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**COMPACTNESS OF THE MODULI SPACE OF THE SOLUTIONS OF THE
GRAVITATIONAL MONOPOLE EQUATIONS.
CONJECTURES ON THE WALL-CROSSING STRUCTURE**

GOURAB BHATTACHARYA

ABSTRACT. We sketched a proof of the compactness of the solution of the Gravitational Monopole equations and conjectured a precise wall-crossing formula for the Gravitational monopole invariant.

1. INTRODUCTION

We briefly sketched the compactness proof of the moduli space of the solutions of the Gravitational monopole equations in a toy case of $U(1)$ Gravitational monopoles. While giving a proof of the Donaldson's theorem in [cf.3] we did not provide the details of the compactness of the moduli space of solutions of the Gravitational monopole equations, hopefully this short note finally provides one. The Theorems are stated and proved in Theorem (2.4), Theorem (2.5).

We also conjectured two Wall crossing formulae in Conjecture (1), and Conjecture (2).

The author hopes, this short note will resolve the ambiguity of the foundation of the Gravitational Monopole equations first appeared in [cf. 1].

2. THE GRAVITATIONAL MONOPOLE EQUATION

2.1. Definition of the ordinary Gravitational Monopole. Let (X^4, g) be a Riemannian spin 4-manifold. Then the Clifford algebra bundle $\mathcal{Cl}(X^4)$ is a vector bundle over X^4 with fibre at $x \in X^4$ is the Clifford algebra $\mathcal{Cl}(T_x X)$. With respect to the metric g , one identifies (isomorphism) $\mathcal{Cl}(T_x X)$ with $\mathcal{Cl}(T_x^* X)$. Therefore, as a vector space, this is isomorphic to $\wedge T_x^* X$. Let us also assume $E \rightarrow X$ is a Clifford module bundle with a covariant derivative ∇^E . Then for each $x \in X$ there is a Clifford action $c : T_x^* X \otimes E_x \rightarrow E_x$ via $c(\alpha \otimes s) = c(\alpha)s$.

Definition 2.1. The twisted Dirac operator associated to (E, ∇^E) is the operator,

$$(2.1) \quad \not{D} := c \circ \nabla^E : C^\infty(X, E) \rightarrow C^\infty(X, E).$$

The equations we wish to consider are (sometimes we omit the mapping c and the dimension 4 for the convenience of computations),

$$(2.2) \quad \begin{aligned} \not{D}\psi &= (d + d^*)\psi = 0, \\ c(W_g^+) &= \frac{1}{4} \langle e_i \cdot e_j \psi, \psi \rangle e^i \wedge e^j, \\ \text{or, } c\left((W_g^+)_{ijkl}\right) e^i \wedge e^j &= \frac{1}{4} \langle e_k \cdot e_l \psi, \psi \rangle. \end{aligned}$$

The Weitzenböck's formula [5]: the decomposition of the Laplace-Beltami operator as a generalized Laplacian is, with the left-Clifford multiplication $\epsilon_l = ext_l - int_l$, and corresponding right Clifford action ϵ_r ,

$$(2.3) \quad (d + d^*)^2 = \Delta^{\wedge T^* X} - \sum_{ijkl} R_{ijkl} \epsilon_l^i \epsilon_l^j \epsilon_r^k \epsilon_r^l + \frac{1}{4} s.$$

Proposition 1. For $i \neq j$, the expression $\langle e_i e_j \psi(x), \psi(x) \rangle$ is purely imaginary for $x \in X$.

2.2. The Perturbed Gravitational Monopole Equations. If a spin bundle exists over X^4 , then it splits into \mathbb{S}^+ and \mathbb{S}^- , the splitting is due to internal grading of the Clifford algebra in question, actually a spin connection stabilises \mathbb{S}^\pm and induces a *gauge equivalent connection* on \sqrt{L} (described below).

The Gravitational Monopole equation can be understood in the following equivalent way in terms of the pair (ω_j^i, ψ) , where ω_j^i is a spin-connection 1-form on X^4 , and ψ is a section of \mathbb{S}^+ , the Dirac operator ∇ in (2.2) depends on the spin connection ω , and we may write ∇ as ∇_ω to show the dependence on ω . The induced connection on L will be denoted by $\widehat{\omega}_j^i$, and the curvature 2-form will be given in terms of $\widehat{\omega}$ and may be denoted by $R(\widehat{\omega})$ and the corresponding Weyl 2-form is denoted by $W(\widehat{\omega})$, and the self-dual part by $W^+(\widehat{\omega})$.

The spinor ψ can be thought of as the section of the squareroot \sqrt{L} (if it exists) of the defining line bundle L , the gauge group \mathcal{G} is defined being the automorphism group of the line-bundle \sqrt{L} (assuming it exists, otherwise we bring the $Spin^C$ -structure), hence $\mathcal{G} := Aut(\sqrt{L}) = (Map(X, U(1)))$ with Map are smooth maps valued in the unit circle group $U(1)$.

The action of \mathcal{G} on the pair (ω, ψ) is given by

$$(2.4) \quad g : (\omega, \psi) \mapsto (\omega - g^{-1}dg, g\psi).$$

and $\widehat{\omega}$ transforms as $\widehat{\omega} - 2ig^{-1}dg$.

Proposition 2. *Above action induces (2.4) induces an action of \mathcal{G} on on the solution space of (2.2).*

Proof.

$$(2.5) \quad \begin{aligned} \nabla_{\omega - g^{-1}dg}(g\psi) &= g\nabla_\omega\psi + dg \cdot \psi - dg \cdot \psi, \\ W^+(\widehat{\omega} - 2ig^{-1}dg) &= W(\widehat{\omega})^+ - 2d^+(g^{-1}dg) = W^+(\widehat{\omega}), \\ \langle e_i e_j g\psi, g\psi \rangle &= |g|^2 \langle e_i e_j \psi, \psi \rangle = \langle e_i e_j \psi, \psi \rangle. \end{aligned}$$

□

Thus we conclude from the above calculations

Corollary 1. *\mathcal{G} acts transitively on the solution space if and only if $\psi \neq 0$.*

The case $\psi = 0$ is called *reducible*. An application of Lichnerowicz formula implies

Theorem 2.2. *If the scalar curvature s of X is non-negative ($s \geq 0$) then we have $\psi = 0$.*

Proposition 3. *If the scalar curvature of X is non-negative and if moreover $b_2^+ > 0$, the scalar curvature s and the Ricci tensor R_{ij} of X vanish, then $\psi = 0$ and $\widehat{\omega}$ is flat.*

Proof. Since $s = R_{ij} = 0$ we get

$$(2.6) \quad W_{ijkl} = R_{ijkl},$$

So, as $W_\omega^+ = R_\omega^+ = 0$, implies

$$(2.7) \quad c_1(L) = \frac{1}{2\pi}[R_\omega]$$

is an integral class modulo torsion. Thus we have

$$(2.8) \quad \frac{1}{2\pi}[R_\omega] \in H^{2,-}(X, \mathbb{R}) \cap H^2(X, \mathbb{Z})/T$$

with T is the torsion subgroup of $H^2(X, \mathbb{Z})$. $H^2(X, \mathbb{Z})/T$ is a lattice inside $H^2(X; \mathbb{R})$, and $b_2^+ > 0$ implies $H^{2,-}(X; \mathbb{R})$ is a proper subspace of $H^2(X; \mathbb{R})$. For a generic choice of the metric g , $H^{2,-}(X; \mathbb{R})$ is in general position with respect to $H^2(X, \mathbb{Z})/T$ and intersect only in $R_\omega = 0$. This however mean $\widehat{\omega}$ is flat.

Corollary 2. *If $c_1(L) \notin T$ inside $H^2(X, \mathbb{Z})$ then in case of Proposition (3), $\widehat{\omega}$ is flat and $\psi = 0$.*

□

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Let $\epsilon^+ \neq 0$ be a self-dual 2-form, then the perturbed Gravitational Monopole is given in the following way,

$$(2.9) \quad \begin{aligned} \nabla \psi &= (d + d^*)\psi = 0, \\ W^+ + \mathbf{i} \epsilon^+ &= \frac{1}{4} \langle e_i \cdot e_j \psi, \psi \rangle e^i \wedge e^j, \end{aligned}$$

The perturbed functional we consider

$$(2.10) \quad \begin{aligned} \mathcal{A}(\widehat{\omega}, \psi) &= \int_X (|W^+|^2 + |\nabla \psi|^2 + \frac{s}{4} |\psi|^2) d(\text{vol}) + \\ &+ \int_X W^+ \wedge \mathbf{i} \epsilon + \int_x \frac{1}{4} \langle e_i e_j \psi, \psi \rangle e^i \wedge e^j - \mathbf{i} \epsilon^+ | d(\text{vol}) \end{aligned}$$

Now we assume X has $s \geq 0$, $R_{ij} = 0$ as in Proposition (3), then we get $\psi = 0$ and $\widehat{\omega}$ is flat, so we get

$$(2.11) \quad \left| \frac{1}{4} \langle e_i e_j \psi, \psi \rangle e^i \wedge e^j - \mathbf{i} \epsilon^+ \right| = 0$$

this however mean

$$(2.12) \quad \epsilon^+ = 0.$$

But this contradicts our assumption that $\epsilon^+ \neq 0$. Hence the reducible solutions of the perturbed Gravitational Monopole equations are of the form

$$(2.13) \quad \begin{aligned} \psi &= 0, \\ W_{\widehat{\omega}}^+ + \mathbf{i} \epsilon^+ &= 0. \end{aligned}$$

So, by a general theory due to Donaldson et al, there is a generic set of 2-forms ϵ can not be carried into elements of $H^{2,+}(X; \mathbb{R})$ of the form $[R_{\widehat{\omega}}] \in H^2(X, \mathbb{Z})$. Therefore we conclude

Proposition 4. *The perturbed Gravitational Monopole equations (2.9) on a smooth manifold X with $b_2^+ > 0$ allows no reducible solutions to (2.9) for all ϵ^+ .*

In deriving the Euler-Lagrange equations, we assume a Ricci-flat metric exists. Therefore

$$(2.14) \quad R_{ijkl} = W_{ijkl}.$$

Then we get a Euler-Lagrange equation of the form

$$(2.15) \quad \begin{aligned} \nabla^2 \psi - \frac{\mathbf{i}}{2} W^+ \cdot \psi - \frac{1}{8} \langle e_i e_j \psi, \psi \rangle e_i e_j \psi &= 0, \\ d^*(W^+ - \frac{1}{4} \langle e_i e_j \psi, \psi \rangle e^i \wedge e^j) + \frac{\mathbf{i}}{2} \text{Im} \langle \nabla \psi, e_i \psi \rangle &= 0. \end{aligned}$$

The Euler-Lagrange equation is independent of R_{ij} , so to trigger the effect of a nonzero Ricci tensor R_{ij} and the scalar curvature, the trick is to use a Bochner-Lichnerowicz-Witzenbock equation, and therefore get the following formula

$$(2.16) \quad \begin{aligned} \nabla^* \nabla \psi + \frac{s}{4} \psi + \frac{1}{4} |\psi|^2 \psi &= 0, \\ d^* W^+ + \frac{\mathbf{i}}{2} \text{Im} \langle \nabla_i \psi, \psi \rangle e^i &= 0. \end{aligned}$$

Using these we get (now we safely assume $R_{ij} \neq 0$),

Proposition 5. *For ψ in (2.9) we get*

$$(2.17) \quad \|\psi\|_{L^\infty}^2 \leq \max\{0, -\min_{x \in X} s(x)\}.$$

The affine space of connection is given by \mathfrak{M} , let $\omega_0 \in \mathfrak{M}$ is fixed, $\omega \in L^2_2(\mathfrak{M})$ the there is a gauge transformation g in the neighbourhood of the identity of \mathcal{G}^2_2 such that the 1-form satisfy the Coulomb gauge condition

$$(2.18) \quad d^*(\omega - g dg - \omega_0) = 0.$$

Lemma 2.3. $\|g_k \psi\|_{L^2_1}$, and $\|\omega_k - g_k dg_k - \omega_0\|_{L^2_1}$ are bounded.

We also have

$$(2.19) \quad \|\omega_k - \omega_0\|_{L^2} \leq \|\omega_k - \omega_0\|_{L^2_1}.$$

The Weyl 2-form is

$$(2.20) \quad W_{ij} = \Omega_{ij} - \frac{1}{2}(R_{ii} + R_{jj})\theta^i \wedge \theta^j + \frac{1}{6}s \theta^i \wedge \theta^j.$$

where Ω is the curvature 2-form for the Riemannian metric

$$(2.21) \quad ds^2 = (\theta^0)^2 + \dots + (\theta^3)^2.$$

If $R_{ij} = s = 0$ in particular, then

$$(2.22) \quad W_{ij} = \Omega_{ij}$$

We also assume

$$(2.23) \quad dW_{\widehat{\omega}_k} = 0,$$

this however means

$$(2.24) \quad d\Omega = 0.$$

If we restrict Ω to $U(1)$ case, we get

$$(2.25) \quad \Omega_{\widehat{\omega}_k} = d\widehat{\omega}_k, \quad d\Omega_{\widehat{\omega}_k} = 0.$$

A choice of connection 1-forms $\widehat{\omega}_k - \widehat{\omega}_0$ has been made such that it satisfies the Coloumb gauge condition

$$(2.26) \quad d^*(\widehat{\omega}_k - \widehat{\omega}_0) = 0, \quad d^*\Omega_{\widehat{\omega}_k} = (d + d^*)\Omega_{\widehat{\omega}_k}$$

and

$$(2.27) \quad d^*d\widehat{\omega}_k = (d^*d + dd^*)\widehat{\omega}_k,$$

with $d + d^*$ and $d^*d + dd^*$ are elliptic operators. We use it to bound the higher Sobolev norms. Use (2.15) and

$$(2.28) \quad *d * W = *dW^+ - *W^-, \quad \& \quad *dW = *dW^+ *dW^- = 0.$$

to get (we triggered back the Ricci tensor and the scalar curvature, i.e., $R_{ij} \neq 0$ and $s \neq$, as they don't appear below explicitly)

$$(2.29) \quad d^*W_{\widehat{\omega}_k} = 2d^*W_{\widehat{\omega}_k}^+ = -\mathbf{i} \sum_j \text{Im}(\langle \nabla_j \psi_k, \psi_k \rangle) e^j$$

Hence a bound on $\|d^*W_{\widehat{\omega}_k}\|_{L^2}$ and so a bound on $\|d^*d\widehat{\omega}_k\|_{L^2}$.

Thus

$$(2.30) \quad \|\widehat{\omega}_k\|_{L^2_2} \leq c(\|d^*W_{\widehat{\omega}_k}\|_{L^2} + \|d^*d\widehat{\omega}_k\|_{L^2}),$$

and therefore it allows us to deliver the similar for higher order Sobolev norms. Thus we record the following theorem,

Theorem 2.4. *A sequence of solutions $\{(\omega_k, \psi_k)\} \subset L^2_1(\mathbb{S}^2) \times L^2_1(\mathfrak{M})$ of (2.9) is given, then there is a subsequence $\{(\omega_{k'}, \psi_{k'})\}$ and a sequence of gauge transformations $\{g_{k'}\} \subset \mathcal{G}^2_2$ such that the sequence $\{(\omega_{k'} - g_{k'} dg_{k'}, g_{k'} \psi_{k'})\}$ converges to a smooth solution (ω, ψ) of (2.9).*

We actually proved a compactness theorem for the moduli space of the solutions

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Theorem 2.5. *Every sequence of points of the space of solutions of the Gravitational Monopole equations modulo the action of the gauge group (called a moduli space of solutions) has a convergent subsequence. Therefore the moduli space is compact.*

2.3. The dimension of the moduli space of solutions. The dimension of the moduli space is calculated in a previous publications [cf.2],[cf.3], (for simplicity, we assume \sqrt{L} a line bundle over X),

Theorem 2.6. *The moduli space of solutions of the Gravitational monopole equation at a regular point has dimension (assuming the index $\sigma(X) \geq 0$)*

$$(2.31) \quad \begin{aligned} & \frac{1}{2}(29\sigma(X) - 15\chi(X)) + c_1(\sqrt{L})^2 - \frac{\sigma}{4} \\ &= c_1(\sqrt{L})^2 + \frac{57\sigma(X) - 30\chi(X)}{4} \\ &= \frac{1}{4} \left[c_1(L)^2 + 57\sigma(X) - 30\chi(X) \right] \end{aligned}$$

This is an infinitesimal result. We denote the dimension by [cf.3]

$$(2.32) \quad d := \frac{1}{4} \left[c_1(L)^2 + 57\sigma(X) - 30\chi(X) \right].$$

2.4. Orientation. We denote the moduli space as M , its embedding in \mathcal{A}/\mathcal{G} with $\psi \neq 0$ is given as in Donaldson theory.

Theorem 2.7. *Let X satisfies $H^1(X; \mathbb{Z}) = 0$. Then the moduli space is orientable.*

Proof. Proof as in Donaldson theory. Actually we use the fibration

$$(2.33) \quad \mathcal{G} \rightarrow \mathcal{A} \rightarrow \mathcal{A}/\mathcal{G}.$$

The subspace of all (ω, ψ) is contractible being product of affine and vector spaces, therefore $\psi \neq 0$, restricts to $\pi_1(\mathcal{A}) = 0$. Therefore

$$(2.34) \quad \cdots \rightarrow \pi_1(\mathcal{G}) \rightarrow \pi_1(\mathcal{A}) \rightarrow \pi_1(\mathcal{A}/\mathcal{G}) \rightarrow \pi_0(\mathcal{G}) \rightarrow \pi_0(\mathcal{A});$$

According to supposition

$$(2.35) \quad \pi_0(\mathcal{G}) = H^1(X, \mathbb{Z}) = 0.$$

Thus we conclude

$$(2.36) \quad \pi_1(\mathcal{A}/\mathcal{G}) = 0.$$

□

2.5. The Gravitational Monopole Invariant.

Definition 2.8. The Gravitational monopole invariant for a choice of a spin structure $\mathfrak{s} \in \mathbb{S}^+$ such that d is positive and even then the invariant $N_{\mathfrak{s}}$ of the moduli space M is given by

$$(2.37) \quad N_{\mathfrak{s}} = \int_M c_1(L)^{\frac{d}{2}}.$$

When d is odd, we set $N_{\mathfrak{s}} = 0$.

Definition 2.9 (Basic class). We call $c_1(L)$ the Gravitational monopole *basic class* that gives non-zero $N_{\mathfrak{s}}$.

3. CONJECTURAL WALL CROSSING STRUCTURE

Theorem 3.1. *The invariants $N_{\mathfrak{s}}(X)$ are diffeomorphism invariants and independent of the ambient metric on X with $b_2^+(X) > 1$.*

Proof. Choose two Riemannian metrics η_0 and η_1 on X with ϵ_0 and ϵ_1 are corresponding perturbations. Let η_t , $0 \leq t \leq 1$ be a smooth curve connecting η_0 and η_1 . Similarly construct ϵ_t , $0 \leq t \leq 1$ be a smooth path of 2-forms. The assumption $b_2^+(X) > 1$ implies the above mentioned paths avoid the reducible solution $\psi = 0$ for a fixed t .

Let us form a bundle $\mathcal{A}/\mathcal{G} \times \{(\eta_t, \epsilon_t)\}$ (infinite dimensional) over X with fibre $\Lambda^{2,+}(X) \oplus \Gamma(X, \mathbb{S}^+)$, section

$$(3.1) \quad f(\omega, \psi) = \left(W^+ + \mathbf{i}\epsilon_t - \frac{1}{4} \langle e_i e_j \psi, \psi \rangle e^i \wedge e^j, \not{D}\psi \right).$$

Define a *universal moduli space* $\mathcal{M}_{\mathfrak{s}} = f^{-1}(0)$. We will use Atiyah-Singer index theorem and transversality of f to the non-reducible locus of $\mathcal{M}_{\mathfrak{s}}$ (we chose $\{(\eta_t, \epsilon_t)\}$ so that it avoids $\psi = 0$). The transversality result we will use is the following: First choose a Fredholm projection $\pi : \mathcal{M}_{\mathfrak{s}} \rightarrow \{(\eta_t, \epsilon_t)\}$, such that the moduli space $\pi^{-1}(\eta_t, \epsilon_t) = M_{\mathfrak{s}, \eta_t, \epsilon_t}$ is a smooth manifold of co-dimension 1 in $\mathcal{M}_{\mathfrak{s}}$.

If for a choice of (η_0, ϵ_0) and (η_1, ϵ_1) we get zero dimensional moduli spaces $M_{\mathfrak{s}, \eta_0, \epsilon_0}$, and $M_{\mathfrak{s}, \eta_1, \epsilon_1}$, then $\mathcal{M}_{\mathfrak{s}}$ is a smooth one-dimensional manifold with boundary. Therefore $\mathcal{M}_{\mathfrak{s}}$ is compact with total oriented boundary vanishes.

Now assume $\dim M_{\mathfrak{s}} = d > 0$, then

$$(3.2) \quad \dim \mathcal{M}_{\mathfrak{s}} = \dim M_{\mathfrak{s}} + 1$$

with oriented boundary

$$(3.3) \quad \partial \mathcal{M}_{\mathfrak{s}} = M_{\mathfrak{s}}(\eta_1, \epsilon_1) - M_{\mathfrak{s}}(\eta_0, \epsilon_0),$$

with invariants

$$(3.4) \quad \int_{M_{\mathfrak{s}}(\eta_1, \epsilon_1)} c_1(L)^{\frac{d}{2}} = \int_{M_{\mathfrak{s}}(\eta_0, \epsilon_0)} c_1(L)^{\frac{d}{2}}.$$

□

For the case of $b_2^+ = 1$ we conjecture a Wall-crossing formula that characterise how the invariant $N_{\mathfrak{s}}$ makes a jump along a reducible solution $\psi = 0$ with respect to a metric and corresponding perturbation (η, ϵ) .

The assumption is the following: the moduli space of solutions are devoid of $\psi = 0$ solutions or the reducible solutions and choose above scenerio but with a "wall"

$$(3.5) \quad H = \{(\eta, \epsilon) | H^{2-}(X, \mathbb{R}) \cap (\epsilon + H^2(X, \mathbb{Z})/T \neq \emptyset\}.$$

The path (η_t, ϵ_t) crosses the wall H transversely only once.

Conjecture 1. *Let X be a smooth, compact 4-manifold with $b_2^+(X) = 1$ and $b_1 = 0$, then*

$$(3.6) \quad N_{\mathfrak{s}}(X(\eta_0, \epsilon_0)) - N_{\mathfrak{s}}(X(\eta_1, \epsilon_1)) = \pm 1.$$

Conjecture 2. *Let X be a smooth, compact 4-manifold with $b_2^+(X) = 1$ and $b_1 \in \mathbb{Z}$, then*

$$(3.7) \quad N_{\mathfrak{s}}(X(\eta_0, \epsilon_0)) - N_{\mathfrak{s}}(X(\eta_1, \epsilon_1)) = \varphi(c_1(\tau), c_1(L))$$

with τ is a tautological line bundle over $\mathbb{T}^{b_1} \times X$, \mathbb{T}^{b_1} is a torus, φ is a Laurent polynomial.

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