a convex function. For $p \neq \infty$, consider arbitrary $v, w \in T_x X$ with $\|v\|^{(p)} = \|w\|^{(p)} = 1$, for any $t \in [0, 1]$,

$$\begin{aligned} & (\|tv + (1-t)w\|^{(p)})^p \\ &= \frac{1}{2} (\|tv + (1-t)w\|^p + \|t(-v) + (1-t)(-w)\|^p) \\ &\leq \frac{1}{2} ((t\|v\| + (1-t)\|w\|)^p + (t\| -v\| + (1-t)\| -w\|)^p). \end{aligned}$$

By Jensen's inequality,

$$(t\|v\| + (1-t)\|w\|)^p \leq t\|v\|^p + (1-t)\|w\|^p.$$

The requisite convexity follows:

$$\begin{aligned} (\|tv + (1-t)w\|^{(p)})^p &\leq \frac{1}{2} (t\|v\|^p + (1-t)\|w\|^p + t\| -v\|^p + (1-t)\| -w\|^p) \\ &= t(\|v\|^{(p)})^p + (1-t)(\|w\|^{(p)})^p = 1. \end{aligned}$$

The case for $p = \infty$ is straightforward. \square

Definition 10.24 (Finsler symmetrisation). Let (X, d) be a Finsler manifold with path metric d induced by a Finsler norm $\|\cdot\|$. We refer to the path metric for the Finsler norm $\|\cdot\|^{(p)}$, where $p \in [1, \infty]$, as the p -th *Finsler symmetrisation* of d . We denote such a path metric by $D^{(p)}$.

Proposition 10.25. *For every $p \in [1, \infty]$, we have $D^{(p)} \geq D^{(1)}$.*

Proof. This is clear for $p = \infty$. For $p \in [1, \infty)$, Jensen's inequality asserts that

$$\begin{aligned} & (\|tv + (1-t)w\|^{(p)})^p \\ &= \frac{1}{2} (\|tv + (1-t)w\|^p + \|t(-v) + (1-t)(-w)\|^p) \\ &\geq \left(\frac{1}{2} (\|tv + (1-t)w\| + \|t(-v) + (1-t)(-w)\|)\right)^p. \end{aligned}$$

Therefore $\|\cdot\|^{(p)} \geq \|\cdot\|^{(1)}$, and hence $D^{(p)} \geq D^{(1)}$. \square

Remark 10.26. The last author and Troyanov gave (in [52]) a geometric description of the transformation of the Finsler norm for the $D^{(1)}$ symmetrisation of the distance function; this is based on an operation called the ‘‘harmonic symmetrisation’’ at the level of the indicatrix. [52, Lemma 4.5] asserts that $D^{(1)} \geq d^{(1)}$. However, we are unaware of any such relationships between $D^{(p)}$ and $d^{(p)}$ for general $p \in (1, \infty]$.

In light of Proposition 10.25, this means that $D^{(p)} \geq d^{(1)}$ for all $p \in [0, \infty]$ and so by Proposition B.17 there is a natural 1-Lipschitz extension of the identity map between the respective FD-completions of $(\mathcal{T}(S), D^{(p)})$ and $(\mathcal{T}(S), d^{(p)})$. We show that such an extension map is homeomorphic:

Theorem 10.27. *The identity map $\text{id} : \mathcal{T}(S) \rightarrow \mathcal{T}(S)$ induces a unique homeomorphic extension between any two of the following:*

- the forward FD-completion of $(\mathcal{T}(S), d_e)$;

- the backward FD-completion of $(\mathcal{T}(S), d_e)$;
- the forward FD-completion of $(\mathcal{T}(S), d_e^\#)$;
- the backward FD-completion of $(\mathcal{T}(S), d_e^\#)$;
- the (Cauchy) completion of $(\mathcal{T}(S), d_{\text{WP}})$;
- the (Cauchy) completion of $(\mathcal{T}(S), d_e^{(p)})$, for any $p \in [1, \infty]$; and
- the (Cauchy) completion of $(\mathcal{T}(S), D_e^{(p)})$, for any $p \in [1, \infty]$.

Proof. We begin with a few observations.

- As Remark B.3 explains, the forward FD-completion of $(\mathcal{T}(S), d_e)$ is the same asymmetric metric space as the backward FD-completion of $(\mathcal{T}(S), d_e^\#)$.
- Similarly, the backward FD-completion of $(\mathcal{T}(S), d_e)$ is identical to the forward FD-completion of $(\mathcal{T}(S), d_e^\#)$.
- By Theorem 10.21, the forward FD-completion of $(\mathcal{T}(S), d_e)$, the backward FD-completion of $(\mathcal{T}(S), d_e)$, and the completion of $(\mathcal{T}(S), d_{\text{WP}})$ all admit unique homeomorphic extensions of the identity map.
- The symmetrised metrics $d_e^{(p)}$, for $p \in [1, \infty]$, are all bi-Lipschitz equivalent to each other, because $d_e^{(\infty)}(x, y) \leq \sqrt[p]{2} d_e^{(p)}(x, y) \leq \sqrt[p]{2} d_e^{(\infty)}(x, y)$. Hence, they all have homeomorphic Cauchy completions.
- Likewise, the symmetrised metrics $D_e^{(p)}$, for $p \in [1, \infty]$, are also all bi-Lipschitz equivalent to each other, because $\|v\|^{(\infty)} \leq \sqrt[p]{2} \|v\|^{(p)} \leq \sqrt[p]{2} \|v\|^{(\infty)}$. Hence, they also all induce homeomorphic Cauchy completions.

All of these things combined mean that we only need to establish unique homeomorphic extension in two cases:

- (1) between $(\mathcal{T}(S), d_e^{(\infty)})$ and $(\mathcal{T}(S), d_{\text{WP}})$, and
- (2) between $(\mathcal{T}(S), D_e^{(\infty)})$ and $(\mathcal{T}(S), d_{\text{WP}})$.

Since $d_e \leq d_e^{(\infty)} \leq C_1 d_{\text{WP}}$ and $d_e \leq D_e^{(\infty)} \leq C_1 d_{\text{WP}}$, and $d_e^{(\infty)}$ and $D_e^{(\infty)}$ are Busemannian (symmetric) metrics, the statements and proofs of Lemma 10.15, Lemma 10.17 and of Theorem 10.21 still hold with the earthquake metric supplanted by either $d_e^{(\infty)}$ or $D_e^{(\infty)}$. \square

11. COARSE COMPARISON WITH THE WEIL–PETERSSON METRIC

Although the earthquake metric is bounded above by some constant multiple of the Weil–Petersson (Proposition 7.6), the two metrics are *not* bi-Lipschitz (Proposition 7.7). Such fine-scale difference is dramatically felt as one approaches the completion locus. Nevertheless, the fact that the completions of these two metrics topologically agree suggests that the coarse geometry of Teichmüller space with respect to the earthquake metric and the Weil–Petersson metric should see no such difference. To be more precise, the following holds.

Theorem 1.31. *The identity map on Teichmüller space is a quasi-isometry between $(\mathcal{T}(S), d_e)$ and $(\mathcal{T}(S), d_{\text{WP}})$. Specifically, there are constants D_1, D_2 which depend only on the topology of S , such that*

$$\text{for any } x, y \in \mathcal{T}(S), \quad D_1 d_{\text{WP}}(x, y) - D_2 \leq d_e(x, y) \leq C d_{\text{WP}}(x, y)$$

where C is the constant from Corollary 1.19.

11.1. Pants graph.

Definition 11.1 (elementary moves for pants decompositions). Two pants decompositions P and P' of S are related by an *elementary move* if P' can be obtained from P by replacing (one isotopy class of) a simple closed curve $\alpha \in P$ by a different one α' which minimally intersects α , and keeping the other curves unchanged.

Definition 11.2 (pants graph). The *pants graph* $\mathcal{P}(S)$ is a graph whose vertices are pants decompositions and where two pants decompositions P and P' are connected by an edge if they are related by an elementary move. We metrize $\mathcal{P}(S)$ with a path metric $d_{\mathcal{P}}$ by setting the length of each edge to be 1.

Brock showed in [9] that the pants graph is a coarse combinatorial model for $(\mathcal{T}(S), d_{\text{WP}})$. We bridge the earthquake metric and the Weil–Petersson metric via the pants graph. We appeal to Brock’s work and his ideas in our arguments.

Let L be a positive constant greater than the Bers constant. Then, every hyperbolic surface $x \in \mathcal{T}(S)$ admits a pants decomposition P such that $\ell_{\alpha}(m) < L$ for every $\alpha \in P$.

Theorem 11.3 ([9], Theorem 1.1). *Let $\pi : \mathcal{T}(S) \rightarrow \mathcal{P}(S)$ be a map which sends $x \in \mathcal{T}(S)$ to any pants decomposition P satisfying $\ell_{\alpha}(m) \leq L$ for every $\alpha \in P$. The map $\pi : (\mathcal{T}(S), d_{\text{WP}}) \rightarrow (\mathcal{P}(S), d_{\mathcal{P}})$ is a quasi-isometry.*

Proving the main result of this section reduces to the following lemma.

Lemma 11.4. *For any $\pi : \mathcal{T}(S) \rightarrow \mathcal{P}(S)$ defined as above, there exist constants C' and D' such that for any $x, y \in \mathcal{T}(S)$,*

$$d_e(x, y) \geq C' d_{\mathcal{P}}(\pi(x), \pi(y)) - D'.$$

Proof of Theorem 1.31 assuming Lemma 11.4. By Corollary 1.19,

$$d_e(x, y) \leq C_1 d_{\text{WP}}(x, y) \quad \text{and} \quad d_e(y, x) \leq C_1 d_{\text{WP}}(y, x) = d_{\text{WP}}(x, y)$$

for any $x, y \in \mathcal{T}(S)$. The converse follows from Lemma 11.4 and Theorem 11.3. \square

11.2. Proof of Lemma 11.4. For each pants decomposition P , we set

$$V_P(L) := \{x \in \mathcal{T}(S) \mid \ell_\alpha(x) < L \text{ for each } \alpha \in P\}.$$

Since L is greater than the Bers constant, $\{V_P(L)\}$ is an open cover of the Teichmüller space.

Lemma 11.5 ([9], Lemma 3.3). *Given $L' > L$, there exists a constant D_2 depending on L' such that if $V_P(L') \cap V_{P'}(L') \neq \emptyset$ then $d_{\mathcal{P}}(P, P') \leq D_2$.*

Lemma 11.6. *Given $L' > L$, there exists an integer J such that for every compact path $\beta \subset \mathcal{T}(S)$ of unit length with respect to the earthquake metric, there exist pants decompositions P_1, \dots, P_J such that $\beta \subset V_{P_1}(L') \cup V_{P_2}(L') \cup \dots \cup V_{P_J}(L')$.*

Proof. By the choice of L and the compactness of β , we see that there exist pants decompositions P_1, P_2, \dots, P_k such that β is covered by the union of $V_{P_1}(L), V_{P_2}(L), \dots, V_{P_k}(L)$. Since $L' > L$, this implies that $\beta \subset \bigcup_{i=1}^k V_{P_i}(L')$. For any pants decomposition P , let $\overline{V_P(L')}$ be the closure of $V_P(L')$ in the Weil-Petersson completion. Let $\mathbf{DT}(P)$ be the subgroup of the mapping class group generated by Dehn twists around simple closed curves in P . Then the quotients $\overline{V_P(L')}/\mathbf{DT}(P)$ and $\overline{V_P(L)}/\mathbf{DT}(P)$ are compact. Since the earthquake metric is mapping class group invariant and extends to the Weil-Petersson completion (Theorem 10.18), it follows that there exists a constant $c > 0$ such that for all $P \in \mathcal{P}(S)$, we have

$$d_e(\partial V_P(L), \partial V_P(L')) \geq c \quad \text{and} \quad d_e(\partial V_P(L'), \partial V_P(L)) \geq c.$$

Therefore, each component of $\beta \cap (V_{P_i}(L') \setminus V_{P_i}(L))$ which meets $\overline{V_{P_i}(L)}$ has length at least c . Thus, if J is the smallest integer larger than $1/c$, then we may cover β by a subcovering of $\{V_{P_i}(L') \mid 1 \leq i \leq k\}$ which has at most J elements. \square

Proof of Lemma 11.4. Let x and y be arbitrary points in $\mathcal{T}(S)$. Let p be a path connecting x to y whose earthquake metric length $L(p)$ satisfies $|L(p) - d_e(x, y)| \leq 1$. Let K be the smallest integer larger than $L(p)$. We subdivide p into segments p_1, p_2, \dots, p_K , each of which has length at most one. By Lemma 11.6 there exists a collection $\{P_j \mid 1 \leq j \leq KJ\}$ of pants decompositions such that $p \subset \bigcup_{1 \leq j \leq KJ} V_{P_j}(L')$.

Combined with Lemma 11.5, this implies that $\pi(P)$ and $\pi(P')$ can be connected by a path in the pants graph with length at most $(KJ + 1)D_2$.

In particular,

$$\begin{aligned}d_{\mathcal{P}}(\pi(x), \pi(y)) &\leq D_2(KJ + 1) \\ &\leq D_2J(\ell(\sigma) + 1) + D_2 \\ &\leq D_2Jd_e(m, m') + (2J + 1)D_2.\end{aligned}$$

This completes the proof. □

APPENDIX A. TOPOLOGY OF FINSLER METRICS OF LOW REGULARITY

For an asymmetric metric space, we can define a forward topology and a backward topology, with sub-bases respectively consisting of forward and backward open balls of the metric. In general these two topologies may be different. Busemann studied these notions in Chapter 1 of [14], and gave conditions under which the forward and backward topologies coincide with the topology of the symmetrised metric $d^{\max}(x, y) = \max\{d(x, y), d(y, x)\}$.

In this appendix, after making precise what we mean by a Finsler (asymmetric) metric, we shall consider the topology of such metrics. We show that the induced topology is the same as the topology of the underlying differentiable manifold. This will show in particular that the topology induced on Teichmüller space by the earthquake metric coincides with the usual topology.

For the remainder of this appendix, we take the following setup:

- X is a C^1 -manifold;
- there is a continuous function $\|\cdot\| : TX \rightarrow [0, \infty)$ called the Finsler metric (see Definition 2.17) on X , and we denote the restriction of $\|\cdot\|$ to a tangent space $T_x X$ by $\|\cdot\|_x$; and
- d is the induced path metric of $\|\cdot\|$ on X .

Let x be an arbitrary point of X and (U, ϕ) a local chart around the point x such that $\phi(x) = 0$ and that the image of $\phi : U \rightarrow \mathbb{R}^m$ is the unit ball. Let $\|\cdot\|'$ be the pullback to TU of the Euclidean norm on \mathbb{R}^m via ϕ . Let d' be the metric induced by $\|\cdot\|'$ on $U \subset M$.

Lemma A.1. *There exists a positive constant $C = C(U, \phi) \geq 1$ such that for any y with $d'(x, y) \leq \frac{3}{4}$ and any $v \in T_y X$, we have*

$$\frac{1}{C} \leq \frac{\|v\|_y}{\|v\|'_y} \leq C.$$

In particular, $d(y, z) \leq C d'(y, z)$ for any $y, z \in U$ with $d'(x, y) \leq \frac{3}{4}$ and $d'(x, z) \leq \frac{3}{4}$.

Proof. The existence of C follows from $\|\cdot\|$ being continuous, the ratio of norms being homogeneous of degree 0, and the compactness of the unit tangent bundle over $\{y \in U \mid d'(x, y) \leq \frac{3}{4}\}$. The distance bound uses the geodesic convexity of $\{y \in U \mid d'(x, y) \leq \frac{3}{4}\}$ with respect to d' . \square

Lemma A.2. *For any $R \in (0, \frac{3}{4})$, if $d'(x, y) \geq R$, then both $d(x, y), d(y, x) \geq \frac{R}{C}$, where C is the constant from Lemma A.1.*

Proof. Consider the open subset

$$V := \{z \in U \mid d'(x, z) < R\}.$$

By Lemma A.1, for any piecewise C^1 path p connecting x to any y such that $d(x, y) \geq \frac{R}{C}$, the length $L(p)$ of p with respect to $\|\cdot\|$ and the length $L'(p)$

with respect to $\|\cdot\|'$ satisfy:

$$L(p) \geq L(p \cap V) \geq \frac{1}{C}L'(p \cap V) \geq \frac{R}{C}.$$

Consequently, $d(x, y) \geq \frac{R}{C}$. Similarly, we have $d(y, x) \geq \frac{R}{C}$. \square

From Lemma A.2, we deduce the following:

Corollary A.3. *A Finsler metric distinguishes points. In other words, for any x and y in X , we have $d(x, y) = 0$ if and only if $x = y$.*

Lemma A.4. *There exists $\epsilon_0 > 0$ such that both $\{y \in X \mid d(x, y) \leq \epsilon_0\}$ and $\{y \in X \mid d(y, x) \leq \epsilon_0\}$ are compact.*

Proof. By Lemma A.2, there exists $\delta > 0$ such that for any $y \in X$ with $d'(x, y) \geq \frac{1}{2}$, we have $d(x, y) \geq \delta$ and $d(y, x) \geq \delta$. This implies that both closed sets $\{y \in X \mid d(x, y) \leq \frac{\delta}{2}\}$ and $\{y \in X \mid d(y, x) \leq \frac{\delta}{2}\}$ are contained in the compact subset $\{y \in U \mid d'(x, y) \leq \frac{1}{2}\}$, hence are also compact. The lemma follows by letting ϵ_0 be any positive constant smaller than $\frac{\delta}{2}$. \square

Lemma A.5. *There exists a neighbourhood $V \subset U$ of x such that*

$$\frac{1}{2C}d'(y, z) \leq d(y, z) \leq Cd'(y, z)$$

holds for any $y, z \in V$, where C is the constant from Lemma A.1.

Proof. Consider the compact subset $K_0 := \{y \in X \mid d(x, y) \leq \epsilon_0\}$, where ϵ_0 is the constant from Lemma A.4. By Lemma A.1, there exists a constant C such that

$$(64) \quad d(y, z) \leq Cd'(y, z)$$

holds for any y, z in K_0 . Now we set

$$K := \{y \in X \mid d(x, y) \leq \frac{\epsilon_0}{5}, d'(x, y) \leq \frac{4\epsilon_0}{5C}\}.$$

Consider two arbitrary points y, z in K . We have

$$d(y, z) \leq Cd'(y, z) \leq \frac{2\epsilon_0}{5}.$$

Any piecewise C^1 -path p in X connecting y to z with length (with respect to d) less than $2d(y, z)$, which is at most $\frac{4\epsilon_0}{5}$, is contained in K_0 , since the distance from x of any point u on p satisfies

$$\epsilon_0 = \frac{\epsilon_0}{5} + \frac{4\epsilon_0}{5} \geq d(x, y) + d(y, u) \geq d(x, u).$$

By Lemma A.1, the length $L'(p)$ of p with respect to d' and the length $L(p)$ of the same path with respect to d satisfy $L'(p) \leq CL(p) \leq 2Cd(y, z)$. Hence,

$$(65) \quad d'(y, z) \leq L'(p) \leq 2Cd(y, z).$$

Let V be the interior of K . The lemma follows from Eq. (64) and Eq. (65). \square

Corollary A.6. *There exists a neighbourhood $V \subset U$ of x such that*

$$\frac{1}{4C^2}d(z, y) \leq d(y, z) \leq 4C^2d(z, y)$$

holds for every $y, z \in V$, where C is the constant from Lemma A.1.

Proposition A.7. *The forward topology induced by d , the backward topology induced by d , and the topology induced by various symmetrisation of d (Definition 10.22 and Definition 10.24) are all the same as the underlying topology of X as a manifold.*

Proof. This follows directly from Lemma A.5. □

Theorem A.8. *Finsler manifolds are Busemannian metric spaces.*

Proof. We need to verify that an arbitrary Finsler metric d satisfies

- for any $x, y \in X$, $d(x, y) = 0 \Leftrightarrow x = y$;
- for any $x, y, z \in X$, $d(x, z) \leq d(x, y) + d(y, z)$; and
- for any $x \in X$ and any sequence (x_n) in X , $d(x_n, x) \rightarrow 0$ if and only if $d(x, x_n) \rightarrow 0$.

The second property is a general property of path metrics, and the first and the third properties follow from the local bi-Lipschitz comparison between any Finsler metric and the Euclidean metric, as given by Lemma A.5. □

APPENDIX B. FD-COMPLETIONS OF ASYMMETRIC METRIC SPACES

In this appendix, we develop the theory of *FD-completions* of asymmetric metric spaces, a notion we defined in Section 10.1.

B.1. Metrising the FD-completion. We first show that the forward FD-completion \overline{X} may be naturally topologised/metrised as follows.

Theorem B.1 (metrising the forward FD-completion). *Let \overline{X} denote the forward FD-completion of an asymmetric metric space (X, d) . Then, d induces a function $\bar{d}: \overline{X} \times \overline{X} \rightarrow [0, \infty]$ defined by*

$$\bar{d}([x_n], [y_m]) := \inf_{(x'_n) \in [x_n], (y'_m) \in [y_m]} \liminf_{k \rightarrow \infty} d(x'_k, y'_k),$$

where the infimum is taken over all forward FD-sequences (x'_n) and (y'_m) respectively representing the forward FD-equivalence classes $[x_n]$ and $[y_m]$. The function \bar{d} defines an asymmetric metric on \overline{X} , and we refer to \bar{d} as the forward metric extension of d .

Before proving the above result, we first establish a useful lemma:

Lemma B.2. *Given $[x_n], [y_m]$, there exist forward FD-sequences (x'_n) and (y'_m) respectively representing $[x_n]$ and $[y_m]$ such that*

$$\bar{d}([x_n], [y_m]) := \inf_{(\hat{x}_n) \in [x_n], (\hat{y}_m) \in [y_m]} \liminf_{k \rightarrow \infty} d(\hat{x}_k, \hat{y}_k) = \lim_{k \rightarrow \infty} d(x'_k, y'_k).$$

Proof. Let $(x_n^1)_{n \in \mathbb{N}}, (x_n^2)_{n \in \mathbb{N}}, \dots$ and $(y_m^1)_{m \in \mathbb{N}}, (y_m^2)_{m \in \mathbb{N}}, \dots$ be two sequences of forward FD-sequences respectively representing $[x_n]$ and $[y_m]$, which realise the infimisation process for $\bar{d}([x_n], [y_m])$, i.e.

$$\bar{d}([x_n], [y_m]) = \lim_{i \rightarrow \infty} \liminf_{k \rightarrow \infty} d(x_k^i, y_k^i).$$

By a straightforward application of the triangle inequality, since subsequences of (forward) FD-sequences are also (forward) FD-sequences, we can replace each pair $(x_k^i), (y_k^i)$ by a pair of subsequences where $\liminf_{k \rightarrow \infty} d(x_k^i, y_k^i) = \lim_{k \rightarrow \infty} d(x_k^i, y_k^i)$. By possibly truncating the head of each sequence, we further ensure that

$$(66) \quad \text{for all } j \geq i, \quad \left| \lim_{k \rightarrow \infty} d(x_k^i, y_k^i) - d(x_k^j, y_k^j) \right| < \frac{1}{i}.$$

We build an “intermediate” FD-sequence (\tilde{x}_n) and (\tilde{y}_n) as follows.

- Step 1. Choose $\tilde{x}_1 = x_{n_1}^1$, $\tilde{x}_2 = x_{n_2}^2$, $\tilde{y}_1 = y_{n_1}^1$, and $\tilde{y}_2 = y_{n_2}^2$ with $n_2 > n_1 > 1$, so that
 - $d(\tilde{x}_1, \tilde{x}_2) < \frac{1}{2}$, $\sum_{k \geq n_2} d(x_k^2, x_{k+1}^2) < \frac{1}{2^2}$, and
 - $d(\tilde{y}_1, \tilde{y}_2) < \frac{1}{2}$, $\sum_{k \geq n_2} d(y_k^2, y_{k+1}^2) < \frac{1}{2^2}$.

This is always possible because (x_n^1) and (x_n^2) (resp. (y_n^1) and (y_n^2)) are FD-equivalent. We informally refer to this as *hopping* from (x_n^1) and (y_n^1) to (x_n^2) and (y_n^2) . In what follows, we always ensure that the n_k in $x_{n_k}^i$ is greater than k .

- Step 2. We hop from (x_n^2) and (y_n^2) back to (x_n^1) and (y_n^1) by choosing $\tilde{x}_3 = x_{n_3}^2$, $\tilde{x}_4 = x_{n_4}^1$, $\tilde{y}_3 = y_{n_3}^2$ and $\tilde{y}_4 = y_{n_4}^1$, with respectively increased indices $n_4 > n_3 > n_2 > n_1$, chosen so that
 - $d(\tilde{x}_3, \tilde{x}_4) < \frac{1}{2^3}$, $\sum_{k \geq n_4} d(x_k^1, x_{k+1}^1) < \frac{1}{2^4}$, and
 - $d(\tilde{y}_3, \tilde{y}_4) < \frac{1}{2^3}$ and $\sum_{k \geq n_4} d(y_k^1, y_{k+1}^1) < \frac{1}{2^4}$.
 Since $n_3 > n_2$, we see that $d(\tilde{x}_2, \tilde{x}_3) \leq \sum_{k \geq n_2} d(x_k^2, x_{k+1}^2) < \frac{1}{2^2}$. Similarly, we have $d(\tilde{y}_2, \tilde{y}_3) \leq \frac{1}{2^2}$.
- Step 3. We hop from (x_n^1) and (y_n^1) to (x_n^3) and (y_n^3) by choosing $\tilde{x}_5 = x_{n_5}^1$, $\tilde{x}_6 = x_{n_6}^3$, $\tilde{y}_5 = y_{n_5}^1$, and $\tilde{y}_6 = x_{n_6}^3$, with $n_6 > n_5 > n_4$, so that
 - $d(\tilde{x}_5, \tilde{x}_6) < \frac{1}{2^5}$, $\sum_{k \geq n_6} d(x_k^3, x_{k+1}^3) < \frac{1}{2^6}$, and
 - $d(\tilde{y}_5, \tilde{y}_6) < \frac{1}{2^5}$ and $\sum_{k \geq n_6} d(y_k^3, y_{k+1}^3) < \frac{1}{2^6}$.
 Since $n_5 > n_4$, we see that $d(\tilde{x}_4, \tilde{x}_5) \leq \sum_{k \geq n_4} d(x_k^1, x_{k+1}^1) < \frac{1}{2^4}$. Similarly, we have $d(\tilde{y}_4, \tilde{y}_5) \leq \frac{1}{2^4}$.
- Step 4. We hop from (x_n^3) and (y_n^3) back to (x_n^1) and (y_n^1) by choosing $\tilde{x}_7 = x_{n_7}^3$, $\tilde{x}_8 = x_{n_8}^1$, $\tilde{y}_7 = y_{n_7}^3$ and $\tilde{y}_8 = y_{n_8}^1$ with increased indices $n_8 > n_7 > n_6$, so that
 - $d(\tilde{x}_7, \tilde{x}_8) < \frac{1}{2^7}$, $\sum_{k \geq n_8} d(x_k^1, x_{k+1}^1) < \frac{1}{2^8}$, and
 - $d(\tilde{y}_7, \tilde{y}_8) < \frac{1}{2^7}$ and $\sum_{k \geq n_8} d(y_k^1, y_{k+1}^1) < \frac{1}{2^8}$.
 Since $n_7 > n_6$, we see that $d(\tilde{x}_6, \tilde{x}_7) \leq \sum_{k \geq n_2} d(x_k^3, x_{k+1}^3) < \frac{1}{2^6}$. Similarly, we have $d(\tilde{y}_6, \tilde{y}_7) \leq \frac{1}{2^6}$.
- We repeat the above steps ad infinitum, doing “rounds” of hops from (x_n^1) and (y_n^1) to new sequences (x_n^i) and (y_n^i) and then hopping back to (x_n^1) and (y_n^1) .

This process yields two forward FD-sequences (\tilde{x}_n) and (\tilde{y}_n) because $\sum_n^\infty d(\tilde{x}_n, \tilde{x}_{n+1}) < \sum_n \frac{1}{2^n} < \infty$ and $\sum_n^\infty d(\tilde{y}_n, \tilde{y}_{n+1}) < \sum_n \frac{1}{2^n} < \infty$.

Define (x'_i) by $x'_i = \tilde{x}_{4i+2} = x_{n_{4i+2}}^{i+2}$ and (y'_j) by $y'_j = \tilde{y}_{4j+2} = y_{m_{4j+2}}^{j+2}$. Since (x'_i) and (y'_j) are subsequences of FD-sequences, they must also be FD-sequences. Moreover, (x'_i) is FD-equivalent to (\tilde{x}_n) , which in turn shares a subsequence with (x_n^1) and is hence FD-equivalent to (x_n^1) . Thus, $(x'_i) \in [x_n]$. Likewise, $(y'_j) \in [y_m]$. Finally, since $n_{i+1} > n_i > \dots > n_1 > 1$, we have $n_i \geq i + 2$. Hence, Eq. (66) ensures that

$$\lim_{i \rightarrow \infty} d(x'_i, y'_i) = \lim_{i \rightarrow \infty} d(x_{n_{4i+2}}^{i+2}, y_{n_{4i+2}}^{i+2}) = \lim_{i \rightarrow \infty} \lim_{k \rightarrow \infty} d(x_k^{i+2}, y_k^{i+2}) = \bar{d}([x_n], [y_m]).$$

□

Proof of Theorem B.1. We verify the three axioms for asymmetric metrics. To begin with, $\bar{d}([x_n], [x_n]) = 0$ because $d(x_n, x_n) = 0$.

(i) We next show that if $\bar{d}([x_i], [y_j]) = \bar{d}([y_j], [x_i]) = 0$, then $[x_i] = [x_j]$. By Lemma B.2, there are forward FD-sequences (x_i) , (x'_i) which both represent $[x_i]$ and forward FD-sequences (y_j) , (y'_j) which both represent $[y_j]$ so that

$$(67) \quad \bar{d}([x_i], [y_j]) = \lim_{k \rightarrow \infty} d(x_k, y_k) = 0, \quad \text{and} \quad \bar{d}([y_j], [x_i]) = \lim_{k \rightarrow \infty} d(y'_k, x'_k) = 0.$$

Since (x_i) and (x'_i) (resp. (y_i) and (y'_i)) are FD-equivalent, after taking subsequences if necessary, we may assume the interlacings $x_1, x'_1, x_2, x'_2, \dots$ and $y_1, y'_1, y_2, y'_2, \dots$ are forward FD-sequences. To see this is possible, we first take an interlacing $x_{i_1}, x'_{i_2}, x_{i_3}, x'_{i_4}, \dots$ so that it is a forward FD-sequence. We next take an interlacing $y_{i_{j_1}}, y'_{i_{j_2}}, y_{i_{j_3}}, y'_{i_{j_4}}, y_{i_{j_5}}, y'_{i_{j_6}}, \dots$ of (y_{i_k}) and (y'_{i_k}) so that the interlacing is a forward FD-sequence. Finally, we replace (x_k) , (x'_k) , (y_k) , and (y'_k) by $(x_{i_{j_{2k}}})$, $(x'_{i_{j_{2k+1}}})$, $(y_{i_{j_{2k}}})$, and $(y'_{i_{j_{2k+1}}})$. The resulting sequences satisfy the desired property. Hence,

$$\lim_{k \rightarrow \infty} d(x_k, y'_k) \leq \lim_{k \rightarrow \infty} d(x_k, y_k) + d(y_k, y'_k) = 0.$$

Similarly,

$$\lim_{k \rightarrow \infty} d(y'_k, x_{k+1}) \leq \lim_{k \rightarrow \infty} d(y'_k, x'_k) + d(x'_k, x_{k+1}) = 0.$$

We now choose a sequence of increasing indices (n_k) such that

$$d(x_{n_k}, y'_{n_k}) < \frac{1}{2^k}, \quad \sum_{n_k \leq j \leq n_{k+1}} d(x_j, x_{j+1}) < \frac{1}{2^k}, \quad d(y'_{n_k}, x_{n_{k+1}}) < \frac{1}{2^k}.$$

Hence $\sum_{k=1}^{\infty} d(x_{n_k}, y'_{n_k}) < \infty$ and

$$\begin{aligned} \sum_{k=1}^{\infty} d(y'_{n_k}, x_{n_{k+1}}) &\leq \sum_{k=1}^{\infty} \left(d(y'_{n_k}, x_{n_{k+1}}) + \sum_{n_k \leq j \leq n_{k+1}} d(x_j, x_{j+1}) \right) \\ &< \sum_{k=1}^{\infty} \left(\frac{1}{2^k} + \frac{1}{2^k} \right) < \infty. \end{aligned}$$

Thus, the interlacing given by $x_{n_1}, y'_{n_1}, x_{n_2}, y'_{n_2}, \dots$ is an FD-sequence, showing that (x_i) and (y'_j) are FD-equivalent.

(ii) Finally, we show that \bar{d} satisfies the triangle inequality. Consider arbitrary elements $[x_i], [y_j], [z_k] \in \bar{X}$. We invoke Lemma B.2 to obtain the following FD-sequences:

- (x_i) and (y_j) are respective representatives for $[x_i]$ and $[y_j]$, so that $\bar{d}([x_i], [y_j]) = \lim_{i \rightarrow \infty} d(x_i, y_i)$;
- (y'_j) and (z'_k) are respective representatives for $[y_j]$ and $[z_k]$, so that $\bar{d}([y_j], [z_k]) = \lim_{i \rightarrow \infty} d(y'_i, z'_i)$.

Since (y_j) and (y'_j) are FD-equivalent, after taking subsequences if necessary, we may assume that the interlacing $y_1, y'_1, y_2, y'_2, \dots$ is a forward FD-sequence. In particular,

$$\lim_{j \rightarrow \infty} d(y_j, y'_j) = 0.$$

The triangle inequality tells us that

$$\begin{aligned} \liminf_{i \rightarrow \infty} d(x_i, z'_i) &\leq \lim_{i \rightarrow \infty} (d(x_i, y_i) + d(y_i, y'_i) + d(y'_i, z'_i)) \\ &= \bar{d}([x_i], [y_j]) + 0 + \bar{d}([y_j], [z_k]). \end{aligned}$$

By definition,

$$\bar{d}([x_i], [z_k]) \leq \liminf_{i \rightarrow \infty} d(x_i, z'_i) \leq \bar{d}([x_i], [y_j]) + \bar{d}([y_j], [z_k]), \text{ as desired.}$$

□

Remark B.3 (backward metric extension). Theorem B.1 holds *specifically* for the forward FD-completion, and there is a similar construction of a standard metric for the backward FD-completion. However, in order to obtain an analogous triangle inequality based on arguments like those in the latter steps of the proof of Theorem B.1 (as well as the latter half of Lemma B.2), we define the *backward metric extension* of d as:

$$\inf_{(x'_n) \in [x_n], (y'_m) \in [y_m]} \liminf_{k \rightarrow \infty} d(y'_k, x'_k).$$

Note in particular, that this makes the backward FD-completion of a metric space (X, d) identical to the forward FD-completion of $(X^\#, d^\#)$.

B.2. Natural inclusion. Recall from Proposition 10.12 that the map $\iota : X \rightarrow \bar{X}$ sending $x \in X$ to the FD-equivalence class represented by the constant FD-sequence $(x)_{n \in \mathbb{N}}$ naturally identifies X with a subset of \bar{X} . We show:

Proposition B.4 (dense inclusion). *The natural inclusion map $\iota : X \rightarrow \bar{X}$ is an isometric embedding from (X, d) to (\bar{X}, \bar{d}) . Moreover, $\iota(X)$ is forward dense in \bar{X} , meaning that every point in \bar{X} is the forward limit of a sequence of points $(\iota(x_n))_{n \in \mathbb{N}}$ in $\iota(X) \subset \bar{X}$.*

Proof. Given two arbitrary FD-equivalence classes $[x] = \iota(x)$ and $[y] = \iota(y)$, let (x_n) and (y_n) denote FD-sequences (obtained via Lemma B.2), which respectively represent $[x]$ and $[y]$ and satisfy

$$\bar{d}([x], [y]) = \lim_{n \rightarrow \infty} d(x_n, y_n).$$

Since the constant sequence $(x)_{n \in \mathbb{N}}$ and the sequence $(x_n)_{n \in \mathbb{N}}$ are FD-equivalent, they have an interlacing which is an FD-sequence, and hence there is a subsequence $(x_{n_k})_{k \in \mathbb{N}}$ such that

$$(68) \quad \sum_k d(x, x_{n_k}), \sum_k d(x_{n_k}, x) < \infty.$$

Replace both (x_n) and (y_n) with their respective subsequences indexed by $(n_k)_{k \in \mathbb{N}}$, and observe that Eq. (68) implies that

$$\lim_{n \rightarrow \infty} d(x, x_n) = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} d(x_n, x) = 0.$$

Since the updated (y_n) is a subsequence of the former, they are FD-equivalent, hence there is an interlacing between $(y_n)_{n \in \mathbb{N}}$ and $(y)_{n \in \mathbb{N}}$ which is an FD-sequence. We similarly find a subsequence $(y_{n_k})_{k \in \mathbb{N}}$ satisfying

$$(69) \quad \sum_k d(y, y_{n_k}), \sum_k d(y_{n_k}, y) < \infty.$$

Update both (x_n) and (y_n) again, and observe that Eq. (69) yields

$$\lim_{n \rightarrow \infty} d(y, y_n) = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} d(y_n, y) = 0.$$

Then,

$$\begin{aligned} d(x_n, y_n) &\leq d(x_n, x) + d(x, y) + d(y, y_n) \\ &\leq d(x_n, x) + d(x, x_n) + d(x_n, y_n) + d(y_n, y) + d(y, y_n). \end{aligned}$$

Taking the limit as $n \rightarrow \infty$, the sandwich theorem tells us that

$$\bar{d}([x], [y]) = \lim_{n \rightarrow \infty} d(x_n, y_n) = \lim_{n \rightarrow \infty} d(x, y) = d(x, y).$$

Thus, ι is an isometric embedding. Density is straightforward: given an arbitrary element $\xi \in \bar{X}$ represented by an FD sequence $(x_k)_{k \in \mathbb{N}}$, we have

$$\lim_{k \rightarrow \infty} \bar{d}(\iota(x_k), \xi) \leq \lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} d(x_k, x_n) \leq \lim_{k \rightarrow \infty} \sum_{j=k}^{\infty} d(x_k, x_{k+1}).$$

The right hand side is equal to 0 because $(x_k)_{k \in \mathbb{N}}$ is an FD-sequence. Thus, $\bar{d}(\iota(x_k), \xi) \rightarrow 0$, as desired. \square

Remark B.5. Analogously applying the above arguments, we obtain that $\iota^\# : X \rightarrow \bar{X}^\#$ is an isometric embedding from $(X, d^\#)$ to $(\bar{X}^\#, \bar{d}^\#)$ with backward dense image.

B.3. FD-Completeness.

Theorem B.6. *The forward FD-completion of any asymmetric metric space (X, d) is forward FD-complete. Likewise, the backward FD-completion is backward FD-complete.*

Proof. We only need to verify this for the forward FD-completion, as the backward FD-completion is equal to the forward FD-completion of the reverse metric space $(X, d^\#)$. Consider an arbitrary (forward) FD-sequence (ξ_k) in (\bar{X}, \bar{d}) . Our goal is to show that there exists $\xi \in \bar{X}$ such that $\bar{d}(\xi_k, \xi) \rightarrow 0$. We do so by constructing a specific FD-sequence in (X, d) and showing that its corresponding FD-equivalence class satisfies the desired property.

By Lemma B.2, for each $i \geq 1$, we find FD-sequences $(\hat{x}_{n,i})$ and $(x'_{n,i})$ representing ξ_i such that

$$\bar{d}(\xi_i, \xi_{i+1}) = \lim_{n \rightarrow \infty} d(\hat{x}_{n,i}, x'_{n,i+1}).$$

Since both $(\hat{x}_{n,i})$ and $(x'_{n,i})$ represent ξ_i , they are FD-equivalent. After taking subsequences, we may assume that for each $i \geq 1$ the interlacing

$$\hat{x}_{1,i}, x'_{1,i}, \hat{x}_{2,i}, x'_{2,i}, \dots$$

is a forward FD-sequence. In particular, $\lim_{n \rightarrow \infty} d(x'_{n,i}, \hat{x}'_{n+1,i}) = 0$. Hence,

$$\begin{aligned} \lim_{n \rightarrow \infty} d(\hat{x}_{n,i}, \hat{x}_{n,i+1}) &\leq \lim_{n \rightarrow \infty} (d(\hat{x}_{n,i}, x'_{n,i+1}) + d(x'_{n,i+1}, \hat{x}_{n+1,i+1})) \\ &= \lim_{n \rightarrow \infty} d(\hat{x}_{n,i}, x'_{n,i+1}) = \bar{d}(\xi_i, \xi_{i+1}). \end{aligned}$$

On the other hand, the definition of \bar{d} implies that

$$\bar{d}(\xi_i, \xi_{i+1}) \leq \lim_{n \rightarrow \infty} d(\hat{x}_{n,i}, \hat{x}_{n,i+1}).$$

Hence, for each $i \geq 1$

$$\bar{d}(\xi_i, \xi_{i+1}) = \lim_{n \rightarrow \infty} d(\hat{x}_{n,i}, \hat{x}_{n,i+1}).$$

To simplify notation, we set $x_{m,k} := \hat{x}_{m,k}$. Take a sequence of (strictly) increasing indices (m_k) satisfying

$$(70) \quad \text{for all } n \geq m_k, \quad |d(x_{n,k}, x_{n,k+1}) - \bar{d}(\xi_k, \xi_{k+1})| \leq 2^{-k},$$

and

$$(71) \quad \sum_{j=m_k}^{\infty} d(x_{j,k}, x_{j+1,k}) \leq 2^{-k}.$$

We verify that the sequence $(x_{m_k,k})_{k \in \mathbb{N}}$ is an FD-sequence. Eq. (70) and Eq. (71) tell us that

$$\begin{aligned} &d(x_{m_k,k}, x_{m_{k+1},k+1}) \\ &\leq d(x_{m_k,k}, x_{m_{k+1},k}) + \dots + d(x_{m_{k+1}-1,k}, x_{m_{k+1},k}) + d(x_{m_{k+1},k}, x_{m_{k+1},k+1}) \\ &\leq 2^{-k} + \bar{d}(\xi_k, \xi_{k+1}) + 2^{-k} = \bar{d}(\xi_k, \xi_{k+1}) + 2^{-k+1}. \end{aligned}$$

Hence,

$$\sum_{k=1}^{\infty} d(x_{m_k,k}, x_{m_{k+1},k+1}) \leq \sum_{k=1}^{\infty} (\bar{d}(\xi_k, \xi_{k+1}) + 2^{-k+1}) < \infty,$$

where we use the assumption that (ξ_k) is a forward FD-sequence in (\bar{X}, \bar{d}) .

Let $\xi \in \bar{X}$ be the FD-equivalence class for $(x_{m_k,k})$. We complete this proof by showing that $\bar{d}(\xi_k, \xi) \rightarrow 0$.

To calculate the desired limit, we observe that

$$\lim_{k \rightarrow \infty} \bar{d}(\xi_k, \xi) \leq \lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} d(x_{m_n,k}, x_{m_n,n})$$

is true because $(x_{m_n,k})$ is a subsequence of $(x_{n,k})$ for each k , and hence is FD-equivalent to $(x_{n,k})$. Finally, since $m_n > n$,

$$\begin{aligned} \lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} d(x_{m_n,k}, x_{m_n,n}) &\leq \lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} \sum_{j=k}^{n-1} d(x_{m_n,j}, x_{m_n,j+1}) \\ &\leq \lim_{k \rightarrow \infty} \sum_{j=k}^{n-1} (\bar{d}(\xi_j, \xi_{j+1}) + 2^{-j}) \quad (\text{by (70)}) \\ &= 0 \quad (\text{since } \sum_{k=1}^{\infty} \bar{d}(\xi_k, \xi_{k+1}) < \infty), \end{aligned}$$

as required. Therefore, (\bar{X}, \bar{d}) is forward FD-complete. \square

Remark B.7 (non-idempotence). Although the forward FD-completion is forward FD-complete, we emphasise that successive applications of FD-completions may indefinitely add more and more points to the metric space. Consider the following example. Let $X = \mathbb{N}$ and

$$d(m, n) = \begin{cases} \frac{1}{m} - \frac{1}{n}, & \text{if } m \leq n, \\ 1, & \text{otherwise.} \end{cases}$$

Successive FD-completions will add a string of “ ∞ ”s, which we denote by $\infty_1, \infty_2, \infty_3, \dots$ so that $d(\infty_m, \infty_n) = 0$ if $m \geq n$, but 1 otherwise. There may be ways of addressing this by, for example, applying infinitely many FD-completions. This is a delicate discussion, however, for one should take into account the purpose that such a completion should serve, and we leave this for future studies.

B.4. FD-completion generalises Cauchy completion. For the purposes of this subsection, we assume that (X, d) is symmetric. The goal is to show that FD-completion naturally generalises Cauchy completion within this restricted context. We first lay down some groundwork:

Lemma B.8. *Any FD-sequence in a symmetric metric space (X, d) is also Cauchy in (X, d) .*

Proof. Let (x_n) be an FD-sequence. For any $\epsilon > 0$, take N to be some integer such that $\sum_{k=N}^{\infty} d(x_k, x_{k+1}) < \epsilon$. Then, for any $m > n \geq N$, the distance $d(x_n, x_m)$ satisfies

$$d(x_n, x_m) \leq d(x_n, x_{n+1}) + \dots + d(x_{m-1}, x_m) \leq \sum_{k=N}^{\infty} d(x_k, x_{k+1}) < \epsilon.$$

\square

Lemma B.9. *Any Cauchy sequence in a symmetric metric space (X, d) has a subsequence which is an FD-sequence in (X, d) .*

Proof. Given a Cauchy sequence (x_n) , let $(m_k)_{k \in \mathbb{N}}$ be an increasing sequence of integers such that $\forall i, j \leq m_k, d(x_i, x_j) < 2^{-k}$. Then, $(x_{m_k})_{k \in \mathbb{N}}$ is an FD-sequence, as $\sum_{k=1}^{\infty} d(x_{m_k}, x_{m_{k+1}}) < 1$. \square

Lemma B.10. *Two FD-sequences (x_n) and (y_n) in a symmetric metric space (X, d) are FD-equivalent if and only if they are Cauchy equivalent.*

Proof. First assume that (x_n) and (y_n) are FD-equivalent. For any $\epsilon > 0$, there exists some N such that the following two inequalities hold:

$$\sum_{k=N}^{\infty} d(x_k, x_{k+1}) < \frac{\epsilon}{3} \quad \text{and} \quad \sum_{k=N}^{\infty} d(y_k, y_{k+1}) < \frac{\epsilon}{3}.$$

For any $m \geq N$, we can find some successive pair of terms x_{i_m}, y_{j_m} in an interlacing for (x_n) and (y_n) such that $i_m, j_m > m$ and $d(x_{i_m}, y_{j_m}) < \frac{\epsilon}{3}$ (since (x_n) and (y_n) are FD-equivalent). Then,

$$d(x_m, y_m) \leq \sum_{k=m}^{i_m-1} d(x_k, x_{k+1}) + d(x_{i_m}, y_{j_m}) + \sum_{k=m}^{j_m-1} d(y_{k+1}, y_k) < \epsilon.$$

Hence, $\lim_{m \rightarrow \infty} d(x_m, y_m) = 0$ and we see that (x_n) and (y_n) (which are necessarily Cauchy, by Lemma B.8) are Cauchy equivalent.

For the converse, assume that $\lim_{k \rightarrow \infty} d(x_k, y_k) = 0$. Choose an increasing sequence of indices $(m_k)_{k \in \mathbb{N}}$ such that

$$\forall n \geq m_k, \quad d(x_n, y_n) < 2^{-k}.$$

Then the sequence $x_{m_1}, y_{m_1}, y_{m_2}, x_{m_2}, x_{m_3}, y_{m_3}, y_{m_4}, x_{m_4}, \dots$ is an FD-sequence because its distance-series is bounded above by

$$\sum_{k=1}^{\infty} (d(x_k, x_{k+1}) + d(y_k, y_{k+1}) + 2^{-k}) < \infty.$$

Hence, this interlacing induces an FD-equivalence between (x_n) and (y_n) . \square

Lemma B.11. *The forward and backward FD-completions of a symmetric metric space (X, d) are canonically isometric symmetric metric spaces.*

Proof. We first note that the symmetry of d as a metric means that the set of forward and backward FD-sequences agree, which in turn means that the two completions canonically identify with each other. Then, it is clear from the definition of \bar{d} in Theorem B.1 and Remark B.3 that the two metric extensions agree and are symmetric. \square

Lemma B.11 allows us to simply refer to “the FD-completion of (X, d) ”, and so allows us to pose the following:

Theorem B.12 (FD generalises Cauchy). *For any symmetric metric space (X, d) , the identity map $\text{id} : X \rightarrow X$ uniquely extends to a continuous map between the FD-completion of (X, d) and its Cauchy completion. Moreover,*

this extension map is an isometric homeomorphism between $(\overline{X}, \overline{d})$ and the Cauchy completion.

Proof. We first introduce a little notation. We denote the FD-equivalence class of an FD-sequence $(x_n) \in \overline{X}$ by $[x_n]_{FD}$. We use $(\widehat{X}, \widehat{d})$ to denote the Cauchy completion, and write Cauchy-equivalence classes as $[x_n]_C$. Since FD-sequences are Cauchy (Lemma B.8), the obvious candidate for an extension map $\bar{\text{id}} : \overline{X} \rightarrow \widehat{X}$ is given by $\bar{\text{id}}([x_n]_{FD}) = [x_n]_C$. For this map to be well-defined, we require that FD-equivalent FD-sequences are also Cauchy equivalent, and this is precisely given by the “only if” direction in Lemma B.10. Furthermore, the “if” direction of Lemma B.10 shows that $\bar{\text{id}}$ is injective, and Lemma B.9 shows that $\bar{\text{id}}$ is surjective, as a Cauchy sequence is Cauchy equivalent to any of its subsequences. Thus, $\bar{\text{id}} : \overline{X} \rightarrow \widehat{X}$ is a well-defined bijection between these two sets.

Theorem B.1 (in conjunction with Lemma B.2, if one so wishes) shows that $\bar{d}([x_n]_{FD}, [y_n]_{FD}) = \widehat{d}([x_n]_C, [y_n]_C)$, and thus $\bar{\text{id}}$ is an isometric homeomorphism. This means in particular that $\bar{\text{id}}$ is a *continuous* extension of id . All that remains is to show that $\bar{\text{id}}$ is the unique continuous extension. Consider a continuous extension $\phi : \overline{X} \rightarrow \widehat{X}$. Take an arbitrary element $\xi = [x_n]_{FD} \in \overline{X}$, represented by an FD-sequence (x_n) . Let \mathbf{x}_k be the constant FD-sequence represented by x_k . Then, the sequence of FD-equivalence classes of constant FD-sequences $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots$ converges to ξ in the sense that

$$\lim_{k \rightarrow \infty} \bar{d}(\mathbf{x}_k, \xi) = 0.$$

Since ϕ is continuous, limits are preserved and

$$\lim_{k \rightarrow \infty} \widehat{d}(\phi(\mathbf{x}_k), \phi(\xi)) = 0.$$

On the other hand, since $[x_n]_C$ is the Cauchy equivalence class represented by the sequence $(x_n) = (\phi(x_n))$ and $\phi(\mathbf{x}_k)$ is the constant Cauchy equivalence class represented by $x_k = \phi(x_k)$, we see that

$$\lim_{k \rightarrow \infty} \widehat{d}(\phi(\mathbf{x}_k), [x_n]_C) = 0.$$

Therefore,

$$\phi(\xi) = [x_n]_C.$$

Consequently, $\phi = \bar{\text{id}}$. This completes the proof. \square

B.5. FD-extensibility. We return now to the general setting where (X, d) is an asymmetric metric space. Thanks to Theorem B.1 and Proposition B.4, FD-completion can be regarded as a process that takes in (X, d) and outputs a larger metric space $(\overline{X}, \overline{d})$. It is natural to wonder if this process can be framed more category theoretically. However, even for symmetric metric spaces, one cannot always extend a continuous map between two (symmetric) metric spaces to a continuous map between their completions. Rather, only maps which take Cauchy sequences to Cauchy sequences produce a well-defined extension. We do something analogous for FD-completion.

Definition B.13 (FD-extensibility). A map $f : (X, d_X) \rightarrow (Y, d_Y)$ is called *forward FD-extensible* if the image of every forward FD-sequence

- contains at least one subsequence which is a forward FD-sequence in (Y, d_Y) , and
- all such subsequences are forward FD-equivalent in (Y, d_Y) .

Backward FD-extensibility is defined analogously.

Proposition B.14 (standard extension). *Any FD-extensible map $f : (X, d_X) \rightarrow (Y, d_Y)$ induces a natural extension map $\bar{f} : \bar{X} \rightarrow \bar{Y}$. We refer to \bar{f} as the standard extension of f .*

Proof. Since f is FD-extensible, we specify the map \bar{f} as follows: given an FD-sequence (x_n) representing an FD-equivalence class $[x_n]$, we define $\bar{f}([x_n])$ by sending $[x_n]$ to the unique FD-equivalence class represented by FD-subsequences of $(f(x_n))_{n \in \mathbb{N}}$. This is the obvious candidate for \bar{f} . We first show that this defines a well-defined map. Consider (x_n) and (x'_n) to be FD-sequences representing the same FD-equivalence class $[x_n]$. Let $(f(x_{n_i}))_{i \in \mathbb{N}}$ be an FD-sequence representing the unique FD-equivalence class among subsequences of $(f(x_n))$ and likewise let $(f(x'_{n_j}))_{j \in \mathbb{N}}$ be an FD-sequence representing the unique FD-equivalence class among subsequences of $(f(x'_n))$. Our present goal is to show that $(f(x_{n_i}))$ and $(f(x'_{n_j}))$ are FD-equivalent. Since (x_{n_i}) and (x'_{n_j}) are subsequences of (x_n) and (x'_n) , they must be FD-equivalent, hence there is some interlacing (\hat{x}_m) of (x_{n_i}) and (x'_{n_j}) . Since f is an FD-morphism, all FD-sequences of $(f(\hat{x}_m))$ are FD-equivalent. In particular, the sequences of $(f(\hat{x}_m))$ comprised of terms in $(f(x_{n_i}))$ and $(f(x'_{n_j}))$ both have finite distance series, hence they are in the same FD-equivalence class. This means that $[f(x_{n_i})] = [f(x'_{n_j})]$, hence f is well-defined. The fact that \bar{f} is an extension is clear from the construction. \square

Given the naturality properties inherent in the definition of the standard extension, one wonders if the standard extension is “canonical” in some formal sense — in the symmetric setting, uniqueness is forced by requiring continuity. However, to attempt to phrase such a question in the full generality of asymmetric metric spaces, one needs to consider the subtleties of *which* topology to use. We skirt this issue by considering Busemannian metric spaces:

Proposition B.15. *Consider asymmetric metric spaces (X, d_X) and (Y, d_Y) whose respective FD-completions $\phi : (\bar{X}, \bar{d}_X), (\bar{Y}, \bar{d}_Y)$ are Busemannian metric spaces. (Note that this also makes (X, d_X) and (Y, d_Y) Busemannian, endowing them also with a standard topology.) Then, any FD-extensible map $f : X \rightarrow Y$ admits at most one continuous extension. Furthermore, whenever such a continuous extension exists, it is necessarily the standard extension \bar{f} .*

Proof. The argument is essentially the same as the proof of uniqueness in Theorem B.12. Let $\phi : \bar{X} \rightarrow \bar{Y}$ denote a continuous extension of $f : X \rightarrow Y$.

Given an arbitrary element $\xi \in \overline{X}$, let (x_n) be an FD-sequence representing ξ . Since f is FD-extensible, we can replace (x_n) with a subsequence so that $(f(x_n))$ is an FD-sequence in (Y, d_Y) . Let $\xi_k \in \overline{X}$ denote the FD-equivalence class represented by the constant FD sequence $[x_k]$. Then $\xi_k \rightarrow \xi$ because

$$\lim_{k \rightarrow \infty} \bar{d}_X(\xi_k, \xi) \leq \lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} d_X(x_k, x_n) \leq \lim_{k \rightarrow \infty} \sum_{j=k}^{\infty} d_X(x_k, x_{k+1}) = 0.$$

Continuous maps between Busemannian spaces preserve limits, hence

$$(72) \quad \lim_{k \rightarrow \infty} \bar{d}_Y(\phi(\xi_k), \phi(\xi)) = 0.$$

Now, since $(f(x_n))_{n \in \mathbb{N}}$ is an FD-sequence in (Y, d_Y) , its FD-equivalence class $[f(x_n)]_{n \in \mathbb{N}}$ defines an element in \overline{Y} . We observe that

$$\begin{aligned} \lim_{k \rightarrow \infty} \bar{d}_Y(\phi(\xi_k), [f(x_n)]_{n \in \mathbb{N}}) &= \lim_{k \rightarrow \infty} \bar{d}_Y([f(\xi_k)]_{n \in \mathbb{N}}, [f(x_n)]_{n \in \mathbb{N}}) \\ &\leq \lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} d_Y(f(x_k), f(x_n)) \\ &\leq \lim_{k \rightarrow \infty} \sum_{j=k}^{\infty} d_Y(f(x_j), f(x_{j+1})) = 0. \end{aligned}$$

Since $(\overline{Y}, \bar{d}_Y)$ is Busemannian, this means

$$(73) \quad \lim_{k \rightarrow \infty} \bar{d}_Y([f(x_n)]_{n \in \mathbb{N}}, \phi(\xi_k)) = 0.$$

Combining Eq. (72) and Eq. (73), we see that

$$\bar{d}_Y([f(x_n)]_{n \in \mathbb{N}}, \phi(\xi)) \leq \lim_{k \rightarrow \infty} (\bar{d}_Y([f(x_n)]_{n \in \mathbb{N}}, \phi(\xi_k)) + \bar{d}_Y(\phi(\xi_k), \phi(\xi))) = 0.$$

Again using the fact that $(\overline{Y}, \bar{d}_Y)$ is Busemannian, we conclude that the continuity of ϕ forces $\phi(\xi_k) = [f(x_n)]_{n \in \mathbb{N}}$, hence ϕ must be the standard extension. \square

Remark B.16. It is possible to weaken the conditions of the previous lemma so that $(\overline{X}, \bar{d}_X)$ is not necessarily Busemannian, and to specify continuity with respect to the backward topology on $(\overline{X}, \bar{d}_X)$.

FD-extensibility is a difficult property to verify without some stronger condition in place, such as Lipschitz continuity. Indeed, Lipschitz continuity is a very natural condition for preserving FD-sequences, hence also FD-equivalence.

Proposition B.17. *Every K -Lipschitz map $f : (X, d_X) \rightarrow (Y, d_Y)$ is FD-extensible, and induces a standard K -Lipschitz map $\bar{f} : (\overline{X}, \bar{d}_X) \rightarrow (\overline{Y}, \bar{d}_Y)$.*

Proof. Since the image of an FD-sequence under a K -Lipschitz map is an FD-sequence and all subsequences are in the same FD-equivalence class, K -Lipschitz maps are FD-extensible. Lemma B.2 then suffices to ensure that \bar{f} is K -Lipschitz. \square

B.6. Categorification of FD-completion. Proposition B.17 motivates the following categorical definition.

Definition B.18 ($\mathcal{AM}\mathcal{L}$ and the FD-completion functor). Let $\mathcal{AM}\mathcal{L}$ denote the category whose objects are asymmetric metric spaces and whose morphisms are Lipschitz maps. For $K \in [1, \infty)$, we similarly define $\mathcal{AM}\mathcal{L}_K$ to be the subcategory consisting of the same objects, and with morphisms given by K -Lipschitz maps. Note that $K \geq 1$ is needed to allow for identity morphisms.

Proposition B.19. *The FD-completion defines an endofunctor on $\mathcal{AM}\mathcal{L}$, as well as on $\mathcal{AM}\mathcal{L}_K$ for any $K \in [1, \infty)$.*

Proof. This is a corollary of Theorem B.1 and Proposition B.17. □

Remark B.20. It is tempting to wonder if one can enrich $\mathcal{AM}\mathcal{L}$ by taking a class of FD-extensible maps as morphisms. This needs a fairly involved discussion which is orthogonal to our present purpose, and deserves independent future investigation.

Question B.21. Constructions based on Remark B.7 might suggest that a naive reading of the universal property of metric completions (either in terms of isometric or uniformly continuous maps) seems to fail because pathological asymmetric metric spaces might admit isometric self-maps which are equal to the identity on a dense subset, but not everywhere. Is there a remedy for this either by adjusting the universal property or by imposing additional conditions on the class of asymmetric metric spaces considered?

Remark B.22. There are other approaches to completing asymmetric metric spaces, see [4, 7, 37, 55, 63, 64], and the closest that we have seen is Algom-Kfir's [4]. Both her construction and ours are motivated by simple geometric considerations, and are more accessible. It would be interesting to see to what extent our constructions differ, and what differences they detect in the geometry of asymmetric metric spaces.

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